

**Design of Progressive Additional Lens with
Wavefront Tracing Method**

**A DISSERTATION
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY**

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**IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
Doctor Of Philosophy**

September, 2010

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Acknowledgements

I would like to thank my advisor Prof. Fadil Santosa for all his advice and discussions. I always feel fortunate that Prof. Santosa introduced me to the area of Progressive Lens Design. His positive attitude toward research, his patience on my research and his broad vision in applied math always inspired me. I am also grateful for his careful reading of this thesis. Without his help, it would not be possible for finish this thesis.

I would like to thank Prof. Robert Gulliver for reviewing my work and serving as the chair of the committee of my thesis defense. The discussions with Prof. Gulliver on Differential Geometry extended my understanding of lens design. I would also like to thank Prof. Bernardo Cockburn and Prof. Hui Zou serving as committee members on my thesis.

I would like to thank Prof. Fadil Santosa, John Dodson, and MCIM for the internship opportunity in Ameriprise Financial, which raised my research interest in Financial Mathematics.

Certainly, I am grateful to my family, especially my parents for consistent support during my study in USTC and UMN. Their encouragement always keeps me going forward. I dedicate this thesis to them. Finally, I would want to take a chance to thank my teachers and friends in the last twenty-two years of study. My life would be a lot different without their help.

Abstract

Progressive addition lenses (PAL) are corrective lenses prescribed to patients with presbyopia. In PAL power varies smoothly over the lens, allowing the wearer to compensate for both far vision and near vision corrections. A PAL lens comprise of at least one complex surface to meet the requirement of having variable power depending on the gaze direction. A typical PAL consists of a large distance zone with low power on the upper portion of the lens, a small near distance zone with higher power on the lower part, and a corridor of increasing power connecting these two zones smoothly and progressively.

In this work, we investigate a variational approach to PAL design problem in which a cost function consists of two parts. The first part, when small, implies that the power distribution is close to the desired one. The second part, when minimized, makes the optical aberrations small. The goal is to minimize the cost function.

Previous work on progressive additional lens uses approximations of the physical optics involved in propagation and refraction of light. These approximations simplify the calculation of power and astigmatism, a first order aberration, to knowledge of the principal curvatures of the lens surfaces, and are not capable of calculating higher order aberrations. To better evaluate the properties of the optical system, we use geometrical optics. This involves tracing an optical ray through the lens and calculating how the incident wavefront is altered through propagation and refraction. The approach is to form a third-order Taylor expansion of the wavefront at the ray. We develop formulas that relate the Taylor coefficients as the wavefront propagates and as it is refracted. These higher order Taylor coefficients are important in understanding optical aberrations.

The formulas we derived can be used to accurately calculate optical properties of a progressive additional lens. For numerical implementation, we use a tensor-product B-spline to represent the lens surfaces. Properties of a lens in a given gaze

direction can be calculated by performing numerical calculations. In this way, the PAL design problem can be formulated as a numerical optimization problem in which lens properties over a set of gaze directions are optimized. The formulation can be applied to front surface or back surface lens design.

The optimization problem can be solved numerically by a number of methods. We consider gradient descent method, Newton's method and the quasi-Newton method. As a demonstration of the approach proposed in this work, we provide an example of a progressive addition lens whose front surface is optimized.

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Chapter 1

Introduction

The human eye is able to alter the curvature of its lens, and thus its power, in order to focus on objects at different distances. However, the ability of the eye to accommodate, i.e. to change the shape of the lens, decreases with age. Such a defect is known as *presbyopia*, and the most common symptom is the inability of the patient to focus on near objects, especially in poor lighting conditions.

Some complicated lenses, such as bifocal lens which consists of two single-vision lenses with different powers, and the trifocal lens which add one more segment as the intermediate vision, has been introduced to deal with presbyopia. The major drawback of these lens is the jump in the image caused by switching between the distance vision and the near vision.

Progressive additional lens (PAL) is often considered as a better solution for presbyopia patients. A PAL has different corrective powers depending on the gaze direction of the wearer's eye. With PAL the patient can focus on objects at various distances by using different parts of the lens.

A PAL consists of a large distance view zone on the upper part of the lens and a small near view zone on the lower part. Typically, the power in the near view zone is higher (less negative for myopic patients) than that in the large distance view zone. Between these two zones, the power increases monotonically and smoothly. The progression of power is the result of a local variation in the curvatures of lens

surface.

An ideal PAL is one with the prescribed smooth progressive power and with zero astigmatism everywhere. Astigmatism, or cylinder is an aberration and is undesirable. But the surface which eliminate the astigmatism everywhere is either a sphere or a plane. Both cases do not provide progressive power. Therefore, the best one can hope for is a lens that has the smooth progressive power and as small astigmatism as possible. Actually, in progressive lens design, certain region are required to have very small astigmatism, while the rest of the lens, the astigmatism should be as small as possible. The regions of very small astigmatism are areas of the lens critical to the eye for distance, intermediate and near zones.

The first PAL was invented in 1967 by Maitenaz [10]. The lens design is based on simple heuristics and makes use of a few free parameters. Since the introduction of the Maitenaz lenses, there have been many patented designs during the past 40 years. The methods by T. Baudaut, F. Ahsbahs, and C. Miede [1] and J. T. Winthrop [21] assign power along a curve and then distribute them in some manner across the rest of the lens. The construction is done in such a way that the cylinder is as small as possible in the critical areas of the lens. However, such a design method is usually not quite satisfactory since there is no real control over the distribution of the cylinder. Another direct approach is to compute a main curve with increasing mean curvature on the front surface. The whole surface is then defined by a conic which is moved along the main curve, also having increasing mean curvature from top to bottom. Instead of considering the lens surface as a whole, it only takes account of some sample points, which may bring large unwanted cylinder in important areas of the lens.

Indirect methods based on variational methods have also been proposed by lens designers. A cost functional balancing the power distribution and the unavoidable cylinder are optimized. One example of such cost functional is given by J. Loos, G. Greiner, and H.P. Seidel [10]. Further improvements over this idea, using simplified formulas from ray tracing, appears in In J. Loos, Ph. Slusallek, and H.P. Seidel [11]. The process is to construct a surface with power close to the

desired distribution, and with small total weighted cylinder over the lens. D. Katzman and J. Rubinstein [9] adapted the same error functional to form the minimization problem. However, instead of solving the minimization, they chose to solve the nonlinear equations corresponding to the Euler-Lagrange equations. Furthermore, they chose a finite element method whereas Loos et al [11] used tensor-product splines.

J. Wang, R. Gulliver and F. Santosa [18] [17], and J. Wang and F. Santosa [19] considered an approximation which makes the functional to be minimized quadratic. They gave a mathematical analysis of the simplified problem and applied a Finite Element Method to solve for the resulting partial differential equation. In their patent [20], the authors extended the method for design of lenses with prescribed astigmatism.

While the results of these type of approaches have been quite good, there are some rooms for improvements. First, the resulting designs are very dependent on the choice of the weights used in matching the desired power distribution and the cylinder. Second, these methods are based on thin lens approximations (with exception of [10]). Third, they does not account for second order optical aberrations.

In previous progressive additional lens designs, the power of the lens is the result of a local variation in the curvatures of the surface. In ophthalmic optics, the power at each point is given by

$$\text{Power} = (1 - n)P^b + \frac{(n - 1)P^f}{1 - d(1 - \frac{1}{n})P^f},$$

where n is the index of refraction of the lens material, d is the thickness of the lens, P^f and P^b are the mean curvatures of the front and back lens surface. For a thin lens design ($d \ll 1$), the formula can be simplified to

$$\text{Power} = (n - 1)(P^f - P^b).$$

Also, if we assume the back surface is a surface of constant mean curvature,

the local cylinder or the astigmatism of the lens is defined as

$$A = (n - 1)|\kappa_1 - \kappa_2|,$$

where κ_1 and κ_2 are the two local principal curvatures of the front lens surface.

Although these formulas are widely used in progressive lens design industry, there are some shortcomings. First, the derivation of the formula only approximate the refraction on the front and back lens surfaces. Also, assuming $d \ll 1$ would limit the design to be for the thin lenses, while it is not suitable for general lens design.

To model the light propagation more accurately, one must consider geometrical optics. In geometrical optics, light propagates along rays, and lenses act to diffract these rays. A wavefront is a surface of points near a ray having the same time phase. To understand the optical properties of a lens, one must study how a wavefront is deformed by a lens and how the curvature of the wavefront is transformed when propagating through a homogeneous medium and refracting on a lens surface. By choosing special coordinates, J. Kneisly [8] gave explicit formulas of the principal curvatures of the propagated wavefront and refracted wavefront.

Wavefront tracing method can then be used to evaluate the exact properties of the optical system. J. Loos et al. [11] applied wavefront tracing method to evaluate the refracted wavefront. However, in their design approach, an approximation is made to simplify the calculation of power and astigmatism, leading to an approximation very much like a thin lens approximation. J. Rubinstein [14] approximates wavefronts as quadratic surfaces and calculate the refracted wavefront. By doing so, he introduced a cost functional that involves prism (a second order aberration) in addition to power and astigmatism. In general wavefront methods, while adding complexity to the design process, allow for much more accurate physics and should, in principle, lead to better lenses.

This thesis is organized as follows. We give a brief review of theory of surfaces in chapter 2. The concept of Third Order Surface coefficients is introduced. Then surface approximation can be obtained by applying the Second Fundamental Form

coefficients and the Third Order Surface coefficients.

Chapter 3 gives an introduction to the geometric optics theory. A wavefront is then used to characterize the lens performance. Zernike polynomials are employed to represent high-order wavefront aberrations. Then a lens design problem is formed by minimizing a design objective functional consisting of Zernike polynomial coefficients.

In Chapter 4, we derive formulas describing how the First Fundamental Form coefficients, the Second Fundamental Form coefficients and the Third Order Surface coefficients are changed during propagation and by a refracting surface with a different index of refraction. In contrast to Kneisly's formula of the Second Fundamental Form coefficients under special coordinates, we extend the formulas to the explicit expression under the general coordinates, which are convenient for calculating the gradient of the curvatures.

Chapter 5 describes the process of ray tracing method, assuming the back surface in nonparametric form $z = b(x_1, x_2)$ and the front surface in nonparametric form $z = f(x_1, x_2)$. We start from the eye center O , pass through the point $P = (x_1, x_2, b(x_1, x_2))$ on the back lens surface. After the refraction on the front surface, the ray intersects with the front surface at $P' = (x'_1, x'_2, f(x'_1, x'_2))$. A planar wavefront is formed at P' , and by tracing the wavefront back, we derive the expression of mean curvature and Gaussian curvature of the wavefront refracted by the back lens surface. We also discuss the design process of front surface design and back surface design. The gradient of the design objective functional with respect to the corresponding surface is calculated. We also introduce tensor-product B-spline functions to represent the lens surface.

In Chapter 6, representing of the lens surface by tensor-product B-spline function, we arrive at a constrained nonlinear optimization problem where the constraints, imply that the front lens surface s^f and the back lens surface s^b do not intersect. After the coordinate change of $s^f = s^b + u^2$, we then transfer the problem into an unconstrained nonlinear optimization problem for u^2 . Numerical schemes such as gradient method, Newton's method, and quasi-Newton method,

are applied to generate a solution. An introduction to Unconstrained Optimization theory is also given in this chapter.

Chapter 7 shows a numerical example where we solve a front lens surface design problem with the back lens surface set to be a sphere surface with power 11.94 diopter. Weight functions used in optimization for power and astigmatism are discussed in this chapter.

In summary, this thesis develops a model of the progressive lens design up to second order aberration by wavefront tracing method. New formulas are given to represent the curvature of the wavefront passing through the lens. An effective numerical scheme has been used for optimization approach to progressive additional lens design.

Chapter 2

Theory of surfaces

2.1 The First Fundamental Form and the Second Fundamental Form

Definition 2.1.1. *The inner product of $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in Euclidean space \mathbb{R}^n is*

$$\langle x, y \rangle = \langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle \triangleq \sum_{i=1}^n x_i y_i = x_1 y_1 + \dots + x_n y_n.$$

Given a parametric surface $R(x_1, x_2) = (r_1(x_1, x_2), r_2(x_1, x_2), r_3(x_1, x_2))$, let

$$g_{ij} = \left\langle \frac{\partial R}{\partial x_i}, \frac{\partial R}{\partial x_j} \right\rangle.$$

Then the First Fundamental Form is defined as

$$I = g_{11} dx_1^2 + 2g_{12} dx_1 dx_2 + g_{22} dx_2^2. \quad (2.1)$$

The values $\{g_{ij}\}_{i,j=1,2}$ are called the coefficients of the First Fundamental Form.

Let

$$n = \frac{\frac{\partial R}{\partial x_1} \times \frac{\partial R}{\partial x_2}}{\left| \frac{\partial R}{\partial x_1} \times \frac{\partial R}{\partial x_2} \right|},$$

be the unit outward normal vector to the surface,

$$L_{ij} = \left\langle \frac{\partial R^2}{\partial x_i \partial x_j}, n \right\rangle = - \left\langle \frac{\partial R}{\partial x_i}, \frac{\partial n}{\partial x_j} \right\rangle = - \left\langle \frac{\partial R}{\partial x_j}, \frac{\partial n}{\partial x_i} \right\rangle.$$

The Second Fundamental Form is defined as

$$II = L_{11}dx_1^2 + 2L_{12}dx_1dx_2 + L_{22}dx_2^2. \quad (2.2)$$

We call $L = \{L_{ij}\}_{i,j=1,2}$ the coefficients of the Second Fundamental Form.

2.2 Christoffel symbols and the equations of Weingarten

Let $R_i = \frac{\partial R}{\partial x_i}$, $R_{ij} = \frac{\partial^2 R}{\partial x_i \partial x_j}$ and $n_i = \frac{\partial n}{\partial x_i}$, since R_{ij} , n_i lie in the space formed by R_i , R_j and n , we have

$$R_{ij} = \Gamma_{ij}^k R_k + L_{ij} n, \quad (2.3)$$

where

$$\Gamma_{ij}^k = \frac{1}{2} g^{lk} \left(\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right)$$

are the Christoffel symbols,

$$(g^{ij}) = (g_{ij})^{-1} = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix},$$

therefore, $g^{ij}g_{jk} = \delta_{ik}$, where

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

The equations of Weingarten give the derivatives of the normal vector,

$$n_j = -g^{ki} L_{ij} R_k. \quad (2.4)$$

2.3 The Third Order Surface coefficients

Definition 2.3.1. *Let*

$$\Lambda_{ijk} = \langle R_{x_i x_j x_k}, n \rangle, \quad (2.5)$$

where $i, j, k = 1, 2$, then we call $\Lambda = \{\Lambda_{ijk}\}_{i,j,k=1}^2$ the Third Order Surface coefficients of R w.r.t. coordinate system x_1, x_2 .

In particular, by $\langle R_i, n \rangle = 0$,

$$\Lambda_{ijk} = -\langle R_{ij}, n_k \rangle - \langle R_{ik}, n_j \rangle - \langle R_i, n_{jk} \rangle. \quad (2.6)$$

From this definition, $\Lambda_{221} = \Lambda_{122} = \Lambda_{212}$, and $\Lambda_{121} = \Lambda_{211} = \Lambda_{112}$.

Theorem 2.3.2. *The Third Order Surface coefficients of a surface are related to the coefficients of the First and Second Fundamental Form through*

$$\Lambda_{ijk} = \frac{\partial L_{ij}}{\partial x_k} + \Gamma_{ik}^n L_{nj}. \quad (2.7)$$

Proof. Consider the derivatives of the coefficients of the Second Fundamental Form

$$\begin{aligned} \frac{\partial L_{ij}}{\partial x_k} &= -\frac{\partial}{\partial x_k} \langle R_{x_i}, n_{x_j} \rangle = -\left\langle \frac{\partial R_{x_i}}{\partial x_k}, n_{x_j} \right\rangle - \left\langle R_{x_i}, \frac{\partial n_{x_j}}{\partial x_k} \right\rangle \\ &= -\langle R_{ik}, n_j \rangle - \langle R_i, n_{jk} \rangle. \end{aligned}$$

Then by 2.6,

$$-\langle R_{ik}, n_j \rangle - \langle R_i, n_{jk} \rangle = \Lambda_{ijk} + \langle R_{ij}, n_k \rangle$$

$$\frac{\partial L_{ij}}{\partial x_k} = \Lambda_{ijk} + \langle R_{x_i x_k}, n_{x_j} \rangle.$$

With the Christoffel symbols and the equations of Weignarten's,

$$\frac{\partial L_{ij}}{\partial x_k} = \Lambda_{ijk} + \langle \Gamma_{ik}^l R_l + L_{ik} n, -g^{mn} L_{nj} R_m \rangle.$$

Since

$$\langle R_l, R_m \rangle = g_{lm}$$

and

$$\langle R_m, n \rangle = 0,$$

we have

$$\frac{\partial L_{ij}}{\partial x_k} = \Lambda_{ijk} - \Gamma_{ik}^l g^{mn} L_{nj} g_{lm}.$$

Using $g^{mn} g_{lm} = \delta_{ln}$, we get

$$\begin{aligned} \frac{\partial L_{ij}}{\partial x_k} &= \Lambda_{ijk} - \Gamma_{ik}^1 L_{nj} \delta_{ln} \\ &= \Lambda_{ijk} - \Gamma_{ik}^n L_{nj}. \end{aligned}$$

□

The third-order derivatives of R and the second-order derivatives of n have a similar representation as the second-order derivatives of R and first-order derivatives of n , except that we need symbols different from the Christoffel symbols:

Lemma 2.3.3.

$$\begin{cases} R_{ijk} = \Omega_{ijk}^l R_l + \Lambda_{ijk} n, \\ n_{ij} = \Delta_{ij}^k R_k + \Pi_{ij} n, \end{cases} \quad (2.8)$$

where

$$\Omega_{ijk}^l = \frac{1}{6} g^{pl} \left(\frac{\partial^2 g_{ip}}{\partial x_j \partial x_k} + \frac{\partial^2 g_{jp}}{\partial x_i \partial x_k} + \frac{\partial^2 g_{kp}}{\partial x_i \partial x_j} - \frac{\partial^2 g_{ij}}{\partial x_k \partial x_p} - \frac{\partial^2 g_{ik}}{\partial x_j \partial x_p} - \frac{\partial^2 g_{jk}}{\partial x_i \partial x_p} \right),$$

$$\Delta_{ij}^k = g^{kl} (\Gamma_{lj}^p L_{pi} + \Gamma_{li}^p L_{pj} - \Lambda_{ijl}),$$

and $\Pi_{ij} = -g^{lk} L_{li} L_{kj}$.

Proof. Multiply

$$R_{ijk} = \Omega_{ijk}^l R_l + \Lambda_{ijk} n,$$

by R_p , we have

$$\langle R_{ijk}, R_p \rangle = \Omega_{ijk}^l g_{lm},$$

so

$$\Omega_{ijk}^l = g^{lm} \langle R_{ijk}, R_p \rangle.$$

It is straight forward to show that

$$\langle R_{ijk}, R_p \rangle = \frac{1}{6} \left(\frac{\partial^2 g_{ip}}{\partial x_j \partial x_k} + \frac{\partial^2 g_{jp}}{\partial x_i \partial x_k} + \frac{\partial^2 g_{kp}}{\partial x_i \partial x_j} - \frac{\partial^2 g_{ij}}{\partial x_k \partial x_p} - \frac{\partial^2 g_{ik}}{\partial x_j \partial x_p} - \frac{\partial^2 g_{jk}}{\partial x_i \partial x_p} \right).$$

Therefore,

$$\Omega_{ijk}^l = \frac{1}{6} g^{pl} \left(\frac{\partial^2 g_{ip}}{\partial x_j \partial x_k} + \frac{\partial^2 g_{jp}}{\partial x_i \partial x_k} + \frac{\partial^2 g_{kp}}{\partial x_i \partial x_j} - \frac{\partial^2 g_{ij}}{\partial x_k \partial x_p} - \frac{\partial^2 g_{ik}}{\partial x_j \partial x_p} - \frac{\partial^2 g_{jk}}{\partial x_i \partial x_p} \right).$$

Similarly, multiplying R_l through both sides of

$$n_{ij} = \Delta_{ij}^k R_k + \Pi_{ij} n,$$

gives

$$\Delta_{ij}^k g_{kl} = \langle n_{ij}, R_l \rangle.$$

Then

$$\Delta_{ij}^k = g^{kl} \langle n_{ij}, R_l \rangle$$

□

2.3.1 Approximation of a surface

To explore the approximate equation of the surface R at point $P = R(x_1, x_2)$, consider the distance $\delta(\Delta x_1, \Delta x_2)$ from $P' = R(x_1 + \Delta x_1, x_2 + \Delta x_2)$ to P [3]:

$$\begin{aligned}
\delta(\Delta x_1, \Delta x_2) &= \overrightarrow{PP'} = R_i \Delta x_i + \frac{1}{2} R_{ij} \Delta x_i \Delta x_j + \frac{1}{6} R_{ijk} \Delta x_i \Delta x_j \Delta x_k + \cdots \\
&= R_i \Delta x_i + \frac{1}{2} (\Gamma_{ij}^k R_k + L_{ij} n) \Delta x_i \Delta x_j \\
&\quad + \frac{1}{6} (\Omega_{ijk}^l R_l + \Lambda_{ijk} n) \Delta x_i \Delta x_j \Delta x_k + \cdots \\
&= (\Delta x_1 + \frac{1}{2} \Gamma_{ij}^1 \Delta x_i \Delta x_j + \frac{1}{6} \Omega_{ijk}^1 \Delta x_i \Delta x_j \Delta x_k + \cdots) R_{x_1} \\
&\quad + (\Delta x_2 + \frac{1}{2} \Gamma_{ij}^2 \Delta x_i \Delta x_j + \frac{1}{6} \Omega_{ijk}^2 \Delta x_i \Delta x_j \Delta x_k + \cdots) R_{x_2} \\
&\quad + (\frac{1}{2} L_{ij} \Delta x_i \Delta x_j + \frac{1}{6} \Lambda_{ijk} \Delta x_i \Delta x_j \Delta x_k + \cdots) n.
\end{aligned}$$

If we adopt $(R_{x_1}/\sqrt{g_{11}}, R_{x_2}/\sqrt{g_{22}}, n)$ as the unit bases (e_1, e_2, n) on the surface R , and point P as the origin, then we can represent R approximately as

$$\delta = X^1 e_1 + X^2 e_2 + Z n,$$

where

$$\begin{aligned}
X^1 &= (\Delta x_1 + \frac{1}{2} \Gamma_{ij}^1 \Delta x_i \Delta x_j + \frac{1}{6} \Omega_{ijk}^1 \Delta x_i \Delta x_j \Delta x_k + \cdots) \sqrt{g_{11}}, \\
X^2 &= (\Delta x_2 + \frac{1}{2} \Gamma_{ij}^2 \Delta x_i \Delta x_j + \frac{1}{6} \Omega_{ijk}^2 \Delta x_i \Delta x_j \Delta x_k + \cdots) \sqrt{g_{22}}, \\
Z &= \frac{1}{2} L_{ij} \Delta x_i \Delta x_j + \frac{1}{6} \Lambda_{ijk} \Delta x_i \Delta x_j \Delta x_k + \cdots.
\end{aligned}$$

Then

$$Z = \frac{1}{2} L_{ij} \frac{X^i}{\sqrt{g_{ii}}} \frac{X^j}{\sqrt{g_{jj}}} + \frac{1}{6} \Lambda_{ijk} \frac{X^i}{\sqrt{g_{ii}}} \frac{X^j}{\sqrt{g_{jj}}} \frac{X^k}{\sqrt{g_{kk}}},$$

if we ignore the terms of $O(\Delta x_1^2 + \Delta x_2^2)^2$ and higher order.

