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OF FINITE PRICE MECHANISMS

by

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ABSTRACT

This paper obtains finite counterparts of previous results that showed the informational efficiency of the Walrasian mechanism among all mechanisms yielding Pareto-optimal individually rational trades in an exchange economy while using a continuum of possible messages. Such mechanisms lack realism, since it is not possible to transmit or announce all points of a continuum, and it generally takes infinite time to find an equilibrium message, among all the messages in a continuum. Accordingly, the paper studies approximations of the continuum Walrasian mechanism, in which the number of messages is finite. It applies general results from a companion paper, which considered finite approximations of continuum mechanisms in general organizations, with exchange economies as a particular example. For classic exchange economies, we compare the continuum Walrasian mechanism with *alternative* continuum mechanisms that also find a Pareto-optimal and individually rational allocation. There are many of them, and some of them, like the continuum Direct Revelation mechanism, do not use prices at all. A finite approximation to a continuum mechanism will have an error. Its *overall error* for a given class of economies is the worst distance (over all members of the class) between the continuum mechanism's final allocation and the approximation's final allocation. We measure a finite mechanism's cost by the number of its (equilibrium) messages. We consider exchange economies in which traders' utility functions are quasi-linear and strictly concave. We find that the overall error of a sufficiently fine finite approximation of the Walrasian mechanism is arbitrarily close to the overall error of a not more costly approximation of an alternative continuum mechanism that has the same number of message variables. The former overall error is *smaller* than the latter if the alternative continuum mechanism has a *larger* number of message variables. A continuum Direct Revelation mechanism is an example of an alternative mechanism with a larger number of message variables than the Walrasian mechanism.

Thus the informational superiority of the Walrasian mechanism emerges again when we approximate it and take the finite number of messages as our cost measure.

I. INTRODUCTION

This paper obtains finite counterparts of previous results that showed the informational efficiency of the Walrasian mechanism among all mechanisms that yield Pareto-optimal individually rational trades in an exchange economy while using a continuum of possible messages. Such mechanisms lack realism, since one cannot transmit or announce all points of a continuum. Moreover finding an equilibrium message, among all the messages in the continuum, generally takes an infinite time. It is therefore of considerable interest to see whether the informational superiority of the Walrasian mechanism again emerges when we approximate it, using a finite set of messages. In our study of this question we apply general results from a companion paper ([5]). That paper considered finite approximations of continuum mechanisms in general organizations, with exchange economies as a particular example.

To set the stage, let us recall the puzzle which gave rise to the previous literature.¹ While the optimality of the Walrasian equilibrium and the conditions under which it exists were well understood, it remained true that many other mechanisms could be constructed, which also yield, at their equilibrium positions, trades that are Pareto optimal and individually rational. Such mechanisms might not use prices at all. One of the alternative mechanisms is Direct Revelation, where a Center gathers complete information about the traders' preferences and announcements, and then calculates and imposes an optimal allocation. Other mechanisms might pool part of the traders' private information but not all of it.

It was natural to conjecture that the Walrasian mechanism had a special property: the information transmitted is no "more" than the task of finding an optimal allocation requires. The alternative mechanisms, on the other hand, generally require the transmission of more information than necessary. That conjecture certainly agreed with the informal but widely shared view that the performance of price mechanisms cannot be matched by "command" mechanisms. It also agreed with more explicit statements in classic writings, by Hayek and others, as to the merits of price mechanisms, with regard to information as well as incentives.²

The difficulty was that some precise modelling had to be done before the conjecture could be checked. "A mechanism achieving the task" and "amount of information transmitted" had to be given exact meanings.

The literature that we seek to supplement took an approach that may be characterized in the following way.

¹A survey of the literature (up to the mid 1980s) is found in [4]. Among the papers that dealt with the informational efficiency (specifically the message-space minimality) of the Walrasian mechanism are [3],[6],[9], and [11].

²While the paper by Hayek ([2]) is frequently cited as an articulate statement of the superiority of price mechanisms, an extensive discussion of the issue is found earlier, in the debate about the possibility of socialism. See, for example, Lange [7].

- It considered an n -person organization (an n -trader exchange economy for example) in which person i privately observes a *local environment* e_i lying in a set E_i . In the exchange-economy case, e_i describes i 's endowment and preferences.
- It defined a *mechanism on* $E \equiv E_1 \times \cdots \times E_n$ with *action space* A as a triple $\langle M, (g_1, \dots, g_n), h \rangle$. Here M is the *message space*, g_i is a function from $M \times E_i$ to a real space, and h is a function, called the *outcome function*, from M to A . In an important special case, M is the Cartesian product $A \times Y$ and h is a projection operator, i.e., for $m = (a, y) \in M$, we have $h(m) = a$. We shall then say that the mechanism is a *projection mechanism*. An example is the Walrasian mechanism (treated in more detail below), where a message is a trade/price pair (a, p) .
- It interpreted the operation of a mechanism in several ways. In one of them, sometimes called the “verification scenario”, a sequence of messages m in M is announced. Person i responds to a given m by computing $g_i(m, e_i)$ and signals “agreement” if the result is zero. In the case of the Walrasian mechanism, with non-numeraire commodities $1, \dots, L$ and a numeraire $L + 1$, the function g_i has real-valued components $g_{i\ell}$, one for each non-numeraire commodity. A message m consists of an allocation a and a price vector $p = (p_1, \dots, p_L)$, and

$$g_{i\ell}((a, p), e_i) = \frac{i\text{'s marginal utility for commodity } \ell \text{ if the trades in } a \text{ were carried out}}{i\text{'s marginal utility for commodity } L + 1 \text{ if the trades in } a \text{ were carried out}} - p_\ell.$$

- It defined an *equilibrium message* for $e = (e_1, \dots, e_n)$ to be a message m to which all agree at e , i.e., $g_i(m, e_i) = 0, i = 1, \dots, n$, and supposed that when an equilibrium message m has been found, then the action $h(m)$, called an *equilibrium action* for e , is taken. In the Walrasian case, h is the projection operator: $h(a, p) = a$.
- It defined a mechanism as *realizing the goal function* $\gamma : E \rightarrow A$ if (i) there is an equilibrium message for every e , and (ii) $h(m) = \gamma(e)$ whenever m is an equilibrium message for e . For the exchange-economy case, we have a goal function γ^* , where $\gamma^*(e)$ is a balanced n -tuple of trade vectors which is Pareto optimal and individually rational for e .
- It chose the *size of the message space* M as a natural measure of a mechanism's informational cost. If M is a differentiable manifold, then its *dimension* is the appropriate size measure.
- It found that for an exchange economy with classic properties (convex preferences) there is no mechanism obeying certain regularity conditions and realizing γ^* which has a message space of smaller dimension than that of the Walrasian mechanism.

