Topological Field Theory and Quantum Master Equation in Two Dimensions

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Hao Yu

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Alexander Voronov

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Dedication

Dedicated to my girlfriend, my parents and all who help me in the last years during my Phd study.

Abstract

In our thesis, I give the analogy of the main results in Kevin Costello's paper [Cos07] for open-closed topological conformal field theory. In other words, I show that there is a Batalin-Vilkovisky algebraic structure on the open-closed moduli space (moduli space of Riemann surface with boundary and marked points) , which is defined by Harrelson, Voronov and Zuniga [HVZ07], and the most important, there is a solution up to homotopy to the quantum master equation of that BV algebra, if the initial condition is given, under the assumption that a new geometric chain theory gives rise to ordinary homology. This solution is hoped to encode the fundamental chain of compactified open-closed moduli space, studied thoroughly by C.-C.Liu, as exactly in the closed case (Deligne-Mumford space in this case). We hope this result can give new insights to the mysterious two dimensional open-closed field theory.

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Chapter 1

Introduction

1.1 Background and Motivation

String theory is currently a candidate for the unified theory or theory of everything (TOE) in theoretical physics. The aim of physicists is to unify all four interactions in nature: electro-magnetic interaction, weak interaction, strong interaction and gravitational interaction, in a single. From the 19th century, quantum field theory has experienced rapid and mysterious development. In the last decades, string theory and quantum field theory have been found to have a mysterious interaction with pure mathematics. A few examples like, Donaldson's theory of four manifolds, Jone's polynomial in knot theory, Gromov-Witten theory,...and even morse homology, are all found to be explainable in the language of quantum field theory (or string theory). This kind of interaction has produced already very fruitful mathematical achievement. From Seiberg-Witten invariant, quantum cohomology to elliptic cohomology, and the geometric Langlands program, mathematics has gained much benefit from this interaction. It is certainly sure that this will continue in the future.

Mirror symmetry is a remarkable physical phenomenon that was found, roughly speaking, in late 1980s.([Dix87],[LVW89]) I briefly talk about the physical origin here.It is mainly based on the observation that a certain kind of quantum field theory, $N = 2$ supersymmetric quantum field theory, has a sign indeterminacy, which suggests that there is a kind of dual theory on a different target space. It is currently called mirror symmetry.

Mirror symmetry was brought to the attention of mathematics world due to a remarkable prediction by physicists of the number of rational curves on a quintic three fold based on the mirror symmetry principle. Now, there are several versions of mathematical mirror symmetry. One of them refers to the equivalence of A model and B model on mirror pairs. A model and B model are two kinds of topological string theory ([Wit91]). A model was mathematically constructed as a theory of Gromov-Witten invariants (see [LT96],[LT98],[Sie96],[Beh97]), but B model, its mirror partner, is still not well established.([Bar99],[Bar00]).

In a sequence of papers [Cos04],[Cos07], Kevin Costello gave an effective approach to construct B model in mathematics. In [Cos04], he shows that if the Homological Mirror Conjecture is true, then A model and B model are equivalent as a corollary of that. Because of this, he gave a construction of something similar to GW invariants out of a categorical data. When applying this to the derived category of coherent sheaves on a Calabi-Yau manifold, it will give the B model for all genus. (The equivalence of his construction for A model to Gromov-Witten invariant is still conjectural). In a subsequent paper [Cos07], Costello further gives construction for all topological conformal field theory(TCFT, for short) something like a Gromov-Witten potential, and also B model potential after applying to TCFT on the B model side. In this thesis, we plan to apply his ideas for general open-closed version of TCFT. This is closely related to open string theory in physics, just as TCFT could be seen as a special kind of topological closed string theory. We want to give similar results and hopefully to pave the way to the better understanding of this subject.

The method used in Costello's paper is to use the BV formalism of physicists Sen and Zwiebach [SZ94],[SZ96], and also investigated by Sullivan [Sul05]. The background is, in GW theory we have cohomology classes on Deligne-Mumford space, GW potential is defined using ψ classes and fundamental class of Deligne-Mumford space. In a TCFT, it gives cochains on (uncompactified) moduli space of complex algebraic curves, if we can extend the operations to the boundary of DM space, then we can define Gromov-Witten potential assoicated to the TCFT in the usual way. But generally we don't know how to do it(Kontsevich has an approach but it will depend on a choice). The formalism of Sen and Zwiebach provides another way. It says there is a BV algebraic structure on the chain complex of moduli space of Riemann surface with punctures, and a solution (they called it "string vertex") to the quantum master equation. Costello rigorously proves this in mathematics and shows there is a solution, up to homotopy, to the master equation. Further, different solutions differ by a BV operator exact term. This "fundamental solution" encodes the structure of fundamental class of Deligne-Mumford space. It also shows that the "fundamental solution" could be mapped to be a ray in the Fock space associated to a periodic cyclic homology which is a symplectic vector space. So it defines the Gromov-Witten potential to be a certain ray in the Fock space, which is a part of the quantum field theory construction.

In our thesis, we find the above construction could be generalized to open-closed TCFT. Because similarly, there is a BV algebraic structure on the moduli space of Riemann surfaces with boundary and punctures also due to Sen and Zwiebach [Zwi98]. We develop the relevant techniques, and show that the some similar results hold in this case. That is, there exists a solution of the quantum master equation coming from moduli space of (uncompactified) Riemann surface with open-closed boundary, and this solution is unique up to homotopy if the initial condition is fixed.

1.2 Organization

The organization of this thesis is as follows.

In Chapter 2, I will review the basics of category theory , this is, of course, the language to formulate functorial topological field theory (which is a kind of axiomatic quantum field theory) proposed by Segal and then Atiyah (see [Seg99],[Seg04],[Ati88] for history and details). And also, it will be used to formulate BV structure for general functors. In this chapter, after introducing some basic concepts, we mainly then discuss symmetric monoidal category and the related functors, where our theory lies in. Then, in Chapter 3, I will give some results on homological algebra, emphasis will be given on the homological property of modules over a differential graded symmetric monoidal category. After these two chapters of preparations , we will give detailed definitions of closed, open and open-closed conformal field theories and their linearized version, topological conformal field theory in Chapter 5. Chapter 4 is one of the main technic results, we give a homological dimension estimate for moduli space of Riemann surface with boundary (and marked points), which is essential in the proof of proposition 9.0.1; After this, I will introduce BV algebra, a new algebraic structure arising originally in theoretical physics, as well as quantum master equation. Then, in Chapter 7, I will introduce BV algebra structure on the geometric chain complex associated to the moduli space of Riemann surfaces with boundary and marked points. It is defined in Harrelson, Voronov and Zuniga's paper [HVZ07], but in a certain twisted coefficients. Chapter 8 mainly treats Weyl algebra and Fock space, in [Cos07] it shows that the partition function of closed TCFT (topological conformal field theory) is a ray in the Fock space for periodic cyclic homology. (Similar thing is expected to hold in open-closed field theory). Finally for Chapter 9 l show there is a unique solution, up to homotopy, to the quantum master equation associated to the BV structure in chapter 7. We call these solutions fundamental chains. Some further questions in this direction are pointed out.

Chapter 2

Category theory

We need the language of category theory to define topological (conformal) field theory, so in this section I will give basics of category theory, especially the definitions of differential graded symmetric monoidal categories as well as, the functors between them.

2.1 Category

"Category" was first introduced by Samuel Eilenberg and Saunders Mac Lane (in order to develop the axiomatic homology theory, (see [SM])). Nowadays, it has became a basic tool and language in mathematics.

Definition 2.1.1. A **category** consists of the following entities:

• A class $ob(C)$, whose elements are called *objects*;

• A class $hom(C)$, whose elements are called *morphisms* or *map* or *arrows*. Each morphism f has a unique *source object* a and *target object* b. We write $f : a \to b$, and we say "f is a morphism from a to b". We write $hom(a, b)$ (or $Hom(a, b)$, or $hom_C(a, b)$, or $Mor(a, b)$, or $C(a, b)$) to denote the *hom-class* of all morphisms from a to b.

• A binary operation $\circ$, called *composition of morphisms*, such that for any three objects a,b, and c, we have $hom(a, b) \times hom(b, c) \rightarrow hom(a, c)$. The composition of $f : a \rightarrow b$ and $g : b \rightarrow c$ is written as $g \circ f$ or gf, governed by two axioms:

• Associativity: If $f : a \rightarrow b, g : b \rightarrow c$ and $h : c \rightarrow d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and

• Identity: For every object x, there exists a morphism $1x : x \rightarrow x$ called the identity morphism for x, such that for every morphism $f : a \rightarrow b$, we have $1_b \circ f = f = f \circ 1_a$.

Note: it can be proved that, using above axioms, there is exactly one identity morphism for each object.

Example.

- The category of Set, denoted **Set** , whose objects are sets and morphisms are maps between sets
- The category of groups, denoted **Group**, whose objects are groups and morphisms are group homomophisms.
- The category of R modules, denoted **R-Module**, whose objects are R modules and morphisms are module homomorphisms.
- The category of topological spaces, denoted **Top**, whose objects are topological spaces and morphisms are continuous maps between two topological spaces

We have various types of morphisms, which abstract the corresponding properties we used in set theory and abstract algebra,etc.

Definition 2.1.2. (properties of morphism) A morphism $f : a \rightarrow b$ is:
• a *monomorphism* if $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : x \rightarrow a$.
• an *epimorphism* if $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2 : b \rightarrow x$.
• an *isomorphism* if there exists a morphism $g : b \rightarrow a$ with $f \circ g = 1b$ and $g \circ f = 1a$
• an *endomorphism* if $a = b$. $end(a)$ denotes the class of endomorphisms of a.

• an *automorphism* if f is both an endomorphism and an isomorphism. $aut(a)$ denotes the class of automorphisms of a.

Some other concepts which will be used are:

• *(Initial Object)* An initial object of a category **C** is an object I in **C** such that for every object X in C, there exists precisely one morphism $I \to X$.

• *(Terminal Object)* A terminal object of a category **C** is an object T in **C** such that for every object x in C, there exists precisely one morphism $X \to I$.

• *(Zero Object)* A zero object of a category is an object which is both initial object and terminal object.

• *(Zero Morphism)* For any two objects X and Y in **C**, there is a fixed morphism $0_{XY} : XY$ such that for all objects X, Y, Z in C and all morphisms $f : YZ, g : XY$, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "0_{XY}"] \arrow[d, "g"'] \arrow[dr, "0_{XZ}"] & Y \arrow[d, "f"] \\ Y \arrow[r, "0_{YZ}"'] & Z \end{tikzcd}$$

Then 0_{XY} is called zero morphism between X and Y. Also C is called the category with zero morphisms.

• *(Kernel)* Given a morphism $f : A \to B$ in a category with zero morphisms C, a *kernel* of f is a morphism $i : X \to A$ such that • $fi = 0$. • For any other morphism $j : X' \to A$ such that $fj = 0$ there exists a unique morphism $j' : X' \to X$ such that the diagram

$$\begin{tikzcd} & X' \arrow[dl, dashed, "j'"'] \arrow[d, "j"] & \\ X \arrow[r, "i"] & A \arrow[r, "f"] & B \end{tikzcd}$$

commutes.

• *(Cokerneal)* Like kernel, a *cokernal* for a given morphism $f : A \to B$ in a category with zero morphisms C is a morphism $p : B \to Y$, such that:

• $pf = 0$

• For any other morphism $j : B \to Y'$ such that $jf = 0$ there exists a unique morphism $j' : Y \to Y'$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "p"] \arrow[d, "j"'] & Y \arrow[dl, dashed, "j'"] \\
& Y' &
\end{tikzcd}$$

commutes.

• *(Pullback)* The pullback of the morphisms f and g consists of an object P and two morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ for which the diagram

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

commutes. Moreover, the pullback (P, p_1, p_2) must be *universal* with respect to this diagram. That is, for any other such triple (Q, q_1, q_2) for which the following diagram commutes, there must exist a unique $\mu : Q \to P$ such that the following diagram commutes:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "u"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

• *(Pushout)* The pushout of the morphisms f and g consists of an object P and two morphisms $i_1 : X \to P$ and $i_2 : Y \to P$ for which the following diagram commutes:

$$\begin{tikzcd}
P & Y \arrow[l, "i_2"'] \\
X \arrow[u, "i_1"] & Z \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

Moreover, the pushout (P, i_1, i_2) must be universal with respect to this diagram. That is, for any other such set (Q, j_1, j_2) for which the following diagram commutes, there must exist a unique $\mu : P \to Q$ such that the following diagram commutes:

$$\begin{tikzcd}
Q & & \\
& P \arrow[ul, dashed, "u"] & Y \arrow[l, "i_2"'] \arrow[ull, bend right, "j_2"'] \\
& X \arrow[u, "i_1"] \arrow[uul, bend left, "j_1"] & Z \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

We give examples illustrating the above concepts.

Example. • Empty set is the unique initial object in the category of set; any one-element set is the terminal object in this category; there are no zero object.

• Empty space is the unique initial object in the category of topological space; any one-point space is the terminal object in this category; there are no zero object.

• In the category of non-empty set, there are no initial objects; the one-element set is not the initial object: while every non-empty set admits a function from it, this function is in general not unique.

• In the category of groups, any trivial group is a zero object. The same is true for the categories of abelian groups, modules over a ring, and vector spaces over a field. This is the origin of the term "zero object"

• Kernels(and cokernels) are familiar in many categories from abstract algebra, such as the category of groups or the category of (left) modules over a fixed ring (including vector spaces over a fixed field). To be explicit, if $f : XY$ is a homomorphism in one of these categories, and K(respectively, L) is its kernel (cokernel) in the usual algebraic sense, then K is a subalgebra of X(L is the quotient algebra of Y) and the inclusion homomorphism from K to$X$(quotient homomorphism from Y to L) is a kernel (cokernel) in the categorical sense.

• In the category of sets, a pullback of f and g is given by the set: $X \times_Z Y = \{(x, y) \in X \times Y | f(x) = g(y)\}$ together with the restrictions of the projection maps π_1 and π_2 to $X \times_Z Y$

• If X, Y, Z are topological spaces, and $f : X \to Z$, $Y \to Z$ are inclusions, then the pullback of f and g is the topological space $Im(X) \cap Im(Y)$ together with maps to X Y induced from f and g..

• Suppose that X and Y are sets. Then if we write Z for their intersection, there are morphisms $f : ZX$ and $g : ZY$ given by inclusion. The pushout of f and g is the union of X and Y together with the inclusion morphisms from X and Y.

Definition 2.1.3. (functor) A **covariant** *functor* from a category C to a category D, written $F : C \to D$, consists of:

- for each object x in C, an object $F(x)$ in D; and
- for each morphism $f : x \to y$ in C, a morphism $F(f) : F(x) \to F(y)$,

A **contravariant** *functor* has a similar properties except it "turns around the arrow". More specifically, every morphism $f : x \to y$ in C must be assigned to a morphism $F(f) : F(y) \to F(x)$ in D.

The examples of functors are:

Example. • Constant functor: The functor $C \to D$ which maps every object of C to a fixed object X in D and every morphism in C to the identity morphism on X. Such a functor is called a constant or selection functor.

• Dual vector space: The map which assigns to every vector space its dual space and to every linear map its dual or transpose is a contravariant functor from the category of all vector spaces over a fixed field to itself.

• Fundamental group: Assign a topological space (with based point) to its fundamental group is a functor from the category of topological space to the category of group.