Therefore, we can represent surface R on a region of P approximately by surface

$$\tilde{R} = (X^1, X^2, \frac{1}{2} L_{ij} \frac{X^i}{\sqrt{g_{ii}}} \frac{X^j}{\sqrt{g_{jj}}} + \frac{1}{6} \Lambda_{ijk} \frac{X^i}{\sqrt{g_{ii}}} \frac{X^j}{\sqrt{g_{jj}}} \frac{X^k}{\sqrt{g_{kk}}}),$$

under unit basis $(R_{x_1}/\sqrt{g_{11}}, R_{x_2}/\sqrt{g_{22}}, n)$ and the origin $P = (0, 0, 0)$.

2.4 Principal curvatures and principle directions

Definition 2.4.1. (Normal Curvatures)[3] Let C be a regular curve in $R(x_1, x_2)$ passing through $p \in R(x_1, x_2)$, κ the curvature of C at p , and $\cos \theta = \langle n, N \rangle$, where n is the normal vector to C and N is the normal vector to $R(x_1, x_2)$ at p . The number

$$\kappa_n = \kappa \cos \theta$$

is then called the normal curvature of $C \subset S$ at p .

Definition 2.4.2. (Principal Curvatures) A tangent vector $X_0 \in T_x R$ is said to be a principal direction if $k(X)$ is extremized in the direction X_0 . The value $k(X_0)$ is called a principal curvature of the surface $R(x_1, x_2)$ at point x .

Theorem 2.4.3. (Principal Curvatures [7])

1. κ is a principal curvature if and only if

$$\left| \begin{pmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{pmatrix} - \kappa \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \right| = 0.$$

2. $X = a_1 R_1 + a_2 R_2$ is a principal direction if and only if $(a_1, a_2)^T$ is an eigenvector for some κ in 1.

Definition 2.4.4. [3] Mean Curvature of the surface R at point x is

$$H = \frac{\kappa_1 + \kappa_2}{2}.$$

Gaussian Curvature of the surface R at point x is

$$K = \kappa_1 \kappa_2.$$

Corollary 2.4.5.

$$H = \frac{1}{2} \frac{L_{11}g_{22} + L_{22}g_{11} - 2g_{12}L_{12}}{g_{11}g_{22} - g_{12}^2}. \quad (2.9)$$

$$K = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}. \quad (2.10)$$

Definition 2.4.6. *If at $x_0 \in R$, $\kappa_1 = \kappa_2$, then x_0 is called an umbilical point of R ; in particular, the planar points ($\kappa_1 = \kappa_2 = 0$) are umbilical points.*

Definition 2.4.7. *A regular curve $c(t)$ on a surface $R(x_1, x_2)$ is called a line of curvature if $c'(t)/|c'(t)|$ is a principal direction for all t .*

Lemma 2.4.8. *[3] Let $R : \Omega \rightarrow \mathbb{R}^3$ be a surface, $x_0 \in R$ is not an umbilical point, then there exists a neighborhood U_0 of x_0 and a change of variable $\phi : V_0 \rightarrow U_0$ such that the coordinate lines of $\bar{U} = U \circ \phi$ are lines of curvatures.*

Such coordinates are called *principal curvature coordinates*.

Theorem 2.4.9. *[3] If $U : \Omega \rightarrow \mathbb{R}^3$ satisfies $g_{12} = L_{12} = 0$, then U is a principal curvature coordinates.*

Corollary 2.4.10. *[3] Under the principal curvature coordinates, the principal curvatures are*

$$\kappa_1 = \frac{L_{11}}{g_{11}}, \quad \kappa_2 = \frac{L_{22}}{g_{22}}.$$

Corollary 2.4.11. *Under the principal curvature coordinates $(x_i)_{i=1,2}$,*

$$\begin{aligned} \Lambda_{111} &= \frac{3\kappa_1}{2} \frac{\partial g_{11}}{\partial x_1} + g_{11} \frac{\partial \kappa_1}{\partial x_1}, & \Lambda_{222} &= \frac{3\kappa_2}{2} \frac{\partial g_{22}}{\partial x_2} + g_{22} \frac{\partial \kappa_2}{\partial x_2}, \\ \Lambda_{221} &= \Lambda_{122} = \Lambda_{212} = \frac{1}{2} \frac{\partial g_{22}}{\partial x_1} \kappa_2 = \frac{3\kappa_1}{2} \frac{\partial g_{11}}{\partial x_2} + g_{11} \frac{\partial \kappa_1}{\partial x_2}, \\ \Lambda_{121} &= \Lambda_{211} = \Lambda_{112} = \frac{1}{2} \frac{\partial g_{11}}{\partial x_2} \kappa_1 = \frac{3\kappa_2}{2} \frac{\partial g_{22}}{\partial x_1} + g_{22} \frac{\partial \kappa_2}{\partial x_1}. \end{aligned}$$

Proof. By Theorem 2.4.9, $g_{12} = L_{12} = 0$, then

$$\begin{aligned} \Lambda_{111} &= \frac{\partial L_{11}}{\partial x_1} + \Gamma_{11}^1 L_{11} = \frac{\partial g_{11} \kappa_1}{\partial x_1} + \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial x_1} g_{11} \kappa_1 = \frac{3\kappa_1}{2} \frac{\partial g_{11}}{\partial x_1} + g_{11} \frac{\partial \kappa_1}{\partial x_1}, \\ \Lambda_{222} &= \frac{\partial L_{22}}{\partial x_2} + \Gamma_{22}^1 L_{22} = \frac{3\kappa_2}{2} \frac{\partial g_{11}}{\partial x_1} + g_{22} \frac{\partial \kappa_2}{\partial x_2}, \\ \Lambda_{122} &= \frac{\partial L_{12}}{\partial x_2} + \Gamma_{12}^2 L_{22} = \frac{1}{2} \frac{\partial g_{22}}{\partial x_1} \kappa_2 = \frac{3\kappa_1}{2} \frac{\partial g_{11}}{\partial x_2} + g_{11} \frac{\partial \kappa_1}{\partial x_2}, \\ \Lambda_{121} &= \frac{\partial L_{21}}{\partial x_1} + \Gamma_{12}^1 L_{11} = \frac{1}{2} \frac{\partial g_{11}}{\partial x_2} \kappa_1 = \frac{3\kappa_2}{2} \frac{\partial g_{22}}{\partial x_1} + g_{22} \frac{\partial \kappa_2}{\partial x_1}. \end{aligned}$$

□

2.5 Rotation of Principal Coordinates

To get the Third Order Surface coefficients in the principal directions $\{R_{x_1}, R_{x_2}\}$, we need a rotation Ξ to transfer the original coordinate system (u_1, u_2) to (x_1, x_2) .

Consider the directions $\{R_{u_1}, R_{u_2}\}$ at point P , the 2nd fundamental form and the First Fundamental Form under (u_1, u_2) are

$$\begin{aligned} I &= g_{11}du_1^2 + 2g_{12}du_1du_2 + g_{22}du_2^2, \\ II &= L_{11}du_1^2 + 2L_{12}du_1du_2 + L_{22}du_2^2, \end{aligned}$$

by Theorem 2.4.3, we have the principal direction

$$\begin{aligned} R_{x_1} &= a_{11}R_{u_1} + a_{12}R_{u_2}, \\ R_{x_2} &= a_{21}R_{u_1} + a_{22}R_{u_2}. \end{aligned}$$

where (a_{11}, a_{12}) , and (a_{21}, a_{22}) are the eigenvectors of the matrix

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1},$$

then the rotation Ξ is the matrix formed by the eigenvectors

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The coefficients of the Second Fundamental Form $\{L_{ij}^r\}$ and the Third Order Surface coefficients $\{\Lambda_{ijk}^r\}$ of a surface on the principal directions $\{R_{x_1}, R_{x_2}\}$ can be transformed from the Second Fundamental Form coefficients $\{L_{ij}\}$ and the 3rd order coefficients Λ_{ijk} of the coordinate system (u_1, u_2) by

$$\begin{aligned} L_{ij}^r &= L_{mn}a_{mi}a_{nj}, \\ \Lambda_{ijk}^r &= \Lambda_{lmn}a_{li}a_{mj}a_{nk}. \end{aligned}$$

Chapter 3

Geometric Optics and Lens Design

3.1 Geometric optics principles

Design and analysis of ophthalmic lens uses numerical calculation based upon geometric optics principles. The geometric optics model is sufficient to describe the properties of an image formed by a lens.

Geometric optics is based on the fundamental assumption that light propagates along rays. Fermat's Principle, known as the principle of the shortest optical path, states that each ray through an optical system follows the path of the shortest time from points in the object space to points located in the image space. Rays in a homogeneous medium follow straight lines.

The velocity of propagation of a light ray in a homogeneous isotropic medium is given by

$$v = \frac{c}{n}$$

where c is the velocity of light in vacuum (3.0×10^8 m/s) and n is the index of refraction of the medium. The length of time of for a ray to propagate from point

P in object space to point P' in the image space is

$$t = \frac{1}{c} \int n(s) ds$$

where s is the length along all the ray vector from P to P' .

Fermat's principle can be derived from Huygens' principle, and can be used to derive Snell's law of refraction, which is a formula used to describe the relationship between the angles of incidence and refraction,

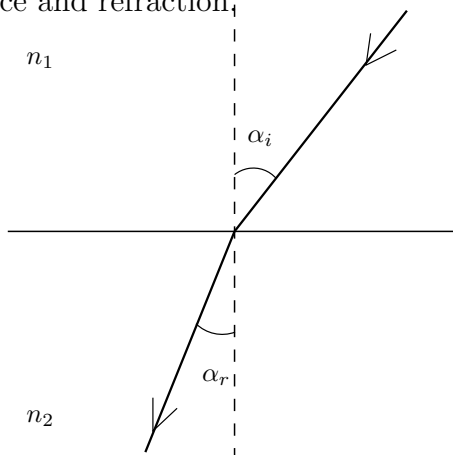


Figure 3.1: Snell's Law

Lemma 3.1.1 (Snell's Law).

$$\frac{\sin \alpha_i}{\sin \alpha_r} = \frac{n_1}{n_2},$$

where α_i , α_r are the incident and refraction angles. n_1 , n_2 are the index of refraction of the two media as shown in Figure 3.1.

Lemma 3.1.2. As shown in figure 3.2, let Q be the incident direction, S be the normal direction at the intersection point, Q' be the refraction direction, by Snell's law,

$$Q' = \mu Q + \gamma S,$$

where $\mu = \frac{n_1}{n_2}$, $\gamma = \cos \alpha_r - \mu \cos \alpha_i$.

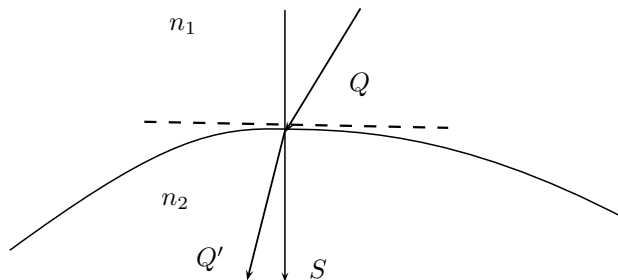


Figure 3.2: The refracted direction

3.2 Characterizing lens performance with a wavefront

The main characteristics of ophthalmic lenses are power of refraction and astigmatism. The method for characterizing these properties are based on wavefronts, and it could be extended to describing properties of lens with arbitrary surfaces.

Consider a point light source that is switched on and off at the time t_0 . At the time $t > t_0$ the photons emitted at t_0 form a surface W_t , the wavefront. The shape of the wavefront W_t changes as it propagates, whether the medium is homogeneous or inhomogeneous and when it is refracted by a surface.

We characterize the wavefront as a time-dependent surface $W_t(x_1, x_2)$. Then we are interested in the local curvatures of W_t along a particular light ray $Q_{t_0}(x_1^0, x_2^0) : t \rightarrow W_t(x_1^0, x_2^0)$. Consider a point at infinity viewed by eye. The point emit a ray Q_{t_0} , with a planar wavefront W_t . After the wavefront is refracted by the lens surface, it arrives at the back side of the lens at time $t = t_1$. The principal curvature κ_1 and κ_2 of the refracted wavefront W'_{t_1} defines the power of refraction P and astigmatism A of the lens along the viewing direction (as shown in figure 3.3):

$$P = \frac{\kappa_1 + \kappa_2}{2},$$

$$A = |\kappa_1 - \kappa_2|.$$

We can write

$$P = H, \quad A = \sqrt{H^2 - K},$$

where H and K are mean curvature and Gaussian curvature of the wavefront.

The higher order aberrations such as Comma, Spherical Abberation, etc., can also be calculated from properties of the wavefront. They depend on a higher derivatives of the wavefront surface.

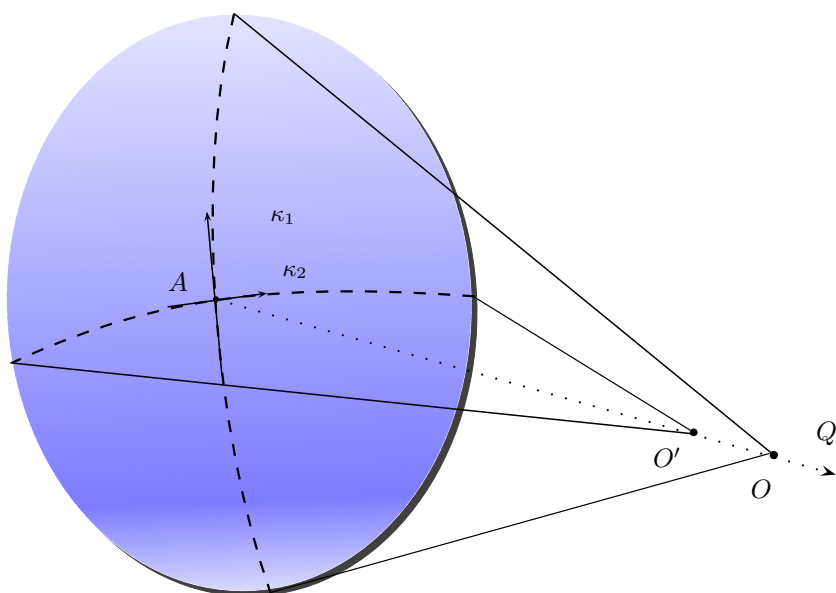


Figure 3.3: Wavefront Curvature Representation

3.3 Wavefront aberration and Zernike polynomials

The deviation of the refracted wavefront after a lens from a desired perfect spherical wavefront is called the wavefront aberration or wavefront distortion.

Consider an orthonormal basis $\{e_1'', e_2'', e_3''\}$, where $e_1'' = \frac{(0,1,0) \times (-\hat{\theta})}{|(0,1,0) \times (-\hat{\theta})|}$, $e_3'' = -\hat{\theta}$, and $e_2'' = e_1'' \times e_3''$. With the origin O'' , let X_i'' be the cartesian coordinates with

direction vectors e_i'' . Let W'' be the refracted wavefront of interest, then W'' can be approximated by a third-order polynomial surface

$$W'' = \frac{1}{2}L''_{ij}X_i''X_j'' + \frac{1}{6}\Lambda''_{ijk}X_i''X_j''X_k'',$$

where L''_{ij} and Λ''_{ijk} are the surface properties with respect to (X_1'', X_2'') parametrization. Note that because of this parametrization, the sum in the above equation, and in the equations below, is carried out for indices 1 and 2 only.

The aberration associated with W'' is the wavefront distortion, and is the wavefront W'' with the part corresponding to the mean curvature contribution removed. Denote the mean curvature by $\kappa(\hat{\theta})$ where

$$\kappa(\hat{\theta}) = \frac{1}{2}(\kappa_1 + \kappa_2),$$

and κ_1 and κ_2 are the principal curvatures. The focal length associated with the mean curvature part is $1/\kappa(\hat{\theta})$ since this wavefront component has the form $\frac{1}{2}\kappa(X_1^2 + X_2^2)$. Therefore, the wavefront aberration is

$$\begin{aligned} \Delta W'' &= \frac{1}{2}(L''_{11} - \kappa(\hat{\theta}))X_1''^2 + L''_{12}X_1''X_2'' + \frac{1}{2}(L''_{22} - \kappa(\hat{\theta}))X_2''^2 \\ &\quad + \frac{1}{6}\Lambda''_{ijk}X_i''X_j''X_k''. \end{aligned} \quad (3.1)$$

Wavefront aberration can be represented in terms of Zernike polynomials whose coefficients can be interpreted as in optical terms (see [2, 12]). Going to polar coordinates (r, ϕ) from (X_1, X_2) , we have

$$\Delta W'' = \sum_{n=0}^p \sum_{m=-n}^n c_n^m(\hat{\theta}) Z_n^m(r, \phi), \quad (3.2)$$

where $Z_n^m(r, \phi) = R_n^m(r)\Theta^m(\phi)$ denotes the Zernike polynomial of order (m, n) . The radial part are polynomials defined as

$$R_n^m(r) = \sum_{s=0}^{(n-|m|)/2} \frac{(-1)^s \sqrt{n+1} (n-s)! r^{n-2s}}{s! [(n+m)/2 - s]! [(n-m)/2 - s]!},$$

and the angular functions are

$$\Theta^m(\phi) = \begin{cases} \sqrt{2} \cos m\phi & m > 0 \\ \sqrt{2} \sin m\phi & m < 0 \\ 1 & m = 0 \end{cases},$$

where $m \leq n$ and $n - m = \text{even}$.

We can now transform Taylor polynomial (3.1) to the Zernike polynomial expansion (up to third order) according to the following table (we drop the double prime notation on L and Λ)

Mode	n	m	Z_m^n	Taylor terms	Meaning
1	0	0	1	1	Piston
2	1	-1	$r \sin \phi$	X_2	Tilt in X-direction
3		1	$r \cos \phi$	X_1	Tilt in Y-direction
4	2	-2	$r^2 \sin 2\phi$	$2X_1X_2$	Astigmatism
5		0	$2r^2 - 1$	$2(X_1^2 + X_2^2) - 1$	Defocus
6		2	$r^2 \cos 2\phi$	$(X_1^2 - X_2^2)$	Astigmatism
7	3	-3	$r^3 \sin 3\phi$	$3X_1^2X_2 - X_2^3$	
8		-1	$(3r^3 - 2r) \sin \phi$	$(3X_1^2X_2 + 3X_2^3 - 2X_2)$	Coma along X-axis
9		1	$(3r^3 - 2r) \cos \phi$	$(3X_1X_2^2 + 3X_1^3 - 2X_1)$	Coma along Y-axis
10		3	$r^3 \cos 3\phi$	$X_1^3 - 3X_1X_2^2$	

Then the relations between the Taylor representation (3.1) coefficients to the Zernike (3.2) coefficients are

Mode	Zernike coefficients
1	$C_0^0 = \frac{1}{4}(L_{11} + L_{22})$
2	$C_{-1}^1 = \frac{1}{12}(\Lambda_{112} + \Lambda_{222})$
3	$C_1^1 = \frac{1}{12}(\Lambda_{111} + \Lambda_{122})$
4	$C_{-2}^2 = \frac{1}{2}L_{12}$
5	$C_0^2 = \frac{1}{8}(L_{11} + L_{22} - 2\kappa)$
6	$C_2^2 = \frac{1}{2}(L_{11} - L_{22})$
7	$C_{-3}^3 = \frac{1}{24}(3\Lambda_{112} - \Lambda_{222})$
8	$C_{-1}^3 = \frac{1}{24}(\Lambda_{112} + \Lambda_{222})$
9	$C_1^3 = \frac{1}{24}(\Lambda_{111} + \Lambda_{122})$
10	$C_3^3 = \frac{1}{24}(\Lambda_{111} - 3\Lambda_{122})$

3.4 Lens design problem

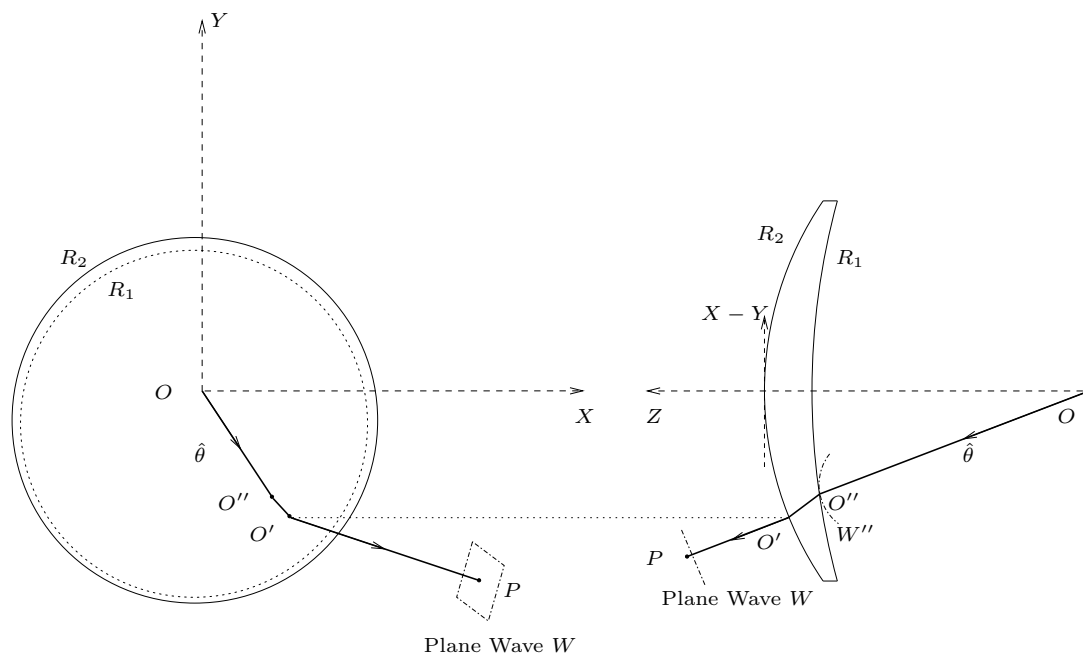


Figure 3.4: The formulation of the lens design

As shown in Figure 3.4, let the gaze direction of the eye is $\hat{\theta}$, the lens surfaces are R_1 and R_2 , the index of refraction of the lens is n while that of air is 1. To obtain the diffractive properties of the lens at that gaze direction, we need to follow the ray at angle $\hat{\theta}$ out from the eye side. A planar wavefront W is set up at point P with the normal direction Q' along the ray, then W follows this direction as it goes back through the lens, generating a refracted wavefront W'' at point O'' at where the ray exits the lens toward the eye. For any $\hat{\theta} \in \Omega$, we can calculate the Second Fundamental Form coefficients $L(\hat{\theta})$ and Third Order Surface coefficients $\Lambda(\hat{\theta})$ of W'' . In the next chapter, we will outline how one can calculate a wavefront after refraction by a single surface. Therefore, in principle, we can perform such a calculation twice in order to calculate how a wavefront is diffracted by a lens.