Thus the approach yielded (probably for the first time) a rigorous version of the informational claims long made informally for the Walrasian mechanism. But the claim dealt only with mechanisms whose message spaces are continua.

In seeking a version of the claim when we require mechanisms to have a *finite* message space, we start by defining a finite approximation of a continuum mechanism, say $\Lambda = \langle M, (g_1, \dots, g_n), h \rangle$ whose message space M is a subset of $\mathbb{R}^D$ and whose action space is a subset of $\mathbb{R}^\alpha$, where $D \geq 1, \alpha \geq 1$. We shall call the function g_i person i 's *agreement function*. Suppose M is compact. Then, in our approach, the message space of a *mesh- ϵ* finite approximation of Λ , where $\epsilon > 0$, is denoted M_ϵ . It is the intersection of M with an unbounded lattice of points in $\mathbb{R}^D$, called the *ϵ -lattice*. The points of the lattice are spaced 2ϵ apart in each coordinate. We can interpret ϵ as determined by the number of digits to which we round off each of the D real message variables when we turn from the continuum mechanism Λ to its finite approximation.

We obtain the k th component of person i 's agreement function in our finite approximation by “softening” the original function g_{ik} . The new function takes the value zero for a given e_i if and only if there is some message in M_ϵ , say m^* , such that $|g_{ik}(m^*, e_i)| \leq \eta$, where η is a positive number called the approximating mechanism's *tolerance*.

Our new “softened” agreement function may be interpreted as the agreement function of the original continuum mechanism, but now computed to a fixed accuracy.

As for the finite approximation's outcome function, we shall consider two variants. In the *exact-outcome* variant, the outcome function is simply the restriction of h to M_ϵ . In defining the *rounded-outcome* variant, we confine attention to an action set A which is an open set of $\mathbb{R}^\alpha$. We specify an *action mesh*, say ν . Then the rounded-outcome function assigns to a message m in M_ϵ that point of the ν -lattice in $\mathbb{R}^\alpha$ which lies closest to the action $h(m)$ selected by the original continuum mechanism. (Distance is defined as the maximum difference over all coordinates).

The *overall error* of a finite approximation of Λ is the largest distance, over all e in E , between an equilibrium action for e in the original mechanism Λ and an equilibrium action for e in the approximation. We shall take the *cost* of a finite approximation to be the number of its equilibrium messages. We shall say that the first of two approximations is *cost-equivalent* to the second if the number of equilibrium messages in the second is not more than the number of equilibrium messages in the first. We shall say that a class of approximations to a given continuum mechanism is *complete* if for some $\epsilon^\#$ the class contains a mesh- ϵ approximation for every ϵ with $0 < \epsilon \leq \epsilon^\#$. Finally, given two regular continuum mechanisms, we shall say that a complete class of finite approximations of the first is *superior* to a complete class of finite approximations of the second if the following holds:

If the mesh of an approximation in the first class is sufficiently fine, then its overall error is less than the overall error of any cost-equivalent member of the second class.

We state four propositions in the present paper. Proposition A considers two regular continuum mechanisms on an environment set E . Each has an action space which is a subset of $\mathbb{R}^\alpha$ and the first has a lower message-space dimension than the second. The Proposition asserts that a complete class of approximations of the first is superior to a complete class of approximations of the second if

- the approximations in both classes are of the rounded-outcome type, with the ratio of action mesh to message mesh meeting a common upper bound and a common lower bound for all members of both classes
- the approximations in the first class are of the rounded-outcome type (with a common upper bound and a common lower bound to the action-mesh/message-mesh ratio), the approximations in the second class are of the exact-outcome type, and the second continuum mechanism has the projection property
- the approximations in the the first class are of the rounded-outcome type (with a common upper bound and a common lower bound to the action-mesh/message-mesh ratio), the approximations in the second class are of the exact-outcome type, and the second continuum mechanism has the Direct Revelation (DR) property: its message space is the environment set E , and each person agrees to a message if and only if it describes her local environment correctly

Proposition B deals with approximations of two continuum mechanisms whose message spaces are of the *same* dimension. We call a complete class of approximations of one of them *indifferent* to a complete class of approximations of the second if we can make the overall error of a member of one class as close as we wish to the overall error of a member of the other class, by choosing the mesh of one of those two approximations to be sufficiently fine and choosing the other to be cost-equivalent. Proposition B says that if the goal function realized by each continuum mechanism meets some regularity properties, then a complete class of rounded-outcome approximations of the first continuum mechanism is indifferent to a complete class of rounded-outcome approximations of the second continuum mechanism. Similarly, the Proposition says that a complete class of exact-outcome approximations of the first continuum mechanism is indifferent to a complete class of exact-outcome approximations of the second continuum mechanism.

Propositions C and D deal with some general issues raised by the striking virtue which the continuum Walrasian mechanism displays in a convex economy: its message-space dimension remains the same no matter how large we make the environment set, i.e., no matter what the dimension of the set of the traders' possible individual characteristics may be. The Walrasian mechanism is superior to the DR mechanism with regard to message-space dimension as long as the environment set's dimension exceeds the dimension of the Walrasian mechanism, and the Walrasian mechanism's superiority over the DR mechanism grows without bound as we increase the dimension of the environment set, since the DR mechanism's message space is (by our definition of DR) the environment set itself.

In Propositions C and D we study this matter in a very general setting, with the Walrasian mechanism in an exchange economy as an example. We consider an infinite sequence of environment sets of increasing dimension. For each set in the sequence we consider a continuum DR mechanism and an *indirect* continuum mechanism, both realizing the same goal function. In all the indirect

continuum mechanisms the message space has *the same* dimension and is contained in a cube which remains the same for all the mechanisms. But while the cost of the indirect continuum mechanism (its message space dimension) remains the same for all environment sets, the cost of the DR mechanism is the dimension of the environment set itself, and so it goes to infinity as the environment set's dimension increases without bound. We seek finite analogues of that continuum statement. Two types of analogues can be studied. The first approach maintains the viewpoint of Proposition A and studies, for each environment set in our sequence, the *indirect-versus-DR error differential*: the difference in overall error between an approximation of the indirect mechanism and a cost-equivalent approximation of the DR mechanism. The second approach is to adopt a viewpoint that is "dual" to that of Proposition A and to study, for each environment set in our sequence, the *indirect-versus-DR cost differential*: the difference in cost between an approximation of the indirect mechanism and an "error-equivalent" approximation of the DR mechanism (an approximation whose error is at least as low).

Pursuing the first approach, we see that if the possible values of the goal-fulfilling actions satisfy a bound that stays unchanged for all environment sets, then the indirect-versus-DR error differential cannot go to infinity as the environment-set dimension goes to infinity.³ Proposition C pursues the second approach and obtains an analogue of the continuum statement: we find that under certain further conditions, the cost differential indeed goes to infinity as the environment set's dimension goes to infinity, even though the possible values of the goal-fulfilling actions satisfy a bound that stays the same for all environment sets.