• Algebra of continuous functions: a contravariant functor from the category of topological spaces (with continuous maps as morphisms) to the category of real associative algebras is given by assigning to every topological space X the algebra $C(X)$ of all real-valued continuous functions on that space. Every continuous map $f : XY$ induces an algebra homomorphism $C(f) : C(Y)C(X)$ by the rule $C(f)(\varphi) = \varphi \circ f$ for every φ in $C(Y)$.

• Tangent and cotangent bundles: The map which sends every differentiable manifold to its tangent bundle and every smooth map to its derivative is a covariant functor from the category of differentiable manifolds to the category of vector bundles. Likewise, the map which sends every differentiable manifold to its cotangent bundle and every smooth map to its pullback is a contravariant functor.

There is another very important concept that relates two functors, telling how one transforms to another one. It is called a natural transformation.

Definition 2.1.4. (Natural transformation) If F and G are (covariant) functors between the categories C and D, then a natural transformation η from F to G that associates to every object x in C a morphism $\eta_x : F(x) \to G(x)$ in D such that for every morphism $f : x \to y$ in C, we have $\eta_y \circ F(f) = G(f) \circ \eta_x$; this means that the following diagram is commutative:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

The two functors F and G are called *naturally isomorphic* if there exists a natural transformation from F to G such that η_x is an isomorphism for every object x in C.

Example. If K is a field, then for every vector space V over K we have a "natural" injective linear map $V \to V^{**}$ from the vector space into its double dual. These maps are "natural" in the following sense: the double dual operation is a functor, and the maps are the components of a natural transformation from the identity functor to the double dual functor.

Remark: If we consider the category whose objects are functors between category C and D, and morphisms are natural transformation, then we call it a functor category . The natural isomorphism is the isomorphism in the functor category.

Finally, we give the definition of the equivalence of two categories.

Definition 2.1.5. (equivalence of categories) Let C and D be two categories, an *equivalence* of categories consists of functors $F : C \to D$, $G : D \to C$ and two natural isomorphism $\varepsilon : FG \to I_C$, $\eta : I_D \to GF$. Here $\varepsilon : FG \to I_C$, $\eta : I_D \to GF$, denote respective composition of F and G, and $I_C : C \to C$, $I_D : D \to D$ denote the *identity functor* of C and D, assigning each object and morphism to itself.

One can show that a functor $F : C \to D$ yields an equivalence of categories if and only if it is:
• full, i.e. for any two objects c_1 and c_2 of C, the map $\mathrm{Hom}C(c_1, c_2) \to \mathrm{Hom}D(Fc_1, Fc_2)$ induced by F is surjective;
• faithful, i.e. for any two objects c_1 and c_2 of C, the map $\mathrm{Hom}C(c1, c2) \to \mathrm{Hom}D(Fc_1, Fc_2)$ induced by F is injective; and
• essentially surjective, i.e. each object d in D is isomorphic to an object of the form Fc, for c in C.
This is a quite useful and commonly applied criterion, because one does not have to explicitly construct the "inverse" G and the natural isomorphisms between FG, GF and the identity functors.

Example. • The category C of finite-dimensional real vector spaces and the category D of all real matricesare equivalent.
• In algebraic geometry, the category of affine schemes and the category of commutative

rings are equivalent.

• In functional analysis, the category of commutative C^*-algebras with identity is contravariantly equivalent to the category of compact Hausdorff spaces.

2.2 Symmetric monoidal category

A **symmetric monoidal category** is a category modelled on the category of R modules with tensor product. Besides the usual category structure, it has a binary operator, called tensor product, with properties to the tensor product of R modules. Or we can say it is a categorification of tensor product.

Definition 2.2.1. A **monoidal category** is a category $\mathbf{C}$ equipped with

- bifunctor $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ called the *tensor product* or *monoidal product*,

- an object $\mathbf{I}$ called the *unit object* or *identity object*,

- three natural isomorphisms subject to certain coherence conditions expressing the fact that the tensor operation

  • is associative: there is a natural isomorphism α, called *associator*, with components $\alpha_{A,B,C} \colon (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$

  • has I as left and right identity: there are two natural isomorphisms λ and ρ, respectively called *left* and *right unitor*, with components $\lambda_A \colon I \otimes A \cong A$ and $\rho_A \colon A \otimes I \cong A$.

The coherence conditions for these natural transformations are:

• for all A, B, C and D in $\mathbf{C}$, the diagram

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "\alpha_{A,B,C \otimes D}"] \arrow[d, "\alpha_{A \otimes B,C,D}"'] & (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B \otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[d, "A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C \otimes D}"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

commutes;

• for all A and B in C, the diagram

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes B"'] & & A \otimes (I \otimes B) \arrow[dl, "A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

commutes;

Example. • The category of sets is a monoidal category with the tensor product as the set theoretic cartesian product, and any one one-element set as the unit object.

• The category of groups is a monoidal category with the tensor product as the cartesian product of groups, and the trivial group as the unit object.

• The category of modules over a commutative ring R, is a monoidal category with the tensor product of modules $\otimes_R$ serving as the monoidal product and the ring R (thought of as a module over itself) serving as the unit.

• For any commutative ring R, the category of R-algebras is monoidal with the tensor product of algebras as the product and R as the unit.

• The category of all functors from a category C to itself is a monoidal category with the composition of functors as the product and the identity functor as the unit.

A **symmetric monoidal category** is a monoidal category with the additional property that the tensor product is symmetric in both its variables.

Definition 2.2.2. (Symmetric monidal category) A **symmetric monoidal category** is a monoidal category $(C, \otimes, I)$, that satisfies:

- $\otimes$ is **symmetric**, i.e. there is a natural isomorphism

$$s_{AB} : A \otimes B \cong B \otimes A$$

that is natural in both A and B such that the following diagrams are commutative

1. (compatible with unit)

$$\begin{tikzcd} A \otimes I \arrow[rr, "s_{AI}"] \arrow[dr, "r_A"'] & & I \otimes A \arrow[dl, "l_A"] \\ & A & \end{tikzcd}$$

commutes.

2. (compatible with the associativity condition)

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "s_{AB}\otimes 1_C"] \arrow[d, "a_{ABC}"'] & (B \otimes A) \otimes C \arrow[d, "a_{BAC}"] \\ A \otimes (B \otimes C) \arrow[d, "s_{A,B\otimes C}"'] & B \otimes (A \otimes C) \arrow[d, "1_B \otimes s_{AC}"] \\ (B \otimes C) \otimes A \arrow[r, "a_{BCA}"'] & B \otimes (C \otimes A) \end{tikzcd}$$

3. (inverse law)

$$\begin{tikzcd} & B \otimes A \arrow[dr, "s_{BA}"] & \\ A \otimes B \arrow[ur, "s_{AB}"] \arrow[rr, equal, "1_{A\otimes B}"'] & & A \otimes B \end{tikzcd}$$

Example. : The first four examples in the "monoidal category" section are all symmetric monoidal category, the fifth one is not.

A category **enriched** over a monoidal category is a refinement of the category in a sense that the space of morphisms $Hom(A,B)$ for two objects A,B will be an object in an auxiliary monoidal category (i.e. it is not merely a set), and that the composition $Hom(A,B)\otimes Hom(B,C)\to Hom(A,C)$ is a morphism itself. For example, if the monoidal category is Top, then the category enriched in Top means the $Hom(A,B)$ is a topological space, and composition $Hom(A,B)\otimes Hom(B,C)\to Hom(A,C)$ is a continuous map from Cartesian product of two topological spaces to another topological space. If we enrich a category in R module category, then we change above "topological space" to "R-module".

A category $\mathbf{Comp_K}$ is defined to be
The objects are complexes: $\to V_{i-1}\to V_i\to V_{i+1}\to$, denoted V, each V_j is K vector space, and d is a linear map satisfying $d^2=0$. The morphisms are linear maps between vector spaces in corresponding degrees which commute with the differential.

$Comp_K$ is a monoidal category, the tensor product of C' and C'' is

$$(C'\otimes C'')_n=\bigoplus_{i+j=n}(C'_i\otimes C''_j)$$

It is easy to check $C'\otimes C''$ is a complex with differential

$$d_n(c'_i\otimes c''_j)=d'_i(c'_i)\otimes c''_j+(-1)^i c'_i\otimes d''_j(c''_j),\forall c'_i\in C',c''_j\in C''(i+j=n)$$

This induces the (symmetric) monoidal category structure on $Comp_K$

Now we define the differential graded symmetric monoidal category, or dgsm category for short.

Definition 2.2.3. (Differential graded symmetric monoidal category) A *differential graded symmetric monoidal category* over K is a symmetric monoidal category enriched over $Comp_K$.

Example. • $Comp_K$ as above is a dgsm category. The morphism space has a structure of complex over K.
• Any category enriched over $K-vect$ (K vector space) can be seen as a dgsm category with trivial grading (the $\mathrm{Hom}(A,B)$ is in degree 0 for any objects A, B, no other degrees) and trivial differential (i.e. $d_i = 0$ for all i).

We have the basic category, we still need functors in order to define topological field theory.

Definition 2.2.4. (Monoidal functor) Let $(C, \otimes, I_C)$ and $(D, \bullet, I_D)$ be monoidal categories. A **monoidal functor** from C to D consists of a functor $F : C \to D$ together with a natural transformation

$$\phi_{A,B} : FA \bullet FB \to F(A \otimes B)$$

and morphisms

$$\phi : I_D \to FI_C$$

called the **coherence maps** or **structure morphisms**, which are such that for every three objects A,B and C of $\mathcal{C}$ the diagrams

$$\begin{tikzcd}
(FA \bullet FB) \bullet FC \arrow[r, "\alpha_{\mathcal{D}}"] \arrow[d, "\phi_{A,B} \bullet 1"'] & FA \bullet (FB \bullet FC) \arrow[d, "1 \bullet \phi_{B,C}"] \\
F(A \otimes B) \bullet FC \arrow[d, "\phi_{A \otimes B, C}"'] & FA \bullet F(B \otimes C) \arrow[d, "\phi_{A, B \otimes C}"] \\
F((A \otimes B) \otimes C) \arrow[r, "F\alpha_{\mathcal{C}}"'] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
FA \bullet I_{\mathcal{D}} \arrow[r, "1 \bullet \phi"] \arrow[d, "\rho_{\mathcal{D}}"'] & FA \bullet FI_{\mathcal{C}} \arrow[d, "\phi_{A, I_{\mathcal{C}}}"] \\
FA & F(A \otimes I_{\mathcal{C}}) \arrow[l, "F\rho_C"]
\end{tikzcd}$$

$$\begin{tikzcd}
I_{\mathcal{D}} \bullet FB \arrow[r, "\phi\bullet 1"] \arrow[d, "\lambda_{\mathcal{D}}"'] & FI_{\mathcal{C}} \bullet FB \arrow[d, "\phi_{I_{\mathcal{C}},B}"] \\
FB & F(I_{\mathcal{C}} \otimes B) \arrow[l, "F\lambda_{\mathcal{C}}"]
\end{tikzcd}$$

commute in the category D.

Similar to the strict monoidal category, we have a *strict monoidal functor*, for which the coherent maps are identities.

If our monoidal category is symmetric, then we have a symmetric monoidal functor whose definiton is obvious.

The topological or conformal field theory we will study in the following sections will be a symmetric monoidal functor from a differential graded symmetric monoidal category over K to $Comp_K$. There are three notions of such functors which will be used in the later part of this thesis.(see [Cos07])

Let Ψ be a dg symmetric monoidal category.

(1) A *weak symmetric monoidal functor* $F : \Psi \to Comp_K$ is a symmetric monoidal functor of enriched category. (which means it is compatible with differentials)

(2) A *lax symmetric monoidal functor* $F : \Psi \to Comp_K$ is a weak symmetric monoidal functor with additional requirement that the maps

$$F(c) \otimes F(c') \to F(c \otimes c')$$

for any $c, c' \in Ob(\Psi)$ are quasi-isomorphisms.

(3) A *strong symmetric monoidal functor* $F : \Psi \to Comp_K$ is a weak symmetric monoidal functor such that the maps

$$F(c) \otimes F(c') \to F(c \otimes c')$$

are isomorphisms.
(This definition is a little different than the one in [Cos07]. He chooses "strict", and we here use "strong" which fits more with the standard defintion).

2.3 Differential graded category

In this part, I briefly discuss differential graded categories and quasi-isomorphisms between them which will be used in the rest of this thesis. We only need the basics of the theory.

Definition 2.3.1. (Differential graded category) A *differential graded category* over K, or *dg* category over K, is a category enriched over $Comp_K$.

In a dg category, we often talk about the quasi-isomorphism between categories. (It is similar to a quasi-isomorphism between complexes)

Definition 2.3.2. (Quasi-isomorphism between dg categories) Let C,D be two differential graded categories. A *quasi-isomorphism* from C to D consists of a functor $F : C \to D$ with the property that for any c',$c'' \in C$, $Hom(c', c'') \to Hom(F(c'), F(c''))$ is a quasi-isomorphism. Any two dg categories are said to be *quasi-equivalence* if they can be connected via a chain of quasi-isomorphisms.

For two functors F, G between dgsm categories C and D, we say F and G are quasi-isomorphic, if there is a natural transformation such that for any object c in C, $F(c) \to G(c)$ is a quasi-isomorphism. We use $F \simeq G$ to denote that F and G are quasi-isomorphic.

Definition 2.3.3. (Quasi-equivalence of dgsm category) Two dgsm categories C and D are said to be *quasi-equivalent* if there exist functors $F : C \to D$, $G : D \to C$ such that $FG \simeq 1_C$ and $GF \simeq 1_D$.

Chapter 3

Homological algebra of modules over a dgsm category

In this section, I will briefly talk about homological algebras, especially homological algebras of modules over a dgsm category. For general treatment of homological algebra theory, see [SY03].

3.1 Homological algebra

Homology algebra treats the complex (see the definition in the preceeding section) over field K or ring R (if so, we call it R module complex) and computes its homology group (or R module), the homology group is defined to be

$$H_n = \frac{Kerd_n}{Imd_{n+1}}$$

over K or R. we have many basic result in homology algebra, like " zig-zag lemma", usually interpreted as "short exact sequence of complex induces long exact sequence of homology group", "five -lemma",...I will not give the details of these basic results here. You can refer the standard textbook for these materials, like [SY03],[CE99].
The more advanced theory of homology algebra is developed by Grothendieck and Verdier, notably derived category and derived functor, which could be seen as a terminal point in the homology ical algebra.

The definition of derived category originates from the fact that the property of an object we want to study usually can't be captured by the object itself. It is encoded in a complex which the object lives in , at least from the homology algebra point of view. In other words, instead of considering a single object, we need to investigate a complex coming from the object, and study its category and functors. This constitutes the theory of derived category.

Definition 3.1.1. (Abelian category) A category $\mathcal{A}$ is a **Abelian category** if it satisfies:

1. For any two objects c and d, the set $Hom(c, d)$ has a structure of an Abelian group. This structure must be arranged so that the compostion is bilinar
2. There is a zero object.
3. Products and coproducts of finite collections of objects always exist.
4. Kernels and cokernels always exist.
5. If $f : E \to F$ is a morphism whose kernel is 0, then f is the kernel of its cokernel. If $f : E \to F$ is a morphism whose cokernel is 0, then f is the cokernel of its kernel.

Example. • The category of all abelian groups is an abelian category. The category of all finitely generated abelian groups is also an abelian category, as is the category of all finite abelian groups.
• If R is a ring, then the category of all left (or right) modules over R is an abelian category.

Definition 3.1.2. (Derived category) Let $\mathcal{A}$ be an abelian category. We obtain its **derived category** $D(A)$ in several steps:

• The basic object is the category of $Kom(A)$ of chain complexes in $\mathcal{A}$. Its objects will be objects of the derived category but its morphisms will be altered.