Therefore, for a given lens with two lens surfaces R_1 and R_2 and a given gaze direction $\hat{\theta}$, we can calculate the Zernike coefficients $c_j^i(\hat{\theta})$ and lens power, which is defined as the mean curvature $\kappa(\hat{\theta})$.

The Progressive Additional Lens (PAL) design problem is described as follows. We want to come close to desired power distribution $P(\hat{\theta})$ over a set of gaze directions Ω . The aberrations in all these directions should also be minimized as well. Therefore, the target functional we want to optimize is

$$\mathcal{J}(R_1, R_2) = \int_{\Omega} \beta |\kappa(\hat{\theta}) - P(\hat{\theta})|^2 d\hat{\theta} + \int_{\Omega} \sum_i \sum_j \alpha_j^i (c_j^i(\hat{\theta}))^2 d\hat{\theta}. \quad (3.3)$$

The weight α_j^i and β are $\hat{\theta}$ dependent and represents the importance of aberration and closeness to desired power in the gaze direction $\hat{\theta}$.

In most PAL design, only one surface, either the front or the back, is designed for progressive power correction. The other surface is typically sphere or toric, depending on whether the wearer has prescribed astigmatism correction. The functional above could be considered with one of the surfaces fixed. The resulting design problem is to find the other surface that minimizes the functional. One could consider designing both surfaces at the same time although the need to make sure that the surfaces do not intersect must be enforced by a constraint.

Chapter 4

Wavefront deformation through smooth refracting lens surface

4.1 Wavefront tracing through optical system

In order to understand how light propagates through a lens, we must first understand how it interacts with a single interface. To do so, we consider the geometrical optics approximation where light travel along rays, and optical energy along rays with the same phase form a wavefront.

Figure 4.1 shows a brief description of wavefront tracing through a refracting surface. Consider a wavefront in three-dimensional space defined by the vector function $W(x, t)$, where $x = (x_1, x_2)$ are surface parameters and t is time. Denote $W(x, t_0)$ as the initial wavefront. Then $W(x, t_0 + \tau)$ is the wavefront after time τ . Let $Q(x)$ be the unit normal vector of $W(x, t)$, which does not depend on t in a homogeneous medium. $W(x, t)$ strikes a refracting surface $R(\bar{x})$ at $\bar{x} = (\bar{x}_1, \bar{x}_2)$ with unit normal $S(\bar{x})$, giving rise to a refracted wavefront $W'(x', t)$ at $x' = (x'_1, x'_2)$, with unit normal $Q'(x')$. Here it is assumed that the index of refraction of the media are constant and equal to n_1 and n_2 , the angle of incidence is α_i and the angle of refraction is α_r .

The corresponding equation can be obtained by considering the physical process: Let $T(x)$ be the travel time between $W(x, t)$ along a ray in the direction $Q(x)$ and the point O on R . Let L be the total time in which $W(x, t)$ travels to W' , then

$$R = W + c_1 T Q, \quad (4.1a)$$

$$W' = W + c_1 T Q + c_2 (L - T) Q', \quad (4.1b)$$

which imply

$$W + c_1 T Q = R = W' - c_2 (L - T) Q', \quad (4.2)$$

where $c_1 = c/n_1$, $c_2 = c/n_2$ are the light speeds in the corresponding media.

We can denote $cT = \phi$, where c is the light speed in vacuum, then the equation is now

$$W + \mu_1 \phi \cdot Q = R = W' - \mu_2 (cL - \phi) \cdot Q', \quad (4.3)$$

where $\mu_1 = \frac{c_1}{c}$ and $\mu_2 = \frac{c_2}{c}$.

When a wavefront passes through an optical system, as described above, two phenomena are responsible for transforming the curvature of the wavefront:

1. Propagation in a homogenous medium.
2. Refraction at an optical boundary.

In the following sections, we will derive formulas for the Second Fundamental Form coefficients and the third-order surface coefficients after the wavefront has gone through the surface R .

4.2 J. Kneisly's Approach

In J. Kneisly's paper [8], he propose a formula of the Second Fundamental Form of the wavefront under propagation.

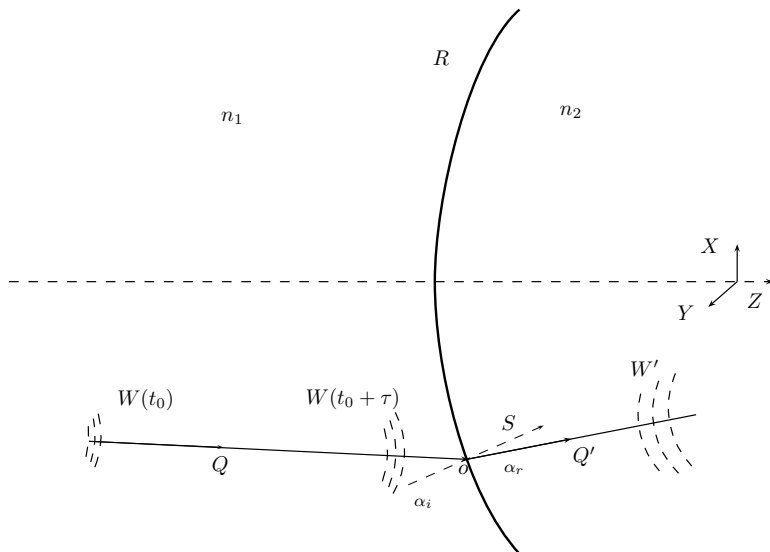


Figure 4.1: Wavefront tracing through a refracting surface.

Consider a wavefront defined by the function $W(x, t)$, where $x = (x_1, x_2)$ are surface parameters and t is time, we can also choose x such that for a given point (x^0, t^0) , we have

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = 1,$$

and

$$L_{12} = 0.$$

Then we have:

Theorem 4.2.1. *The Second Fundamental Form coefficients of the wavefront $W(x, t_0 + \tau)$ after propagation are:*

$$L_{11}(t_0 + \tau) = (1 - dL_{11}(t_0))L_{11}(t_0),$$

$$L_{22}(t_0 + \tau) = (1 - dL_{22}(t_0))L_{22}(t_0),$$

$$L_{12}(t_0 + \tau) = 0.$$

In particular, the principal curvatures evolve as

$$\kappa_1(t_0 + \tau) = \frac{\kappa_1(t_0)}{1 - d\kappa_1(t_0)}, \quad \kappa_2(t_0 + \tau) = \frac{\kappa_2(t_0)}{1 - d\kappa_2(t_0)}.$$

where $d = c\tau$ the distance W traveled during time τ .

In [8], J. Kneisly also proposed a formula of representing Second Fundamental Form of the refracted wavefront by the coefficients of the initial wavefront and the refracting lens surface.

Consider point O where three surfaces meet, then

$$O = W(x_1^0, x_2^0) = R(\bar{x}_1^0, \bar{x}_2^0) = W'(x_1'^0, x_2'^0),$$

Snell's law gives the relation between the normal vectors Q' and Q, S .

$$Q'(x_1'^0, x_2'^0) = \mu Q(x_1^0, x_2^0) + \gamma S(\bar{x}_1^0, \bar{x}_2^0),$$

where $\gamma = \cos \alpha_r - \mu \cos \alpha_i$, where α_i and α_r are the angles of incidence and refraction, respectively, $\mu = n_1/n_2 = c_1/c_2$ is the ratio of refraction.

By the law of refraction, the normals Q, S, Q' are all coplanar. Define

$$P = (Q \times S)/|Q \times S| = (Q \times S)/\sin \alpha_i,$$

which is perpendicular to the plane of refraction and tangent to all the surfaces.

Let $T = Q \times P, \bar{T} = S \times P$, and $T' = Q' \times P$, which are tangent to the corresponding surfaces, respectively, and lie in the plane of refraction, as shown in Figure 4.2. Then we can select coordinates system of (u_1, u_2) such that at point O , we have

$$\begin{aligned} P &= W_{u_1}, & T &= W_{u_2}, \\ P &= R_{\bar{u}_1}, & \bar{T} &= R_{\bar{u}_2}, \\ P &= W'_{u'_1}, & T' &= W'_{u'_2}, \end{aligned} \tag{4.4}$$

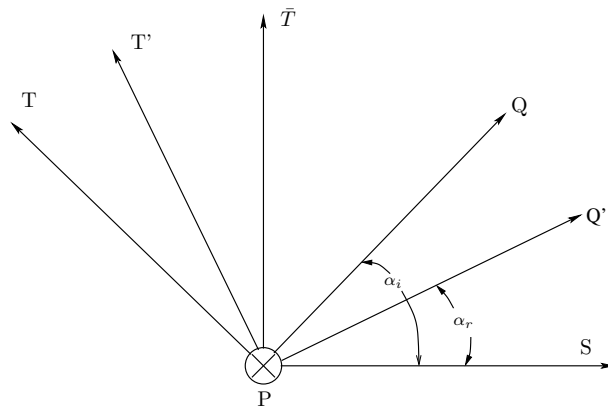


Figure 4.2: Relations between P, Q, T, S, \bar{T}, Q' .

Theorem 4.2.2. *With coordinate system $\{u_1, u_2\}$, $\{\bar{u}_1, \bar{u}_2\}$, and $\{u'_1, u'_2\}$, we can represent the refracted second order fundamental form of W' , $\{L'_{ij}\}_{i,j=1,2}$, by that of the initial wavefront W and the lens surface R :*

$$\begin{aligned}
 L'_{11} &= \mu L_{11} + \gamma \bar{L}_{11}, \\
 L'_{12} &= \mu L_{12} \frac{\cos \alpha_i}{\cos \alpha_r} + \gamma \bar{L}_{12} \frac{1}{\cos \alpha_r}, \\
 L'_{22} &= \mu L_{22} \left(\frac{\cos \alpha_i}{\cos \alpha_r} \right)^2 + \gamma \bar{L}_{22} \left(\frac{1}{\cos \alpha_r} \right)^2,
 \end{aligned} \tag{4.5}$$

where α_i and α_r are the angles of incidence and refraction.

Kneisly's method gives simple formulas of evaluating curvatures of the wavefront under propagation and refraction through lens system with some special coordinates. However, it involves coordinate changes, which brings a lot of complexity when trying to evaluate the gradient of the curvatures. An explicit and direct way to evaluating the curvature is needed. Also, we want generalized formulas to describing the deformation of the wavefront in Second Fundamental Form coefficients and the Third Order Surface coefficients.

In the following sections, we are going to extend Kneisly's formulas to general coordinates. Also, First Fundamental Form coefficients and Third Order Surface

coefficients on propagation and on refraction under general coordinates are also presented. Note that all the equations are carried out for indices 1 and 2 only.

4.3 Wavefront surface coefficients on propagation

4.3.1 The Second Fundamental Form coefficients on propagation

Let v be the local speed of light in a homogenous medium, then

$$W(x, t_0 + \tau) = W(x, t_0) + \tau v Q(x), \quad \text{and} \quad W_t = vQ,$$

where τ is the travel time.

Let $d = \tau v$ be the distance light traveled during time τ , then take the first derivative on x_i , we have

$$W_{x_i}(t_0 + \tau) = W_{x_i}(t_0) + dQ_i.$$

By the equation of Weingarten (2.4),

$$Q_i = -g^{kj}(t_0)L_{ji}(t_0)W_k(t_0),$$

then

$$W_{x_i}(t_0 + \tau) = W_{x_i}(t_0) - dg^{kj}(t_0)L_{ji}(t_0)W_{x_k}(t_0). \quad (4.6)$$

Therefore, by definition of g_{ij}

$$g_{ij}(t_0 + \tau) = \langle W_{x_i}(t_0 + \tau), W_{x_j}(t_0 + \tau) \rangle,$$

by (4.6),

$$g_{ij}(t_0 + \tau) = \langle W_{x_i}(t_0) - dg^{kl}(t_0)L_{li}(t_0)W_{x_k}(t_0), W_{x_j}(t_0) - dg^{nm}(t_0)L_{mj}(t_0)W_{x_n}(t_0) \rangle.$$

Expanding this equation gives

$$\begin{aligned} g_{ij}(t_0 + \tau) &= g_{ij}(t_0) - dg^{nm}(t_0)g_{in}(t_0)L_{mj}(t_0) - dg^{kl}(t_0)g_{kj}(t_0)L_{li}(t_0) \\ &\quad + d^2g^{kl}(t_0)g^{nm}(t_0)g_{kn}(t_0)L_{li}(t_0)L_{mj}(t_0). \end{aligned} \quad (4.7)$$

Similarly, by the definition of L_{ij} ,

$$L_{ij}(t_0 + \tau) = -\langle W_{x_i}(t_0 + \tau), Q_{x_j} \rangle,$$

Expanding the expression gives

$$L_{ij}(t_0 + \tau) = L_{ij}(t_0) - dg^{kj}(t_0)L_{ji}(t_0)L_{kj}(t_0),$$

which is

$$L_{ij}(t_0 + \tau) = (1 - dg^{kj}(t_0)L_{kj}(t_0))L_{ij}(t_0).$$

Theorem 4.3.1. *The Second Fundamental Form coefficients on propagation is given by*

$$L_{ij}(t_0 + \tau) = (1 - dg^{kj}L_{kj}(t_0))L_{ij}(t_0),$$

where d is the distance the wavefront propagates during time τ . In particular, under unit orthogonal coordinate system, where $g_{11} = g_{22} = 1$ and $g_{12} = 0$, $L(t_0 + \tau)$ have representations,

$$L_{11}(t_0 + \tau) = (1 - dL_{11}(t_0))L_{11}(t_0),$$

$$L_{22}(t_0 + \tau) = (1 - dL_{22}(t_0))L_{22}(t_0),$$

$$L_{12}(t_0 + \tau) = L_{21}(t_0 + \tau) = 0.$$

Corollary 4.3.2. *The principal curvature of the wavefront on propagation is given by*

$$\kappa_i(t_0 + \tau) = \frac{\kappa_i(t_0)}{1 - d\kappa_i(t_0)},$$

where d is the distance the wavefront propagates during time τ .

Corollary 4.3.3. *Mean curvature and Gaussian Curvature of the wavefront on propagation is given by*

$$\begin{aligned} H(t_0 + \tau) &= \frac{H(t_0) - dK(t_0)}{1 - 2dH(t_0) + d^2K(t_0)} \\ K(t_0 + \tau) &= \frac{K(t_0)}{1 - 2dH(t_0) + d^2K(t_0)} \end{aligned}$$

Proof. By definition 2.4.5,

$$H(t_0 + \tau) = \frac{1}{2}(\kappa_1(t_0 + \tau) + \kappa_2(t_0 + \tau)),$$

By Corollary 4.3.2

$$H(t_0 + \tau) = \frac{1}{2} \left(\frac{\kappa_1(t_0)}{1 - d\kappa_1(t_0)} + \frac{\kappa_2(t_0)}{1 - d\kappa_2(t_0)} \right).$$

Combine these two fractions, we obtain

$$H(t_0 + \tau) = \frac{1}{2} \frac{\kappa_1(t_0) + \kappa_2(t_0) - 2d\kappa_1(t_0)\kappa_2(t_0)}{1 - d(\kappa_1(t_0) + \kappa_2(t_0)) + d^2\kappa_1(t_0)\kappa_2(t_0)}.$$

At the end, applying the definition of $H(t_0)$ and $K(t_0)$, we have

$$H(t_0 + \tau) = \frac{H(t_0) - dK(t_0)}{1 - 2dH(t_0) + d^2K(t_0)}. \quad (4.8)$$

Similarly, we have

$$K(t_0 + \tau) = \frac{K(t_0)}{1 - 2dH(t_0) + d^2K(t_0)}. \quad (4.9)$$

□

Definition 4.3.4. *The astigmatism (or local cylinder) A of a wavefront is defined as*

$$A = \sqrt{H^2 - K} = |\kappa_1 - \kappa_2|, \quad (4.10)$$

where κ_1, κ_2 are principal curvatures of the wavefront.

Corollary 4.3.5. *The astigmatism A of the wavefront on propagation is given by*

$$A(t_0 + \tau) = \frac{A(t_0)}{1 - 2dH(t_0) + d^2K(t_0)}$$

Proof. By definition 4.3.4,

$$A^2(t_0 + \tau) = H^2(t_0 + \tau) - K(t_0 + \tau).$$

By Corollary 4.3.3, expanding $H(t_0 + \tau)$ and $K(t_0 + \tau)$ gives

$$A^2(t_0 + \tau) = \left(\frac{H(t_0) - dK(t_0)}{1 - 2dH(t_0) + d^2K(t_0)} \right)^2 - \frac{K(t_0)}{1 - 2dH(t_0) + d^2K(t_0)}.$$

Combining the expression gives

$$A^2(t_0 + \tau) = \frac{(H(t_0) - dK(t_0))^2 - K(t_0)(1 - 2dH(t_0) + d^2K(t_0))}{(1 - 2dH(t_0) + d^2K(t_0))^2}.$$

Hence,

$$A^2(t_0 + \tau) = \frac{H(t_0)^2 - K(t_0)}{(1 - 2dH(t_0) + d^2K(t_0))^2}.$$

By the definition of $A(t_0) = \sqrt{H(t_0)^2 - K(t_0)}$,

$$A^2(t_0 + \tau) = \frac{(A(t_0))^2}{(1 - 2dH(t_0) + d^2K(t_0))^2},$$

then, $A(t_0 + \tau) = \frac{A(t_0)}{1 - 2dH(t_0) + d^2K(t_0)}$. □

4.3.2 The Third Order Surface coefficients on propagation

By definition (2.3.1),

$$\begin{aligned} \Lambda_{ijk}(t_0 + \tau) &= -\langle W_{x_i x_j}(t_0 + \tau), Q_{x_k} \rangle - \langle W_{x_i x_k}(t_0 + \tau), Q_{x_j} \rangle \\ &\quad - \langle W_{x_i}(t_0 + \tau), Q_{x_j x_k} \rangle. \end{aligned}$$

Since

$$W_{x_i x_j}(t_0 + \tau) = W_{x_i x_j}(t_0) + dQ_{x_i x_j},$$

then

$$\begin{aligned}\Lambda_{ijk}(t_0 + \tau) &= -\langle W_{x_i x_j}(t_0) + dQ_{x_i x_j}, Q_{x_k} \rangle - \langle W_{x_i x_k}(t_0) + dQ_{x_i x_k}, Q_{x_j} \rangle \\ &\quad - \langle W_{x_i}(t_0) + dQ_{x_i}, Q_{x_j x_k} \rangle.\end{aligned}$$

Also,

$$\Lambda_{ijk}(t_0) = -\langle W_{x_i x_j}(t_0), Q_{x_k} \rangle - \langle W_{x_i x_k}(t_0), Q_{x_j} \rangle - \langle W_{x_i}(t_0), Q_{x_j x_k} \rangle.$$

We have

$$\Lambda_{ijk}(t_0 + \tau) = \Lambda_{ijk}(t_0) - d(\langle Q_{x_i x_j}, Q_{x_k} \rangle + \langle Q_{x_i x_k}, Q_{x_j} \rangle + \langle Q_{x_j x_k}, Q_{x_i} \rangle).$$

With lemma 2.3.3,

$$\langle Q_{x_i x_j}, Q_{x_k} \rangle = \langle \Delta_{ij}^l R_{x_l}, Q_{x_k} \rangle = -\Delta_{ij}^l L_{lk}.$$

Overall, we have the formula describing the Third Order Surface coefficients on propagation.

Theorem 4.3.6. *The Third Order Surface coefficients on propagation are given by*

$$\begin{aligned}\Lambda_{ijk}(t_0 + \tau) &= \Lambda_{ijk}(t_0) + d[\Delta_{ik}^l(t_0)L_{lj}(t_0) + \Delta_{jk}^l(t_0)L_{li}(t_0) \\ &\quad + \Delta_{ij}^l(t_0)L_{lk}(t_0)],\end{aligned}$$

where d is the distance the wavefront propagates during time τ .

4.4 Wavefront surface coefficients on refraction

4.4.1 The derivative of ϕ

Recall equation (4.3), if we assume that $\mu_1 = 1$ and $\mu_2 = \mu$, then

$$W + \phi Q = R = W' - \mu(l - \phi)Q'.$$

Since at the strike point O ,

$$O = W(x_1^0, x_2^0) = R(\bar{x}_1^0, \bar{x}_2^0) = W'(x_1'^0, x_2'^0),$$

then $\phi = 0$ at point O , and $l = 0$ for this refracted wavefront W' .