While the first approach (requiring cost equivalence and then examining error) fails to yield an analogue of the continuum statement when actions have to satisfy an unchanging bound, we may still ask whether the error differential goes to a limit as the environment set's dimension goes to infinity. Proposition D provides a partial answer. It says that as the environment set's dimension goes to infinity, the overall error of the cost-equivalent DR approximation goes to a limit and the error differential is bounded from below by a positive number.

We apply Proposition A to a class of exchange economies. We let the low-dimensional continuum mechanism be the Walrasian mechanism and we let each e_i in E_i consist of i 's endowment and a vector of parameters defining a concave quasi-linear utility function. We let the high-dimensional continuum mechanism be a Direct Revelation mechanism which has E itself as its message space and has trade vectors as its actions. We show that the Walrasian mechanism indeed has the regularity required for Proposition A. We conclude that a sufficiently fine finite approximation of the DR mechanism has higher overall error than a cost-equivalent approximation of the continuum Walrasian mechanism.

³In the exchange-economy example with fixed endowments, the actions are trades and those are bounded by the fixed endowments for all environment sets. So both the overall error of an approximation of the Walrasian mechanism, and the overall error of an approximation of a DR mechanism realizing the same goal as the Walrasian mechanism, are bounded as the environment set dimension grows without bound.

Applying Proposition B, we consider the (possibly empty) class of regular continuum mechanisms that realize the same goal function as the continuum Walrasian mechanism (for our class of exchange economies), are distinct from the Walrasian mechanism, and have a message-space dimension equal to that of the continuum Walrasian mechanism. We find that nothing is to be gained by considering approximations of such a continuum mechanism: a complete class of approximations of such a continuum mechanism is indifferent (in our sense) to a complete class of approximations of the continuum Walrasian mechanism.

Next we apply Propositions C and D to obtain finite counterparts of the continuum Walrasian mechanism's increasing superiority over the continuum DR mechanism as the size of the environment set E grows.

Finally, we recapture in a new way the "winning" status of the Walrasian mechanism among all regular mechanisms that yield Pareto-optimal individually rational trade-vector n -tuples. If one seeks finite mechanisms for an exchange economy, if one confines attention to finite mechanisms that are approximations of regular continuum mechanisms, and if one wants the overall error of any sufficiently fine approximation to be as small as possible for a given cost, then one cannot do better than to approximate the Walrasian mechanism.

The remainder of this paper is organized in the following way. In Section II we present the concepts and definitions that we need, including the regularity requirement that we impose on the continuum mechanisms to be considered. The discussion includes an extensive remark (Remark R1) on conditions that imply regularity. In Section III we summarize our four general propositions A-D. They are obtained from our companion paper ([5]). Section IV describes our exchange economy with quasi-linear utilities and presents the continuum Walrasian mechanism for that economy. We show (following the program presented in Remark R1) that the mechanism obeys our regularity conditions, so that the four propositions of Section III can indeed be applied.

In Section V we apply the propositions of Section III and obtain the statements about approximations of the continuum Walrasian mechanism that were just sketched. Section VI offers brief concluding remarks.

In the Appendix of this paper we supply some proofs that were omitted (to save space) from our published companion paper ([5]), entitled "Comparing Finite Mechanisms".

II. CONTINUUM MECHANISMS AND THEIR FINITE APPROXIMATIONS: CONCEPTS AND DEFINITIONS

We suppose throughout that Person i 's environment set $E_i, i = 1, 2$ is a compact set in $\mathbb{R}^{J_i}$. Its typical element is

$$e_i = (e_{i1}, \dots, e_{iJ_i}).$$

We shall suppose throughout that E_i contains an open set in $\mathbb{R}^{J_i}$. We define

$$J \equiv \sum_{i=1}^n J_i.$$

We shall also suppose that the action space, denoted A , is a subset of $\mathbb{R}^\alpha$, where α is a positive real number.

A mechanism on $E = E_1 \times \cdots \times E_n$ with action space A is a triple

$$\Lambda = \langle M, (g_1, \dots, g_n), h \rangle,$$

where M is the *message space*; $g_1, \dots, g_n$ are the *individual agreement functions*; and h is the *outcome function*.

They have the following properties:

- For some D , with $2 \leq D \leq J$, M is a subset of $\mathbb{R}^D$. Its typical element is the message $m = (m_1, \dots, m_D)$.
- $g_i : M \times E_i \rightarrow \mathbb{R}^{D_i}, i = 1, \dots, n$, where $D_1 + \cdots + D_n = D$.
- For $i = 1, \dots, n$, the function g_i has components $g_{i1}, \dots, g_{iD_i}$, where $g_{ik} : M \times E_i \rightarrow \mathbb{R}, k = 1, \dots, D_i$.
- (“Coverage”) For every $e = (e_1, \dots, e_n) \in E$, there exists $m \in M$ such that $g_1(m, e_1) = g_2(m, e_2) = \cdots = g_n(m, e_n) = 0$.
- $h : M \rightarrow A$.

We shall sometimes use the statement⁴

$$g(m) = 0$$

⁴We may interpret the D_i -tuple $(g_{i1}, \dots, g_{iD_i})$ using an *iterative adjustment-process* scenario rather than the “agreement” (or “verification”) scenario of the Introduction. In the iterative adjustment-process scenario, there is a sequence of messages, each belonging to M . Person i responds to the t th message, denoted

$$m(t) = (m_1(t), \dots, m_D(t)),$$

by broadcasting to everyone else the message variables

$$m_{(\sum_{j=1}^{i-1} D_j)+1}(t+1) = f_{i1}(m(t), e_1), \dots, m_{(\sum_{j=1}^i D_j)}(t+1) = f_{iD_i}(m(t), e_i),$$

where

$$f_{i1} - m_{(\sum_{j=1}^{i-1} D_j)+1} = g_{i1}, \dots, f_{iD_i} - m_{(\sum_{j=1}^i D_j)} = g_{iD_i}.$$

Thus an equilibrium message for e for the functions $g_1, \dots, g_n$ is a *stationary* message for e in the iterative adjustment process defined by the functions $\{f_{ij}\}_{i=1, \dots, n, j=1, \dots, D_i}$.

as an abbreviation for the statement

$$g_i(m, e_i) = 0, i = 1, \dots, n.$$

If M is a continuum, then Λ is a *continuum mechanism*. If M is finite, then Λ is a *finite mechanism*.