•Pass to *homotopy category of chain complexes*$K(A)$ by identifying morphisms which are chain homotopic.

• Pass to the derived category $D(A)$ by *localizing* at the set of quasi-isomorphisms. Morphisms in the dervied category may be explicitly described as *roofs* $A \leftarrow A' \to B$, where s is a quasi-isomorphism and f is any morphism of chain complexes. Here "localize" is a similar term in ring theory. Roughly speaking, localize at a set of morphisms means adding the inverse to these morphisms so that the morphisms become isomorphisms, and this "adding" operation must be in the universal way.

Now if we have two category C and D, and a functor $F : C \to D$, we can apply F to each term (which is an object of C) of the complex in C to get another complex in D. This can pass to the derived category $D(C)$ and $D(D)$, so we get a functor from $D(C)$ to $D(D)$. We call it a *total derived functor*, denote RF.

3.2 Some homological algebras of the dgsm category

In this part, I will present some homological algebra of the dgsm category. We will talk about the module over a dgsm category, which can be seen as a categorical version of a module over a ring. And we will talk about derived tensor products of these modules. We will use the results in the following parts of this paper.

Definition 3.2.1. (Module over dgsm category) Let C be a dgsm category. A *module* over C is a functor $F : C \to Comp_K$ as a functor between dgsm categories.

Note: if C has only one object, so that C can be viewed as a differential graded algebra over K, then the module over C is simply the usual concept in abstract algebra. So the definition of module over a dgsm category can be seen as a categorical generalization of concepts in ordinary algebra.

Now I define the derived tensor product of modules over dgsm categories. This will be used in the future to construct a homotopy coinvariant of a complex under a group action.

Let M be a $B - A$ bimodule. Let N be a left A module Then we can form a left B module $M \otimes_A N$. For each $b \in B$, $M \otimes_A N(b)$ is defined to be the universal complex with maps $M(b,a) \otimes_K N(a) \to M \otimes_A N(b)$, such that the diagram commutes. One can check that $M \otimes_A N$ is again a monoidal nfunctor from B to complexes. Thus $M \otimes_A -$ defines a functor A - mod $\to$ B - mod.

Definition 3.2.2. (Derived tensor product [Cos07]) If M is a $A - B$ bimodule, and N is a left B modulem we define a left A module by

$$M \otimes_B^L N = M \otimes_B Bar_B N$$

Here $Bar_B N$ is a certain flat resolution of B. For details see [Cos07]. Note that any other flat resolution of N will give a quasi-isomorphism answer; as, suppose N', N'' are flat resolutions of N, and M' is a B flat resolution of M. Then $M \otimes_B N' \simeq M' \otimes BN' \simeq M' \otimes_B N'' \simeq M \otimes_B N'$.

Chapter 4

On the facts about mapping class group and teichmuller space

In this section, we give two estimates for homological dimension of moduli space of Riemann surface with boundary respectively, it turns out that the latter is stronger than the former.

4.1 Estimate of homological dimension of moduli space

Definition 4.1.1. A **Riemann surface** is a 2 dimensional manifold X with a coordinate charts (U_i, φ_i, V_i), $i = 1, \ldots, n$ with
1. $U_i \in X$, $V_i \in C$, and $\varphi_i : U_i \to V_i$ a homeomorphism, for all $1 \leq i \leq n$.
2. $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is biholomorphism, for any $1 \leq i < j \leq n$.

For example, complex plane $\mathcal{C}$ is a Riemann surface with one coordinate chart $(\mathcal{C}, Id, \mathcal{C})$ where Id defines the identity map.
Another example is **Riemann sphere** $S = \mathcal{C} \cup \{\infty\}$. We can choose the charts to be $(Sn\{\infty\}, f(z), \mathcal{C})$, $(Sn0, g(z), \mathcal{C})$, where $f(z) = z$, $g(z) = 1/z, g(\infty) = 0$ (It is easy to check condition 2 satisfies). So S is a Riemann surface.

From hyperbolic geometry and theory of Riemann surface, we know that Riemann surface with r boundary components and n punctures is equivalent to the surface equipped with a complete, finite area hyperbolic metric with geodesic boundary (to see this, double the surface, and we have a unique hyperbolic geometric structure corresponding to the complex structure of the double, and the boundary is invariant under the involution, so it is geodesic). Let $\mathfrak{H}_S$ be the space of such metric on the surface S, let $Diff(S)$ denote the group of oriented preserving differmophism of S which fixes each puncture and boundary components, and $Diff^1(S)$ be the subgroup consisting of those differmophisms which is istropic via such diffemorphisms to the identity. And $Mod(S) = Diff(S)/Diff^1(S)$ is known as **Mapping class group** of S. $Diff(S)$ and $Diff^1(S)$ acts on $\mathfrak{H}_S$ (via metric pull back) and the quotient space $\mathcal{T}_S = \mathfrak{H}_S/Diff^1(S)$ is called *Teichmuller space* of S, the space $\mathcal{M}_S = \mathfrak{H}_S/Diff(S)$ is the *Moduli space* of S, whose point parametrize the isomorphism class of complete,finite area hyperbolic metric on S with ordered punctures and boundary components. From the definition, we also know $\mathcal{T}_S/Mod(S) = \mathcal{M}_S$.

In the following, we will define $S_{g,n}^{b,\overrightarrow{m}}$ to be a surface of genus g with b boundary components, labelled $1, 2, \ldots, b$ and n interior punctures/marked points, $m = (m_1, m_2, \ldots, m_b)$ punctures/marked points on the boundary, with m_i punctures/marked points on the ith boundary component. If $\overrightarrow{m} = 0$, we denote it as $S_{g,n}^b$, and further if $b = 0$ or $n = 0$ or both are 0 we denote it $S_{g,n}, S_g^b$ or S_g respectively.

Theorem 4.1.2. *Let $\mathcal{M}_{g,n}^{b,\overrightarrow{m}}$ be moduli space of Riemann surface of genus g with b boundary components, labelled $1, 2, \ldots, b$ and n interior punctures which is also labelled, $m = (m_1, m_2, \ldots, m_b)$ punctures on the boundary, with m_i punctures on the ith boundary component, then we have the following estimates for the homology:*

$$H_i(\mathcal{M}_{S_{g,n}^{b,\overrightarrow{m}}}, Q) = 0 \qquad \textit{for} \quad i \geq 6g - 7 + 2n + 3b + m$$

except $(g, n, b, \overrightarrow{m}) = (0, 3, 0, 0), (0, 2, 1, 0), (0, 2, 1, 1), (0, 0, 2, ((1, 0))$
$(0, 0, 2, (1, 1)), (0, 1, 1, 1), (0, 1, 1, 2), (0, 0, 1, 3), (0, 0, 1, 4), (0, 4, 0, 0)$

For oriented surface of genus g (with or without boundary and punctures), it is well known that Teichmuller space is contractible (it is homeomorphism to a Euclean space). Specifically, if S is an oriented surface of genus g with b boundary components and n punctures in the interior, then [FM08]

$$\mathcal{T}_S \cong R^{6g-6+2n+3b}$$

And we know ([FM08]) the mapping class group $Mod(S)$ acts properly discontinuously on Teichmuller space T_S, and the stabilizer is finite for each point, so by the *Borel construction* $E := T_S \times_{Mod(S)} EMod(S)$, $EMod(S)$ is a universal principle $Mod(S)$ bundle and $BMod(S) = EMod(S)/Mod(S)$ is the *classifying space* of $Mod(S)$ (which classifies all principle $Mod(S)$ bundle on any topological space). The above $Mod(S)$ acts via diagonal action. Since T_S is contractible, E is homotopic to $BMod(S)$, and the projection $E \to T_S/Mod(S) = M_S$ is a fibration with fiber $EMod(S)/stab(x) = Bstab(x)$. Since $stab(x)$ is finite group, the cohomology of the fiber vanishes over Q. That is, we have $H_*(Mod(S), Q) = H_*(BMod(S), Q) = H_*(\mathcal{M}_S, Q)$.

In the following, I will consider the surface with possibly punctures/marked points on the boundary.

For mapping class group, we have the Birman exact sequence:

$$1 \to \pi_1(S^b_{g,n-1}) \to \mathcal{M}_{S^b_{g,n}} \to \mathcal{M}_{S^b_{g,n-1}} \to 1 \tag{4.1}$$

except the degenerate cases: $g = 0, b + n \leq 3$, $g = 1, b + n \leq 1$.
(forgetting multiple punctures)

$$1 \to \pi_1 C(S^b_{g,n-1}, k) \to \mathcal{M}_{S^b_{g,n+k-1}} \to \mathcal{M}_{S^b_{g,n-1}} \to 1$$

where for any surface S, $C(S, k)$ is the configuration space of k distinct,ordered points in S. That is,

$$C(S, k) = S^{\times k} - BigDiag(S^{\times k})$$

where $S^{\times k}$ is the k-fold cartesian product of S and BigDiag($S^{\times k}$) is the *Big Diag* of $S^{\times k}$, that is, the subset of $S^{\times k}$ with at least two coordinates are equal.

and an variant of it

$$1 \to \pi_1(UTS^{b-1}_{g,n}) \to \mathcal{M}_{S^b_{g,n}} \to \mathcal{M}_{S^{b-1}_{g,n}} \to 1 \tag{4.2}$$

$UTS^{b-1}_{g,n}$ is the unit tangent bundle (spherized tangent bundle) of $S^{b-1}_{g,n}$. Also excludes the cases above.

Those sequences are important tool for our calculation of mapping class group. I will derive a sequence similar in spirit, for mapping class group of surface with punctures/marked points on the boundary. More precisely, assume S is a oriented surface of genus g with b boundary components and n punctures/marked points in the interior and $m = (m_1, m_2, \ldots, m_b)$ punctures/marked points on the boundary, with m_i punctures/marked points on the ith boundary component. $Mod(S)$ is the mapping class group, i.e. the group of equivalence classes of oriented-preserving homeomorphism of S which fixes boundary component setwise and fixes each punctures individually, the equivalence relation is given by isotropy of the same type between them.

proposition

$$\mathcal{M}_{S^{b,(m_1,m_2,\ldots,m_i+1,\ldots,m_b)}_{g,n}} \cong \mathcal{M}_{S^{b,(m_1,m_2,\ldots,m_i,\ldots,m_b)}_{g,n}} \qquad \text{if} \quad m_i > 0 \tag{4.3}$$

so we can reduce the homology of moduli space of surface with $m = (m_1, m_2, \ldots, m_b)$ punctures/marked points to that with each m_i at most 1. For this kind of surface we have the following there is a fibration sequence

$$(S^1)^r \to \mathcal{M}_{S^{b,(m_1,m_2,\ldots,m_b)}_{g,n}} \to \mathcal{M}_{S^b_{g,n}} \quad \text{if} \quad \chi(S^b_{g,n}) < 0 \tag{4.4}$$

r is the number of i for which $m_i = 0, m_i = 0$ or 1 for $i = 1, 2, \ldots, b$

In order to achieve the desired homological dimension estimate, we need the following two key indigrients:

(i) The short exact sequence of group corresponding to the fibration sequence of classifying spaces.

If G_1, G_2, G_3 are groups,

$$1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow !$$

then the indueced sequence

$$BG_1 \longrightarrow BG_2 \longrightarrow BG_3$$

is a fibration. BG_i are classifying space of G_i.

This is the standard facts in classifying space theory.

(ii) If $F \longrightarrow E \longrightarrow B$ is a serre fibration, $H_i(B,Q) = 0, H_j(F,Q) = 0$ when $i \geq n, j \geq m+1$, and $H_*(B,Q)$ is of finite rank, then $H_k(E,Q) = 0$ if $k \geq n+m$. In fact, by the Serre spectral sequence,

$$E_2^{p,q} = H_p(F, H_q(B,Q)) \Longrightarrow H_{p+q}(E,Q)$$

so the homology we are going to compute is $p+q = k \geq n+m$. so at least $p \geq n$ or $q \geq m+1$ exists, use the basic property of homology group and rank finiteness of $H_*(B,Q)$ we get the conclusion.

Now we can prove the theorem.

proof $S_{g,n}^{b,\overrightarrow{m}}$ is a surface with possibly punctures/marked points on the boundary.

(1) there are some i such that $m_i \geq 2$. In this case, we can use (3) to reduce it to the case that $m_i = 0$, or 1 for all i. i.e., $\mathcal{M}_{S_{g,n}^{b,\overrightarrow{m}}} \cong \mathcal{M}_{S_{g,n}^{b,\overrightarrow{m}'}}$, where $m_i' = 1$, if $m_i \geq 1$;
(2) all the $m_i = 0$ or 1. In this case, we use the short exact sequence (4) , provided the Eulr class of punctures "fill in" surface is negative, and the induced fibration of classifying spaces mentioned before. Since the classifying space of group Z^r is $(S^1)^r$ (i.e, $K(Z^r,1)$), we have a fibration

$$(S^1)^r \longrightarrow \mathcal{M}_{S_{g,n}^{b,\overrightarrow{m}}} \longrightarrow \mathcal{M}_{S_{g,n}^b} \tag{4.5}$$

Since $(S^1)^r$ is an r dimensional manifold, we have $H_i((S^1)^r, Q) = 0$ if $i \geq r+1$. Use (*), we have an induction
if $H_i(\mathcal{M}_{S_{g,n}^b}, Q) = 0$, for $i \geq 6g-7+2n+3b$, then $H_j(\mathcal{M}_{S_{g,n}^{b,\overrightarrow{m}}}, Q) = 0$, for $j \geq$

$6g-7+2n+3b+r$, where the condition $\chi(S_{g,n}^b)<0$ holds

(3) there are no punctures on the boundary, and surface has boundary . In this case, we can use (2), provided the Euler characteristic of the disk "patching" surface is negative, and the same the induced fibration of classifying spaces. The classifying space of group $\pi_1(UTS_{g,n}^{b-1})$ is $UTS_{g,n}^{b-1}$ (since $UTS_{g,n}^{b-1}$ is $K(\pi_1,1)$), so we have a fibration sequence

$$UTS_{g,n}^{b-1} \longrightarrow \mathcal{M}_{S_{g,n}^b} \longrightarrow \mathcal{M}_{S_{g,n}^{b-1}} \tag{4.6}$$

Because $UTS_{g,n}^{b-1}$ is 3 dimensional manifold, $H_i(UTS_{g,n}^{b-1},Q)=0$, for $i\geq 4$. Use (I), we also get an induction
if $H_i(\mathcal{M}_{S_{g,n}^{b-1}},Q)=0$, for $i\geq 6g-7+2n+3(b-1)$, then $H_j(\mathcal{M}_{S_{g,n}^b},Q)=0$, for $j\geq 6g-7+2n+3b$, where the condition $\chi(S_{g,n}^{b-1})<0$ holds.

Finally, $b=0$. In this case, in [Cos07], it has established that $H_i(\mathcal{M}_{S_{g,n}},Q)=0$, for $i\geq 6g-7+2n$ except $(g,n)=(0,3)$.