Lemma 4.4.1.

$$W'_{x_i} = W_{x_i} + \phi_{x_i}(Q - \mu Q') = R_{x_i} - \mu \phi_{x_i} Q',$$

where

$$\phi_{x_i} = \frac{\langle R_{x_i}, Q' \rangle}{\mu} = \langle R_{x_i}, Q \rangle.$$

Proof. First differentiation on x gives

$$W_{x_i} + \phi_{x_i} Q = R_{x_i} = W'_{x_i} + \mu \phi_{x_i} Q'.$$

Then we apply inner product with S through both sides, since $W'_{x_i} \cdot Q' = 0$,

$$\phi_{x_i} = \frac{\langle R_{x_i}, Q' \rangle}{\mu}.$$

and

$$\frac{\langle R_{x_i}, Q' \rangle}{\mu} = \frac{\langle R_{x_i}, (\mu Q + \gamma S) \rangle}{\mu} = \langle R_{x_i}, Q \rangle.$$

□

Lemma 4.4.2.

$$\begin{aligned} W'_{x_i x_j} &= W_{x_i x_j} + \phi_{x_i x_j}(Q - \mu Q') + \phi_{x_i}(Q_{x_j} - \mu Q'_{x_j}) + \phi_{x_j}(Q_{x_i} - \mu Q'_{x_i}) \\ &= R_{x_i x_j} - \mu \phi_{x_i x_j} Q' - \mu \phi_{x_i} Q'_{x_j} - \mu \phi_{x_j} Q'_{x_i} \end{aligned}$$

where

$$\phi_{x_i x_j} = \frac{\langle R_{x_i x_j}, Q' \rangle}{\mu} - \frac{L_{ij}^{W'}}{\mu} = \langle R_{x_i x_j}, Q \rangle + L_{ij}^W.$$

Proof. Taking derivative of equation 4.3 with respect to x_i twice, we have

$$\begin{aligned} R_{x_i x_j} &= W_{x_i x_j} + \phi_{x_i x_j} Q + \phi_{x_i} Q_{x_j} + \phi_{x_j} Q_{x_i} \\ &= W'_{x_i x_j} + \mu \phi_{x_i x_j} Q' + \mu \phi_{x_i} Q'_{x_j} + \mu \phi_{x_j} Q'_{x_i}. \end{aligned}$$

Since $\langle Q'_{x_i}, Q' \rangle = 0$, then

$$\langle W'_{x_i x_j}, Q' \rangle = \langle R_{x_i x_j}, Q' \rangle - \phi_{x_i x_j}.$$

Then,

$$\phi_{x_i x_j} = \frac{\langle R_{x_i x_j}, Q' \rangle}{\mu} - \frac{L_{ij}^{W'}}{\mu}.$$

By lemma 3.1.2,

$$Q' = \mu Q + \gamma S,$$

where $\gamma = \cos \alpha_r - \mu \cos \alpha_i$, then

$$\langle R_{x_i x_j}, Q' \rangle = \langle R_{x_i x_j}, \mu Q + \gamma S \rangle = \mu \langle R_{x_i x_j}, Q \rangle + \gamma \langle R_{x_i x_j}, S \rangle.$$

Recall that $\langle R_{x_i x_j}, S \rangle = L_{ij}^R$, we have

$$\phi_{x_i x_j} = \frac{\langle R_{x_i x_j}, \mu Q \rangle}{\mu} + \gamma \frac{L_{ij}^R}{\mu} - \frac{L_{ij}^{W'}}{\mu}.$$

By theorem 4.4.4,

$$L_{ij}^W = \frac{L_{ij}^{W'}}{\mu} - \gamma \frac{L_{ij}^R}{\mu},$$

then $\phi_{x_i x_j}$ is

$$\phi_{x_i x_j} = \langle R_{x_i x_j}, Q \rangle + L_{ij}^W.$$

□

4.4.2 The Fundamental Form coefficients on refraction

We are now ready to derive the First Fundamental Form coefficients and the Second Fundamental Form coefficients of the refracted wavefront.

Theorem 4.4.3. *The First Fundamental Form coefficients $g^{W'}$ of W' under coordinate system $\{x_i\}_{i=1}^2$ are*

$$\begin{aligned} g_{ij}^{W'} &= g_{ij}^R - \mu^2 \phi_{x_i} \phi_{x_j} \\ &= g_{ij}^W - (2 - \mu^2) \phi_{x_i} \phi_{x_j}. \end{aligned}$$

Proof. By definition 2.1,

$$g_{ij}^{W'} = \langle W'_{x_i}, W'_{x_j} \rangle.$$

According to lemma 4.4.1,

$$W'_{x_i} = R_{x_i} - \phi_{x_i} \mu Q',$$

then

$$g_{ij}^{W'} = \langle R_{x_i} - \phi_{x_i} \mu Q', R_{x_j} - \phi_{x_j} \mu Q' \rangle.$$

Expanding the above expression gives

$$g_{ij}^{W'} = \langle R_{x_i}, R_{x_j} \rangle - \mu \phi_{x_i} \langle R_{x_j}, Q' \rangle - \mu \phi_{x_j} \langle R_{x_i}, Q' \rangle - \mu^2 \phi_{x_i} \phi_{x_j}. \quad (4.11)$$

By the definition of g_{ij}^R , we have

$$\langle R_{x_i}, R_{x_j} \rangle = g_{ij}^R.$$

Also

$$\langle R_{x_i}, Q' \rangle = \langle R_{x_j}, \mu Q + \gamma S \rangle = \mu \langle R_{x_j}, Q \rangle + \gamma \langle R_{x_j}, S \rangle.$$

Recall lemma 4.4.1,

$$\phi_{x_i} = \langle R_{x_i}, Q' \rangle / \mu.$$

Then the third term of equation (4.11) turns out to be

$$-\mu \phi_{x_j} \langle R_{x_i}, Q' \rangle = -\mu \phi_{x_j} (\mu \phi_{x_i}) = -\mu^2 \phi_{x_i} \phi_{x_j}.$$

Overall, we have

$$\begin{aligned} g_{ij}^{W'} &= g_{ij}^R + \mu^2 \phi_{x_i} \phi_{x_j} - \mu \phi_{x_i} \phi_{x_j} - \mu \phi_{x_j} \phi_{x_i} \\ &= g_{ij}^R - \mu^2 \phi_{x_i} \phi_{x_j}. \end{aligned}$$

□

Theorem 4.4.4. *The Second Fundamental Form coefficients $L^{W'}$ of W' under coordinate system $\{x_i\}_{i=1}^2$ are*

$$L_{ij}^{W'} = \mu L_{ij}^W + \gamma L_{ij}^R.$$

Proof. By definition 2.2,

$$L_{ij}^{W'} = \langle W'_{x_i x_j}, Q' \rangle.$$

Expanding Q' gives

$$L_{ij}^{W'} = \langle W'_{x_i x_j}, \mu Q + \gamma S \rangle = \mu \langle W'_{x_i x_j}, Q \rangle + \gamma \langle W'_{x_i x_j}, S \rangle.$$

By lemma 4.4.2,

$$\langle W'_{x_i x_j}, Q \rangle = \langle W_{x_i x_j} + \phi_{x_i x_j} (Q - \mu Q') + \phi_{x_i} (Q_{x_j} - \mu Q'_{x_j}) + \phi_{x_j} (Q_{x_i} - \mu Q'_{x_i}), Q \rangle,$$

and

$$\langle W'_{x_i x_j}, S \rangle = \langle R_{x_i x_j} - \phi_{x_i x_j} \mu Q' - \phi_{x_i} \mu Q'_{x_j} - \phi_{x_j} \mu Q'_{x_i}, S \rangle.$$

Expanding the equations gives

$$\begin{aligned} \langle W'_{x_i x_j}, Q \rangle &= \langle W_{x_i x_j}, Q \rangle + \phi_{x_i x_j} \langle (Q - \mu Q'), Q \rangle + \phi_{x_i} \langle (Q_{x_j} - \mu Q'_{x_j}), Q \rangle \\ &\quad + \phi_{x_j} \langle (Q_{x_i} - \mu Q'_{x_i}), Q \rangle, \end{aligned}$$

and

$$\langle W'_{x_i x_j}, S \rangle = \langle R_{x_i x_j}, S \rangle - \phi_{x_i x_j} \langle \mu Q', S \rangle - \phi_{x_i} \langle \mu Q'_{x_j}, S \rangle - \phi_{x_j} \langle \mu Q'_{x_i}, S \rangle.$$

Recall that

$$\langle W_{x_i x_j}, Q \rangle = L_{ij}^W, \quad \langle Q_{x_i}, Q \rangle = 1,$$

then

$$\langle W'_{x_i x_j}, Q \rangle = L_{ij}^W + \phi_{x_i x_j} - \mu \phi_{x_i x_j} \langle Q', Q \rangle - \mu \phi_{x_i} \langle Q'_{x_j}, Q \rangle - \mu \phi_{x_j} \langle Q'_{x_i}, Q \rangle,$$

Also

$$\langle R_{x_i x_j}, S \rangle = L_{ij}^R,$$

we have

$$\langle W'_{x_i x_j}, S \rangle = L_{ij}^R - \mu \phi_{x_i x_j} \langle Q', S \rangle - \mu \phi_{x_i} \langle Q'_{x_j}, S \rangle - \mu \phi_{x_j} \langle Q'_{x_i}, S \rangle.$$

Combining them gives

$$\begin{aligned} L_{ij}^{W'} &= \mu(L_{ij}^W + \phi_{x_i x_j} - \mu \phi_{x_i x_j} \langle Q', Q \rangle - \mu \phi_{x_i} \langle Q'_{x_j}, Q \rangle - \mu \phi_{x_j} \langle Q'_{x_i}, Q \rangle) \\ &\quad + \gamma(L_{ij}^R - \mu \phi_{x_i x_j} \langle Q', S \rangle - \mu \phi_{x_i} \langle Q'_{x_j}, S \rangle - \mu \phi_{x_j} \langle Q'_{x_i}, S \rangle). \end{aligned}$$

Recall that $Q' = \mu Q + \gamma S$, then

$$\begin{aligned} \mu^2 \phi_{x_i x_j} \langle Q', Q \rangle + \gamma \mu \phi_{x_i x_j} \langle Q', S \rangle &= \mu \phi_{x_i x_j} \langle Q', \mu Q + \gamma S \rangle \\ &= \mu \phi_{x_i x_j} \langle Q', Q' \rangle \\ &= \mu \phi_{x_i x_j}, \end{aligned}$$

given that $\langle Q', Q' \rangle = 1$.

Similarly,

$$\mu^2 \phi_{x_i} \langle Q'_{x_j}, Q \rangle + \gamma \mu \phi_{x_i} \langle Q'_{x_j}, S \rangle = \mu \phi_{x_i} \langle Q'_{x_j}, Q' \rangle,$$

the expression is eliminated since $\langle Q'_{x_j}, Q' \rangle = 0$. Similar derivation leads to

$$\mu^2 \phi_{x_j} \langle Q'_{x_i}, Q \rangle + \gamma \mu \phi_{x_j} \langle Q'_{x_i}, S \rangle = \mu \phi_{x_j} \langle Q'_{x_i}, Q' \rangle = 0.$$

Overall, we have

$$L_{ij}^{W'} = \mu L_{ij}^W + \gamma L_{ij}^R.$$

□

If we adapter the coordinates in Kneisly's paper [8], as in equations (4.4), $\{u_1, u_2\}$, $\{\bar{u}_1, \bar{u}_2\}$, and $\{u'_1, u'_2\}$, according to 4.4.4, we have

$$(L^{W'})_{ij}^r = \mu(L^W)_{mn}^r \frac{\partial u_m}{\partial u'_i} \frac{\partial u_m}{\partial u'_j} + \gamma(L^R)_{mn}^r \frac{\partial \bar{u}_m}{\partial u'_i} \frac{\partial \bar{u}_m}{\partial u'_j}.$$

Recall that

$$du_1 = d\bar{u}_1 = du'_1, \quad \text{and} \quad du_2 = \cos \alpha_i d\bar{u}_2 = \frac{\cos \alpha_i}{\cos \alpha_r} du'_2,$$

in particular, $\frac{\partial u'_i}{\partial u_j} = \frac{\partial u'_i}{\partial \bar{u}_j} = 0$, if $i \neq j$, then we have

$$\begin{aligned} (L^{W'})_{11}^r &= \mu(L^W)_{11}^r + \gamma(L^R)_{11}^r, \\ (L^{W'})_{12}^r &= \mu(L^W)_{12}^r \frac{\cos \alpha_i}{\cos \alpha_i} + \gamma(L^R)_{12}^r \frac{1}{\cos \alpha_r}, \\ (L^{W'})_{22}^r &= \mu(L^W)_{22}^r \left(\frac{\cos \alpha_i}{\cos \alpha_i} \right)^2 + \gamma(L^R)_{22}^r \left(\frac{1}{\cos \alpha_i} \right)^2. \end{aligned}$$

Therefore, with coordinate systems $\{u_1, u_2\}$, $\{\bar{u}_1, \bar{u}_2\}$, and $\{u'_1, u'_2\}$, our result coincides with Kneisly's formula (4.5).

From theorem 4.4.4, the Second Fundamental Form coefficients of the refracted wavefront W' can be fully represented by the Second Fundamental Form coefficients of the initial wavefront and the refract surface.

Also, by Corollary 2.4.5, we have the direct formula of mean curvature and Gaussian curvature of the refracted wavefront W' ,

Theorem 4.4.5.

$$\begin{aligned} H^{W'} &= \mu H^W \frac{\Delta_W}{\Theta_W} - \frac{1}{2} \frac{\Pi_W}{\Theta_W} + \gamma H^R \frac{\Delta_R}{\Theta_R} - \frac{1}{2} \frac{\Pi_R}{\Theta_R}, \\ K^{W'} &= \mu^2 K^W \frac{\Delta_W}{\Theta_W} + \gamma^2 K^R \frac{\Delta_R}{\Theta_R} + \mu \gamma \frac{\sigma}{\Theta_R}. \end{aligned}$$

where

$$\begin{aligned}
\Delta_W &= g_{11}^W g_{22}^W - (g_{12}^W)^2; \\
\Delta_R &= g_{11}^R g_{22}^R - (g_{12}^R)^2; \\
\Theta_W &= (g_{11}^W g_{22}^W - (g_{12}^W)^2) - (2 - 2\mu^2)(\phi_{x_i} \phi_{x_j} g_{ij}^W); \\
\Theta_R &= (g_{11}^R g_{22}^R - (g_{12}^R)^2) - \mu^2(\phi_{x_i} \phi_{x_j} g_{ij}^W); \\
\Pi_W &= (2 - 2\mu^2)(\phi_{x_i} \phi_{x_j} L_{ij}^W); \\
\Pi_R &= \mu^2(\phi_{x_i} \phi_{x_j} L_{ij}^R); \\
\sigma &= L_{11}^R L_{22}^W + L_{11}^R L_{22}^W - 2L_{12}^R L_{12}^W.
\end{aligned}$$

Proof. By corollary 2.4.5,

$$H^{W'} = \frac{1}{2} \frac{L_{11}^{W'} g_{22}^{W'} + L_{22}^{W'} g_{11}^{W'} - 2g_{12}^{W'} L_{12}^{W'}}{g_{11}^{W'} g_{22}^{W'} - (g_{12}^{W'})^2}$$

then by theorem 4.4.4,

$$L_{11}^{W'} g_{22}^{W'} = (\mu L_{11}^W + \gamma L_{11}^R) g_{22}^{W'}.$$

Also by theorem 4.4.3,

$$L_{11}^{W'} g_{22}^{W'} = \mu L_{11}^W (g_{22}^W - (2 - \mu^2)\phi_{x_2}^2) + \gamma L_{11}^R (g_{22}^R - \mu^2\phi_{x_2}^2).$$

Expanding the above expression gives

$$L_{11}^{W'} g_{22}^{W'} = \mu L_{11}^W g_{22}^W - \mu(2 - \mu^2)\phi_{x_2}^2 L_{11}^W + \gamma L_{11}^R g_{22}^R - \mu^2\gamma\phi_{x_2}^2 L_{11}^R.$$

Similarly, we can deduce that

$$L_{12}^{W'} g_{12}^{W'} = \mu L_{12}^W g_{12}^W - \mu(2 - \mu^2)\phi_{x_1}\phi_{x_2} L_{12}^W + \gamma L_{12}^R g_{12}^R - \mu^2\gamma\phi_{x_1}\phi_{x_2} L_{12}^R.$$

and

$$L_{22}^{W'} g_{11}^{W'} = \mu L_{22}^W g_{11}^W - \mu(2 - \mu^2)\phi_{x_1}^2 L_{22}^W + \gamma L_{22}^R g_{11}^R - \mu^2\gamma\phi_{x_1}^2 L_{22}^R.$$

Also, by theorem 4.4.3,

$$g_{11}^{W'} g_{22}^{W'} - (g_{12}^{W'})^2 = (g_{11}^W - (2 - \mu^2)\phi_{x_1}^2)(g_{22}^W - (2 - \mu^2)\phi_{x_2}^2) - (g_{12}^W - (2 - \mu^2)\phi_{x_1}\phi_{x_2})^2. \quad (4.12)$$

and

$$g_{11}^{W'} g_{22}^{W'} - (g_{12}^{W'})^2 = (g_{11}^R - \mu^2 \phi_{x_1}^2)(g_{22}^R - \mu^2 \phi_{x_2}^2) - (g_{12}^R - \mu^2 \phi_{x_1} \phi_{x_2})^2. \quad (4.13)$$

Simplifying (4.12) and (4.13) gives

$$\begin{aligned} g_{11}^{W'} g_{22}^{W'} - (g_{12}^{W'})^2 &= (g_{11}^W g_{22}^W - (g_{12}^W)^2) - (2 - 2\mu^2)(\phi_{x_i} \phi_{x_j} g_{ij}^W) \\ &= (g_{11}^R g_{22}^R - (g_{12}^R)^2) - \mu^2(\phi_{x_i} \phi_{x_j} g_{ij}^W). \end{aligned}$$

Therefore, we have

$$\begin{aligned} H^{W'} &= \frac{1}{2} \frac{\mu(g_{11}^W L_{22}^W + g_{22}^W L_{11}^W - 2g_{12}^W L_{12}^W)}{(g_{11}^W g_{22}^W - (g_{12}^W)^2) - (2 - 2\mu^2)(\phi_{x_i} \phi_{x_j} g_{ij}^W)} \\ &\quad + \frac{1}{2} \frac{(2 - 2\mu^2)(\phi_{x_i} \phi_{x_j} L_{ij}^W)}{(g_{11}^W g_{22}^W - (g_{12}^W)^2) - (2 - 2\mu^2)(\phi_{x_i} \phi_{x_j} g_{ij}^W)} \\ &\quad + \gamma \frac{1}{2} \frac{g_{11}^R L_{22}^R + g_{22}^R L_{11}^R - 2g_{12}^R L_{12}^R}{(g_{11}^R g_{22}^R - (g_{12}^R)^2) - \mu^2(\phi_{x_i} \phi_{x_j} g_{ij}^W)} \\ &\quad + \frac{1}{2} \frac{\mu^2(\phi_{x_i} \phi_{x_j} L_{ij}^R)}{(g_{11}^R g_{22}^R - (g_{12}^R)^2) - \mu^2(\phi_{x_i} \phi_{x_j} g_{ij}^W)}. \end{aligned}$$

With the notations defined above, we have

$$H^{W'} = \mu H^W \frac{\Delta_W}{\Theta_W} - \frac{1}{2} \frac{\Pi_W}{\Theta_W} \gamma H^R \frac{\Delta_R}{\Theta_R} - \frac{1}{2} \frac{\Pi_R}{\Theta_R}.$$

Similarly, by Corollary 2.4.5,

$$K^{W'} = \frac{L_{11}^{W'} L_{22}^{W'} - (L_{12}^{W'})^2}{g_{11}^{W'} g_{22}^{W'} - (g_{12}^{W'})^2}.$$

Then by theorem 4.4.4

$$L_{11}^{W'} L_{22}^{W'} - (L_{12}^{W'})^2 = (\mu L_{11}^W + \gamma L_{11}^R)(\mu L_{22}^W + \gamma L_{22}^R) - (\mu L_{12}^W + \gamma L_{12}^R)^2$$

Expanding the expression gives

$$\begin{aligned} L_{11}^{W'} L_{22}^{W'} - (L_{12}^{W'})^2 &= \mu^2(L_{11}^W L_{22}^W - (L_{12}^W)^2) + \gamma^2(L_{11}^R L_{22}^R - (L_{12}^R)^2) \\ &\quad - 2\mu\gamma(L_{11}^R L_{22}^W + L_{11}^W L_{22}^R - 2L_{12}^R L_{12}^W). \end{aligned}$$

Then

$$\begin{aligned}
K^{W'} &= \mu^2 \frac{L_{11}^W L_{22}^W - (L_{12}^W)^2}{(g_{11}^W g_{22}^W - (g_{12}^W)^2) - (2 - 2\mu^2)(\phi_{x_i} \phi_{x_j} g_{ij}^W)} + \\
&\quad \gamma^2 \frac{L_{11}^R L_{22}^R - (L_{12}^R)^2}{(g_{11}^R g_{22}^R - (g_{12}^R)^2) - \mu^2(\phi_{x_i} \phi_{x_j} g_{ij}^W)} \\
&\quad + \mu\gamma \frac{L_{11}^R L_{22}^W + L_{11}^R L_{22}^W - 2L_{12}^R L_{12}^W}{(g_{11}^R g_{22}^R - (g_{12}^R)^2) - \mu^2(\phi_{x_i} \phi_{x_j} g_{ij}^W)},
\end{aligned}$$

which is

$$K^{W'} = \mu^2 K_W \frac{\Delta_W}{\Theta_W} + \gamma^2 K^R \frac{\Delta_R}{\Theta_R} + \mu\gamma \frac{\sigma}{\Theta_R}.$$

□

4.4.3 The Third Order Surface coefficients on refraction

The Third Order Surfaces coefficients of W' on refraction involves the derivative of the traveling distance ϕ , the Second Fundamental Form and the Third Order Surfaces coefficients of W and R .