Definition D1

We shall call the mechanism $\Lambda^* = \langle M^*, g_1^*, \dots, g_n^*, h^* \rangle$ on E with outcome space A a *Direct Revelation mechanism* if

- $M^* = E$ and each $m^* \in M^*$ is a J -tuple

$$\left(m_1^*, \dots, m_{J_1}^*, m_{(\sum_{k=1}^{i-1} J_k)+1}^*, \dots, m_{\sum_{k=1}^i J_k}^*, \dots, m_J^* \right)$$

- for $i = 1, \dots, n$, we have $g_i^* : M^* \times E_i \rightarrow \mathbb{R}^{J_i}$ and

$$g_i^*(m^*, e_i) = \left(m_{(\sum_{k=1}^{i-1} J_k)+1}^*, \dots, m_{(\sum_{k=1}^i J_k)}^* \right) - e_i.$$

Thus, for a DR mechanism, we have $D_i = J_i, i = 1, \dots, n$, and $D = J_1 + \dots + J_n = J$.

An important class of mechanisms has the property that a portion of each message is a proposed action, and the outcome function is a projection operator, which assigns that proposed action to the message.

Definition D2

We shall call Λ a *projection mechanism* if there is a set Y such that $M = A \times Y$ and, for every $m = (a, y)$, with $a \in A, y \in Y$, we have $h(m) = a$.

Definition D3

A message $m \in M$ is said to be an *equilibrium message* of Λ for the environment $e = (e_1, \dots, e_n) \in E$ if $g_i(m, e_i) = 0, i = 1, \dots, n$. An *outcome (action)* $a \in A$ is said to be an *equilibrium outcome (action)* of Λ for the environment $e \in E$ if $a = h(m)$ and m is an equilibrium message for e . The *equilibrium message set* for Λ is the set

$$\{m \in M : m \text{ is an equilibrium message for some } e \in E\}.$$

The *equilibrium outcome (action) set* for Λ is the set

$$\{a = h(m) : m \text{ is an equilibrium message for some } e \in E\}.$$

Definition D4

The mechanism Λ is said to *realize the goal function* $\gamma : E \rightarrow A$ if we have

$$a = \gamma(e)$$

whenever the following two statements hold:

- $a = h(m)$
- m is an equilibrium message for e .

Note that the goal function realized by a DR mechanism $\Lambda^* = \langle M^*, (g_1^*, \dots, g_n^*), h^* \rangle$ (where $M^* = E$) is identical to the outcome function h^* .

We next state regularity conditions that we shall henceforth impose on the continuum mechanism that we consider.

Definition D5

We shall call a continuum mechanism $\Lambda = \langle M, (g_1, \dots, g_n), h \rangle$ on the environment set E *regular* if it satisfies the following conditions:

- (a) M is compact.
- (b) For each $i \in \{1, \dots, n\}$ and each $k \in \{1, 2, \dots, D_i\}$, the function g_{ik} is $\mathcal{C}^2$.
- (c) There exists a real number $\bar{\delta} > 0$ and a $\mathcal{C}^2$ function

$$\bar{t} : E \times \underbrace{[-\bar{\delta}, \bar{\delta}] \times [-\bar{\delta}, \bar{\delta}] \times \dots \times [-\bar{\delta}, \bar{\delta}]}_{D \text{ times}} \rightarrow M,$$

called an *approximate solution function*, such that for all $e \in E$, and all $\delta_{11}, \dots, \delta_{1D_1}, \dots, \delta_{n1}, \dots, \delta_{nD_n}$ in $[-\bar{\delta}, \bar{\delta}]$, the message

$$m = \bar{t}(e, \delta_{11}, \dots, \delta_{1D_1}, \dots, \delta_{n1}, \dots, \delta_{nD_n})$$

is the only message in M satisfying the equation system

$$g_{i1}(m, e_i) = \delta_{i1}, \quad g_{i2}(m, e_i) = \delta_{i2}, \quad \dots, \quad g_{iD_i}(m, e_i) = \delta_{iD_i}, \quad i = 1, \dots, n.$$

In particular, for any $e \in E$, the only message in M satisfying $g(m, e) = 0$ is $m = t(e)$, where the function $t : E \rightarrow M$, is called the *thread* of the mechanism Λ , and is defined by:

$$t(e) \equiv \bar{t}(e, \underbrace{0, 0, \dots, 0}_{D \text{ times}}).$$

(d) The image set $t(E)$ includes an open set of $\mathbb{R}^D$.

Note that a regular mechanism realizes the goal function

$$\gamma = h \circ t$$

and no other goal function.

Note also that if Λ is a DR mechanism, then t is the identity function.

Remark R1 (concerning Regularity condition (c))

The regularity condition (c) appears to be a demanding one. In particular, while one might expect a mechanism to have a smooth thread t and a smooth approximate-solution function $\bar{t}$, the *uniqueness* of those functions may be difficult to guarantee.

Hence it is natural to ask whether, once we impose Regularity Conditions (a) and (b), we can conclude that (c) holds as well if the Jacobian

$$Q(m, e) \equiv \left(\left(\frac{\partial g_{ik_i}(m, e_i)}{\partial m_j} \right) \right)_{i=1, \dots, n; k_i=1, \dots, D_i; j=1, \dots, D_i}$$

has appropriate properties. In particular, one might require that for every e in the interior of the compact set E and every m in the interior of the compact set M , the matrix $Q(m, e)$ has nonzero determinant. That would imply that the agreement rules g_i have no redundancy.

It turns out that we can indeed obtain all of Condition (c), including uniqueness, if our mechanism has some further properties, including some restrictions on $Q(m, e)$. Those properties will be exhibited, in particular, by the exchange-economy mechanism that we construct in Section IV below. The further properties are as follows.

(1). The compact message space M contains *in its interior* a set $\bar{M}$ such that for every $e \in E$, there is a message $m_e \in \bar{M}$ which solves $g(m, e) = 0$. (Since the mechanism covers E we already know that M contains such a solution).

(2). The functions g_i obey the following *extension condition*:

There exist

- (i) open sets $\hat{E}_i \subset \mathbb{R}^{J_i}$ such that E_i is in the interior of $\hat{E}_i$, $i=1, \dots, n$;
- (ii) an open set $\hat{M} \subset \mathbb{R}^D$ such that M is in the interior of $\hat{M}$ and $\hat{M}$ is a *rectangular region* ⁵
- (iii) an extension of g , namely a $\mathcal{C}^2$ function $\hat{g} : \hat{M} \times \hat{E} \rightarrow \mathbb{R}^D$ (where $\hat{E} \equiv \hat{E}_1 \times \dots \times \hat{E}_n$) such that $\hat{g}(m, e) = g(m, e)$ for all $m \in M, e \in E$.

(3). For every $(m, e) \in \hat{M} \times \hat{E}$, the Jacobian

$$\hat{Q}(m, e^*) \equiv \left(\left(\frac{\partial \hat{g}_{ik_i}(m, e_i^*)}{\partial m_j} \right) \right)_{i=1, \dots, n; k_i=1, \dots, D_i; j=1, \dots, D_i}$$

has nonzero determinant.