Now assume our $S_{g,n}^{b,\overrightarrow{m}}$ has $\chi(S_{g,n}^{b,\overrightarrow{m}})<0$.

i There are no punctures on the boundary of $S_{g,n}^{b,\overrightarrow{m}}$. Then use induction (as above) we conclude that $H_i(\mathcal{M}_{S_{g,n}^{b,\overrightarrow{m}}},Q)=0$ for $i\geq 6g-7+2n+3b$, except $(g,n,b)=(1,1,0),(1,0,1),(1,0,0),(0,3,0),(0,2,0),(0,1,0),(0,0,0),(0,2,1),(0,1,1),(0,0,1),$ $(0,1,2),(0,0,2),(0,0,3)$, consider Euler characteristic should be less than zero, we have $(g,n,b)=(1,1,0),(1,0,1),(0,3,0),(0,2,1),(0,1,2),(0,0,3)$
ii For other cases, the mapping clss group is the same as the one with fewer punctures on the boundary(see (3) for precise meaning of this), and then we can still reduce it to the surface without punctures on the boundary at all if $\chi(S_{g,n}^b)=2-2g-(b+n)<0$, this includes all the cases except $(g,n,b)=(0,1,1),(0,0,2),(0,0,1)$.Note that $b>0$. So if (g,n,b) are not those numbers, we reduce the surface with punctures on the boundary to the one without punctures on the boundary, by **(i)**, except those "bad" cases, other surfaces all satisfy the homological dimension estimates. Then by induction, the original surface satisfies the homological dimension estimate.
So we are left the cases $(g,n,b,\overrightarrow{m})=(0,1,1,\overrightarrow{m}),(0,0,2,\overrightarrow{m}),(0,0,1,\overrightarrow{m})$,and $(1,0,1,\overrightarrow{m})$,

$(0,2,1,\overrightarrow{m})$,$(0,1,2,\overrightarrow{m})$,$(0,0,3,\overrightarrow{m})$, we need to treat them individually.

Let's start from $g = 1$.
If $(g,n,b,\overrightarrow{m}) = (1,0,1,\overrightarrow{m})$, Since the dimension of the moduli space $\mathcal{M}_{S^1_{1,0}}$ is 3,(it is the complement of trileaf knot) and $H_2(\mathcal{M}_{S^1_{1,0}},Q) = H_3(\mathcal{M}_{S^1_{1,0}},Q) = 0$, (see [MK02]), so in this case the homological dimension estimate is satisfied.

For $g = 0$, if $(g,n,b,\overrightarrow{m}) = (0,0,3,\overrightarrow{m})$, Since the moduli space of $S^3_{0,0}$,i.e.,a pair of pants, is the same as the Teichmuller space, and $T_{S^3_{0,0}} \cong R^3$, so the homological dimension estimate is obviously satisfied, then for all $\overrightarrow{m}$, the homological dimension is satisfied. The same is true for $(g,n,b,\overrightarrow{m}) = (0,1,2,\overrightarrow{m})$.

if $(g,n,b,\overrightarrow{m}) = (0,2,1,\overrightarrow{m})$, then since $\mathcal{M}_{S^1_{0,2}} \cong R$, $H_i(M_{S^1_{0,2}},Q) = 0$ when $i \geq 1$, so when $m \geq 2$, the homological dimension estimate is satisfied. The only cases left is $(0,2,1,0),(0,2,1,1)$

if $(g,n,b,\overrightarrow{m}) = (0,0,1,\overrightarrow{m})$, then the $\chi(S^{1,\overrightarrow{m}}_{0,0}) < 0$ requires $m \geq 3$, and the dimension of moduli space of $S^{1,3}_{0,0}$ is 0, so when $m > 4$, the homological dimension estimate is satisfied. This left the cases $(g,n,b,\overrightarrow{m}) = (0,0,1,3),(0,0,1,4)$ (these don't satisfy the homological dimension estimate by dimension counting)

if $(g,n,b,\overrightarrow{m}) = (0,0,2,\overrightarrow{m})$, then the moduli space of $S^{2,(1,0)}_{0,0}$ is an interval $(0,1)$ ([Liu02]), its homology $H_i(\mathcal{M}_{S^{2,(1,0)}_{0,0}},Q) = 0$ when $i \geq 1$. And the moduli space of $S^{2,(1,1)}_{0,0}$ is of dimension 2 and homotopic to S^1, so the homology $H_i(\mathcal{M}_{S^{2,(1,1)}_{0,0}},Q) = 0$ when $i \geq 2$. Thus when $m \neq (0,1),(1,0),(1,1)$, the homological dimension estimate is satisfied. This only left $(g,n,b,\overrightarrow{m}) = (0,0,2,(1,0)),(0,0,2,(1,1))$ (which doesn't satisfy the homology dimension estimate)

if $(g,n,b,\overrightarrow{m}) = (0,1,1,\overrightarrow{m})$, because $m \geq 1$ and the dimension of the moduli space $\mathcal{M}_{S^{1,1}_{0,1}}$ is 0, we have that when $m \geq 3$, then homological dimension estimate is satisfied. This left $(g,n,b,\overrightarrow{m}) = (0,1,1,1),(0,1,1,2)$.(these two don't satisfy the homological dimension estimate)

If there are no punctures on the boundary, then when $(g,n,b)=(1,1,0),(1,0,1),(0,1,2),$ $(0,0,3)$ it is shown in [Cos07] and the result above that the homological dimension estimate is satisfied. When $(g,n,b)=(0,3,0),(0,2,1)$, the homological dimension estimate is not satisfyied by dimension counting.

In summary, $H_i(\mathcal{M}_{S^{b,\overrightarrow{m}}_{g,n}},Q)=0$, for $i\geq 6g-7+2n+3b+m$, except $(g,n,b,\overrightarrow{m})=$ $(0,3,0,0),(0,2,1,0),(0,2,1,1),(0,0,2,(1,0)),(0,0,2,(1,1)),(0,1,1,1),(0,1,1,2),$ $(0,0,1,3),(0,0,1,4)$.

Remark: In fact, it is already from the above analysis that the homological dimension estimates can be improved $H_i(\mathcal{M}_{S^{b,\overrightarrow{m}}_{g,n}},Q)=0$ when $i\geq 6g-7+2n+3b+q$, q is the number of nonzero entires in $\overrightarrow{m}$.

4.2 Sharper estimate of homological dimension

In this part, we will get a homological dimension estimate for moduli space over coefficient Q, which is stronger than the one we get above.

Denote

$$d(g,n,b)=\begin{cases}4g-5 & \text{for } n=b=0\\ 4g+2b+n-4 & \text{for } g>0 \text{ and } n+b>0\cdot\\ 2b+n-3 & \text{otherwise.}\end{cases}$$

as above, q is the number of nonzero entries in $\overrightarrow{m}$. Then, we have

Theorem 4.2.1. $H_i(\mathcal{M}_{S^{b,\overrightarrow{m}}_{g,n}},Q)=0$ *when* $i>d(g,n,b)+q$.
except$(g,n,b,\overrightarrow{m})=(0,1,1,\overrightarrow{m}),(0,0,2,\overrightarrow{m})$

proof (1) It is easy to see that the moduli space $\mathcal{M}_{S^{b,\overrightarrow{m}}_{g,n}}$ is homotopy equivalent to $\mathcal{M}_{S^{b,\overrightarrow{m}'}_{g,n}}$, where $m'_i=0$or1, depending on whether $m_i=0$ or >0. And $\mathcal{M}_{S^{b,\overrightarrow{m}'}_{g,n}}$ is a fibration with fiber homeomorphic to $(S^1)^p$ over moduli space $\mathcal{M}_{S^b_{g,n}}$, where p is the number of non-zero entires in $\overrightarrow{m}'$. (2) In [Har86], it is shown that when $\chi(S^b_{g,n}) < 0$ then the $Mod(S^b_{g,n})$ is a virtual duality group of dimension $d(g,n,b)$, in particular, the virtual cohomological dimension of $Mod(S^b_{g,n})$ is $d(g,n,b)$. Since for a duality group the cohomological dimension and homological dimension agrees ([BE73]), we have $H_i(\mathcal{M}_{S^b_{g,n}}, Q) = 0$, if $i > d(g,n,b)$. Summarize the above two, and use the Serre spectral sequence and (ii), we get the conclusion. We still have some cases which need to be treated separately, they are $(g,n,b,\overrightarrow{m}) = (0,1,1,\overrightarrow{m}), (0,0,2,\overrightarrow{m}), (0,0,1,\overrightarrow{m})$

When $(g,n,b,\overrightarrow{m}) = (0,1,1,\overrightarrow{m})$ we need only consider $m > 0$. In this case, since $H_*(\mathcal{M}_{S^{b,\overrightarrow{m}}_{g,n}}, Q) = H_*(\mathcal{M}_{S^{b,1}_{g,n}}, Q)$ and $S^{b,1}_{g,n}$ is of dimension 0, so $H_i(\mathcal{M}_{S^{1,\overrightarrow{m}}_{0,1}}, Q) = 0$ when $i \geq 1$. Since $d(0,1,1) = 0, d(0,1,1) + q = 1$, the result is stronger.

When $(g,n,b,\overrightarrow{m}) = (0,0,2,\overrightarrow{m})$, $\overrightarrow{m}$ must has at least one nonzero entry. From the result in the proof of theorem 1, we have $H_i(\mathcal{M}_{S^{2,(m,0)}_{0,0}}, Q) = 0$ when $i \geq 1$, it is stronger. And we have $H_i(\mathcal{M}_{S^{2,(m_1,m_2)}_{0,0}}, Q) = 0$ $(i.e., q = 2)$ when $i \geq 2$, which is also stronger.

When $(g,n,b,\overrightarrow{m}) = (0,0,1,\overrightarrow{m})$, then we have $m \geq 3$. When $m = 3$, $S^{1,\overrightarrow{m}}_{0,0}$ is of dimension 0, so $H_i(\mathcal{M}_{S^{1,\overrightarrow{m}}_{0,0}}, Q) = 0$ when $i \geq 1$, which is the same as the above estimate.

In this thesis, we still need to discuss moduli space of surfaces with unordered punctures and boundary components. We similarly let $\tilde{S}^{b,\overrightarrow{m}}_{g,n}$ be a surface with unordered punctures and boundary component.

$$H_i(\mathcal{M}_{\tilde{S}^{b,\overrightarrow{m}}_{g,n}}, Q) = 0 \quad \text{when} \quad i > d(g,n,b) + q \tag{4.7}$$

This can be shown from the fact that

It is an exact sequence

$$0 \longrightarrow \mathcal{M}_{S^{b,\overrightarrow{m}}_{g,n}} \longrightarrow \mathcal{M}_{\tilde{S}^{b,\overrightarrow{m}}_{g,n}} \longrightarrow S_n \times S_b \ltimes (Z_{m_1} \times \cdots \times Z_{m_b}) \longrightarrow 0$$

The map $\mathcal{M}_{S^{b,\overrightarrow{m}}_{g,n}} \longrightarrow \mathcal{M}_{\tilde{S}^{b,\overrightarrow{m}}_{g,n}}$ is the canonical one, and $\mathcal{M}_{\tilde{S}^{b,\overrightarrow{m}}_{g,n}} \to S_n \times S_b \ltimes (Z_{m_1} \times \cdots \times Z_{m_b})$ records how the homeomorphism permutes punctures.

The third nonzeron term of exact sequence is a finite group implies that if the homology over Q of the second nonzeron term is 0 then so it is for the first nonzero term (this can be deduced from the "transfer map" for group homology associated to a group G and its finite-index subgroup H.([Brown82],[AJ04]) The restriction map composite with the transfer map is: $Tr_H^G \circ Res_H^G(x) = [G : H]x$, for $x \in H_*(G, Q)$, because $H_*(G, Q)$ is a Q vector space, it means the restrction map, $Res_H^G : H_*(G, Q) \to H_*(H, Q)$, is injective)

Chapter 5

Topological Conformal Field Theory

5.1 Definition of a TCFT

Segal category [Seg04]. Let S be Segal's category of Riemann surfaces. The objects of S are finite sets. For sets I, J, the morphism space $S(I, J)$ is the moduli space of Riemann surfaces with I incoming and J outgoing parametrized boundaries. These surfaces are not necessarily connected. Also there may be a connected component without boundary. S is a symmetric monoidal category, under disjoint union.

Let $\pi_0(S)$ denote the category with the same objects as S, but whose morphisms are $\pi_0(S(I, J))$. The *Vect* will be denoted the category of vector spaces over K.

Definition 5.1.1. (Topological field theory [Ati88]) A *strict topological field theory* (TFT) is a strong symmetric monoidal functor

$$\pi_0(S) \to Vect.$$

A folk theorem says the category of naive topological field theory is the same as the category of commutative Frobenius algebra.

A naive topological field theory does not see any interesting properties of the moduli space of Riemann surfaces. We are interested in a variant of this notion, which includes the chains on the moduli spaces of Riemann surfaces into the defintion.

Let C_* be a symmetric monoidal functor from the category of topological spaces to that of complexes of K vector spaces, which computes homology groups. The category S has a discrete set of objects, but the spaces of morphisms are topological spaces. Applying C_* to the topological category S yields a differential graded category $C_*(S)$. The objects of $C_*(S)$ are, as before, finite sets; the morphisms of $C_*(S)$ are defined by

$$Mor_{C_*(S)}(a,b) = C_*(Mor_S(a,b))$$

Define

$$\mathcal{L} = C_*(S)$$

$\mathcal{L}$, like S, is a symmetric monoidal category.

Definition 5.1.2. (Topological conformal field theory [Seg04]) A full *topological conformal field theory* over K is a lax symmetric monoidal functor

$$F : \mathcal{L} \to Comp_K.$$

Let S_+ be the subcategory of S with the same objects, but whose morphisms are surfaces, each of whose connected components has at least one incoming boundary.

Definition 5.1.3. (Positive-boundary) A *positive-boundary topological conformal field theory* over K is a lax symmetric monoidal functor

$$C_*(S_{,+}) \to Comp_K$$

We will twist the definiton of topological conformal field theory by a local system. Let det be the locally constant sheaf of K lines on the morphism spaces of the category S whose fibre at a surface $\sum$ is

$$det(\sum) = (detH^*(\sum))[-\chi(\sum)]$$

This is suited in degree $\chi(\sum)$. If $\sum_1, \sum_2$ are two surfaces with the incoming boundaries of $\sum_2$ identified with the outgoing boundaries of $\sum_1$, then there is a natural isomorphism

$$det(\sum_1 \circ \sum_2) \cong det(\sum_1) \otimes det(\sum_2)$$

This shows that $C_*(S(I,J), det^{\otimes d})$ are morphisms of a symmetric monoidal category. we define

$$\mathcal{L}^d = C_*(S, det^{\otimes d})$$

Similarly, define $\mathcal{L}$ to be

$$\mathcal{L} = C_*(S_+, det^{\otimes d})$$

Definition 5.1.4. (TCFT of dimension d) A full (respectively, positive-boundary) topological conformal field theory over K of dimension d is a lax symmetric monoidal functor

$$C_*(S, det^{\otimes d}) \longrightarrow Comp_K$$

(respectively, $C_*(S_+, det^{\otimes d})$)

An amazing theorem of Costello ([Cos04]) gives a description of topological conformal field theory:

Theorem 5.1.5 ([Cos04], Theorem C)**.** *. Let C be a Calabi-Yau A_∞ category of dimension d over K. Then there is a positive-boundary topological conformal field theory F, of dimension d, with a natural quasi-isomorphism $CH_*(C)^{\otimes n} \cong F(n)$, where CH_* refers to the Hochschild chain complex of the category C*

5.2 open and open-closed TCFTs

The above definition of TCFT can be modified to an open and, more generally, open-closed version. We consider the open or open-closed versions because of string theory in

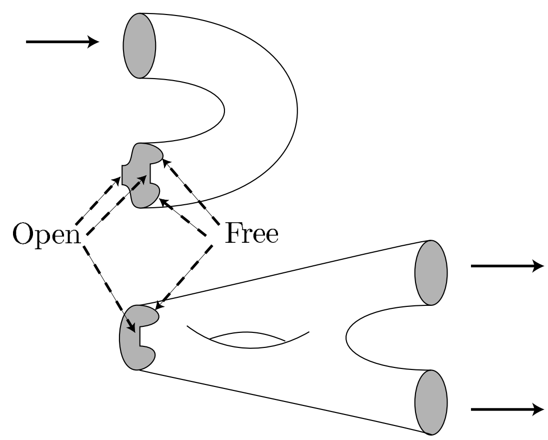

Figure 5.1: A Riemann surface with open-closed boundary. The open boundaries can be either incoming or outgoing boundaries, but this is not illustrated

physics. The open or open-closed construction naturally includes the open string into the theory.