Theorem 4.4.6. *The Third Order Surface coefficients Λ of W' under coordinate system $\{x_i\}_{i=1}^2$ are*

$$\Lambda_{ijk}^{W'} = \mu \Lambda_{ijk}^W + \gamma \Lambda_{ijk}^R + \mu(\Pi_{ik}^{W'} - \Pi_{ik}^W) + \mu \phi_{x_j} (\Pi_{ik}^{W'} - \Pi_{ik}^W) + \mu \phi_{x_k} (\Pi_{ik}^{W'} - \Pi_{ik}^W).$$

Π_{ij}^W is defined as

$$\Pi_{ij}^W = (g^W)^{lm} L_{il}^W L_{jm}^W, \quad \Pi_{ij}^{W'} = (g^{W'})^{lm} L_{il}^{W'} L_{jm}^{W'}.$$

Proof. By definition 2.3.1,

$$\Lambda_{ijk}^{W'} = \langle W'_{ijk}, Q' \rangle.$$

By $Q' = \mu Q + \gamma S$,

$$\Lambda_{ijk}^{W'} = \langle W'_{ijk}, (\mu Q + \gamma S) \rangle.$$

Since

$$\begin{aligned} W'_{ijk} &= W_{ijk} + (\phi Q)_{ijk} - (\mu\phi Q')_{ijk} \\ &= R_{ijk} - \mu(\phi Q')_{ijk}, \end{aligned}$$

then

$$\Lambda_{ijk}^{W'} = \langle (W_{ijk} + (\phi Q)_{ijk} - (\mu\phi Q')_{ijk}), (\mu Q) \rangle + \langle (R_{ijk} - (\mu\phi Q')_{ijk}), (\gamma S) \rangle,$$

which is

$$\Lambda_{ijk}^{W'} = \mu \langle (W_{ijk}, Q) \rangle + \gamma \langle (R_{ijk}, S) \rangle + \mu \langle (\phi Q)_{ijk}, Q \rangle - \langle (\mu\phi Q')_{ijk}, \mu Q + \gamma S \rangle.$$

Notice that

$$\Lambda_{ijk}^W = \langle (W_{ijk}, Q) \rangle, \quad \Lambda_{ijk}^R = \langle (R_{ijk}, S) \rangle,$$

then

$$\begin{aligned} \Lambda_{ijk}^{W'} &= \mu \Lambda_{ijk}^W + \gamma \Lambda_{ijk}^R + \mu \langle (\phi Q)_{ijk}, Q \rangle \\ &\quad - \langle (\mu\phi Q')_{ijk}, \mu Q + \gamma S \rangle. \end{aligned}$$

Expand $\langle (\phi Q)_{ijk}, Q \rangle$, we have

$$\begin{aligned} \langle (\phi Q)_{ijk}, Q \rangle &= \langle \phi Q_{ijk}, Q \rangle + \langle \phi_i Q_{jk}, Q \rangle + \langle \phi_j Q_{ik}, Q \rangle + \langle \phi_k Q_{ij}, Q \rangle \\ &\quad + \langle \phi_{ij} Q_k, Q \rangle + \langle \phi_{ik} Q_j, Q \rangle + \langle \phi_{jk} Q_i, Q \rangle \\ &\quad + \langle \phi_{ijk} Q, Q \rangle, \end{aligned}$$

Given the fact that at the intersection point,

$$\phi = 0,$$

and Q is an unit normal vector,

$$\langle Q, Q \rangle = 1, \quad \langle Q_i, Q \rangle = 0,$$

$$\langle (\phi Q)_{ijk}, Q \rangle = \langle \phi_i Q_{jk}, Q \rangle + \langle \phi_j Q_{ik}, Q \rangle + \langle \phi_k Q_{ij}, Q \rangle + \phi_{ijk}.$$

By lemma 2.3.3,

$$\langle Q_{ij}, Q \rangle = \langle (\Delta_{ij}^k)^W W_k + \Pi_{ij}^W Q, Q \rangle = \Pi_{ij}^W,$$

where $\Pi_{ij}^W = -(g^W)^{lk} L_{li}^W L_{kj}^W$. Therefore,

$$\langle (\phi Q)_{ijk}, Q \rangle = \phi_i \Pi_{jk}^W + \phi_j \Pi_{ik}^W + \phi_k \Pi_{ij}^W + \phi_{ijk}.$$

Similarly,

$$\langle (\phi Q')_{ijk}, Q' \rangle = \phi_i \Pi_{jk}^{W'} + \phi_j \Pi_{ik}^{W'} + \phi_k \Pi_{ij}^{W'} + \phi_{ijk},$$

where $\Pi_{ij}^{W'} = -(g^{W'})^{lk} L_{li}^{W'} L_{kj}^{W'}$.

With all the information, the simplified form of $\Lambda_{ijk}^{W'}$ is

$$\Lambda_{ijk}^{W'} = \mu \Lambda_{ijk}^W + \gamma \Lambda_{ijk}^R + \mu (\Pi_{ik}^{W'} - \Pi_{ik}^W) + \mu \phi_{x_j} (\Pi_{ik}^{W'} - \Pi_{ik}^W) + \mu \phi_{x_k} (\Pi_{ik}^{W'} - \Pi_{ik}^W).$$

□

From theorem 4.4.6, the Third Order Surface coefficients of the refracted wavefront W' involves the Third Order Surface coefficients of W and R , the Second Fundamental Form of W and R .

Chapter 5

Lens design problem

The lens design problem can be stated as an optimization problem as follows: we want to find the front lens surface R^f and the back lens surface R^b such that the design objective functional \mathcal{J} is

$$\mathcal{J}(R^f, R^b) = \int_{\Omega} \beta |H(\theta) - \hat{P}(\theta)|^2 d\theta + \int_{\Omega} \sum_j \alpha_j c_j^2(\theta) d\theta.$$

where Ω is the lens design region, θ is the gaze direction, \hat{P} is the prescribed power distribution, H is the power of the refracted wavefront after the lens system of $\{R^f, R^b\}$, β and $\{\alpha_j\}$ are corresponding weights for the power and high-order aberration terms.

In this chapter, we will focus on the power and the astigmatism of the refracted wavefront W_{rr} , so the objective functional \mathcal{J} is

$$\mathcal{J}(R^f, R^b) = \int_{\Omega} \beta |H(\theta) - \hat{P}(\theta)|^2 d\theta + \int_{\Omega} \alpha A^2(\theta) d\theta.$$

5.1 Ray tracing method and curvature of wavefront

As shown in Figure 5.1, we will show the ray tracing process from eye center O to the front lens surface R^f . All the perimeters will be evaluated according to

the formulas given in Chapter 4 along the ray tracing route. The relations of the wavefronts through the lens is also illustrated in Figure 5.2.

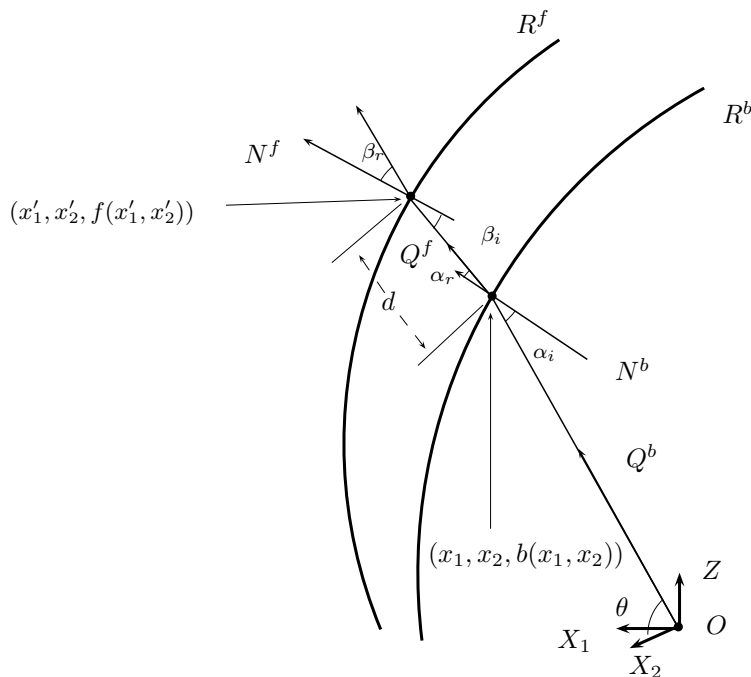


Figure 5.1: Ray tracing through a lens

Let's consider the back lens surface R^b given by $z = b(X_1, X_2) : \Omega \rightarrow \mathbb{R}$, where the domain $\Omega \in \mathbb{R}^2$. For a given gaze direction θ , let (x_1, x_2) be the corresponding X_1 - X_2 coordinates of θ , therefore, the incident direction Q^b is

$$\left(\frac{x_1}{\sqrt{x_1^2 + x_2^2 + b(x_1, x_2)^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2 + b(x_1, x_2)^2}}, \frac{b(x_1, x_2)}{\sqrt{x_1^2 + x_2^2 + b(x_1, x_2)^2}} \right) \quad (5.1)$$

The unit tangent vectors of the back surface R^b are

$$\left(\frac{1}{\sqrt{1 + b_{x_1}(x_1, x_2)^2}}, 0, \frac{b_{x_1}(x_1, x_2)}{\sqrt{1 + b_{x_1}(x_1, x_2)^2}} \right)$$

and

$$\left(\frac{1}{\sqrt{1 + b_{x_2}(x_1, x_2)^2}}, 0, \frac{b_{x_2}(x_1, x_2)}{\sqrt{1 + b_{x_2}(x_1, x_2)^2}} \right).$$

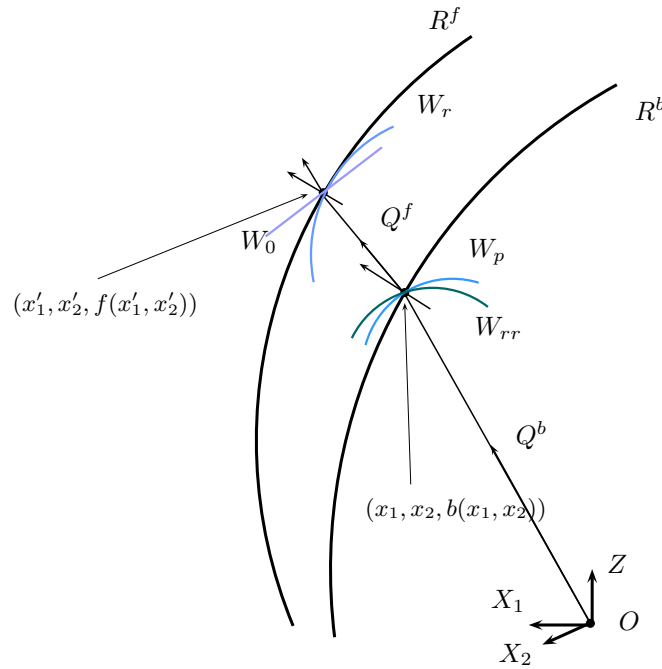


Figure 5.2: Wavefronts W_0 , W_r , W_p , W_{rr} through a lens

The normal direction N^b on the back surface R^b is the cross product of the two unit tangent vectors

$$\left(\frac{-b_{x_1}}{\sqrt{1 + b_{x_1}^2 + b_{x_2}^2}}, \frac{-b_{x_2}}{\sqrt{1 + b_{x_1}^2 + b_{x_2}^2}}, \frac{1}{\sqrt{1 + b_{x_1}^2 + b_{x_2}^2}} \right).$$

Then by lemma 3.1.2 the refracted direction Q^f on the back surface R^b is given by

$$Q^f = \mu_1 Q^b + \gamma^b N^b, \quad (5.2)$$

where $\mu_1 = n_1/n_2$ and $\gamma^b = \cos \alpha_r - \mu_1 \cos \alpha_i$, where α_i and α_r are the corresponding incident angle and refraction angle.

In addition,

$$\cos \alpha_i = \langle Q^b, N \rangle = \frac{-x_1 b_{x_1}(x_1, x_2) - x_2 b_{x_2}(x_1, x_2) + b(x_1, x_2)}{\sqrt{1 + b_{x_1}(x_1, x_2)^2 + b_{x_2}(x_1, x_2)^2} \sqrt{x_1^2 + x_2^2 + b(x_1, x_2)^2}},$$

We have

$$\cos \alpha_r = \sqrt{1 - \sin^2 \alpha_r},$$

by Snell's law,

$$\sqrt{1 - \sin^2 \alpha_r} = \sqrt{1 - \mu_1^2 \sin^2 \alpha_i},$$

therefore

$$\cos \alpha_r = \sqrt{1 - \mu_1^2(1 - \cos^2 \alpha_i)},$$

which also is

$$\cos \alpha_r = \sqrt{1 - \mu_1^2(1 - \langle Q^b, N \rangle^2)}.$$

Assume the refracted direction Q^f intersects the front lens surface R^f at point $(x_1', x_2', f(x_1', x_2'))$ with travel distance d , then the normal direction of R^f at coordinates (x_1', x_2') is

$$\frac{(-f_{x_1}(x_1', x_2'), -f_{x_2}(x_1', x_2'), 1)}{\sqrt{1 + f_{x_1}(x_1', x_2')^2 + f_{x_2}(x_1', x_2')^2}}.$$

Also, x_1, x_2, d, x_1', x_2' should satisfy the equation

$$(x_1, x_2, b(x_1, x_2)) + d \cdot Q^f = (x_1', x_2', f(x_1', x_2')). \quad (5.3)$$

Assume the incident angle and the refracted angle on the front lens surface are β_i and β_r , then

$$\begin{aligned} \cos \beta_i &= \langle Q^f, N^f \rangle, \\ \cos \beta_r &= \sqrt{1 - \mu_1^2(1 - \langle Q^f, N^f \rangle^2)}, \\ \gamma^f &= \cos \beta_i - \mu_2 \cos \beta_r \end{aligned}$$

By definition, the First Fundamental Form coefficients and the Second Fundamental Form coefficients of the back lens surface

$$\begin{aligned} g_{ij}^b &= \langle (R^b)_{x_i}, (R^b)_{x_j} \rangle = e_m^i e_m^j + b_{x_i}(x_1, x_2) b_{x_j}(x_1, x_2), \\ L_{ij}^b &= \langle (R^b)_{x_i x_j}, -N^b \rangle = -\frac{b_{x_i x_j}}{\sqrt{1 + b_{x_1}(x_1, x_2)^2 + b_{x_2}(x_1, x_2)^2}}, \end{aligned}$$

where

$$e_j^i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

In particular, the mean curvature and the Gaussian curvature of R^b at point (x_1, x_2) are given by

$$H^b = \frac{(1 + b_{x_1}^2)b_{x_2x_2} + (1 + b_{x_2}^2)b_{x_1x_1} + 2b_{x_1}b_{x_2}b_{x_1x_2}}{2(\sqrt{1 + b_{x_1}^2 + b_{x_2}^2})^{3/2}},$$

$$H^b = \frac{b_{x_1x_1}b_{x_2x_2} - b_{x_1x_2}^2}{1 + b_{x_1}^2 + b_{x_2}^2}.$$

Similarly, the coefficients of the front lens surface are

$$g_{ij}^f = \langle (R^f)_{x_i}, (R^f)_{x_j} \rangle = e_{1i}e_{1j} + e_{2i}e_{2j} + f_{x_i}(x'_1, x'_2)f_{x_j}(x'_1, x'_2),$$

$$L_{ij}^f = \langle (R^f)_{x_ix_j}, N^f \rangle = -\frac{f_{x_ix_j}(x'_1, x'_2)}{\sqrt{1 + f_{x_1}(x_1, x_2)^2 + f_{x_2}(x_1, x_2)^2}}.$$

Consider the initial plane wavefront W_0 on the refracted ray direction of the front lens surface, the First Fundamental Form coefficients and Second Fundamental Form coefficients are:

$$g_{11}^{W_0} = g_{12}^{W_0} = 1, \quad g_{22}^{W_0} = 0,$$

$$L_{11}^{W_0} = L_{12}^{W_0} = L_{22}^{W_0} = 0.$$

Then by theorem 4.4.3 and 4.4.4, the coefficients of the first refracted wavefront are

$$g_{ij}^{W_r} = g_{ij}^f - \mu_1 \phi_i \phi_j, \quad (5.4)$$

$$L_{ij}^{W_r} = \gamma^f L_{ij}^f, \quad (5.5)$$

where

$$\phi_1 = -\langle (R^f)_{x_1}, Q^f \rangle / \mu_2, \quad \phi_2 = -\langle (R^f)_{x_2}, Q^f \rangle / \mu_2.$$

and

$$\gamma^f = -1/\mu_1 \cos \beta_r + \cos \beta_i.$$

By theorem 4.4.5, for the curvature of the first refracted wavefront, we have

$$\begin{aligned}
H^{W_0} &= 0, \\
K^{W_0} &= 0, \\
\Delta_{W_0} &= 1, \\
\Delta^f &= 1 + f_{x_1}(x_1', x_2')^2 + f_{x_2}(x_1', x_2')^2, \\
\Theta_{W_0} &= 1 - (2 - 2\mu^2)(\phi_{x_1}^2 + \phi_{x_2}^2), \\
\Theta^f &= 1 + f_{x_1}(x_1', x_2')^2 + f_{x_2}(x_1', x_2')^2 - \mu^2(\phi_{x_1}^2 g_{11}^f + \phi_{x_2}^2 g_{22}^f + 2\phi_{x_1}\phi_{x_2}g_{12}^f), \\
\Pi_W &= 0, \\
\Pi^f &= \mu^2 \frac{\phi_{x_1}^2 f_{x_1 x_1}(x_1', x_2') + \phi_{x_2}^2 f_{x_2 x_2}(x_1', x_2') + 2\phi_{x_1}\phi_{x_2}f_{x_1 x_2}(x_1', x_2')}{\sqrt{1 + f_{x_1}(x_1, x_2)^2 + f_{x_2}(x_1, x_2)^2}}, \\
\sigma &= 0.
\end{aligned}$$

then,

$$H^{W_r} = \gamma H^R \frac{\Delta^f}{\Theta^f} - \frac{1}{2} \frac{\Pi^f}{\Theta^f}, \quad (5.6)$$

$$K^{W_r} = \gamma^2 K^R \frac{\Delta^f}{\Theta^f}. \quad (5.7)$$

By theorem 4.3.3, we can get a simple representation of the curvatures of the propagated wavefront before the back lens surface,

$$H^{W_p} = \frac{H^{W_r} - dK^{W_r}}{1 - 2dH^{W_r} + d^2K^{W_r}}, \quad (5.8)$$

$$K^{W_p} = \frac{K^{W_r}}{1 - 2dH^{W_r} + d^2K^{W_r}}. \quad (5.9)$$

To calculate the curvature of the refracted wavefront W_{rr} after the back lens surface, we need the inverse matrix of the First Fundamental Form coefficients

$$\begin{aligned}
g^{11} &= \frac{g_{22}^{W_r}}{g_{11}^{W_r} g_{22}^{W_r} - (g_{12}^{W_r})^2}, \\
g^{12} &= -\frac{g_{12}^{W_r}}{g_{11}^{W_r} g_{22}^{W_r} - (g_{12}^{W_r})^2}, \\
g^{22} &= \frac{g_{11}^{W_r}}{g_{11}^{W_r} g_{22}^{W_r} - (g_{12}^{W_r})^2}.
\end{aligned}$$

and the derivative of the propagated wavefront under the coordinate system (x_1', x_2') ,

$$(W_p)_{x_i'} = (W_r)_{x_j'} - d[g^{kj} L_{ji}^{W_r} (W_r)_{x_k'}]. \quad (5.10)$$

The First Fundamental Form coefficients and the Second Fundamental Form coefficients of the propagated wavefront W_p are

$$g_{ij}^{W_p} = C_{ik} C_{jl} g_{kl}^{W_r}, \quad (5.11)$$

and

$$L_{ij}^{W_p} = L_{ij}^{W_r} - dg_{kl}^{W_r} L_{ik}^{W_r} L_{jl}^{W_r}, \quad (5.12)$$

where

$$C_{ij} = e_j^i - dg^{i1} L_{1i}^{W_r} - dg^{i2} L_{2i}^{W_r},$$

Now consider the refraction of the wavefront on the back lens surface. The derivatives of the propagated wavefront under the coordinate system (x_1, x_2) are

$$(W_p)_{x_i} = (R^b)_{x_i} - \eta_{x_i} Q^f,$$

where $\eta_{x_i} = -\langle (R^b)_{x_i}, Q^b \rangle / \mu_1$.

With the derivatives of the propagated wavefront W_p under the coordinate system (x_1, x_2) by equation (5.10), the coordinate change matrix from (x_1', x_2') to (x_1, x_2) is

$$\frac{\partial x_i'}{\partial x_j} = \hat{g}^{im} b_{mj}, \quad (5.13)$$

where

$$b_{ij} = \langle (W_p)_{x_i}, (W_p)_{x_j'} \rangle,$$

and $\{\hat{g}^{ij}\}_{i,j=1,2}$ is the inverse matrix of the First Fundamental Form coefficient matrix g^{W_p}

$$\begin{aligned}\hat{g}^{11} &= \frac{g_{22}^{W_p}}{g_{11}^{W_p} g_{22}^{W_p} - (g_{12}^{W_p})^2}, \\ \hat{g}^{12} &= -\frac{g_{12}^{W_p}}{g_{11}^{W_p} g_{22}^{W_p} - (g_{12}^{W_p})^2}, \\ \hat{g}^{22} &= \frac{g_{11}^{W_p}}{g_{11}^{W_p} g_{22}^{W_p} - (g_{12}^{W_p})^2}.\end{aligned}$$

The Second Fundamental Form coefficients of the propagated wavefront W_p under the coordinate system (x_1, x_2) is

$$\bar{L}_{ij}^{W_p} = L_{mn}^{W_p} \frac{\partial x_m'}{\partial x_i} \frac{\partial x_n'}{\partial x_j}, \quad (5.14)$$

Then by theorem 4.4.3 and 4.4.4, the coefficients of the second refracted wavefront W_{rr} are

$$g_{ij}^{W_{rr}} = g_{ij}^b - \mu_1^2 \eta_i \eta_j, \quad (5.15)$$

$$L_{ij}^{W_{rr}} = \mu_1 \bar{L}_{ij}^{W_p} + \left(-\frac{\gamma^b}{\mu_1}\right) L_{ij}^b. \quad (5.16)$$

Therefore, the mean curvature and the Gaussian curvature of the second refracted wavefront W_{rr} are

$$H^{W_{rr}} = \frac{1}{2} \frac{L_{11}^{W_{rr}} g_{22}^{W_{rr}} + L_{22}^{W_{rr}} g_{11}^{W_{rr}} - 2L_{12}^{W_{rr}} g_{12}^{W_{rr}}}{g_{11}^{W_{rr}} g_{22}^{W_{rr}} - (g_{12}^{W_{rr}})^2}, \quad (5.17)$$

$$K^{W_{rr}} = \frac{L_{11}^{W_{rr}} L_{22}^{W_{rr}} - (L_{12}^{W_{rr}})^2}{g_{11}^{W_{rr}} g_{22}^{W_{rr}} - (g_{12}^{W_{rr}})^2}. \quad (5.18)$$

5.2 Front surface design

While we can manipulate both the front lens surface R^f and the back lens surface R^b to reach the minimum functional value, in practical design, we can either fix the back lens surface R^b as a sphere or a toroidal surface to have a front surface

design, or we can make a back lens surface design by fixing the front lens surface R^f .

If we fix the back lens surface as a sphere or a toroid, the design objective functional is

$$\mathcal{J}(R^f) = \int_{\Omega} \beta(H^{W_{rr}}(\theta) - P(\theta))^2 d\theta + \alpha \int_{\Omega} (A^{W_{rr}})^2(\theta) d\theta,$$

where $H^{W_{rr}}$ and $K^{W_{rr}}$ are calculated through formulas (5.4), (5.5), (5.11), (5.12), (5.13), (5.14), (5.15), (5.16), (5.17) and (5.18).