(4). For every (m, e) in $\hat{M} \times \hat{E}$, the Jacobian $\hat{Q}(m, e)$ obeys a suitable “univalence” condition, selected from the work of Gale and Nikaido. Such a condition guarantees, for any e^* in E , that if m belongs to the open rectangular region $\hat{M}$ and solves an equation system for which $\hat{Q}(m, e^*)$ is the associated Jacobian, then $\hat{M}$ contains no other solution to that system.

Now property (1) tells us that for every e in E , the message m_e , which solves $g(m, e) = 0$, lies in the interior of M . Then properties (2) and (3), together with the Implicit Function Theorem, tell us that for every e^* in E , there is an open neighborhood $U(e^*) \subset \hat{E}$, a neighborhood $V(m_{e^*}) \subset M$, and a $\mathcal{C}^2$ function $\rho_{e^*} : U(e^*) \rightarrow M$, such that for all $e \in U(e^*)$:

- (i) the message $\rho_{e^*}(e) = m_{e^*}$ solves $g(m, e) = 0$
- (ii) $m_{e^*} \in V(m_{e^*})$
- (iii) for every $\bar{m} \neq m_{e^*}$ in $V(m_{e^*})$, we have $g(\bar{m}, e) \neq 0$.

⁵That means that $\hat{M} = \{m = (m_1, \dots, m_D) : \bar{m}_j < m_j < \bar{\bar{m}}_j, \text{ all } j \in \{1, \dots, D\}\}$, where $\bar{m}_j < \bar{\bar{m}}_j$.

Moreover, since m_{e^*} is in the interior of M and solves $g(m, e^*) = 0$, the Implicit Function Theorem has a broader implication.⁶ For $i = 1, \dots, n$, let $\vec{\delta}_i$ denote the vector $(\delta_{i1}, \dots, \delta_{iD_i})$. There is a positive number r_{e^*} , and a C^2 function

$$\bar{\rho}_{e^*} : U(e^*) \times \underbrace{[-r_{e^*}, r_{e^*}] \times \dots \times [-r_{e^*}, r_{e^*}]}_{D \text{ times}} \rightarrow M,$$

such that for all $\vec{\delta}_1, \dots, \vec{\delta}_n$ with components in $[-r_{e^*}, r_{e^*}]$, and all $e \in U(e^*)$, the message

$$m_{e^* \vec{\delta}_1, \dots, \vec{\delta}_n} = \bar{\rho}_{e^*}(e, \vec{\delta}_1, \dots, \vec{\delta}_n)$$

solves the equation system

$$(*) \quad g_{ik}(m, e_i) = \delta_{ik}, k = 1, \dots, D_i, i = 1, \dots, n,$$

and no other message in a neighborhood $\bar{V}\left(m_{e^* \vec{\delta}_1, \dots, \vec{\delta}_n}\right)$ of $m_{e^* \vec{\delta}_1, \dots, \vec{\delta}_n}$ solves that system. (Thus $\bar{\rho}_{e^*}(e, \underbrace{0, \dots, 0}_{D \text{ times}}) = \rho_{e^*}(e)$ and $\bar{V}(m_{e^* \underbrace{0, \dots, 0}_{D \text{ times}}}) = V(m_{e^*})$).

The Jacobian associated with the equation system (*), for $(m, e) \in M \times E$, is again $\hat{Q}(m, e)$. By (4), that Jacobian satisfies a Gale/Nikaido condition. Such a condition, together with the fact that $\hat{M}$ is an open rectangular region, assures us that $m_{e^* \vec{\delta}_1, \dots, \vec{\delta}_n}$ is the only solution to the equation system (*) to be found in all of $\hat{M}$ and, *a fortiori*, in all of M .

To summarize what we have so far obtained:

(+) for all $e^* \in \hat{E}$ and for all $\vec{\delta}_1, \dots, \vec{\delta}_n$ with components in $[-r_{e^*}, r_{e^*}]$, the compact set M contains just one solution to the equation system

$$\hat{g}_{ik}(m, e_i^*) = \delta_{ik}, k = 1, \dots, D_i, i = 1, \dots, n,$$

namely the message $m_{e^* \vec{\delta}_1, \dots, \vec{\delta}_n}$.

⁶To obtain the broader implication, one uses the facts that (1) for every $m \neq m_{e^*}$ in $V(m_{e^*})$, we have $g(m, e^*) \neq 0$, and (2) the set $g(V(m_{e^*})) \subset \mathbb{R}^D$ has dimension D (since $Q(m, e^*)$ has full rank for all $m \in M$).

Finally, we can construct the positive constant $\bar{\delta}$ of Regularity Condition (c). The procedure is as follows. The collection of open sets $\{U(e^*) : e^* \in \hat{E}\}$ covers the compact set E (including its boundary points). By the Heine-Borel theorem, that collection contains a *finite* sub-collection, say Σ , which also covers E . Now for each set $U(e^*)$ in Σ , consider the associated positive number r_{e^*} in (+). Take the smallest such positive number, over all the sets in the finite collection Σ . In view of (+), that smallest positive number meets the requirements of $\bar{\delta}$ in Condition (c).

Now define the function

$$\bar{t} : E \times \underbrace{[-\bar{\delta}, \bar{\delta}] \times \cdots \times [-\bar{\delta}, \bar{\delta}]}_{D \text{ times}} \rightarrow M,$$

required in Condition (c), by

$$\bar{t}(e^*, \bar{\delta}_1, \dots, \bar{\delta}_n) \equiv m_{e^*, \bar{\delta}_1, \dots, \bar{\delta}_n}.$$

Since the function $\bar{\rho}_{e^*}$ is a $\mathcal{C}^2$ function on $U(e^*) \times \underbrace{[-r_{e^*}, r_{e^*}] \times \cdots \times [-r_{e^*}, r_{e^*}]}_{D \text{ times}}$, it follows that $\bar{t}$ is also a $\mathcal{C}^2$ function, as Condition (c) requires.

We conclude that if our continuum mechanism obeys Regularity Conditions (a) and (b), and has properties (1) - (4), then it also obeys Condition (c).

The message space of a finite approximation to a continuum mechanism $\Lambda = \langle M, g_1, g_2, h \rangle$ is the intersection of the continuum $M \subset \mathbb{R}^D$ with an unbounded lattice of separated and evenly spaced points in $\mathbb{R}^D$. Each agreement function of the approximation is a “softening” of the corresponding agreement function in the continuum mechanism: it equals zero if and only if the absolute value of the continuum agreement function does not exceed a positive number called the *tolerance*.