In order to define these theories, we need a modified version of a Segal category which should includes Riemann surfaces not just with closed boundary but also open boundary. This version of the theory was first axiomatised by Moore and Segal [MS].

A Riemann surface with open-closed boundary is a Riemann surface $\sum$, some of whose boundary components are parameterised, and labelled as closed (incoming or outgoing); and with some intervals (the open boundaries) embedded in the remaining boundary components. These are also parameterised and labelled as incoming and outgoing. The boundary of such a surface is partitioned into three types: the closed boundaries, the open boundary intervals, and the free boundaries. The free boundaries are the complement of the closed boundaries and the open boundary intervals, and are either circles or intervals.

To defined an open or open-closed topological conformal field theory, we need a set Λ of D-branes. Define a category S_Λ, whose objects are pairs O, C of finite sets and maps $s, t : O \to \Lambda$. The morphisms in this category are Riemann surfaces with open-closed boundary, whose free boundaries are labelled by D-branes. To each open boundary o of $\sum$ is associated an ordered pair $s(o), t(o)$ of D-branes, where it starts and where it ends.

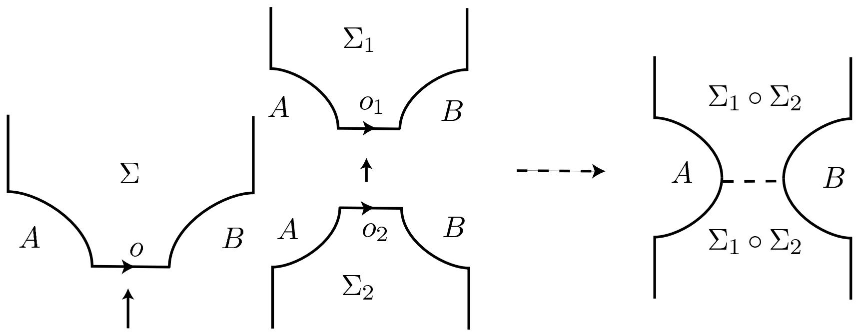

Figure 5.2: Open gluing, corresponding to composition. o_1 on $\sum_1$ is incoming, o_2 on $\sum_2$ is outgoing, and $s(o_1) = s(o_2) = A, t(o_1) = t(o_2) = B$. Note incoming and outgoing boundaries are parameterised in the opposite sense

Composition is given by gluing of surfaces; we glue all the outgoing open boundaries of $\sum_1$ to the incoming open boundaries of $\sum_2$, and similarly for the closed boundary, to get $\sum_1 \circ \sum_2$. Disjoint union makes S_Λ into a symmetric monoidal category. Like the usual TCFT, we can define $S_{\Lambda+}$ in the same way, to be the subcategory of S_Λ whose morphisms are Riemann surfaces each of whose connected boundary components has at least one free or incoming closed boundary. Define

$$\mathcal{O}C_\Lambda = C_*(S_\Lambda)$$

The defintion of open-closed TCFT is

Definition 5.2.1. (Open-closed TCFT) A *full (respectively, positive-boundary) open-closed topological conformal field theory* over K is a lax symmetric monoidal functor

$$F : OC \longrightarrow Comp_K$$

(respectively, $F_+ : C_*(S_{\Lambda+} \longrightarrow Comp_K)$

We can twist the definition of open-closed TCFT by a local system det, and defined as

$$\mathcal{O}C^d_\Lambda = C_*(S_\Lambda, det^{\otimes d})$$

An open-closed TCFT of dimension d is a lax symmetric monoidal functor

$$\mathcal{O}C^d_\Lambda \longrightarrow Comp_K$$

Open TCFT is similar to this, except there are no closed boundaries. we define $\mathcal{O}_{\Lambda}^{d}$ (respectively, $\mathcal{O}_{\Lambda+}^{d}$) as the full subcategory of $\mathcal{OC}_{\Lambda}^{d}$ (respectively, $\mathcal{OC}_{\Lambda}^{d}$) whose objects are pure open; so they are of the form (C, O) where $C = \varnothing$. Morphisms in $\mathcal{O}_{\Lambda}^{d}$ are chains on moduli of surfaces with no closed boundary. The definition of open TCFT (of dimension d with twist coefficent) is modified straightforwardly.

Chapter 6

BV algebra and Quantum Master Equation

In this section, I study the Batlain-Vilkovisky algebra and its homotopy theory, the quantum master equation is also introduced.

In the study of moduli spaces of maps, physicists came up with the following BV formalism which allows one to encode geometric data algebraically. The Quantum Master Equation (QME) arose in Batalin-Vilkovisky Quantization, also called Batalin-Vilkovisky (BV) formalism, of gauge field theory in theoretical physics. Batalin-Vilkovisky formalism was developed as a method for determining the ghost structure for theories, such as gravity and supergravity, whose Hamiltonian formalism has constraints not related to a Lie algebra action. The formalism, based on a Lagrangian that contains both fields and "antifields", can be thought of as a very complicated generalization of the BRST formalism.

Mathematically, BV algebra is

Let V be a graded linear space over field k. A dg-BV algebraic structure on V is a quadruple $(V, \bullet, d, \Delta)$,satisfying the following three conditions:
1. $(V, \bullet, d)$ is a differential, graded, (graded)commutative, (graded)associative algebra over k. The differential d is of degree 1 and $d(1) = 0$.

2. Δ is a second order differential operator with respect to $\bullet$, i.e. the degree of Δ is 1, $\Delta^2 = 0, \Delta(1) = 0$, and for any given $a, b, c \in V$,

$$\begin{aligned}\Delta(a \bullet b \bullet c) &= \Delta(a \bullet b) \bullet c + (-1)^{|a|} a \bullet \Delta(b \bullet c) + (-1)^{(|a|+1)|b|} b \bullet \Delta(a \bullet c)\\ &\quad - (\Delta a) \bullet b \bullet c - (-1)^{|a|} a \bullet (\Delta b) \bullet c - (-1)^{|a|+|b|} a \bullet b \bullet (\Delta c).\end{aligned}$$

where $||$ is the degree of an element.
3. graded commutator $[d, \Delta] = d\Delta + \Delta d = 0$

condition 2 is equivalent to the fact that the deviation of the derivative Δ from being derivation, which is defined by

$$\{,\} := \Delta(ab) - \Delta(a)b - (-1)^{|a|} a\Delta(b),$$

is a (graded)Lie bracket and $\{,\}$ is a (graded)derivation for each variables,i.e.

$$\begin{aligned}\{a, bc\} &= b\{a, c\} + (-1)^{\{|b|\}}\{a, b\}c\\ \{ab, c\} &= a\{b, c\} + (-1)^{\{|a|\}}\{a, c\}b \qquad (6.1)\end{aligned}$$

The condition 3 is equivalent to d being a (graded)derivation for the Lie bracket$\{,\}$, i.e.

$$d\{a, b\} = \{da, b\} + (-1)^{|a|}\{a, db\}.$$

Examples (Most known examples are from mathematical physics) :

- Let M be a odd symplectic manifold (i.e., A manifold with a closed, non-degenerated two forms of odd parity), assume $C^\infty(M)$ is the set of smooth functions on M. It has a natural graded commutative associative algebraic structure, denote its multiplication by $\bullet$. Let $(x_1, \ldots, x_n; \eta_1, \ldots, \eta_n)$ be an array of Darboux coordinate, and let

$$\Delta = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \frac{\partial}{\partial \eta^i},$$

Then it is proved that $(C^\infty(M), \bullet, \Delta)$ is a BV algebra.

• The homology groups of the free loop space of a manifold M, i.e., $Map_{S^1}(S, M)$, has a natural BV algebraic structure (we don't give the details here because it is delicate, the interested reader should refer

• In quantum field theory, E. Getzler shows that there exists a natural BV algebra structure on the homology groups of a 2D topological conformal field theory. (also interested reader should refer

Definition 6.0.2. (BV algebra) Let k be an even number. A *Batalin-Vilkovisky* algebra of degree k, or a BV_k algebra, is a differential graded commutative algebra B, together with an operator $\Delta : B \to B$, which is of degree $k-1$, is of order two as a differential operator, and satisfies

$$\Delta^2 = [d, \Delta] = \Delta(1) = 0.$$

We let $\hat{d} = d + \Delta$.

If B is a BV algebra, then it acquires a Poisson bracket of degree $k-1$. The bracket is defined by

$$\{f, g\} = \Delta(fg) - (-1)^{|f|}\Delta(g) - \Delta(f)g.$$

This satisfies the Jacobi identity; Because Δ is a second order differential operator, which means

$$\Delta\{f, g\} = \{\Delta f, g\} + (-1)^{|f|}\{f, \Delta g\}$$

Also, it can be shown d is a derivation of the bracket

$$d\{f, g\} = \{df, g\} + (-1)^{|f|}\{f, dg\}$$

For the purpose of this thesis, I need an additional parameter $\hbar$ of degree $-k$ added into the BV algebra, which means I will deal with another BV algebra $B[[\hbar]]$ over $K[[\hbar]]$; The BV structure is obtained from B by $K[[\hbar]]$-linear extension. We modify $\hat{d} = d + \hbar\Delta$ in this situation.

The Maurer-Cartan equation in B is the equation

$$\hat{d}S + \frac{1}{2}\{S, S\} = 0$$

Definition 6.0.3. (Quantum Master Equation) The *quantum master equation* associated to a BV algebra B is the equation

$$\hat{d}exp(S/\hbar) = 0$$

(whenever this expression makes sense in the algebra).

In fact, this is equivalent to the Maurer-Cartan equation. Indeed, we have

$$exp(-S/\hbar)\hat{d}exp(S/\hbar) = (\hat{d}S + \frac{1}{2}\{S, S\})/\hbar$$

From the above, A BV algebra has an associated Lie algebra. In general,

Definition 6.0.4. (Differential graded Lie_n algebra) A *differential graded Lie_n algebra* is a chain complex V equipped with a Lie bracket of degree n, which satisfies the Jacobi identity and is compatible with the bracket.

If V is a dg Lie_n algebra, then $V[n]$ is a dg Lie algebra in the usual sense.
If B is a BV_k algebra, then the complex $B[[\hbar]]$, with the differential $\hat{d} = d + \hbar\Delta$ and the bracket $-, -$, is a differential graded Lie_{k-1} algebra over the ring $K[[\hbar]]$.
In general, if g is a dg Lie_n algebra, a solution to the Maurer-Cartan element in g is an element

$$X \in g_{-n-1}$$

satisfying

$$dX + \frac{1}{2}[X, X] = 0.$$

5.1 **Homotopies between solutions of the master equation**. Consider the differential graded algebra $K[t, \epsilon]$, where t is of degree 0 and ϵ is of degree -1, with differential $\epsilon\frac{d}{dt}$. Let g be a differential graded Lie_n algebra, with differential of degree -1. A solution of the Maurer-Cartan equation in g is an element $S \in g_{-n-1}$ satisfying

$$dS + \frac{1}{2}[S, S] = 0.$$

A homotopy between solutions S_0, S_1 of the Maurer-Cartan equation in g is an element

$$S(t,\epsilon) \in g[t,\epsilon]$$

which satisfies the Maurer-Cartan equatin:

$$dS + \epsilon\frac{dS}{dt} + \frac{1}{2}\{S,S\} = 0$$

and such that $S(0,0) = S_0$, and $S(1,0) = S_1$.

Note that we can write

$$S(t,\epsilon) = S_a(t) + \epsilon S_b(t)$$

The Maurer-Cartan equation for S implies that S_a satisfies the Maurer-Cartan equation, and that

$$\frac{dS_a(t)}{dt} = -[S_b(t), S_a(t)] - dS_b(t)$$

so that the path in g_{-n-1} given by $S_a(t)$ is tangent to the action of g_0 on solutions of Maurer-Cartan in g_{-n-1}.

We need an important concept *simplicial set* for the following part. Simplicial set is a kind of categorical version of usual simplicial set we learn, for example, in topology.

Definition 6.0.5. (simplicial set) A **simplicial set** X is a contravariant functor

$$X : \Delta \to set$$

where Δ denotes the *simplicial category* whose objects are finite strings of ordinal numbers of the form

$$0 \to 1 \to \ldots \to n$$

and whose morphisms are order-preserving functions between them.

The set of Maurer-Cartan elementsin g has a natrual enrichment to be a simplicial set. Let $\Omega^*(\Delta^n) = K[t_1,\ldots,t_n,dt_1,\ldots,dt_n]/(\sum t_i = 1, \sum dt_i = 0)$ denote the differential graded algebra of polynomial forms on the K simplex. We change the degree to be its negative, so that the space of i form is in degree $-i$. We define a simplicial set $MC(g)$ whose l simplices are elements

$$\alpha \in g \otimes \Omega^*(\Delta^l) \tag{6.2}$$

of degree $-n-1$, satisfying the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \tag{6.3}$$

The face and degeneracy maps arise from those relating $\Omega^*(\Delta^l)$ and $\Omega^*(\Delta^{l'})$. Let $\pi_0(MC(g))$ be the quotient of this by the equivalence relation generated by homotopy.

If B is a BV algebra, let $BV(B)$ be the set of homotopy classes of solutions of the master equation in B, that is the set of solutions of the Maurer-Cartan equation in B considered as a dg Lie_n algebra. Let $\pi_0 BV(B)$ be the set of homotoy classes of solutions of the master equation, defined as above.

There is an obvious notion of homotopy between maps $f_0, f_1 : g \to g'$ of dg Lie_n algebras. This is a map $F : g \to g'[t, \epsilon]$ of dg Lie_n algebras, such that $F(0,0) = f_0$ and $F(1,0) = f_1$. Clearly homotopic maps induce the same map $\pi_0 MC(g) \to= \pi_0 MC(g')$. It follows that a homotopy equivalence $g \to g'$ (i.e. a map which has an inverse up to homotopy) induces an isomorphism on $\pi_0 MC$.

In some cases, quasi-isomorphisms of dg Lie_n algebra also induce isomorphisms on the set of homotopy classes of solutions of the Maurer-Cartan equation. Suppose g is a dg Lie_n algebra with a filtration $g = F^1 g \supset F^2 g \supset \ldots$ such that g is complete with respect to the filtration, and such that $[F^i g, F^j g] \subset F^{i+j} g$. In particular $g/F^2 g$ is Abelian and $g/F^i g$ is nilpotent. Then we say g is a filtered pro-nilpotent Lie_n algebra.

Lemma 6.0.6. *([HL04]) Let g, g' be filtered pro-nilpotent dg Lie_n algebras, and let $f : g \longrightarrow g'$ be a filtration preserving map. Suppose that the induced map*

$$H_i(Grg) \longrightarrow H_i(Grg')$$

is an isomorphism, if $i \geq -n$.

Then the map

$$MC(g) \longrightarrow MC(g')$$

is a weak homotopy equivalence. In particular, the map $\pi_0 MC(g) \to \pi_0 MC(g')$ is an isomorphism.