5.2.1 Gradient of the design objective functional

To calculate the gradient of the design objective functional with respect to the front lens surface, We add a perturbation δf on the front lens surface f , By equation 5.3,

$$b(x_1, x_2) + (d + \delta d)Q_z^b = (f + \delta f)(x_1 + (d + \delta d)Q_{x_1}^b, x_2 + (d + \delta d)Q_{x_2}^b),$$

Taylor expansion gives

$$\begin{aligned} b(x_1, x_2) + dQ_z^b + \delta dQ_z^b &= f(x_1 + dQ_{x_1}^b, x_2 + dQ_{x_2}^b) \\ &+ f_{x_1}(x_1 + dQ_{x_1}^b, x_2 + dQ_{x_2}^b)\delta dQ_{x_1}^b \\ &+ f_{x_2}(x_1 + dQ_{x_1}^b, x_2 + dQ_{x_2}^b)\delta dQ_{x_2}^b \\ &+ \delta f(x_1 + dQ_{x_1}^b, x_2 + dQ_{x_2}^b), \end{aligned}$$

therefore,

$$\delta d = \frac{\delta f}{Q_z^b - f_{x_1}(x_1', x_2')Q_{x_1}^b + f_{x_2}(x_1', x_2')Q_{x_2}^b}, \quad (5.19)$$

then,

$$\begin{aligned} \delta x_1' &= \delta dQ_{x_1}^b, \\ \delta x_2' &= \delta dQ_{x_2}^b. \end{aligned}$$

Similarly, we can have the perturbation of the coefficients of the front lens surface with respect to δf ,

$$\delta g_{ij}^f = 2f_{x_i}(x_1', x_2')f_{x_ix_m}(x_1', x_2')\delta x_m' + f_{x_j}(x_1', x_2')f_{x_jx_m}(x_1', x_2')\delta x_m',$$

and

$$\begin{aligned} \delta L_{ij}^f &= \frac{\delta f_{x_ix_j}(x_1', x_2') + f_{x_ix_jx_m}(x_1', x_2')\delta x_m'}{\sqrt{1 + f_{x_1}^2(x_1', x_2') + f_{x_2}^2(x_1', x_2')}} \\ &+ \frac{f_{x_ix_j}(x_1', x_2')}{(1 + f_{x_1}^2(x_1', x_2') + f_{x_2}^2(x_1', x_2'))^{3/2}} (2f_{x_m}(x_1', x_2')\delta f_{x_m}(x_1', x_2') \\ &+ f_{x_mx_n}(x_1', x_2')\delta x_n'). \end{aligned}$$

Then the perturbation of mean curvature and Gaussian curvature of the front lens surface are

$$\begin{aligned} \delta H^f &= \frac{1}{2} \frac{\delta g_{11}^f L_{22}^f + g_{11}^f \delta L_{22}^f + \delta g_{22}^f L_{11}^f + g_{22}^f \delta L_{11}^f - 2\delta g_{12}^f L_{12}^f - 2g_{12}^f \delta L_{12}^f}{g_{11}^f g_{22}^f - (g_{12}^f)^2} \\ &+ \frac{g_{11}^f L_{22}^f + g_{22}^f L_{11}^f - 2g_{12}^f L_{12}^f}{(g_{11}^f g_{22}^f - (g_{12}^f)^2)^2} (\delta g_{11}^f g_{22}^f + g_{11}^f \delta g_{22}^f - (2g_{12}^f \delta g_{12}^f)), \quad (5.20) \end{aligned}$$

$$\begin{aligned} \delta K^f &= \frac{\delta L_{11}^f L_{22}^f + L_{11}^f \delta L_{22}^f - 2L_{12}^f \delta L_{12}^f}{g_{11}^f g_{22}^f - (g_{12}^f)^2} \\ &+ \frac{L_{11}^f L_{22}^f - (L_{12}^f)^2}{(g_{11}^f g_{22}^f - (g_{12}^f)^2)^2} (\delta g_{11}^f g_{22}^f + g_{11}^f \delta g_{22}^f - 2g_{12}^f \delta g_{12}^f) \quad (5.21) \end{aligned}$$

According to equations (5.4) and (5.5), the perturbation of the First Fundamental Form coefficients and the Second Fundamental Form coefficients of the first refracted wavefront W_r are

$$\begin{aligned} \delta g_{ij}^{W_r} &= \delta g_{ij}^f - \mu_1^2 (\phi_i \delta \phi_j + \phi_j \delta \phi_i) \\ \delta L_{ij}^{W_r} &= \delta \gamma^f L_{ij}^f + \gamma^f \delta L_{ij}^f, \end{aligned}$$

where

$$\begin{aligned}
\delta\phi_i &= -\langle\delta(R^f)_i Q^f\rangle/\mu_2, \\
\delta\gamma^f &= -1/\mu_1\delta\cos\beta_r - \delta\cos\beta_i, \\
\delta\cos\beta_i &= \langle Q^f, \delta N^f\rangle, \\
\delta\cos\beta_r &= \frac{1}{2}\sqrt{1-\mu_1(1-\langle Q^f, N^f\rangle)}(1-\mu_1(1-2\langle Q^f, N^f\rangle\langle Q^f, \delta N^f\rangle)).
\end{aligned}$$

Therefore, the perturbation of mean curvature and Gaussian curvature of the first refracted wavefront are

$$\begin{aligned}
\delta H^{Wr} &= \delta\gamma^f H^f \frac{\Delta^f}{\Theta^f} + \gamma^f \delta H^f \frac{\Delta^f}{\Theta^f} + \gamma H^R \frac{\delta\Delta^f \Theta^f + \Delta^f \delta\Theta^f}{(\Theta^f)^2} \\
&\quad - \frac{1}{2} \frac{\delta\Pi^f \Theta^f - \Pi^f \delta\Theta^f}{(\Theta^f)^2}, \tag{5.22}
\end{aligned}$$

$$\delta K^{Wr} = 2\gamma^f \delta\gamma^f K^f \frac{\Delta^f}{\Theta^f} + \gamma^2 \delta K^f \frac{\Delta^f}{\Theta^f} + \gamma^2 K^f \frac{\delta\Delta^f \Theta^f - \Delta^f \delta\Theta^f}{(\Theta^f)^2}. \tag{5.23}$$

where

$$\begin{aligned}
\delta\Delta^f &= 2f_{x_1}(x_1', x_2')(f_{x_1 x_1}(x_1', x_2')\delta x_1' + f_{x_1 x_2}(x_1', x_2')\delta x_2' + \delta f_{x_1}(x_1', x_2')) \\
&\quad + 2f_{x_2}(x_1', x_2')(f_{x_1 x_1}(x_1', x_2')\delta x_1' + f_{x_1 x_2}(x_1', x_2')\delta x_2' + \delta f_{x_2}(x_1', x_2')), \\
\delta\Theta^f &= \delta\Delta^f - \mu^2(2\phi_{x_1}\delta\phi_{x_1}g_{11}^f + \phi_{x_1}^2\delta g_{11}^f + 2\phi_{x_2}\delta\phi_{x_2}g_{22}^f + \phi_{x_2}^2\delta g_{22}^f + 2\delta\phi_{x_1}\phi_{x_2}g_{12}^f \\
&\quad + 2\phi_{x_1}\delta\phi_{x_2}g_{12}^f + 2\phi_{x_1}\phi_{x_2}\delta g_{12}^f), \\
\delta\Pi^f &= \mu_1^2 \left(\frac{\delta\phi_i\phi_j f_{x_i x_j}(x_1', x_2') + \phi_i\delta\phi_j f_{x_i x_j}(x_1', x_2')}{\sqrt{1+f_{x_1}(x_1, x_2)^2+f_{x_2}(x_1, x_2)^2}} \right. \\
&\quad + \frac{\phi_i\phi_j(\delta f_{x_i x_j}(x_1', x_2') + f_{x_i x_j x_m}(x_1', x_2')\delta x'_m)}{\sqrt{1+f_{x_1}(x_1, x_2)^2+f_{x_2}(x_1, x_2)^2}} \\
&\quad \left. - \frac{1}{2} \frac{\phi_i\phi_j f_{x_i x_j}(x_1', x_2')}{(1+f_{x_1}(x_1, x_2)^2+f_{x_2}(x_1, x_2)^2)^{\frac{3}{2}}} \delta\Delta^f \right).
\end{aligned}$$

By (5.11),

$$\delta g_{ij}^{Wp} = C_{jl}g_{kl}^{Wr}\delta C_{ik} + C_{ik}g_{kl}^{Wr}\delta C_{jl} + C_{ik}C_{jl}\delta g_{kl}^{Wr}, \quad ,$$

where

$$\begin{aligned}\delta C_{ij} &= -dg^{i1}\delta L_{1i}^{W_r} - dL_{1i}^{W_r}\delta g^{i1} - g^{i1}L_{1i}^{W_r}\delta d \\ &\quad - dg^{i2}\delta L_{2i}^{W_r} - dL_{2i}^{W_r}\delta g^{i2} - g^{i2}L_{2i}^{W_r}\delta d,\end{aligned}$$

and by (5.12),

$$\begin{aligned}\delta L_{ij}^{W_p} &= \delta L_{ij}^{W_r} - dg_{kl}^{W_r}L_{ik}^{W_r}\delta L_{jl}^{W_r} - dg_{kl}^{W_r}L_{jl}^{W_r}\delta L_{ik}^{W_r} \\ &\quad - dL_{ik}^{W_r}L_{jl}^{W_r}\delta g_{kl}^{W_r} - g_{kl}^{W_r}L_{ik}^{W_r}L_{jl}^{W_r}\delta d,\end{aligned}$$

By (5.14),

$$\delta \bar{L}_{ij}^{W_p} = \frac{\partial x_m'}{\partial x_i} \frac{\partial x_n'}{\partial x_j} \delta L_{mn}^{W_p} + L_{mn}^{W_p} \frac{\partial x_m'}{\partial x_i} \delta \frac{\partial x_n'}{\partial x_j} + L_{mn}^{W_p} \frac{\partial x_n'}{\partial x_j} \delta \frac{\partial x_m'}{\partial x_i}.$$

The perturbation on the front lens should not be related to the back lens surface, so the perturbation of the First Fundamental Form coefficients and the Second Fundamental Form coefficients of the back lens surface should all be 0, which are

$$\begin{aligned}\delta g_{ij}^b &= 0, \\ \delta L_{ij}^b &= 0.\end{aligned}$$

Then by (5.15), the perturbation of the First Fundamental Form coefficients are

$$\delta g_{ij}^{W_{rr}} = 0.$$

and by the Second Fundamental Form coefficients are

$$\delta L_{ij}^{W_{rr}} = \mu_1 \delta \bar{L}_{ij}^{W_p}.$$

Then the perturbation of the mean Curvature and the Gaussian Curvature of the second refracted wavefront W_{rr} are

$$\begin{aligned}\delta H^{W_{rr}} &= \frac{1}{2} \frac{g_{22}^{W_{rr}} \delta L_{11}^{W_{rr}} + g_{11}^{W_{rr}} \delta L_{22}^{W_{rr}} - 2g_{12}^{W_{rr}} \delta L_{12}^{W_{rr}}}{g_{11}^{W_{rr}} g_{22}^{W_{rr}} - (g_{12}^{W_{rr}})^2}, \\ \delta K^{W_{rr}} &= \frac{L_{22}^{W_{rr}} \delta L_{11}^{W_{rr}} + L_{11}^{W_{rr}} \delta L_{22}^{W_{rr}} - 2L_{12}^{W_{rr}} \delta L_{12}^{W_{rr}}}{g_{11}^{W_{rr}} g_{22}^{W_{rr}} - (g_{12}^{W_{rr}})^2}.\end{aligned}$$

5.2.2 Optimization on the wavefront before the back lens surface

From subsection 5.1, we may use formulas (5.6), (5.7), (5.8) and (5.9) to calculate the mean curvature and the Gaussian curvature of the wavefront before the back lens surface. We can see that the representation is much simpler than the wavefront after the back lens surface, which is using (5.4), (5.5), (5.11), (5.12), (5.13), (5.14), (5.15), (5.16), (5.17) and (5.18).

If we change the design objective functional to be optimizing on the wavefront before the back lens surface,

$$\mathcal{J}(R^f) = \int_{\Omega} \beta(H^{W_p}(\theta) - \hat{P}(\theta))^2 d\theta + \int_{\Omega} \alpha(A^{W_p})^2(\theta) d\theta,$$

where \hat{P} is the approximated power before the back lens surface. We need to find the approximated power before the back lens surface based on the desired power of the wavefront after the back lens surface. The front lens surface obtained by optimizing on this approximated power should be efficient.

Assume the desired power of the lens is P , we want to find the desired power of the wavefront before the back lens surface \hat{P} . Since the perfect shape of the refracted wavefront should be, at each point (x_1, x_2) , a sphere with power $P(x_1, x_2)$ and the astigmatism 0. Assume the principle curvature of the desired wavefront at point (x_1, x_2) is κ_1 and κ_2 , then

$$\begin{aligned} \frac{\kappa_1 + \kappa_2}{2} &= P(x_1, x_2), \\ \kappa_1 - \kappa_2 &= 0, \end{aligned}$$

which means $\kappa_1 = \kappa_2 = P(x_1, x_2)$.

Since the principal curvature κ_1 and κ_2 are the roots of equations

$$\det \left(\begin{bmatrix} L_{11}^{W_{rr}} & L_{12}^{W_{rr}} \\ L_{21}^{W_{rr}} & L_{22}^{W_{rr}} \end{bmatrix} - \kappa \begin{bmatrix} g_{11}^{W_{rr}} & g_{12}^{W_{rr}} \\ g_{21}^{W_{rr}} & g_{22}^{W_{rr}} \end{bmatrix} \right) = 0,$$

κ_1 and κ_2 satisfies

$$\begin{aligned} (L_{11}^{W_{rr}} L^{W_{rr}} - (L_{12}^{W_{rr}})^2) - \kappa_1 (L_{11}^{W_{rr}} g_{22}^{W_{rr}} + L_{11}^{W_{rr}} g_{22}^{W_{rr}} - 2L_{12}^{W_{rr}} g_{12}^{W_{rr}}) \\ + \kappa_1^2 (g_{11}^{W_{rr}} g_{22}^{W_{rr}} - (L_{12}^{W_{rr}})^2) = 0 \\ (L_{11}^{W_{rr}} L^{W_{rr}} - (L_{12}^{W_{rr}})^2) - \kappa_2 (L_{11}^{W_{rr}} g_{22}^{W_{rr}} + L_{11}^{W_{rr}} g_{22}^{W_{rr}} - 2L_{12}^{W_{rr}} g_{12}^{W_{rr}}) \\ + \kappa_2^2 (g_{11}^{W_{rr}} g_{22}^{W_{rr}} - (L_{12}^{W_{rr}})^2) = 0 \end{aligned}$$

Since at every point the wavefront is a spherical shape, we can make the assumption that the coordinate system (X_1, X_2) is the principal coordinate system. Under the principal coordinate system (X_1, X_2) , $L_{12}^{W_{rr}}$ and $g_{12}^{W_{rr}}$ are equal to 0. Then we have

$$L_{11}^{W_{rr}} = \kappa_1 g_{11}^{W_{rr}}, \quad L_{22}^{W_{rr}} = \kappa_2 g_{22}^{W_{rr}}.$$

By theorem 4.4.4, we have the desired Second Fundamental Form of the wavefront before the back lens surface R^b under coordinate system x_1, y

$$L_{ij}^{W_p} = \mu_1 L_{ij}^{W_{rr}} + \gamma_1 L_{ij}^b.$$

By theorem 4.4.3, the desired First Fundamental Form of the wavefront before the back lens surface R^b under coordinate system x_1, y is

$$g_{ij}^{W_p} = g_{ij}^b - \phi_i \phi_j.$$

Therefore the desired power of the wavefront before the back lens surface can be calculated by the formula

$$\hat{P} = \frac{L_{11}^{W_p} g_{22}^{W_p} + L_{22}^{W_p} g_{11}^{W_p} - 2L_{12}^{W_p} g_{12}^{W_p}}{g_{11}^{W_p} g_{22}^{W_p} - (g_{11}^{W_p})^2},$$

then by (5.11) and (5.12),

$$\hat{P} = \frac{(\mu_1 \kappa_1 g_{11}^{W_{rr}} + \gamma_1 L_{ij}^b) g_{22}^{W_p} + L_{22}^{W_p} g_{11}^{W_p} - 2L_{12}^{W_p} g_{12}^{W_p}}{g_{11}^{W_p} g_{22}^{W_p} - (g_{11}^{W_p})^2} \quad (5.24)$$

The representation of the gradient of the mean curvature and the Gaussian curvature of W_p are much simpler than the mean curvature and Gaussian curvature of W_{rr} ,

$$\begin{aligned}\delta H^{W_p} &= \frac{[\delta H^{W_r} - (K^{W_r} \delta d + d \delta K^{W_r})](1 - 2dH^{W_r} + d^2 K^{W_r})}{(1 - 2dH^{W_r} + d^2 K^{W_r})^2} \\ &\quad - \frac{(H^{W_r} - dK^{W_r})[-2(d\delta H^{W_r} + H^{W_r} \delta d) + (2dK^{W_r} \delta d + d^2 \delta K^{W_r})]}{(1 - 2dH^{W_r} + d^2 K^{W_r})^2}, \\ \delta K^{W_p} &= \frac{\delta K^{W_r}(1 - 2dH^{W_r} + d^2 K^{W_r})}{(1 - 2dH^{W_r} + d^2 K^{W_r})^2} \\ &\quad - \frac{K^{W_r}[-2(d\delta H^{W_r} + H^{W_r} \delta d) + (2dK^{W_r} \delta d + d^2 \delta K^{W_r})]}{(1 - 2dH^{W_r} + d^2 K^{W_r})^2},\end{aligned}$$

where δH^{W_r} , δK^{W_r} and δd can be calculated by (5.22), (5.23), and (5.19).

5.3 Back surface design

As we fix the front lens surface as a sphere or a toroid, the design objective functional is

$$\mathcal{J}(R^b) = \int_{\Omega} \beta(H^{W_{rr}}(\theta) - P(\theta))^2 d\theta + \int_{\Omega} \alpha(A^{W_{rr}})^2(\theta) d\theta.$$

Consider a fixed gaze direction $\theta = (\varphi, \psi)$, a different back lens surface may give a different intersection point, hence it should be reasonable to use (φ, ψ) as the coordinate rather than (x, y) coordinates of the intersect point. If we assume the distance from the eye center O to the back lens surface R^b is $\rho(\varphi, \psi)$, then

$$R^b = (\rho(\varphi, \psi) \cos \varphi \cos \psi, \rho(\varphi, \psi) \cos \varphi \sin \psi, \rho(\varphi, \psi) \sin \varphi).$$

Then the mean curvature and Gaussian curvature of the second refracted wavefront W_{rr} can be calculated by (5.4), (5.5), (5.11), (5.12), (5.15) and (5.16).

5.4 Use of travel distance to represent lens surface

In previous sections, we show a way to calculate the curvature with direct surface representation, this evaluation requires to compute a line-surface intersection for every gaze direction, which has to be done numerically and time-consuming. While in [11], J. Loost. shows a representation of the surface by the travel distance. In this way, we can calculate the intersection directly.

Consider the sample point $(x_1, x_2, b(x_1, x_2))$ on the back lens surface, we can parameterize the front surface R^f by the travel distance $\phi(x_1, x_2)$ from the back lens surface R^b and the front lens surface R^f under coordinate system (X_1, X_2) of R^b ,

$$R^f(x_1, x_2) = R^b(x_1, x_2) + \phi(x_1, x_2)Q^f(x_1, x_2).$$

5.5 Lens surface representation by splines

5.5.1 An abstracted linear interpolation scheme

Most of the linear approximation schemes (such as interpolation, least squares approximation) fit into the following abstract framework, for a given function g , we attempt to construct an approximation $\mathbb{P}g = \sum_{j=1}^n \alpha_j f_j$ with $\{f_j\}_{j=1}^n$ certain fixed functions, on the domain Ω on which g is defined. We construct the specific approximation by interpolation with respect to linear functionals $\{\lambda_i\}_{i=1}^n$, i.e. by the requirement that

$$\lambda_i(\mathbb{P}g) = \lambda_i(g), \quad i = 1, \dots, n,$$

for certain fixed linear interpolation conditions or linear functional $\lambda_1, \dots, \lambda_n$.

We might choose

$$\lambda_i(g) = g(t_i), \quad i = 1, \dots, n$$

for some $t_1 < \dots < t_n$, this is ordinary interpolation, which is used in our numerical scheme.

Definition 5.5.1. *Assume the linear span*

$$\mathbb{F} = \text{span} (f_j)_{j=1}^n := \left\{ \sum_{j=1}^n \alpha_j f_j : \alpha \in \mathbb{R}^n \right\}$$

of the f_j 's and linear span

$$\Lambda = \text{span} (\lambda_i)_{i=1}^n := \left\{ \sum_{i=1}^n \alpha_i \lambda_i : \alpha \in \mathbb{R}^n \right\},$$

We can define linear interpolation problem (LIP) as follows: Find an $f \in \mathbb{F}$ for a given g so that

$$\lambda(f) = \lambda(g), \quad \text{for all } \lambda \in \Lambda.$$

We say that the LIP given by \mathbb{F} and Λ is correct if it has exactly one solution for every $g \in \mathbb{U}$. Here, \mathbb{U} is some linear space of functions, all defined on the same domain of g , and $\mathbb{U} \supseteq \mathbb{F}$. In our numerical scheme, $\mathbb{C}^{k \times k}(X \times Y)$ is a space of all k -th continuous functions on $X = [a, b] \times [c, d]$.

Definition 5.5.2. [5] *Let \mathbb{U} be a linear space of functions, all defined on some set X into \mathbb{R} , and \mathbb{V} be, similarly, a linear space of functions defined on some set Y into \mathbb{R} . For each $u \in \mathbb{U}$ and $v \in \mathbb{V}$, the rule*

$$w(x, y) := u(x)v(y) \quad \text{all } (x, y) \in X \times Y,$$

defines a function on $X \times Y$ called the tensor product of u with v and denoted by

$$u \otimes v.$$

The set of all finite linear combination of functions on $X \times Y$ of the form $u \otimes v$ for some $u \in \mathbb{U}$ and some $v \in \mathbb{V}$ is called the tensor product of \mathbb{U} with \mathbb{V} and is denoted by $\mathbb{U} \otimes \mathbb{V}$. Thus,

$$\mathbb{U} \otimes \mathbb{V} := \left\{ \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} (u_i \otimes v_j) : \alpha_{ij} \in \mathbb{R}, u_i \in \mathbb{U}, v_j \in \mathbb{V}, i = 1, \dots, n \right\},$$

and $\mathbb{U} \otimes \mathbb{V}$ is a linear space of functions on $X \times Y$.

The basic facts of the tensor product of two univariate interpolation schemes are contained in the following theorem.

Theorem 5.5.3. [5] *Suppose that the Gramian matrix $A := (\lambda_i f_j)_{i=1, \dots, m, j=1, \dots, m}$ for the sequence f_1, \dots, f_m in \mathbb{U} and the sequence $\lambda_1, \dots, \lambda_m$ of linear functionals on \mathbb{U} is invertible, so that the LIP given by*

$$F := \text{span} (f_j)_1^m \quad \text{and} \quad \Lambda := (\lambda_i)_1^m$$

is correct.

Similarly, assume that $B := (\mu_i g_j)_{i=1, \dots, n, j=1, \dots, n}$ is invertible, with g_1, \dots, g_n in \mathbb{V} and the sequence μ_1, \dots, μ_n of linear functionals on \mathbb{V} , and set

$$G := \text{span} (g_i)_1^n \quad \text{and} \quad M := (\mu_i)_1^n.$$

Finally, assume that $(v_{ij})_{n \times n}$ is a matrix of linear functionals on some linear space W containing $U \otimes V$ so that

$$v_{ij}(u \otimes v) = (\lambda_i u)(\mu_j v) \quad \text{for all } i, j \text{ and all } (u, v) \in U \times V.$$

Then

(i) $(f_i \otimes g_j)$ is a basis for $F \otimes G$, hence

$$\dim F \otimes G = (\dim F)(\dim G) = mn;$$

(ii) The LIP on W given by $F \otimes G$ and $\text{span} (\epsilon_{ij})_{n \times n}$ is correct.