Definition D6

(i) Given a continuum mechanism $\Lambda = \langle M, (g_1, \dots, g_n), h \rangle$ on E , we shall call the mechanism $\Lambda_{\eta\epsilon} = \langle M_\epsilon, (g_1^{\eta\epsilon}, \dots, g_n^{\eta\epsilon}), h^0 \rangle$, with $\epsilon > 0, \eta > 0$, the *finite exact-outcome approximation of Λ with message mesh ϵ and tolerance η* if

- $M_\epsilon = M \cap S_\epsilon^D$, where S_ϵ^D is called the ϵ -lattice and is defined as follows:

$$S_\epsilon^D \equiv \text{the } D\text{-fold Cartesian product of}$$

$$S_\epsilon \equiv \{\dots, -2(\ell+1)\epsilon, -2\ell\epsilon, \dots, -4\epsilon, -2\epsilon, 0, 2\epsilon, 4\epsilon, \dots, 2\ell\epsilon, 2(\ell+1)\epsilon, \dots\};$$

- for $i = 1, \dots, n$ we have $g_i^{\eta^\epsilon} : M_\epsilon \times E_i \rightarrow \mathbb{R}^{D_i}$;
- the function $g_i^{\eta^\epsilon}$ has components $g_{i1}^{\eta^\epsilon}, \dots, g_{iD_i}^{\eta^\epsilon}$, where, for $m \in M_\epsilon, e = (e_1, \dots, e_n) \in E, k \in \{1, 2, \dots, D_i\}$

$$g_{ik}^{\eta^\epsilon}(m, e_i) = \begin{cases} 0 & \text{if } |g_{ik}(m, e_i)| \leq \eta \\ 1 & \text{otherwise;} \end{cases}$$

- the coverage requirement is met, i.e., for every $e \in E$, there exists $m \in M_\epsilon$ such that $g_i^{\eta^\epsilon}(m, e_i) = 0, i = 1, \dots, n$;
- for $m \in M_\epsilon$ we have $h^0(m) = h(m)$.

(ii) A *finite rounded-outcome approximation* of Λ with message mesh ϵ , action mesh ν and tolerance η is a quadruple $\Lambda_{\eta^\epsilon}^\nu = \langle M_\epsilon, g_1^{\eta^\epsilon}, \dots, g_n^{\eta^\epsilon}, h^\nu \rangle$, where $M_\epsilon, g_1^{\eta^\epsilon}, \dots, g_n^{\eta^\epsilon}$ have the definitions just given, while the outcome function $h^\nu : M_\epsilon \rightarrow A$ is defined as follows:

$$h^\nu(m) = \begin{cases} \text{the element } a = (a_1, \dots, a_\alpha) \text{ of } A \cap S_\nu^\alpha \\ \text{which is closest to } h(m), \text{ where distance is }^7 \\ \text{measured by } \max_{r \in \{1, \dots, \alpha\}} (|h(m) - a_r|) \text{ and} \\ \text{ties are broken downward.}^6 \end{cases}$$

We now define an approximation's error.

Definition D7

Consider a continuum mechanism $\Lambda = \langle M, (g_1, \dots, g_n), h \rangle$ on the environment set $E = E_1 \times \dots \times E_n$ with action set $A \in \mathbb{R}^\alpha$. Consider an approximation of Λ whose mesh is ϵ and whose tolerance is η . Let the type of outcome function be unspecified: it is either of the exact type or of the rounded type with some action mesh ν . To avoid repetition, denote the mechanism $\mathcal{L} = \langle M_\epsilon, (g_1^{\epsilon\eta}, \dots, g_n^{\epsilon\eta}), h^\# \rangle$, where $h^\#$ stands for h^0 if the mechanism is of the exact-outcome type and stands for h^ν if the mechanism is of the rounded-outcome type with action mesh ν . The domain of h and of $h^\#$ is M_ϵ and the range of both functions is $\mathbb{R}^\alpha$. So h has α real-valued component functions, denoted $h_1, \dots, h_\alpha$ and $h^\#$ has α real-valued component functions, denoted $h_1^\#, \dots, h_\alpha^\#$.

For the environment $e = (e_1, \dots, e_n) \in E$, the error at e of $\mathcal{L}$ is defined by

$$\sup \left\{ |h_j^\#(\tilde{m}) - h_j(m)| : m \in M, \tilde{m} \in M_\epsilon; g_i(m, e_i) = 0, g_i^{\epsilon\eta}(\tilde{m}, e_i) = 0, i = 1, \dots, n; j \in \{1, \dots, \alpha\} \right\}.$$

⁷ That means that if the actions a', a'' , with $a' \neq a''$, are equally close to $h(m)$, then $h^\nu(m)$ equals the action whose k th component is lower, where the k th component is the first one in which a' and a'' differ.

The *overall error* of $\mathcal{L}$ is defined by

$$\sup_{e \in E} (\text{error at } e \text{ of } \mathcal{L}).$$

Notation for overall error: For any finite approximation of a continuum mechanism Λ , say the approximation $\mathcal{L}$, we shall let the symbol

$$\mathbb{E}(\mathcal{L})$$

denote the overall error of $\mathcal{L}$.

III. FOUR PROPOSITIONS ABOUT THE MESSAGE-SPACE DIMENSION OF A CONTINUUM MECHANISM AND THE ERROR OF ITS FINITE APPROXIMATIONS

We now present four general propositions that we shall be applying to the Walrasian mechanism in a class of exchange economies.

The first of our general Propositions considers a regular continuum mechanism $\Lambda^{(1)}$. The mechanism's message space has dimension $D(1)$, and⁸ Its action space is a subset of $\mathbb{R}^\alpha$ for some $\alpha \geq 1$. The Proposition also considers three other regular continuum mechanisms. Their action spaces are also subsets of $\mathbb{R}^\alpha$, but their message spaces have dimension higher than $D(1)$. The three other mechanisms are denoted $\Lambda^{(2)}$, $\Lambda^{(3)}$, and $\Lambda^{(4)}$. The mechanism $\Lambda^{(2)}$ has no further restrictions. The mechanism $\Lambda^{(3)}$ has the projection property and the mechanism $\Lambda^{(4)}$ is a DR mechanism.

The Proposition asserts the *superiority* of finite approximations of $\Lambda^{(1)}$ over finite approximations of the other three mechanisms: if the mesh used to approximate any of the three is sufficiently fine, then a not more costly approximation of $\Lambda^{(1)}$ has smaller overall error, where cost is measured by the number of equilibrium messages. The strongest superiority result concerns approximations of $\Lambda^{(1)}$ versus approximations of $\Lambda^{(3)}$ and $\Lambda^{(4)}$. We obtain the superiority of $\Lambda^{(1)}$ even if we “handicap” it by insisting that its approximations have rounded-outcome functions, while we “favor” the other two by permitting their approximations to have exact outcome functions. On the other hand rounded-outcome approximations of $\Lambda^{(1)}$ are superior to rounded-outcome approximations of the unrestricted mechanism $\Lambda^{(2)}$.

To state the Proposition formally, we first review several terms that were presented in the Introduction and we introduce several new terms.