For dg Lie algebra (i.e.,$n = 0$), In [Cos07], Costello shows that the condition of Lemma 6.5 can be weakened, that is, he proves

Lemma 6.0.7. *([Cos07]) Let g, g' be filtered pro-nilpotent dg Lie algebras, and let $f : g \to g'$ be a filtration preserving map. Suppose the map $Grg \to Grg'$ induces an isomorphism on H_i for $i = 0, -1, -2$. Then the map*

$$\pi_0 MC(g) \to \pi_0 MC(g')$$

is an isomorphism.

The elementary homotopy theory of the BV_k algebra is as follows.

Definition 6.0.8. Let B, B' be BV_k algebras. A map $f : B \to B'$ is a quasi-isomorphism if it is a quasi-isomorphism of chain complexes $(B, d) \to (B', d)$.

If $B \to B'$ is a quasi-isomorphism, then the map

$$B[[\hbar]] \longrightarrow B'[[\hbar]]$$

is also a quasi-isomorphism, where each side is equipped with the differential $d + \hbar\Delta$.

Any quasi-isomorphism $B \to B'$ induces a weak equivalence of simplicial sets

$$MC(\lambda B[[\lambda, \hbar]]) \longrightarrow MC(\lambda B'[[\lambda, \hbar]]).$$

This follows from Lemma 5.1.

If B, B' are BV_k algebras, we say that a map $B \to B'$ in the homotopy category of BV_k algebras is a map $B'' \to B'$, where B'' and B are connected by a chain of quasi-isomorphisms. Any map in the homotopy category of BV_k algebras induces a map

$$MC(\lambda B[[\lambda, \hbar]])) \longrightarrow (MC(\lambda B'[[\lambda, \hbar]]))$$

in the homotopy category of simplicial sets.

Chapter 7

The BV structure on moduli spaces

In [HVZ07], Harrelson, Voronov and Zuniga gives a construction of BV algebraic structure on the geometric chain complex associated to the moduli space.

Definition 7.0.9. (**geometric chain**) A *geometric chain* on a topological space X is a formal linear combination over Q of continuous maps

$$f : P \to X$$

where P is a compact connected oriented (smooth) orbifold with corners, module the equivalence relation induced by isomorphisms between the source orbifold P. Here, an *orientation* on an orbifold with corners is a trivialization of the determinant of its tangent bundle.

Note that geometric chains form a graded Q-vector space $C_*^{geom}(X, Q)$, graded by the dimension of P. The boundary of a chain is given by $(\partial P, f|_{\partial P})$, where ∂P is the sum of codimension one faces of P with the induced orientation (Locally, in positively oriented coordinates near ∂P, the manifold P is given by the equation "the last coordinate is nonnegative.") and $f|_{\partial P}$ is the restriction of f to the ∂P of P We will work on singular chains with local coefficients. In other words, If F is a locally constant sheaf of Q-vector spaces on X, *a chain with coefficients in* F will be a (finite) formal sum $c = P_i(\Delta_i, f_i, c_i)$, where f_is are continuous maps from a standard simplex Δ_i to X and c_is are global sections: $c_i \in \Gamma(P_i; f_i^* F)$. The differential is defined as $dc := \sum_i (P_i, f_i|_{\partial P_i}, c_i|_{\partial P_i})$. We will use $C_*(X; F)$ to denote this complex. We simply call them chains in the following (unless otherwise specified).

If M is a compact connected oriented orbifold with corners, then its *fundamental chain* $[M]$ is by definition the identity map $id : M \to M$, understood as a geometric chain $(M, id) \in C_d^{geom}(M; Q) = C_0^{geom}(M; Q[d])$, where $d = dimM$ and $Q[d]$ is the constant sheaf Q shifted by d in degree, regarded as a graded local system concentrated in degree $-d$. If M is not necessarily oriented and $p : M^* \to M$ is the orientation cover, then we define the*fundamental chain* $[M] \in C_0^{geom}(M; Q^\varepsilon)$ of M to be $(M^*, p, \frac{or}{2})$, where $Q^\varepsilon = Q_2 M^*[d]$ is the *orientation local system* (in particular, a locally constant sheaf of rational graded vector spaces of rank one, concentrated in degree $-d$) on M, with M^* thought of as a principle bundle over the multiplicative group $_2 = 1$ of changes of orientation and $or \in \Gamma(M^*; p^* Q^\varepsilon)$ being the canonical orientation on M^*. If $M = M'/G$, where M' is an oriented compact connected orbifold with corners and G a finite group acting on M, then the fundamental chain of M may be obtained from the natural projection $\pi : M' \to M$ as $(M', \pi, \frac{or}{|G|}) \in C_0^{geom}(M; Q^\varepsilon)$, where Q^ε is the orientation local system of M. Note that a *geometric chain with coefficients in the orientation local system* on an orbifold M may be understood as a linear combination of geometric chains $f : P \to M$ with a (continuous) choice of *local orientation* on M along P.

We need a notion of stable bordered Riemann surface, and the moduli space of them (more precisely, stable bordered Riemann surface with decorations, see the definition of this concept below) is the space which our geometric chain lies on.

A *bordered Riemann surface* means a complex curve with real boundary, i.e., a compact, connected, Hausdorff topological space, locally modeled on the upper half-plane $H = z \in C | Imz = 0$ using analytic maps. A *prestable bordered Riemann surface* is a bordered Riemann surface with at most a finite number of singularities of nodal type at points other than the punctures. The allowed types of nodes are denoted X, E, and H, where X means an interior node (locally isomorphic to a neighborhood of 0 on $\{xy = 0\} \subset C^2$), E a boundary node, when the whole boundary component is shrunk to a point (locally modeled on a neighborhood of 0 on $\{x^2 + y^2 = 0 \subset C^2\}/\sigma$, where $\sigma(x, y) = (\overline{x}, \overline{y})$ is the complex conjugation), and H a boundary node at which a boundary component intersects itself or another boundary component (locally modeled on a neighborhood of 0 on $\{x^2 - y^2 = 0 \subset C^2\}/\sigma$). A prestable bordered Riemann surface is *stable*, if its automorphism group is discrete. Here an automorphism must map the boundaries to the boundaries and the punctures to the punctures, respecting the labels. The stability condition is equivalent to the condition that the *Euler characteristic* (in a certain generalized sense, see below) of each component of the surface obtained by removing all the punctures is negative. Note that the Euler characteristic is by definition one half of the Euler characteristic of the double. Thus, for a nondegenerate bordered surface $\sum$, its Euler characteristic is given by

$$\chi(\sum) = 2 - 2g - b - n - m/2.$$

The stability condition thereby excludes a finite number of types $(g, b; n, \overrightarrow{m})$, namely $g = b = 0$ with $n = 2$; $g = 1, b = 0$ with $n = 0$; $g = 0, b = 1$ with $n = 1, m = 0$ or $n = 0, m = 2$; and $g = 0, b = 2$ with $m = n = 0$. The spaces $\overline{M}_{g,n}^{b,\overrightarrow{m}}$ have been thoroughly studied by M. Liu in [Liu02]. They are compact, Hausdorff topological spaces with the structure of a smooth orbifold with corners of dimension $6g - 6 + 2n + 3b + m$, where $m = \sum_{i=1}^{b} m_i$ is the total number of boundary punctures.

The space we are using is the moduli space of stable bordered Riemann surface with decorations. More precisely, $\underline{M}_{g,n}^{b,\overrightarrow{m}}$ is the moduli space of isomorphism classes of stable bordered Riemann surfaces of type (g, b) with $(n, \overrightarrow{m})$ punctures and certain extra data, namely, decorations by a real tangent direction, i.e., a ray, in the complex tensor product of the tangent spaces on each side of each interior node. The space $\underline{M}_{g,n}^{b,\overrightarrow{m}}$ can be obtained by performing real blowups along the divisors of $\overline{M}_{g,n}^{b,\overrightarrow{m}}$ corresponding to the interior nodes, as in [**?**]. The dimension of $\underline{M}_{g,n}^{b,\overrightarrow{m}}$ is the same as that of $\overline{M}_{g,n}^{b,\overrightarrow{m}}$: $\dim \underline{M}_{g,n}^{b,\overrightarrow{m}} = \dim \overline{M}_{g,n}^{b,\overrightarrow{m}} = 6g - 6 + 2n + 3b + m$.

We will concentrate on the moduli space

$$\underline{M}_{g,n}^{b,m}/\varrho = (\Pi_{\overrightarrow{m}:\sum m_i = m} \underline{M}_{g,n}^{b,\overrightarrow{m}}/Z_{m_1} \times \cdots \times Z_{m_b}) \risingdotseq \varrho_b \times \varrho_n \tag{7.1}$$

of stable bordered Riemann surfaces as above with *unlabeled* boundary components and punctures, that is, the quotient of the disjoint union $\underline{M}_{g,n}^{b,\overrightarrow{m}} = \Pi_{\overrightarrow{m}:\sum m_i=m}\underline{M}_{g,n}^{b,\overrightarrow{m}}$ of moduli spaces with labeled boundaries and punctures by an appropriate action of the permutation group $\varrho = (\Pi_{\overrightarrow{m}:\sum m_i=m} Z_{m_1} \times \cdots \times Z_{m_b}) \rtimes \varrho_b \times \varrho_n$
We will work on the chain of this moduli space with twisted coefficients, i.e., a one-dimensional local system Q^ε obtained from a certain sign representation $\rho : \varrho \to EndL = Q^*$ of the permutation group ϱ in a one-dimensional rational graded vector space L concentrated in degree $-d := -dim(\underline{M})_{g,n}^{b,m}$. This representation is defined as follows: ρ is a trivial representation of ϱ_n;

$$\rho(\zeta_i) = (-1)^{m_i - 1}$$

for the generator $\zeta_i(p) = p + 1 \bmod m_i$ of the group Z_{m_i} of cyclic permutations of the punctures on the ith boundary component (where $m_i \geq 1$); and

$$\rho(\tau_{ij}) = (-1)^((m_i - 1)(m_j - 1))$$

for the transposition $\tau_{ij} \in \varrho_b$ interchanging (the labels of) the ith and jth boundary components. Then Q^ε is the locally constant sheaf $\underline{M}_{g,n}^{b,m} \times_\varrho L$ over $\underline{M}_{g,n}^{b,m}/\varrho$ Note that an ordering of the boundary components and an ordering of the boundary punctures on each boundary component compatible with the cyclic ordering thereof on a given Riemann surface determines a section of the local system over the point in the moduli space $\underline{M}_{g,n}^{b,m}/\varrho$ corresponding to that Riemann surface. A change of these orderings will change this section by a sign factor as defined by the representation ρ.
It is shown in [HVZ07] that the space $\underline{M}_{g,n}^{b,\overrightarrow{m}}$ is an orientable orbifold with corners and the local system Q^ε is the orientation sheaf for the orbifold $\underline{M}_{g,n}^{b,m}/\varrho$ with corners. Thus, the *fundamental chain* $[\underline{M}_{g,n}^{b,m}/\varrho] \in C_0^{geom}(\underline{M}_{g,n}^{b,m}/\varrho; Q^\varepsilon)$ is well defined.

Consider the space

$$U = \bigoplus_{g,n,b,m} C_*(\underline{M}_{g,n}^{b,m}/\varrho; Q^\varepsilon)$$

of chains. Since the coefficient system carries a degree shift by the dimension of $\underline{M}_{g,n}^{b,m}$, the space U carries a natural grading by the negative codimension of the chain in $\underline{M}_{g,n}^{b,m}/\varrho$. We will take the opposite grading on U, i.e., the *grading by the codimension* of the chain in $\underline{M}_{g,n}^{b,m}/\varrho$. The differential d of chains will then have degree 1 and make U into a complex of rational vector spaces.

The space V on which we will introduce a dg BV algebra structure will be defined as follows:

$$V := \bigoplus_{b,m,n} C_*(\underline{M}_n^{b,m}/\varrho; Q^\varepsilon)$$

where $\underline{M}_n^{b,m}/\varrho$ is the moduli space of stable bordered Riemann surfaces with b boundary components, n interior punctures, and m boundary punctures, just like the Riemann surfaces in $\underline{M}_{g,n}^{b,m}/\varrho$, but in general having multiple connected components of various genera. The grading on V is given by codimension in the corresponding connected component of $\underline{M}_n^{b,m}/\varrho$, and the dot product is induced by disjoint union of Riemann surfaces. We formally add a copy of the ground field Q to V , and the unit element $1 \in Q \subset V$ might be interpreted as the fundamental chain of the one-point moduli space comprised by the empty Riemann surface.

To define a dg BV algebra structure on V , it remains to define a BV operator $\Delta : V \to V$ satisfying required properties. It will consist of three components:

$$\Delta = \Delta_c + \Delta_o + \Delta_{co},$$

each of degree 1, square zero, and (graded) commuting with each other.

The operator Δ_c is induced on chains by twist-attaching at each pair of interior punctures. To achieve this, we define a bundle $ST M_n^{b,m}/\varrho$ over $\underline{M}_n^{b,m}/\varrho$ of triples $(\sum, P, r)$, where $\sum \in \underline{M}_n^{b,m}/\varrho$, P is a choice of an unordered pair of interior punctures on $\sum$, and r is one of the S^1 ways of attaching them (i.e., a real ray in the tensor product (over C) of the tangent spaces of $\sum$ at these two punctures). So the fiber is homeomorphic to $\Pi^{n(n-1)/2} S^1$. Then we have a diagram

$$\underline{M}_n^{b,m}/\varrho \xleftarrow{\pi} ST\underline{M}_n^{b,m}/\varrho \xrightarrow{a_c} \underline{M}_{n-2}^{b,m}/\varrho \qquad (7.2)$$

where π is the bundle projection map and a_c is obtained by attaching the two chosen punctures P on $\sum$ and decorating the resulting node with the chosen real ray r. (One can view this diagram as a morphism realizing twist-attaching in the category of correspondences.) Then twist-attaching for chains is defined as the corresponding push-pull, giving us the closed part Δ_c of the BV operator:

$$\Delta_c := (a_c) * \pi^! : C_*(\underline{M}_n^{b,m}/\varrho; Q^\varepsilon) \to C_{*+1}(\underline{M}_{n-2}^{b,m}/\varrho; Q^\varepsilon)$$

Here the pullback $\pi^!$ for chains is simply the geometric pre-image. More precisely, to define the pullback of a chain (Δ, f, c), we take the pullback f^*ST of the fiber bundle ST along f and the chain $(f^*ST, \tilde{f})$, where f^*ST is the total space and $\tilde{f} : f^*ST \to ST$ is the pullback of f (if f^*ST is disconnected we regard $\tilde{f}$ as a sum of maps). To define what $\pi^!$ does to a section c of Q^ε, lift the diagram (3) to

$$\underline{M}_n^{b,m} \leftarrow ST\underline{M}_n^{b,m} \to \underline{M}_{n-2}^{b,m}$$

defined before taking the quotient by the symmetric groups. Here, $STM_n^{b,m}$ is the bundle whose fiber over $\sum \in M_n^{b,m}$ consists of all the $n(n-1)/2$ possible choices of unordered pairs $\{i, j\}$ of labeled punctures along with the S^1 ways of attaching them. The fiber of π (isomorphic to $n(n-1)/2$ copies of S^1) has a natural orientation coming from the counterclockwise orientation on the tensor product (over C) of the tangent spaces at the punctures i and j. The orientation on the total space $ST\underline{M}_n^{b,m}$ is then defined (locally) as the orientation on the base $\underline{M}_n^{b,m}$ times the orientation of the fiber of π, and the orientation sheaf on $ST\underline{M}_n^{b,m}/\varrho$ determines a local system. Now recall that a section c of Q^ε is a rational number c' multiplied by the orientation or of the moduli space $\underline{M}_n^{b,m}$. Since an orientation of $\underline{M}_n^{b,m}$ determines an orientation on $ST\underline{M}_n^{b,m}$, as we have just described, we use that orientation, multiplied by the same number c', to get a section of the local system on $ST\underline{M}_n^{b,m}/\varrho$.