(iii) For given $w \in W$, the interpolant Rw can be computed as

$$Rw = \sum_{i,j} \gamma_{ij} f_i \otimes g_j \tag{5.25}$$

with

$$\gamma := \gamma_w := A^{-1} L_w (B^T)^{-1} \tag{5.26}$$

and

$$L_w(i, j) := \epsilon_{ij} w, \quad \text{all } i, j. \tag{5.27}$$

5.5.2 Tensor-product B-spline function

Tensor-product B-Spline function is employed to represent the lens surface. Although applicability of tensor product method is limited, but when their use can be justified, then these method should be used since they are extremely efficient compared to other surface approximation techniques.

First, let's introduce the univariate B-spline function.

Definition 5.5.4. [5] Given m real value t_i called knots, with

$$t_0 \leq t_1 \leq \dots \leq t_m.$$

A B-spline of degree n is a parametric curve

$$S : [t_0, t_m] \mapsto \mathbb{R}$$

composed of a linear combination of basis B-spline function B_i^n of degree n ,

$$S(t) = \sum_{i=1}^{m-n-2} a_i B_i^n(t), \quad t \in [t_n, t_{n-m-1}]$$

The $m-n-1$ basis B-splines of degree n can be defined, for $n = 0, 1, \dots, m-2$, using the Cox-de Boor recursion formula. ([5])

$$B_j^0 = \begin{cases} 1, & \text{if } t_j \leq t \leq t_{j+1} \\ 0, & \text{otherwise} \end{cases} \quad j = 1, \dots, m-2$$

and

$$B_j^n(t) = \frac{t - t_j}{t_{j+n} - t_j} B_j^{n-1}(t) + \frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}} B_{j+1}^{n-1}(t), \quad j = 0, \dots, m-n-2$$

In particular, suppose $t = \{t_i\}_{i=1}^m$ are the uniform knots, the non-recursive definition of the $m-n-1$ basis B-splines for a given value n is

$$B_j^n(t) = \frac{n+1}{n} \sum_{i=0}^{n+1} \omega_{i,n}(t - t_j - t_i)_+^n,$$

with

$$\omega_{i,n} = \prod_{j=0, j \neq i}^{n+1} \frac{1}{t_j - t_i},$$

where

$$(t - t_i)_+^n = \begin{cases} (t - t_i - t_j)^n, & \text{if } t \geq t_i + t_j \\ 0, & \text{otherwise} \end{cases}$$

We may denote the linear space of B-splines of order n with knot sequence t as

$$\mathbb{B}_{n,t} = \text{span} (B_j^n)_1^n.$$

Without loss of generality, suppose Ω is the unit square domain

$$\Omega = [0, 1] \otimes [0, 1].$$

The uniform partition is the partition of Ω by the following knot lines

$$mx - i = 0, \quad ny - j = 0,$$

where m and n are the numbers of the rectangles in the x and y direction respectively and $i = 1 \dots m, j = 1 \dots n$.

Now we consider the following situation in Theorem 5.5.3 of tensor product B-spline functions.

Suppose we have

$$F = \mathbb{B}_{h,s}, \text{ with } s = (s_i)_1^{m+h}, \text{ and } \lambda_i = \sigma_i, i = 1, \dots, m$$

with $\sigma_1 < \dots < \sigma_m$.

This problem is correct if and only if $s_i < \sigma_i < s_{i+h}$ for all i based on this following theorem

Theorem 5.5.5. [4] (Schoenberg-Whitney Theorem) *The matrix $(B_j(\tau_i))_{i=1, \dots, n, j=1, \dots, n}$ of linear system*

$$\sum_{j=1}^n \alpha_j B_j(\tau_i) = g(\tau_i), \quad i = 1, \dots, n.$$

is invertible if and only if

$$B_i(\tau_i) \neq 0, i = 1, \dots, n,$$

i.e. if and only if $t_i < \tau_i < t_{i+k}$, all i .

Also, we take

$$X = [s_1, s_{m+h}], \text{ and } \mathbb{U} = C(X),$$

the linear space of continuous functions on the interval X .

Similarly, we take

$$G = \mathbb{B}_{k,t}, \text{ with } t = (t_j)_1^{n+k}, \text{ and } \mu_j = [\tau_j], j = 1, \dots, n$$

with τ strictly increasing and $t_j < \tau_j < t_{j+k}$ for all j .

Also,

$$Y = [t_1, t_{n+k}], \text{ and } \mathbb{V} = C(Y).$$

Then, $\mathbb{U} \otimes \mathbb{V}$ is contained in $W := C(X \times Y)$, the linear space of continuous functions on the rectangle $X \times Y$, and $\lambda_i \otimes \mu_j$ coincides (on $\mathbb{U} \otimes \mathbb{V}$) with the linear functional

$$v_{ij} : w \longmapsto w(\sigma_i, \tau_j)$$

at point $(\sigma_i, \tau_j) \in X \times Y$.

Theorem 5.5.3 therefore gives us that, for a given $w \in C(X \times Y)$, there exists exactly one spline function $Rw \in F \otimes G$ which agrees with w at points (σ_i, τ_j) , $i = 1, \dots, m$, $j = 1, \dots, n$ of the given rectangular mesh. Further, this interpolant can be written in form

$$Rw = \sum_{i,j} \gamma(i,j) B_i^m \otimes B_j^n,$$

with

$$\gamma = (B_j^m(\sigma_i))^{-1} (w(\sigma_i, \tau_j)) (B_i^n(\tau_j))^{-1}.$$

Chapter 6

Numerical optimization for lens design

Let the front and back surfaces represented as tensor-product B-spline functions

$$f(x_1, x_2) = \sum_{i=1}^n \sum_{j=1}^n f_{ij} B_i^n(x_1) \otimes B_j^n(x_2), \quad b(x_1, x_2) = \sum_{i=1}^n \sum_{j=1}^n b_{ij} B_i^n(x_1) \otimes B_j^n(x_2),$$

and

$$s^b = \{b_{ij}\}_{i,j=1,\dots,n}, \quad s^f = \{f_{ij}\}_{i,j=1,\dots,n},$$

given that the back lens surface b and the front lens surface f should be not intersected, we form the optimization problem of obtaining the minimum value of the sum of the desired design functional at the samples points $\{(x_1, x_2)_l\}_{l=1,\dots,m}$ as following

$$\min_{\{s^b, s^f\}} J = \min_{\{s^b, s^f\}} \sum_{l=1}^m \beta_l (H_l^{W_{rr}}(s^b, s^f) - \hat{P}_l)^2 + \alpha_l (A_l^{W_{rr}}(s^b, s^f))^2, \quad (6.1a)$$

$$s^b \leq s^f \quad (6.1b)$$

where \hat{P}_l is the desired power at point $(x_1, x_2)_l$. $H_l^{W_{rr}}$ is the power of the lens at point $(x_1, x_2)_l$. $A_l^{W_{rr}}$ is the astigmatism of the lens at point $(x_1, x_2)_l$. α_l, β_l are the weights of the lens design at point $(x_1, x_2)_l$.

In a front lens surface design, we usually fix the back lens surface as a spherical surface or a toric surface, then want to find $(s^f)^* = \{f_{ij}^*\}_{i,j=1,\dots,n}$ such that

$$J((s^f)^*) = \min_{s^f} \sum_{l=1}^m \beta_l (H_l^{W_{rr}}(s^f) - \hat{P}_l)^2 + \alpha_l (A_l^{W_{rr}}(s^f))^2, \quad (6.2a)$$

$$s^b \leq s^f. \quad (6.2b)$$

In a back lens surface design, we fix the front lens surface as the spherical surface and the toric surface, then try to solve for a minimum design objective functional $J(s^b)$ in constrained optimization problem

$$\min_{s^b} \sum_{l=1}^m \beta_l (H_l^{W_{rr}}(s^b) - \hat{P}_l)^2 + \alpha_l (A_l^{W_{rr}}(s^b))^2, \quad (6.3a)$$

$$s^b \leq s^f. \quad (6.3b)$$

We will focus on the front lens surface design problem (6.2) in the following sections, let

$$u^2 = s^f - s^b, \text{ where } u = \{u_{ij}\}_{i,j=1,\dots,n}, \quad (6.4)$$

then rearranging the equation gives

$$s^f = s^b + u^2 \geq s^b. \quad (6.5)$$

Therefore, substituting (6.4) into optimization problem (6.2), the constrain (6.2b) is satisfied, then we have an unconstrained optimization problem

$$\min_u J(u) = \min_u \sum_{l=1}^m \beta_l (H_l^{W_{rr}}(u^2 + s^b) - \hat{P}_l)^2 + \alpha_l (A_l^{W_{rr}}(u^2 + s^b))^2, \quad (6.6)$$

where $u \in \mathbb{R}^{n \times n}$.

We need to find a minimizer for this nonlinear unconstrained optimization problem (6.6). Assume J is a twice continuously differentiable, we may be able to tell that a point u^* is a local minimizer by examining just the gradient $\nabla J(u^*)$ and the Hessian $\nabla^2 J(u^*)$.

In the following sections, we will give some introduction of the unconstrained optimization theory. Gradient, Hessian and the approximated Hessian of the objective design functional will be discussed and numerical optimization methods such as Gradient method, Newton's method and Quasi Newton's method will be employed on obtaining the minimizer of the optimization problem.

6.1 Unconstrained Optimization

Definition 6.1.1. [13]

1. A point u^* is a global minimizer of J if $J(u^*) < J(u)$ for all u .
2. A point u^* is a local minimizer of J if there is a neighborhood \mathcal{N} of u^* such that $J(u^*) \leq J(u)$ for $u \in \mathcal{N}$.
3. A point u^* is a strict local minimizer of J if there is a neighborhood \mathcal{N} of u^* such that $J(u^*) < J(u)$ for all $u \in \mathcal{N}$ with $u \neq u^*$.

Taylor's theorem is the key tool to our analysis of the unconstrained optimization problem.

Theorem 6.1.2. Suppose $J : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable and that $p \in \mathbb{R}^n$. Then we have that

$$J(u + p) = J(u) + \nabla J(u + tp)^T p.$$

for some $t \in (0, 1)$. Moreover, if J is twice continuously differentiable, we have that

$$\nabla J(u + p) = \nabla J(u) + \int_0^1 \nabla^2 J(u + tp) p dt.$$

and that

$$J(u + p) = J(u) + \nabla J(u)^T p + \frac{1}{2} p^T \nabla^2 J(u + tp) p,$$

for some $t \in (0, 1)$.

By Taylor's theorem, if x^* is a local minimizer and f is continuously differentiable in an open neighborhood of x^* , then $\nabla f(x^*) = 0$. This condition is called First-Order Necessary Conditions. By adding one more condition that $\nabla^2 f$ is continuous in an open neighborhood of x^* to First-Order Necessary Conditions, we have the Second-Order Necessary Conditions: $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite.

Theorem 6.1.3. ([6]) *Suppose that $\nabla^2 f(x^*)$ is continuous in an open neighborhood of x^* and that $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite. Then x^* is a strict local minimizer of f .*

6.2 Gradients

The gradient $\nabla J(u^*)$ of the design objective functional J then is

$$\nabla J(s^b, s^f) = \sum_{l=1}^m 2\beta_l (H_l^{W_{rr}}(s^b, s^f) - \hat{P}_l) \nabla H_l^{W_{rr}}(s^b, s^f) + 2\alpha_l (A_l^{W_{rr}}(s^b, s^f)) \nabla (A_l^{W_{rr}}(s^b, s^f)),$$

$\nabla H^{W_{rr}}(s^b, s^f)$ and $\nabla A^{W_{rr}}(s^b, s^f)$ can be calculated according to Section 5.2.1 depending on the front surface lens design or the back surface lens design.

6.3 Approximated Hessian

Calculating the Hessian $\nabla^2 J$ directly would be time consuming and unstable, so we use BFGS method to give an approximation of the inverse matrix of the Hessian on every iteration of our optimization process (see [13]).

The derivation of the approximated Hessian matrix starts from approximating the design objective functional $J(u)$ at the current iterate $\{u_k\}$:

$$m_k(p) = J_k + \nabla J_k^T p + \frac{1}{2} p^T D_k J p.$$

Here D_k is an $n \times n$ symmetric positive definite matrix that will be updated at every iteration.

The minimizer u_k of this convex quadratic model is

$$p_k = -(D_k)^{-1} \nabla J,$$

which can be used as the search direction, and the new search iterate is

$$u_{k+1} = u_k + p_k.$$

D_k is the approximated Hessian.

In order to calculate the approximated Hessian, suppose we have a new iterate x_{k+1} , then the new quadratic model is

$$m_{k+1}(p) = J_{k+1} + \nabla J_{k+1}^T p + \frac{1}{2} p^T D_{k+1} J p.$$

One requirement for D_{k+1} is that the gradient of m_{k+1} should match the gradient of the desired design functional J at the latest two iterates x_k and x_{k+1} . Therefore, we have

$$\begin{aligned} \nabla m_{k+1}(0) &= \nabla J_{k+1}, \\ \nabla m_{k+1}(-p_k) &= \nabla J_{k+1} - D_{k+1} p_k = \nabla J_k. \end{aligned}$$

By rearranging, we obtain

$$D_{k+1} p_k = \nabla J_{k+1} - \nabla J_k, \tag{6.7}$$

To simplify the notation, we define the vectors,

$$s_k = x_{k+1} - x_k, \quad y_k = \nabla J_{k+1} - \nabla J_k,$$

so that (6.7) becomes

$$D_{k+1} s_k = y_k. \tag{6.8}$$

The equation is referred to as the secant equations.

Given the displacement s_k and the change of the gradient y_k , the secant equation requires that the symmetric positive definite matrix D_{k+1} map s_k into y_k . Therefore, s_k and y_k should satisfy the curvature condition

$$s_k^T y_k > 0, \quad (6.9)$$

When the curvature condition is satisfied, the secant equation will always have a solution D_{k+1} , since there are $n(n+1)/2$ degrees of freedom in a symmetric matrix, and the secant equation represents only n conditions. The n inequalities given by the positive definiteness do not absorb the remaining degrees of freedom.

The inverse of D_k , which we denote by

$$H_k = (D_k)^{-1}. \quad (6.10)$$

The updated approximation H_{k+1} must also be symmetric and positive definite, and must satisfy the secant equation 6.8, now is

$$H_{k+1} y_k = s_k, \quad (6.11)$$

Similarly, H_{k+1} , like D_{k+1} , is not unique in (6.11). To determine H_{k+1} uniquely, then, we impose the additional condition that among all symmetric matrices satisfying (6.11), H_{k+1} is, in some sense, closest to the current matrix H_k . We solve the problem

$$\min_H \|H - H_k\| \quad (6.12a)$$

$$\text{subject to } H = H^T, \quad H y_k = s_k. \quad (6.12b)$$

where H_k is symmetric and positive definite. The weighted Frobenius norm is used to give rise to a scale-invariant optimization method,

$$\|A\|_W = \|W^{1/2} A W^{1/2}\|_F \quad (6.13)$$

where $\|\cdot\|_F$ is defined by $\|C\|_F^2 = \sum_{i=1}^n \sum_{j=1}^n c_{ij}^2$. The weight

$$W = \tilde{G}_k^{-1},$$

where \tilde{G}_k is the average Hessian defined by

$$\tilde{G}_k = \left[\int_0^1 \nabla^2 J(x_k + \tau p_k) d\tau \right]$$

. The unique solution H_{k+1} to 6.12 is given by

$$(BFGS) \quad H_{k+1} = (1 - \rho_k s_k y_k^T) H_k (1 - \rho_k s_k y_k^T) + \rho_k s_k s_k^T, \quad (6.14)$$

where

$$\rho_k = \frac{1}{y_k^T s_k}.$$

The update formula for the approximate Hessian D_k can be obtained by applying Sherman-Morrison-Woodbury formula [16] to 6.14.

Lemma 6.3.1. *Assume A is a square nonsingular matrix. Let U and V be matrices in $\mathbb{R}^{n \times p}$. If*

$$\tilde{A} = A + UV^T,$$

then

$$\tilde{A}^{-1} = A^{-1} - A^{-1}U(I + V^T A^{-1}U)^{-1}V^T A^{-1}. \quad (6.15)$$

Proof. Consider the following block matrix

$$\begin{bmatrix} A & U \\ V & -I \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

.

Expanding of the above equation gives

$$AX + UY = I, \quad (6.16a)$$

$$VX - Y = 0, \quad (6.16b)$$

which is equivalent to

$$(A + UV)X = I.$$

Therefore,

$$X = (A + UV)^{-1} = (\tilde{A})^{-1}.$$

Eliminating (6.16a) gives

$$X = A^{-1}(I - UY). \quad (6.17)$$

Substituting (6.17) to (6.16b), we obtain

$$Y = (I + VA^{-1}U)^{-1}VA^{-1}. \quad (6.18)$$

Then 6.16a can be transformed into

$$AX + U(I + VA^{-1}U)^{-1}VA^{-1} = I,$$

therefore,

$$(\tilde{A})^{-1} = X^{-1} = A^{-1} - A^{-1}U(I + V^T A^{-1}U)^{-1}V^T A^{-1}.$$

□

The approximated Hessian by BFGS method is therefore

$$(BFGS) \quad B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k}. \quad (6.19)$$

6.4 Outline of a computational algorithm

A computational algorithm is stated as in 6.4,

6.5 Method of optimization

In unconstrained optimization, we minimize an objective function that depends on real variables, with no restriction on the values of these variables. The mathematical formulation is

$$\min_u J(u), \quad (6.20)$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function.

Set the initial front lens surface u_0 as a spherical surface or a toric surface which is close to the back surface, convergence tolerance $\epsilon > 0$;

$k \leftarrow 0$;

while $\|\nabla J\| \leq \epsilon$; **do**

 Obtain the minimizer search step p_k by Gradient method, Newton's method or Quasi-Newton's method;

 Set $x_{k+1} = x_k + p_k$;

$k \leftarrow k + 1$;

end while

All algorithms for unconstrained optimization require to start with a starting point x_0 , which should be a reasonable estimate of the solution. Beginning at x_0 , optimization algorithm generates a sequence of iterates $\{x_k\}_{k=1}^{\infty}$ that terminates when $\|\nabla J\|$ is small enough. In order to decide how to move from x_k to a new iterate x_{k+1} , the algorithm use information of J at x_k to find a new iterate x_{k+1} with a lower function value $J(x_{k+1})$ than $J(x_k)$.

6.5.1 Gradient method

From the k -th iterate x_k , for any unit search direction d and the step size parameter α , by Taylor Theorem (6.1.2), we have

$$J(x_k + \alpha d) = J(x_k) + \alpha d^T \nabla J(x_k) + \frac{1}{2} \alpha^2 d^T \nabla^2 J(x_k + td) d, \text{ for some } t \in (0, \alpha) \quad (6.21)$$

The rate of change in f along d at x_k is the coefficient of α , $d^T \nabla J(x_k)$. Hence, to get the most rapid decrease search direction d , we want to solve the problem

$$\min_d d^T \nabla J(x_k), \quad \text{subject to } \|d\| = 1. \quad (6.22)$$

Since

$$d^T \nabla J(x_k) = \|d\| \|\nabla J(x_k)\| \cos \theta,$$

where θ is the angle between d and $\nabla J(x_k)$, also with

$$\|d\| = 1,$$

we have

$$d^T \nabla J(x_k) = \|\nabla J(x_k)\| \cos \theta.$$

The objective function in 6.22 is minimized when $\theta = \pi$, in other words, the solution to 6.22 is

$$p = -\nabla J(x_k) / \|\nabla J(x_k)\|,$$

as claimed.

The steepest descent method is a line search method that moves along $d_k = -\nabla J(x_k)$ at every k -th step. The step size α_k should be chosen by a variety ways. Wolfe conditions is a efficient requirement on performing an inexact line search.

Lemma 6.5.1. (*Wolfe condition*)

$$J(x_k + \alpha_k d_k) \leq J(x_k) + c_1 \alpha_k \nabla J^T(x_k) d_k, \quad (6.23a)$$

$$|\nabla J(x_k + \alpha_k d_k)^T d_k| \leq c_2 |\nabla J(x_k)^T d_k|, \quad (6.23b)$$

with $0 < c_1 < c_2 < 1$.

6.5.2 Newton's method

Newton's method is derived from the second-order Taylor series approximation to $J(x_k + d)$, which is

$$J(x_k + p) \approx J(x_k) + p^T \nabla J(x_k) + \frac{1}{2} p^T \nabla^2 J(x_k) p \triangleq m_k(p). \quad (6.24)$$

Notice that the derivative of $m_k(p)$ should be eliminated on p_k that minimizes $m_k(p)$, we obtain

$$\nabla_p m_k(p_k) = 0,$$

which is

$$\nabla^2 J(x_k) p_k = -\nabla J(x_k).$$

Assume that $\nabla^2 J(x_k)$ is positive definite, then

$$p_k = -(\nabla^2 J(x_k))^{-1} \nabla J(x_k). \quad (6.25)$$

Newton's method can achieve a quadratic convergence rate. However, the main drawback is the need for the Hessian $\nabla^2 J(x_k)$. Explicit computation of the second derivatives is a time-consuming process.

6.5.3 Quasi-Newton method

Quasi-Newton method is an alternative method of Newton's method in that they don't require computation of the Hessian and attain the superlinear rate of convergence.