⁸More accurately, the message space is a subset of $\mathbb{R}^{D(1)}$ and therefore, by regularity, it includes an open set of $\mathbb{R}^{D(1)}$. Recall that if the message space M of a regular continuum mechanism on E is a subset of $\mathbb{R}^D$, then it has a thread t such that the image set $t(E)$ contains an open set of $\mathbb{R}^D$ (regularity condition (d)). Since $t(E) \subseteq M$, the message space also contains an open set of $\mathbb{R}^D$.

We shall say that a finite approximation to a continuum mechanism is *nontrivial* if it has two or more equilibrium messages.

We view the cost of a finite approximation as the number of its equilibrium messages. If $\mathcal{L}$ is a finite approximation of a continuum mechanism Λ , the symbol $C(\mathcal{L})$, called *the cost of $\mathcal{L}$* , denotes the number of equilibrium messages in $\mathcal{L}$. Given two finite approximations, it will be convenient to call the first *cost-equivalent* to the second if the cost of the first does not exceed the cost of the second. (Thus “cost-equivalent” is a convenient shorthand for “not more costly than”).

Given a continuum mechanism Λ , we shall call a class $\mathcal{B}$ of finite approximations of Λ *complete*, if there exists $\epsilon^\# > 0$ such that for every ϵ with $0 < \epsilon \leq \epsilon^\#$, $\mathcal{B}$ contains at least one ϵ -mesh approximation of Λ .

Given two regular continuum mechanisms, a complete class of finite approximations to the first will be called *superior* to a complete class of finite approximations to the second if the following holds:

If the mesh of an approximation in the first class is sufficiently fine, then its overall error is less than the overall error of any cost-equivalent member of the second class.

We shall say that a complete class $\mathcal{S}$ of rounded-outcome approximation $\Lambda_\epsilon^{\nu(\epsilon)}$ is *controlled* if the set of action-mesh/message-mesh ratios

$$\left\{ \frac{\nu(\epsilon)}{\epsilon} : \Lambda_\epsilon^{\nu(\epsilon)} \in \mathcal{S} \right\}$$

is (i) bounded from above, and (ii) bounded from below by a positive number.

Finally we shall say that a goal function $\gamma : E \rightarrow A$, where $A \subseteq \mathbb{R}^\alpha$, is *well-behaved* if

- (i) γ is $\mathcal{C}^1$.
- (ii) $\gamma(E)$ contains an open set of $\mathbb{R}^\alpha$.

Our first proposition now follows. It is proved in our companion paper.⁹

Proposition A

For $i = 1, \dots, n$, let

$$E_i = \{e_i = (e_{i1}, \dots, e_{iJ_i}) : \bar{e}_{ij} \leq e_{ij} \leq \bar{\bar{e}}_{ij}, j = 1, \dots, J_i\},$$

⁹This proposition combines Propositions 1, 2, and 3 of our companion paper ([5]). Those are proved for two persons and a single action variable (and an arbitrary number of message variables), but, as remarked in [5], the proof is easily extended to many persons and many action variables.

where $\bar{e}_{ij} < \bar{\bar{e}}_{ij}, j = 1, \dots, J_i$.

Suppose $\Lambda^{(1)}, \Lambda^{(2)}, \Lambda^{(3)}, \Lambda^{(4)}$ are all regular continuum mechanisms on $E \equiv E_1 \times \dots \times E_n$. Suppose each realizes a well-behaved goal function whose range is $\mathbb{R}^\alpha$. Suppose:

- the message space of $\Lambda^{(1)}$ is a subset of $\mathbb{R}^{D(1)}$, where $D(1) < J_1 + \dots + J_n$.
- the message space of $\Lambda^{(2)}$ is a subset of $\mathbb{R}^{D(2)}$, where $D(2) > D(1)$.
- $\Lambda^{(3)}$ has the projection property and its message space is a subset of $\mathbb{R}^{D(3)}$ where $D(3) > D(1)$.
- $\Lambda^{(4)}$ is a DR mechanism (and hence its message space is E).

Then any complete class of nontrivial controlled rounded-outcome approximations of $\Lambda^{(1)}$ is superior to:

- any complete class of nontrivial controlled rounded-outcome approximations of $\Lambda^{(2)}$
- any complete class of nontrivial exact-outcome approximations of $\Lambda^{(3)}$
- any complete class of nontrivial exact-outcome approximations of $\Lambda^{(4)}$.

The next proposition concerns approximations of distinct continuum mechanisms whose message spaces are of *equal* dimension.

To state it, we shall need to consider two distinct complete classes of approximations, $\mathcal{C}$ and $\mathcal{D}$, such that the difference between the $\mathcal{C}$ -error and the $\mathcal{D}$ -error can be made arbitrarily close to zero by choosing the $\mathcal{C}$ -mesh sufficiently small and the $\mathcal{D}$ -mesh not more costly, and also by choosing the $\mathcal{D}$ -mesh sufficiently small and the $\mathcal{C}$ -mesh not more costly. More formally:

Given two regular continuum mechanisms, a complete class $\mathcal{C}$ of finite approximations to the first will be called *indifferent* to a complete class $\mathcal{D}$ of finite approximations to the second if the following holds:

Let Γ_ϵ denote the ϵ -mesh approximation in $\mathcal{C}$ and let Δ_ϵ denote the ϵ -mesh approximation in $\mathcal{D}$. Then there exist $V > 0, \epsilon^\# > 0$ such that for any ϵ with $0 < \epsilon \leq \epsilon^\#$:

- (i) $\mathcal{D}$ contains an approximation Γ' , such that Γ' is cost-equivalent to Γ_ϵ and

$$|\mathbb{E}(\Gamma_\epsilon) - \mathbb{E}(\Gamma')| \leq V \cdot \epsilon.$$

- (ii) $\mathcal{C}$ contains an approximation Δ' , such that Δ' is cost-equivalent to Δ_ϵ and

$$|\mathbb{E}(\Delta_\epsilon) - \mathbb{E}(\Delta')| \leq V \cdot \epsilon.$$

We can now state the next proposition.

Proposition B

Suppose that Λ and Λ' are regular continuum mechanism on the environment set E defined in Proposition A. Suppose that the message space of each is a subset of $\mathbb{R}^D$. Suppose that each realizes a well-behaved goal function whose range is $\mathbb{R}^\alpha$. Then

- any complete class of nontrivial controlled rounded-outcome approximations of Λ is indifferent to any complete class of nontrivial controlled rounded-outcome approximations of Λ'
- any complete class of nontrivial exact-outcome approximations of Λ is indifferent to any complete class of nontrivial exact-outcome approximations of Λ'

The proof of Proposition B is obtained by modifying certain steps in the proofs of our companion paper.¹⁰

We now turn to two final propositions, which concern the informational performance of DR mechanisms relative to indirect mechanisms when the environment set grows.