The operator Δ_o is induced on chains by attaching at each pair of boundary punctures. To describe this procedure precisely, we form the bundle $B'\underline{M}_n^{b,m}/\varrho$ where the fiber over a point $\sum \in \underline{M}_n^{b,m}/\varrho$ consists of all possible choices of pairs of punctures on $\sum$ which both lie on the same boundary component. Similarly, form the bundle $B''\underline{M}_n^{b,m}/\varrho$ whose fibers are all possible pairs of punctures lying on different boundary components. Then we get the following diagrams:

$$\underline{M}_n^{b,m}/\varrho \xleftarrow{\pi} B'\underline{M}_n^{b,m}/\varrho \xrightarrow{a'_o} \underline{M}_n^{b+1,m-2}/\varrho$$

and

$$\underline{M}_n^{b,m}/\varrho \xleftarrow{\pi} B'\underline{M}_n^{b,m}/\varrho \xrightarrow{a''_o} \underline{M}_n^{b-1,m-2}/\varrho$$

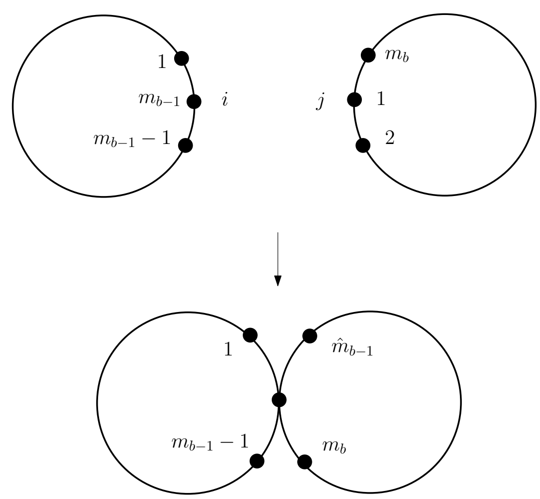

Figure 7.1: Attaching two punctures on different boundary components. The resulting new boundary component has $\hat{m}_{b-1} = m_{b-1} + m_b - 2$ punctures.

where a'_o and a''_o are the obvious attaching maps . We perform the push-pull again to obtain chain maps Δ'_o and Δ''_o. The bundles before quotienting, $B'\underline{M}_n^{b,m}$ and $B''\underline{M}_n^{b,m}$, are just direct products of $\underline{M}_n^{b,m}$ with finite discrete sets and are thus orientable, and we define the pullback (in this case also known as the transfer homomorphism) of geometric chains as in the closed case. The pushforward of sections of the local system is defined in the next paragraph. We will then define the corresponding component of the BV operator as

$$\Delta_o := \Delta'_o + \Delta''_o.$$

Now let us define the pushforward of sections of the local system via a'_o and a''_o . Recall that an ordering of the boundary components and a cyclic ordering of the punctures on each boundary, for a given surface $\sum \in \underline{M}_n^{b,m}/\varrho$, gives a section of the local system over the point $\sum$. The same can be said for $(\sum, P)B'\underline{M}_n^{b,m}/\varrho$ or $B''\underline{M}_n^{b,m}/\varrho$, where P is the choice of a pair of boundary punctures on $\sum$. Thus to define the pushforward it suffices to explain how attaching acts on the labeling of boundaries and boundary punctures. If the punctures i and j in the pair P lie on different boundary components, see Figure 1, first change the ordering of the boundary components and boundary punctures in a way that the puncture i is the last puncture on the $b-1$st boundary component and the puncture j is the first puncture on the bth boundary component. Then, after

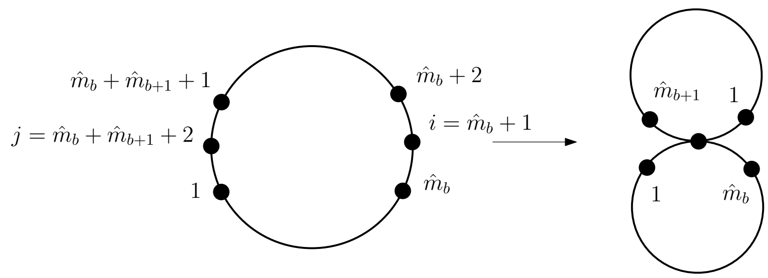

Figure 7.2: Attaching two punctures on the same boundary component with $m_b = \hat{m}_b + \hat{m}_{b+1} + 2$ punctures.

the punctures are attached to form a single boundary component, order the boundary components so that this new one goes last (i.e., becomes number $b-1$), with the same ordering of the old boundary components. Order the punctures on the new boundary component by placing the punctures coming from the old $b-1$st boundary component first, preserving their order, followed by the punctures coming from the old bth boundary component, in their old order. Keep the old ordering of punctures on the boundary components not affected by attaching.

In the case when the punctures i and j happen to be on the same boundary component, see Figure 2, change the ordering of the boundary components so that this component goes last and the puncture j is the last puncture on that boundary component. Out of the two boundary components obtained by the pinching, order the one following j counterclockwise first, the other new boundary component next, preceded by the old boundary components in the old order. Order the punctures on the two new boundary components, declaring the puncture going after the double point on the first new boundary component to be first, followed by the other punctures in the counterclockwise order, and the puncture going after the double point on the second boundary component in the counterclockwise manner first on that boundary component. Again, keep the old ordering of punctures on the boundary components not affected by attaching. Finally, define Δ_{co} as follows. Let $B\underline{M}_n^{b,m}/\varrho$ be the bundle over $\underline{M}_n^{b,m}/\varrho$ whose fiber over $\sum$ consists of all choices of an interior puncture on $\sum$. Then we get:

$$\underline{M}_n^{b,m}/\varrho \leftarrow B\underline{M}_n^{b,m}/\varrho \rightarrow \underline{M}_{n-1}^{b+1,m}/\varrho$$

where a_{co} is the map induced by declaring the chosen interior puncture to be a degenerate boundary component with no open punctures on it. Then the chain map Δ_{co} is defined as the corresponding push-pull where the pushforward of the section of our local system is defined by putting this boundary component after the other boundary components in the ordering.

It is shown in [HVZ07] that above operator is really a BV operator.

Theorem 7.0.10. *([HVZ07]) The operator $\Delta = \Delta_c + \Delta_o + \Delta_{co}$ is a graded second-order differential on the dg graded commutative algebra V and thereby defines the structure of a dg BV algebra on V.*

Proof. The fact that $\Delta(1) = 0$ follows tautologically from the definition of the three components of Δ. It will be enough to check the following identities:

$$[\Delta_c, d] = [\Delta_o, d] = [\Delta_{co}, d] = 0,$$

$$\Delta_c^2 = \Delta_o^2 = \Delta_{co}^2 = 0$$

$$[\Delta_c, \Delta_o] = [\Delta_c, \Delta_{co}] = [\Delta_o, \Delta_{co}] = 0$$

$$\Delta_c \quad \text{and} \quad \Delta_o \quad \text{are second-order differential operators,}$$

$$\Delta_{co} \quad \text{is a derivation.}$$

The operator Δ_c and the differential d commute, because the copy of S^1 acquired by twist-attaching is a closed manifold. The commutation of d with the other components of Δ is more obvious.

The fact that Δ_c is a differential, i.e., $\Delta_c^2 = 0$, comes from our definition of orientation: we place the extra component S^1 last in orientation, so that each term in $\Delta_c^2 C$ will have two extra twists S^1, as compared to the original chain C, and will be canceled by another term in $\Delta_c^2 C$, in which the two twists come in the opposite order. The property $\Delta_o^2 = 0$ is also true because of our choice of orientation. Each term in $\Delta_o^2 C$ is obtained by attaching one pair of boundary punctures together and then another pair. This term will be canceled by the term in which those two pairs of punctures are attached in the opposite order. It is a straightforward calculation to see that the signs coming from the our local system work out to cancel those pairs of terms in $\Delta_o^2 C$.

A more conceptual explanation of the same phenomenon may be done using the interpretation of the local system as the orientation sheaf of our orbifold. Note that the choice of orientation under attaching a pair of open punctures is performed in a way that we remove one factor in the top wedge power of the tangent bundle to the moduli space $\underline{M}_n^{b,m}$, leaving the other factors intact. The corresponding pairs of terms in $\Delta_o^2 C$ in which the same two pairs of punctures are attached in the opposite order will cancel each other, because the orders in which the corresponding factors are removed from the wedge product will be opposite. The same argument applies to showing $\Delta_{co}^2 = 0$ and, in fact, the graded commutation

$$[\Delta_c, \Delta_o] = [\Delta_c, \Delta_{co}] = [\Delta_o, \Delta_{co}] = 0.$$

The fact that Δ_{co} is a graded derivation of the dot product,

$$\Delta_{co}(ab) = \Delta_{co}(a)b + (-1)^{|a|} a\Delta_{co}(b),$$

is obvious: transformation of an interior puncture into a degenerate boundary component on a disjoint union of two Riemann surfaces happens on either one surface or the other.

The fact that Δ_c is a second-order derivation is equivalent to the following statement. Define a bracket

$$\{a, b\}c := (-1)^{|a|}\Delta_c(ab) - (-1)^{|a|}\Delta_c(a)b - a\Delta_c b.$$

Then this bracket is a graded derivation in each (or equivalently, one) of its variables, that is

$$\{a, bc\}_c = \{a, b\}_c c + (-1)^{(|a|+1)|b|} b\{a, c\}_c. \tag{7.3}$$

What is clear from the definition, the geometric meaning of the bracket $\{a, b\}_c$ is $(-1)^{|a|}$ multiplied by the alternating sum of twist-attachments over all pairs of closed punctures for the chains a and b, respectively. Given three geometric chains a, b, and c, Equation (13) is obvious, as it just says that twist-attaching of closed punctures in a with those in the disjoint union of b and c breaks into twist-attaching with punctures in b and c

and then taking the disjoint union. The signs come out right, because of our definition of orientation under twist-attaching and disjoint union.

The same argument applies to Δ_o. Consider a bracket

$$\{a, b\}_o := (-1)^{|a|}\Delta_o(ab) - (-1)^{|a|}\Delta_o(a)b - a\Delta_o b$$

and show it satisfies the derivation property

$$\{a, bc\}_o = \{a, b\}_o + (-1)^{(|a|+1)|b|}b\{a, c\}_o.$$

Remark. Note that the part Δ_o' of Δ_o corresponding to attaching punctures lying on the same boundary component is actually a derivation, and therefore the open part of the antibracket comes only from Δ_o'' corresponding to attaching punctures lying on different boundary components:

$$\{a, b\}_o' = 0, \qquad \{a, b\}_o = \{a, b\}_o''.$$

As for geometric chain complex of above, in [HVZ07], it was discussed that If we impose extra equivalence relations, such as some kind of a suspension isomorphism, as in [Jak00], or work with piecewise smooth geometric chains and treat them as currents, in the spirit of [FOOO00], we may obtain a complex whose homology is isomorphic to the ordinary real homology. A similar work, which shows that the Kuranish homology, defined via Kuranish space rather than simplical complex, is isomorphic to singular homology, was done by Dominic Joyce ([Joy07]). Thus, we make a conjecture, *our geometric chains gives to ordinary homology, possibly under some extra equivalent relations.*

Chapter 8

The Weyl algebra and the Fock space associated to a functor

I describe briefly the construction of Weyl algebra and the Fock space of Costello ([Cos04]).

Let $F : H_*A \to Comp_K^{Z/2}$ be a symmetric monoidal functor. We will construct an associated Weyl algebra and Fock space.

8.1 The construction in a simplified case.

Let us first consider the simplified case when F is split. Let $V = F(1)$. Then V has a circle action, and we have the auxiliary equivariant chain complexes, V_{hS^1}, V^{hS^1} and V_{Tate}.

Define an antisymmetric form Ω on V_{Tate} by

$$\Omega(vf(t), wg(t)) = \langle v, w\rangle Resf(-t)g(t)dt$$

This is the same as the form used in the work of Givental and Coates [**?**, **?**, **?**]. In the case when the inner product on V is non-degenerate this is symplectic. Note that Ω is compatible with the differential, that is

$$\Omega(d(vf(t)), wg(t)) + (-1)^{|v|}\Omega(vf(t), d(wg(t))) = 0$$

This follows from the fact that on V, d is skew self adjoint and D is self adjoint with respect to the pairing $\langle\rangle$. Thus, we have an associated Weyl algebra $\mathcal{W}(V_{\text{Tate}})$. V_{Tate} is polarised, as $V_{\text{Tate}} = V_{hS^1} \oplus V^{hS^1}$. The differential on V_{Tate} preserves V^{hS^1}, but not in general V_{hS^1}. Let $\mathcal{V}_{\text{Tate}}$ be the associated Fock space. This is defined to be the quotient of $\mathcal{W}(V_{\text{Tate}})$ by the left ideal generated by V^{hS^1}. We can identify

$$\mathcal{V}_{\text{Tate}} = Sym^* V_{hS^1}$$

As, we can consider $Sym^* V_{hS^1}$ as a subalgebra of $\mathcal{W}(V)$, using the splitting of the map $V_{\text{Tate}} \to V_{hS^1}$. Then the action on this on the element $1 \in \mathcal{V}_{\text{Tate}}$ gives the isomorphism. This is not an isomorphism of complexes, however, because $V_{hS^1} \subset V_{\text{Tate}}$ is not a subcomplex.

Let us write the natural differential on $\mathcal{V}_{\text{Tate}}$ as $\widehat{\mathrm{d}}$. This is the differential obtained by realising it as a quotient of $\mathcal{W}(V_{\text{Tate}})$. This is an order 2 differential operator. Let d denote the usual differential on $Sym^* V_{hS^1}$, which we identify with $\mathcal{V}_{\text{Tate}}$. Then we can write

$$\widehat{\mathrm{d}} = \mathrm{d} + \triangle$$

where $\triangle$ is an odd order 2 differential operator on $\mathcal{V}_{\text{Tate}}$, and satisfies

$$[\mathrm{d}, \triangle] = \triangle^2 = 0$$

We can describe $\triangle$ explicitly. It is an order 2 differential operator on $Sym^* V_{hS^1}$. Such an operator is uniquely characterised by its behaviour on $Sym^{\leq 2} V_{hS^1}$. $\triangle$ is zero on $Sym^{\leq 1} V_{hS^1}$, and for $(v_1 f_1(t_1))(v_2 f_2(t_2)) \in Sym^2 V_{hS^1}$, we have

$$\triangle((v_1 f_1(t_1))(v_2 f_2(t_2))) = \langle \mathrm{D}\, v_1, v_2 \rangle \operatorname{Res} f_1(t_1) f_2(t_2) \, \mathrm{d}\, t_1 \, \mathrm{d}\, t_2$$

8.2 The construction in general

We want to mimic this construction in general. Let $F : H_*(A) \to Comp_K^{Z/2}$ be any symmetric monoidal functor. On $F(n)$ there are n commuting circle actions, that is operators D_i for $1 \leq i \leq n$, which (super)-commute and square to zero. Thus we can form the various auxiliary complexes,

$$\begin{aligned}F_{\mathrm{Tate}}(n) &= F(n)\otimes\mathbb{K}((t_1,\dots,t_n))\\ F^{hS^1}(n) &= F(n)\otimes\mathbb{K}[[t_1,\dots,t_n]]\\ F_{hS^1}(n) &= F(n)\otimes\mathbb{K}((t_1))/\mathbb{K}[[t_1]]\otimes\dots\mathbb{K}((t_n))/\mathbb{K}[[t_n]]\end{aligned}$$

with differential

$$\mathrm{d}+\sum t_i\,\mathrm{D}_i$$

Let $G_{ij} : F(n) \to F(n-2)$ be the gluing map, coming from the class of a point in $H_0(A(2,0))$. For $1 \le i < n$ denote by

$$\Omega_i : F_{\mathrm{Tate}}(n) \to F_{\mathrm{Tate}}(n-2)$$

the map defined by

$$a\otimes f(t_1,\dots,t_n)\mapsto G_{i,i+1}(a)\otimes \mathrm{Res}_{z=0}\, f(t_1,\dots,t_{i-1},-z,z,t_i,\dots,t_{n-2})\,\mathrm{d}\,z$$

For each $1 \le i < n$, let $\sigma_i \in S_n$ be the transposition of i with $i+1$. Recall S_n acts on $F(n)$; this action extends to each of the auxiliary complexes mentioned above.