In place of the true Hessian $\nabla^2 J(x_k)$, they use an BFGS Hessian approximation B_k , which is updated after each step. The calculation of the approximated Hessian B_k and the inverse of the approximated Hessian H_k is stated in the section 6.3. The formulas are

$$(BFGS) \quad B_k = B_{k-1} - \frac{B_{k-1}s_{k-1}s_{k-1}^T B_{k-1}}{s_{k-1}^T B_{k-1} s_{k-1}} + \frac{y_{k-1}y_{k-1}^T}{y_{k-1}^T s_{k-1}}, \quad (6.26)$$

$$(BFGS) \quad H_{k-1} = (1 - \rho_{k-1}s_{k-1}y_{k-1}^T)H_{k-1}(1 - \rho_{k-1}s_{k-1}y_{k-1}^T) + \rho_{k-1}s_{k-1}s_{k-1}^T, \quad (6.27)$$

where $s_{k-1} = x_k - x_{k-1}$, $y_{k-1} = \nabla J_k - \nabla J_{k-1}$, and $\rho_{k-1} = \frac{1}{y_{k-1}^T s_{k-1}}$

The basic algorithm of Quasi-Newton method using BFGS approximation is

6.5.3

Algorithm 1 BFGS Quasi-Newton Method

Set the initial front lens surface u_0 as a spherical surface or a toric surface which is close to the back surface, convergence tolerance $\epsilon > 0$, inverse Hessian approximation H_0 ;

$k \leftarrow 0$;

while $\|\nabla J\| \leq \epsilon$; **do**

 Compute the search direction

$$d_k = -H_k \nabla J(x_k); \tag{6.28}$$

 Set $x_{k+1} = x_k + \alpha_k d_k$ where α_k is computed from a line search procedure satisfy the Wolfe conditions 6.5.1;

 Define $s_k = x_{k+1} - x_k$ and $y_k = \nabla J_{k+1} - \nabla J_k$;

 Compute H_{k+1} by the formula of 6.14;

$k \leftarrow k + 1$;

end while

Chapter 7

Numerical Result

In practise, lenses are designed on a circular region of diameter 80mm with grid size of 2mm or 4mm. In our numerical example, we choose the design region to be a square domain $[-22, 22] \otimes [-22, 22]$, and we set the sample region to be $[-20, 20] \otimes [-20, 20]$. With grid size 2mm, n is set to be 23 and m is set to be 21.

To start we first assign the fixed parameters involved in the optimization problem (6.6),

$$\min_u J(u) = \min_u \sum_{l=1}^m \beta_l (H_l^{W_{rr}}(u^2 + s^b) - \hat{P}_l)^2 + \alpha_l (A_l^{W_{rr}}(u^2 + s^b))^2, \quad (7.1)$$

The weighted function value α_l, β_l at point x_l and the desired power distribution value \hat{P}_l at point x_l .

To assign these values, we divided the square region into five subregions as shown in Figure 7.1.

The large region, distance zone, is the upper portion of the lens. It provides the specified distance prescription. The smaller region, near zone, is the lower portion of the lens and it provides the specified add power. The progressive corridor is a corridor of increasing power connects these two zones and provides intermediate vision. The rest of the region is divided into 2 subdomains for easy assignment of these functions.

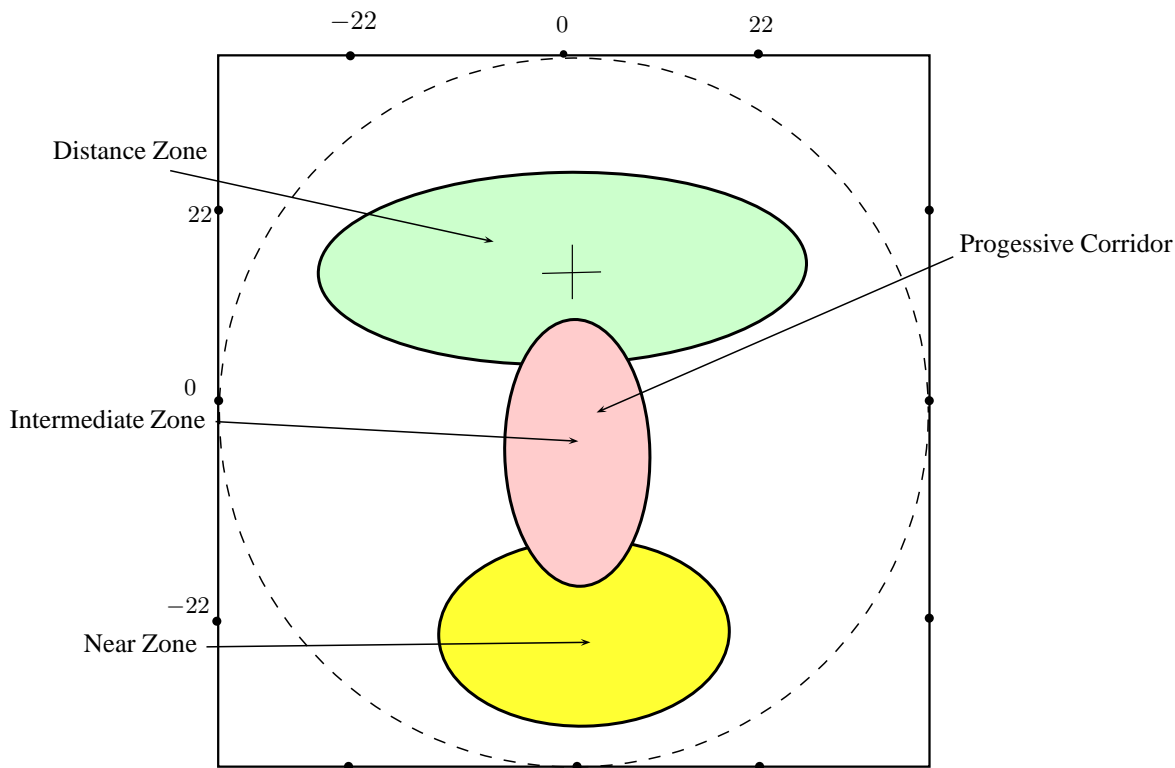


Figure 7.1: Partition of the design region

In each subdomain, the weight functions are assigned based on the importance to the lens. In a progressive lens design, the center line crosses distance zone, progressive corridor and near zone should be weighted the most. Therefore, we put more weight on the center line, distance zone and near zone. One example of weight function for power distribution $\beta(x_1, x_2)$ and weight function for astigmatism distribution $\alpha(x_1, x_2)$ is shown in Figure 7.2. Note that $\alpha(x_1, x_2)$ is much larger than $\beta(x_1, x_2)$ along the center line, distance zone and near zone, since we expect to have very small astigmatism in these areas.

For the front lens surface design, the desired power distribution depends on the base radius, which is the radius of the spherical back lens surface, the add power and the index of refraction of the lens material (see [15]).

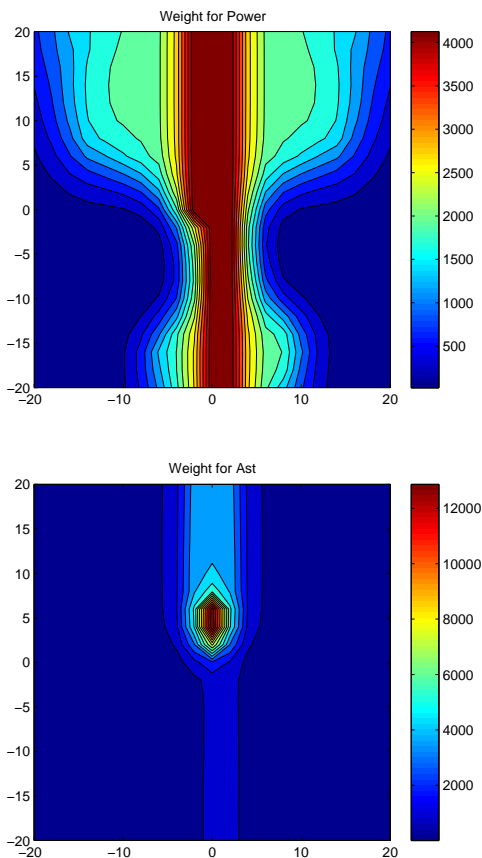


Figure 7.2: Contour map of the weight functions

In the lens industry, the base radius is generally between 70mm and 350mm. Small radius is used for high prescription design (e.g. power level of 2.5), and large radius is for low prescription design (e.g. power level of -10). The base radius in our numerical example is 85mm since the power level is around 2.00 diopter. For the add power, it is usually between 0.75 diopter to 3.5 diopter. The difficulty of the design increases as the add power increases. The index of refraction of the lens material is from 1.4 and 1.7.

Figure 7.3 shows a sample desired power distribution in our numerical results. We assign power of -2.00 diopter on the distant zone, while add power of $+2.25$

diopter for the near zone. The index of 1.6 is employed in our calculation.

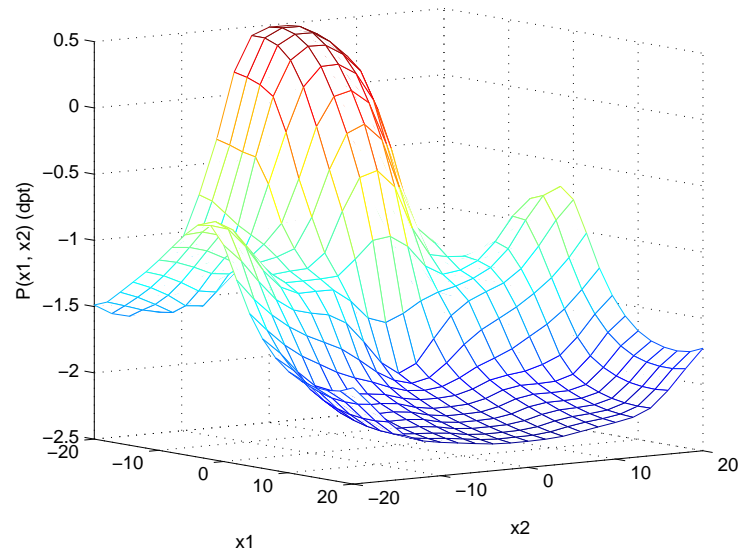


Figure 7.3: Desired power distribution

For the numerical result in this work, we consider an optimization problem of the front surface design. Tensor-product cubic B-spline functions on the control mesh of size 23×23 is applied to represent the front lens surface. The back lens surface is set to be a spherical surface with power $+11.94$ diopter and its distance to the eye center is set to be 27mm . A spherical surface of power $+11.14$ is set to be the initial estimate.

Numerical results are shown in Figure 7.4 and 7.5. Figure 7.5 and 7.6 shows that the calculated progressive front surface and its difference between the front surface and the back surface.

Figure 7.5 and 7.6 shows power contour and astigmatism contour of our result, the optimization process produces a nice progressive lens design where the power is close to the desired power distribution on distance zone and near zone, the power transforms progressively on progressive corridor, and the astigmatism is relatively small (< 0.25 diopter) in distance zone, near zone and progressive corridor.

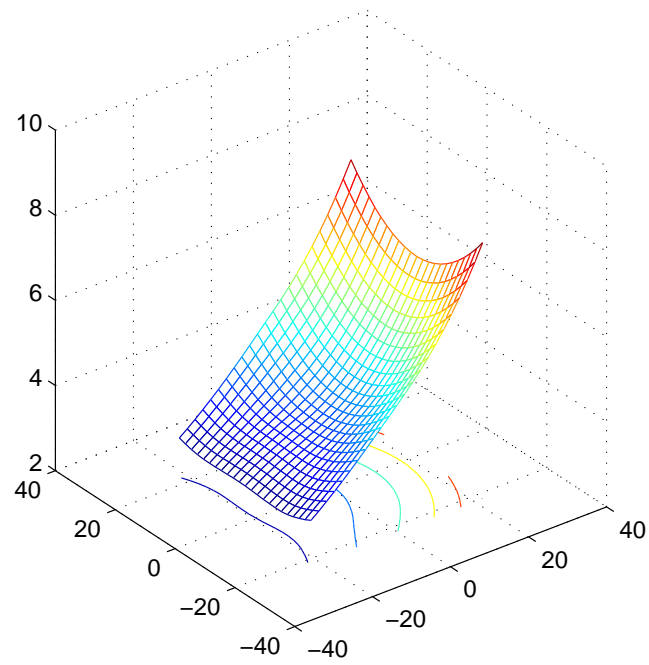
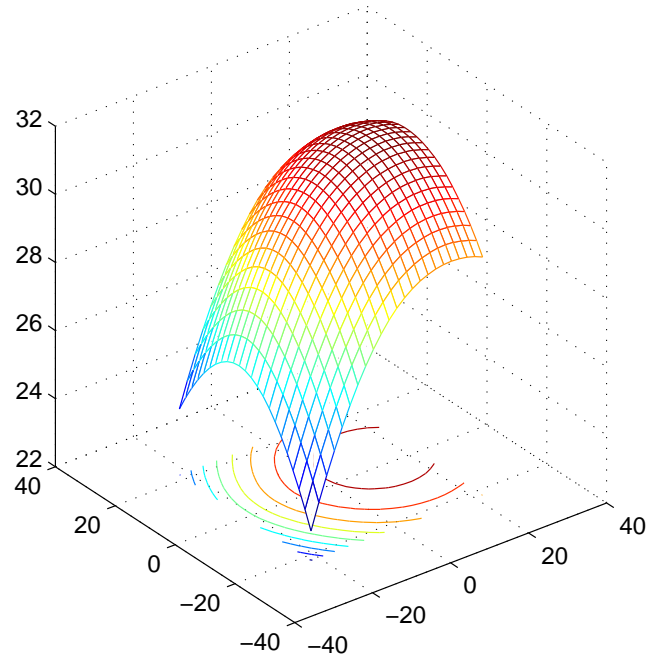


Figure 7.4: Calculated Progressive Surface

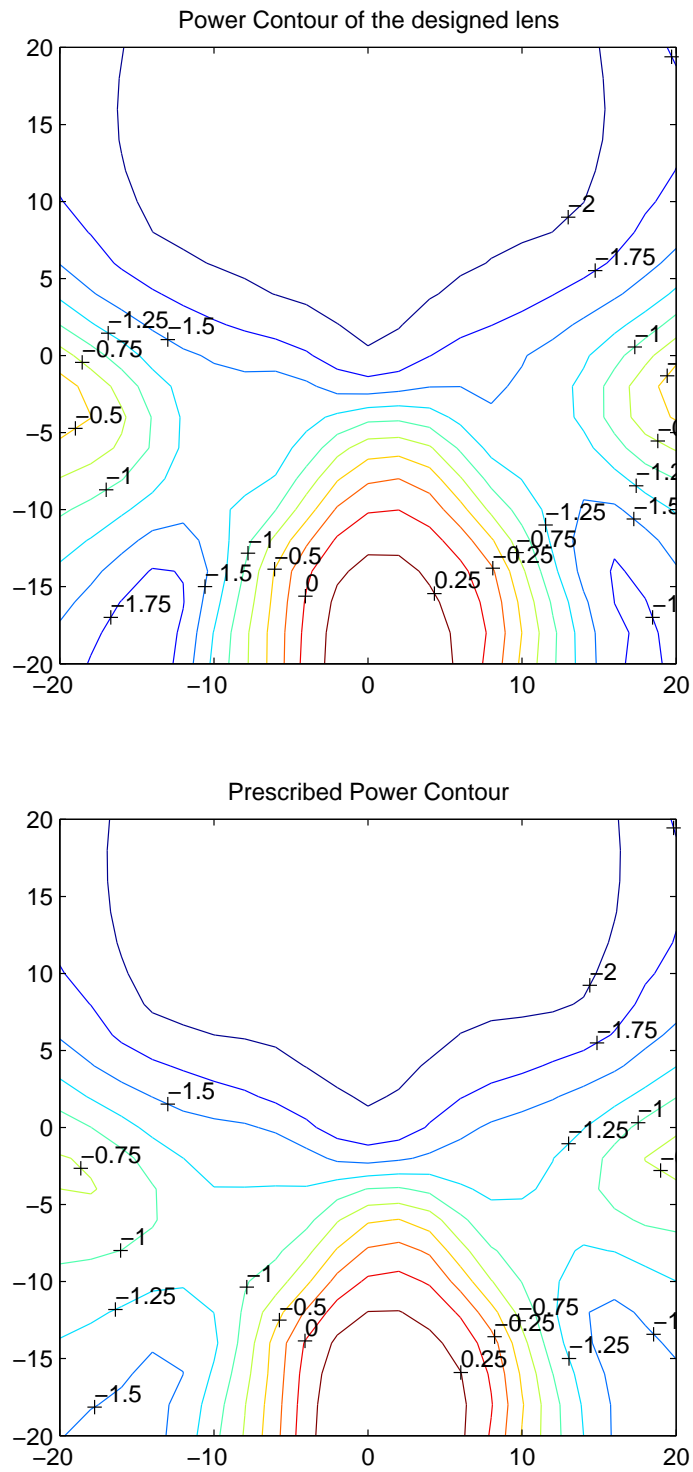


Figure 7.5: Contour map of power of the designed lens and the prescribed power

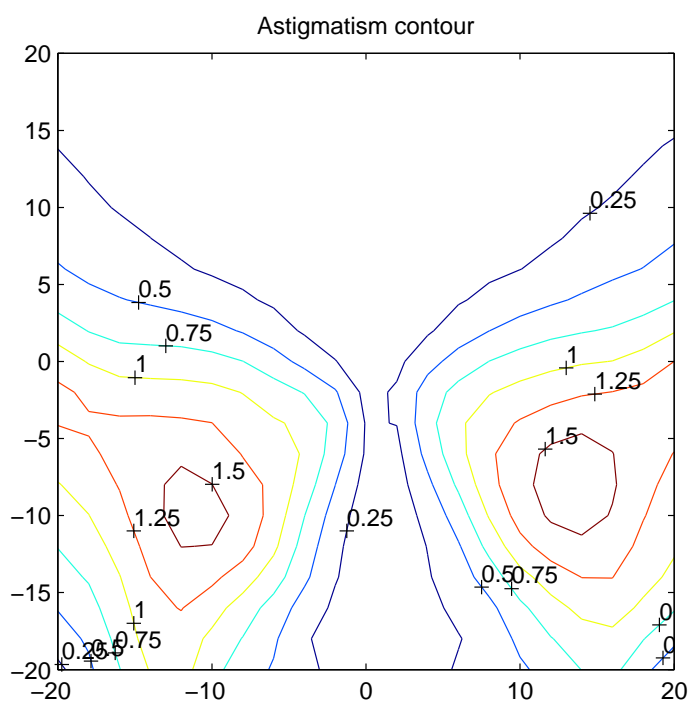


Figure 7.6: Contour map of astigmatism

Chapter 8

Discussion

In this work, we studied the problem of progressive addition lens design with wavefront tracing method. The design problem is to obtaining a front lens surface, or a back lens surface or both surfaces such that the power distribution of the refracted wavefront is close to a desired power distribution and the astigmatism of the refracted wavefront is minimized over the designated regions. When posed as a variational problem, the design problem is a nonlinear optimization problem.

To evaluate power, astigmatism and higher order aberrations of the refracted wavefront, we developed new formulas of First Fundamental Form Coefficients, Second Fundamental Form Coefficients and Third Order Surface Coefficients of the wavefront on propagation in a homogeneous medium and on refraction at a lens surface. With these formulas, a complete model of describing how the curvature changes when a planar wavefront passing through a lens is obtained by ray tracing method. The new formulas also allow for somewhat straight-forward calculation of the gradients of the design objective.

To solve the problem numerically tensor-product B-splines are employed to represent the surfaces. Several optimization methods for nonquadratic optimization are considered, among them are the gradient method, Newton's method and the quasi-Newton. The quasi-Newton method is implemented in this work.

As an example, we demonstrate our method on the problem of designing

a front progressive lens surface when the back surface is a spherical surface of a fixed radius. In the calculation, weight functions are chosen so that the prescribed power is close to the desired one in important areas of the lens while at the same time astigmatism in these areas are made small. We obtained a lens with good properties using our method.

Future research should focus on several improvements. First, there is a need to speed up the computation. The evaluation of the cost function in the optimization is necessarily complex. However, the wavefront in each gaze is independent and thus a parallel approach is possible. Second, if we extend the design mesh to larger size, say $[-30, 30] \times [-30, 30]$, which is the industrial standard for lens design mesh, the memory needs for storing the Hessian or the approximate Hessian becomes excessive so we should consider limited-memory Quasi-Newton methods. Third, when we designing over a large region, the Hessian may become ill-conditioned. This means that regularization methods must be considered. The ill-conditioning may be explained by the sensitivity of the rays to small changes in the surface, or rays that leave the region when the surface is perturbed. Finally, minimizing second order aberrations should be considered in the future.

Overall, we believe that this work, in which the light propagation is accurately described using geometrical optics, holds a lot of promise for the lens design problem. With further development, the method presented here can potentially find use in industrial lens design.

References

- [1] T. Baudaut, F. Ahsbahs, and C. Miege. Multifocal ophthalmic lens. *United States Patent*, 6102544, 2000.
- [2] M. Born and E. Wolf. *Principles of Optics-Electromagnetic Theory of Propagation, Interference and Diffraction of Light*. Pergamon Press, 1975.
- [3] M. P. Do Carmo. *Differential Geometry of Curves and Surface*. Prentice-Hall, 1976.
- [4] C. de Boor. Total positivity of the spline collocation matrix. *Indiana University Journal of Math*, 25:541–551, 1976.
- [5] Carl de Boor. *A Practical Guide to Splines*. Springer-Verlag New York, Inc., 1978.
- [6] J. E. Dennis. *Numerical methods for unconstrained optimization and nonlinear equations*. Englewood Cliffs, N.J. : Prentice-Hall, 1983.
- [7] J. H. Gallier. *Curves and surfaces in geometric modeling : theory and algorithms*. San Francisco, Calif. : Morgan Kaufmann Publishers, 2000.
- [8] J. Kneisly II. Local curvature of wavefronts in an optical system. *Journal of The Optical Society of America*, 54(2):229–235, 1964.
- [9] D. Katzman and J. Rubinstein. Method for the design of multifocal optical elements. *United States Patent*, 6302540, 2001.

- [10] J. Loost, G. Greiner, and H.P. Seidel. A variational approach to progressive lens design. *Computer Aided Design*, 30:595–602, 1998.
- [11] J. Loost, Ph. Slusallek, and H.P. Seidel. Using wavefront tracing for the visualization and optimization of progressive lens. *EUROGRAPHICS'98*, 17(3):c-255–265, 1998.
- [12] D. Malacara. *Optical Shop Testing*. John Wiley and Sons, Inc., 1978.
- [13] J. Nocedal and S. J. Wright. *Numerical optimization*. New York: Springer, 1999.
- [14] J. Rubinstein and G. Wolansky. Wavefront method for designing optical elements. *United States Patent*, 6655803, 2003.
- [15] J. Sheedy, R. Hardy, and J. Hayes. Progressive addition lenses c measurements and ratings. *Optometry*, 77:34–36, 2006.
- [16] A. Sherman. On Newton-iterative methods for the solution of systems non-linear equations. *SIAM Journal on Numerical Analysis*, 15:755–771, 1978.
- [17] J. Wang. Design of progressive lenses-mathematical analysis and numerical methods. *UMI Microform 3058674*, UMN Thesis, 2002.
- [18] J. Wang, R. Gulliver, and F. Santosa. Analysis of a variational approach to progressive lens design. *SIAM J. Appl. Math.*, 64:277–296, 2003.
- [19] J. Wang and F. Santosa. A numerical method for progressive lens design. *Mathematical Models and Methods in Applied Sciences (M3AS)*, 14:619–640, 2004.
- [20] J. Wang, F. Santosa, and R. Gulliver. Multifocal optical device design. *United States Patent*, 7044601, 2006.
- [21] J. T. Winthrop. Progressive addition spectacle lens. *United States Patent*, 4861153, 1989.