There are tensor product maps $F_{\mathrm{Tate}}(n)\otimes F_{\mathrm{Tate}}(m) \to F_{\mathrm{Tate}}(n+m)$, and similarly for F_{hS^1} and F^{hS^1}. The space $\oplus_n F_{\mathrm{Tate}}(n)$ is an associative algebra, with product coming from these tensor product maps.

We define the *Weyl algebra* $\mathcal{W}(F)$ to be the quotient of $\oplus_n F_{\mathrm{Tate}}(n)$ by the two-sided ideal generated by the relation,

$$x-\sigma_i(x)=\Omega_i(x)$$

for each $x \in F_{\mathrm{Tate}}(n)$.

The *Fock space* $\mathfrak{F}(F)$ is defined to be the quotient of $\mathcal{W}(F)$ by the left ideal spanned by those elements $x \in F_{\mathrm{Tate}}(n)$ which contain no negative powers of t_n.

We can consider

$$\oplus F_{hS^1}(n)_{S_n}$$

to be a subalgebra of $\mathcal{W}(F)$, using the standard splitting of the map $F_{\mathrm{Tate}}(n) \to F_{hS^1}(n)$. Here the subscript S_n refers to coinvariants, so that $\oplus F_{hS^1}(n)_{S_n}$ is a commutative algebra. The action of $\oplus F_{hS^1}(n)_{S_n}$ on the vector $1 \in \mathfrak{F}(F)$ generates $\mathfrak{F}(F)$, and induces an isomorphism

$$\mathfrak{F}(F) \cong \oplus F_{hS^1}(n)_{S_n}$$

As before, this is *not* an isomorphism of complexes. We will refer to the natural differential on the left hand side as $\widehat{\mathrm{d}}$, and that on the right hand side as d. The differential $\widehat{\mathrm{d}}$ is an order 2 differential operator, whereas d is a derivation.

It is easy to see that, as before,

$$\triangle \stackrel{\text{def}}{=} \widehat{\mathrm{d}} - \mathrm{d}$$

is an order two operator which satisfies $\triangle^2 = [\underset{\cdot}{,}\triangle] = 0$. As before, we can write this operator down explicitly. For $1 \leq i < j \leq n$, define a map $\triangle_{ij} : F_{hS^1}(n) \to F_{hS^1}(n-2)$ by

$$\begin{aligned}\triangle_{ij}(a \otimes f(t_1, \dots, t_n)) = G_{ij}(\mathrm{D}_i\, a)\otimes \\ \operatorname{Res}_{z=0} \operatorname{Res}_{w=0} f(t_1, \dots, t_{i-1}, z, t_i, \dots, t_{j-2}, w, t_{j-1}, \dots, t_{n-2})\, \mathrm{d}\, z\, \mathrm{d}\, w\end{aligned}$$

(In this expression z is in the i'th position and w is in the j'th position, and the remaining places are filled with $t_1, \dots, t_{n-2}$ in increasing order).

Then, $\sum_{i<j} \triangle_{ij}$ commutes with symmetric group actions, and so descends to give a map

$$\triangle : \oplus F_{hS^1}(n)_{S_n} \to \oplus F_{hS^1}(n)_{S_n}$$

It is now not difficult to check that $\widehat{\mathrm{d}} = \mathrm{d} + \triangle$.

We will need these constructions when F is the functor $C'_*(\mathcal{M})$ associated to moduli spaces of curves. In that case we use the notation $\mathcal{W}(\mathcal{M})$, $\mathfrak{F}(\mathcal{M})$.

Note that if $F \to G$ is a natural transformation of functors $H_*(A) \to \mathrm{Comp}_{\mathbb{K}}^{\mathbb{Z}/2}$, there is an associated homomorphism of Weyl algebras $\mathcal{W}(F) \to \mathcal{W}(G)$, and a map $\mathfrak{F}(F) \to \mathfrak{F}(G)$ of $\mathcal{W}(F)$ modules.

Chapter 9

Existence and uniqueness of solutions to quantum master equation

In this section, I will show there is a unique (up to homotopy) solution to our Quantum Master Equation.

A Riemann surface with open-closed boundary has been introduced in section 3.

Let's recall the definition of category S_Λ in section 3. The objects are pairs of finite sets O, C and maps $s, t : O \to \Lambda$, the morphism space $S(O_1, C_1, s_1, t_1; O_2, C_2, s_2, t_2)$ are, the moduli space of Riemann surface with open-closed boundary, i.e., some boundary components are parameterised and labelled as closed (incoming or outgoing); some intervals embedded in the remaining boundary components which are also paremeterised and labelled as incoming and outgoing; The complement of closed boundaries and open intervals are called free boundary, which are either circles or intervals. The closed boundaries, open boundaries and free boundaries are labelled by sets C, O, Λ respectively and compatible with the maps s, t. In this section, we don't need to care about how open boundary ends in D-brane, so we simply omit s, t. We define the space $M_{O,C,\Lambda} \subset S(0, 0, 0; O, C, \Lambda)$ as the moduli space of Riemann surfaces with O, C, Λ type of (outgoing) open-closed boundary. Such surfaces may be disconnected; also they may have connected components with no boundary. Define $n = |C|, m = |O|$ and b=number of boundary components which contains open intervals. Let $M_{g,n}^{m,b} \subset M_{O,C,\Lambda}$ be the subspace of connected surfaces of genus g and with corresponding parameters n, m, b. And for the following part of the paper, for simplicity, we denote $\mathfrak{F}H := S_n \times S_b \ltimes Z_1 \times \cdots \times Z_{m_b}$

Consider the complex

$$\mathcal{F}(M) = \oplus C_*^{geom}(M(O, C, \Lambda))$$

$$= \oplus_{n,m,b} C_*^{geom}(M_n^{b,m})$$

The space $M_n^{b,m}$ is the moduli space of Riemann surface with parameter n, m, b, just like the space $M_{g,n}^{b,m}$, but in general having multiple connected components of various genuses. A *type* of such a chain are the parameters b, m, g, n so that when we forget the parametrization of closed boundary component (i.e.,S^1 parametrization) our chain fall into $M_{g,n}^{b,m}$. Our degree of chain is not the actual degree, we define it to be the difference between the dimension of the type space of each corresponding connected component and the actual degree. Hence, the differential is of degree 1. $F(M)$ has a commutative algebra structure induced by disjoint union. We formally add a copy of the ground field Q to $F(M)$, and the unit element $1 \in Q \subset V$ is the fundamental chain of the one-point moduli space comprised by the empty Riemann surface

We have an inclusion $C_*^{geom}(M_{g,n}^{b,m}) \longrightarrow F(M)$. Denote by $F_{g,n}^{b,m}$ this subspace.

Similar to proposition 9.0.2 of [Cos07], we have the following propositon:

Proposition 9.0.1. *For each g, n, m, b with $2 - 2g - n - b - m/2 < -\frac{1}{2}$, there exists an element $S_{g,n}^{m,b} \in F_{g,n}^{m,b}$ of degree 0, with the following properties.*

(1)$S_{0,3}^{0,0}$ is a 0-chain in the moduli space of Riemann spheres with 3 unparameterised, unordered closed boundaries and with no open or free boundaries.

(2)Form the generating function

$$S = \sum_{\substack{g,n,m,b>=0 \\ 2g+n+b+m/2-2>0}} \hbar^{p-\chi} \lambda^{-2\chi} S_{g,n}^{m,b} \in \lambda F(M)[[\sqrt{\hbar}, \lambda]].$$

here $p = 1 - \frac{m+n}{2}, \chi = 2 - 2g - n - b - m/2$.

S satisfies the Batalin-Vilkovisky quantum master equation:

$$\hat{d} e^{S/\hbar} = 0$$

(Remember that in dg BV algebra $B[[\hbar, \lambda]]$*, we let* $\hat{d} = d + \hbar\Delta$*)*

Equivalently,

$$\hat{d} S + \frac{1}{2}\{S, S\} = 0$$

Further, such an S is unique up to homotopy through such elements.

A homotopy of such elements is a solution of the master equation in $F(M) \otimes K[t, \epsilon]$, satisfying the analogous conditions. Here t has degree 0 and ϵ has degree -1, and $dt = \epsilon$.

Proof. Let $M_{g,n}^{b,m}$ be the moduli space of Riemann surface of genus g and b boundaries and with n marked points in the interior, m_i marked points on the ith boundary and of genus g, this is first studied by C.C.Liu [Liu02]. It is an orbifold with corners.

From section 4, we know the following homology estimate of $M_{g,n}^{b,m}/\mathfrak{F}H$

$$H_i(M_{g,n}^{b,m}/\varrho, Q) = 0, \quad \text{for} \quad i \geq 6g - 7 + 2n + 3b + m$$

except $(g, n, b, \vec{m}) = (0, 3, 0, 0), (0, 2, 1, 0), (0, 0, 2, (1, 0)), (0, 0, 2, (1, 1)), (0, 1, 1, 1)$ $(0, 1, 1, 2), (0, 0, 1, 3), (0, 0, 1, 4)$.

$$H_i(M_{g,n}^{b,m}/\varrho, Q) = 0, \quad \text{for} \quad i \geq 6g - 7 + 2n + 3b + m$$

with excpetional cases above.

We indeed need a homological dimension estimate for twisted coefficient in section 7. We have ([Vor10]) also

$$H_i(M_{g,n}^{b,m}/\varrho; Q^{\varepsilon}) = 0, \quad \text{for} \quad i \geq 6g - 7 + 2n + 3b + m \tag{9.1}$$

now the exceptional cases are $(g, n, b, m) = (0, 0, 1, 3), (0, 0, 2, (1, 1)), (0, 1, 1, 1)$, $(0, 2, 1, 1)$, and $(0, 3, 0, 0)$

This can be seen from a fact that the left side is equal to the homology of invariant chains in labelled moduli space tensored with the representation of ϱ defining the local coefficient by using the similar reasoning as the corresponding result concerning the relation between twisted homology and usual homology of the universal covering space ([Hat02]), which in turn can be computed as the invariants of the action of ϱ on the homology of labelled moduli space tensored the representation.

Now define a dg Lie algebra g. The space g_i is the set of

$$S = \sum \hbar^{p-\chi} \lambda^{-2\chi} S_{g,n}^{b,m} \in \lambda F(M)[[\hbar, \lambda]]$$

such that $S_{g,n}^{b,m} \in F_{g,n}^{b,m}(M)$, and $S_{g,n}^{b,m}$ is of degree $-1-i$. The bracket $[,]_g$ is $\{\}$ and the differential is $d_g = \hat{d}$.

It is easy to see, by definition, the set of homotopy equivalence classes of solutions of the Maurer-Cartan equation in g_i is the same as the set of homotopy equivalence classes of solutions S of the master equation in $F(M)$ satisfying $S_{g,n}^{b,m} \in F_{g,n}^{b,m}(M)$ and $S_{g,n}^{b,m}$ is of degree $-1-i$. Filter g by saying $F^k g$ is the set of those S such that $S_{g,n}^{b,m}$ is zero for $2g-2+n+b+m/2 < k$. Then

$$g = F^1 g \supset F^2 g \ldots$$

is a descending filtration by dgla ideals. g is complete with respect to this filtration.

The bounds on the homological dimensions of moduli spaces with twisted coefficients, together with the fact that all the exceptional cases above have Euler characteristic $-\frac{1}{2}, -1$ or $-\frac{3}{2}$ tell us that

$$H_i(F^k g / F^{k+1} g) = 0 \quad \text{for} \quad i = 0, -1, -2 \quad \text{and} \quad k > 1$$

$$\oplus_{i=-2}^{0} H_i(g/F^2 g) \neq 0$$

(e.g.,$M_{0,3}^{0,0}$ contributes a nonzero element in $H_{-1}(g/F^2 g)$)

Therefore the map $g \to g/F^2 g$ satisfies the conditions of lemma 6.6. The result follows immediately.

Chapter 10

Further Question

There are two further questions in this direction.

1. If the open-closed Gromov-Witten theory is defined in the future, then just as in the closed TCFT constructed by Costello, there may also have an open-closed Gromov-Witten potential for OC TCFT, which should alos be a ray in the Fock space associated to a periodic cyclic homology which is a dg symplectic vector space.

2. As mentioned by Costello, the fundamental chain constructed here gives operations between spaces of morphisms in the Calabi-Yau A_∞ category and the o Hochschild complex. This structure is a kind of quantisation of the A_∞ structure. For the A model, these operators should correspond to "counting" surfaces with Lagrangian boundary conditions and with marked points constrained to lie in certain cycles.

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Appendix A

Group (co)homology Assume G is a group, a G-**module** is an abelian group M together with an action of G on M, with every element of G acting as an endomorphism of M. In the following we write G multiplicatively and M additively.

Given a G-module M, the subgroup of G-invariant elements is:

$$M^G = \{x \in M : \forall g \in G \quad gx = x\}$$

Now, if N is a submodule of M (i.e. a subgroup of M mapped to itself by the action of G), it is general not true that the invariants in M/N are found as the quotient of the invariants in M by N: being invariant 'up to something in N ' is broader. The first group cohomology $H^1(G, N)$ precisely measures this difference. The group cohomology functors H^n in general measure the extent to which taking invariants doesn't respect exact sequences.

Now we give a formal definition of group cohomology. Before that, we need the following fact: the set of all G-module is an abelian category. (the morphisms are group homomorphisms f with the property $f(gx) = g(f(x))$ for all g in G and x in M). This category is an abelian category since it is isomorphic to the category of modules over the group ring $Z[G]$ (use the fact a G-module is equivalent to a $Z[G]$-module, with $Z[G]$ is rather a ring (and Z-module)). Denote this category as **Gmod**. And let **Gr** be the category of abelian group.

Definition A.0.2. Let f : **Gmod** $\rightarrow$ **Gr** be a functor from **Gmod** to **Gr** which takes a G-module M to the group of its invariant M^G. This functor is left exact. The nth right derived functor is called the nth cohomology of G with M coefficient, denoted $H^n(G, M)$.

Dually, we can define group homology. Let M be a G-module as above, and define DM to be the submodule generated by elements of the form $gm - m, g \in G, m \in M$. The *coinvariant* of M by the action of G is defined as:

$$M_G := M/DM$$

Taking M to M_G is a right exact functor. The left derived functors are defined to be group homology.

Definition A.0.3. Let $g : \mathbf{Gmod} \to \mathbf{Gr}$ be a functor from **Gmod** to **Gr** which takes a G-module M to the group of its coinvariants M_G. This functor is right exact. The nth left derived functor is called the nth homology of G with M coefficient, denoted $H_n(G, M)$.