The informational deficiency of continuum DR mechanisms, as compared to continuum “indirect” mechanisms that realize the same goal function, reveals itself most strikingly when we vary the dimension of the environment set E . That is illustrated by the convex exchange economy with n traders and a fixed number of commodities, which we will study in Part IV below. The environment set E is the set of possible individual characteristics (endowments and utility-function parameters). Each environment e in E defines a convex exchange economy. We are interested in a goal function which assigns to every e in E a trade vector that is Pareto optimal and individually rational for the economy described by e . We can realize the goal function by a continuum DR mechanism, whose message space is E itself, and we can also realize it by an indirect continuum mechanism, namely the Walrasian mechanism.

As the dimension of E rises, so does the message-space dimension of the associated DR mechanism (E itself). But no matter how large the dimension of E , the Walrasian mechanism retains the same message-space dimension, since that dimension depends only on the number of traders and commodities, and those are fixed.

Thus the “informational differential” (dimensional difference) between the DR and the indirect mechanisms goes to infinity as the environment set’s dimension grows without bound. That remains

¹⁰In the proof of Lemma 1 of that paper, one modifies the chain of inequalities following statement (i) of Step 3, by letting $D = D^*$ and then dividing the chain into two parts. The first part provides a lower bound to the first overall error appearing in the chain and the second part provides an upper bound to the second overall error. One then considers the absolute value of the difference between the two bounds. The absolute value is bounded from above by a constant times the mesh. That fact can then be used to obtain a version of Lemma 1 which deals with the case $D^* = D$ and becomes part of the proof of Proposition B.

the case if, for all sets E , the endowments stay the same, or satisfy the same bounds, so that only the dimension of the set of possible utility-function parameters grows without bound.

Generalizing from the exchange-economy example, we have the following statement about continuum mechanisms:

- (§) For each environment set E in a family of environment sets, consider a continuum DR mechanism on E and a continuum indirect mechanism on E , where both realize the same goal function, with α real-valued components, and the indirect mechanism has a message space that is a subset of $\mathbb{R}^D$ for every E . As we increase without bound the dimension of the environment set E , we also increase without bound the “informational differential” (difference in message-space dimension) between the two mechanisms. That is true, in particular, if (a) the indirect mechanism’s message space is not all of $\mathbb{R}^D$ but is contained, for all E , in the same closed (nondegenerate) “message cube” in $\mathbb{R}^D$, or (b) for both mechanisms, the equilibrium actions are contained, for all E , in the same closed (nondegenerate) “action cube” in $\mathbb{R}^\alpha$.

We ask whether there is a finite analogue of (§). In Proposition A, we found that higher message-space dimension for continuum mechanisms implied higher overall error for cost-equivalent finite approximations. It is therefore natural to consider first a finite analogue of (§) which says

- (§§) If the environment set’s dimension grows without bound, then so does the DR-versus-indirect “error differential”, i.e., the difference in overall error between an approximation of the indirect continuum mechanism and a cost-equivalent exact-outcome approximation of the corresponding continuum DR mechanism.

But it is immediately clear that (§§) cannot hold if the possible values of the goal function that both continuum mechanisms realize, or (equivalently) the possible values of an equilibrium action of either continuum mechanism, are indeed contained in the same “action cube” for all environment sets. For then the overall error of an approximation of either mechanism cannot exceed the difference between the upper and lower action bounds defined by the cube and so the overall error cannot go to infinity as the dimension of the environment set increases without bound.

In the Walrasian example, the messages are price/trade pairs and the outcome function is a projection operator which selects the trade portion of each message. If the messages are contained in the same closed message cube for all environment sets E , then the possible trades are bounded from above and from below by bounds which remain the same for all E , and hence the equilibrium actions (trades) are contained in an unchanging action cube.¹¹ (A special case is that in which endowments

¹¹More generally, if the indirect mechanism has, for every E , the projection property (as the Walrasian mechanism does), then the existence of an unchanging message cube implies the existence of an unchanging action cube.

never change). So neither the overall error of approximations of the Walrasian mechanism, nor the overall error of approximations of DR mechanisms that realize the same goal function as the Walrasian mechanism, can go to infinity as we vary E .

Thus (§§), our first attempt at a finite analogue of (§), fails to hold when there is an unchanging action cube for all environment sets.

Since (§§) fails when there is an unchanging action cube, it is natural to make a second attempt at a finite analogue of (§) by investigating a “dual” of (§§). We require, in the dual, that the DR approximation and the indirect approximation be “error-equivalent”, i.e., the overall errors are the same or the former has lower overall error. We study the “DR-versus-indirect cost differential”, i.e., the difference between the DR approximation’s *cost* (number of equilibrium messages) and the indirect approximation’s cost, as we vary the environment set’s dimension. Our next Proposition,¹² Proposition C, formally establishes the following informally stated “dual” finite analogue of (§).

(§§§) As the environment sets’ dimension grows without bound, the DR-versus-indirect cost differential also grows without bound. It does so, in particular, if the continuum indirect mechanism has a message space that is contained, for every environment set E , in the same closed message cube in $\mathbb{R}^D$ and, for both mechanisms, the equilibrium actions are contained, for all E , in the same closed action cube in $\mathbb{R}^\alpha$.

In Proposition D we return to the viewpoint of Proposition A. We again impose cost-equivalence and examine the DR-versus-indirect error differential. If actions lie in an unchanging cube, then that differential cannot go to infinity as the environment set grows without bound. But Proposition D shows that it is nevertheless the case (under several further conditions) that the differential is bounded from below by a positive number and that the DR approximation’s overall error goes to a limit.

To state Propositions C and D, we consider a *sequence* of environment sets, $\{E^v : v = 1, 2, \dots\}$. The environment set E^v has a larger dimension than E^{v-1} and the dimension goes to infinity as v increases without bound. For each v , we consider an indirect continuum mechanism on E^v , denoted Λ^v , and a DR continuum mechanism on E^v , denoted Λ^{*v} . For all v , both mechanisms have actions in the same Euclidean space $\mathbb{R}^\alpha$. In particular, it may be the case that for all v , the mechanisms Λ^v and Λ^{*v} realize the same goal function, but (as in Proposition A) that will not be needed for the Propositions we obtain. If the action sets lie in a common Euclidean space $\mathbb{R}^\alpha$, then that is enough to allow us to compare the overall error of an approximation of the indirect mechanism relative to the continuum indirect mechanism itself with the overall error of an approximation of the DR mechanism relative to the continuum DR mechanism itself.

We now impose conditions on the environment-set sequence $\{E^v = E_1^v \times \dots \times E_n^v : v = 1, 2, \dots\}$, and on the continuum-mechanism sequences $\{\Lambda^v\}, \{\Lambda^{*v}\}$.

¹²Proposition C, which follows, corresponds to Part B of Proposition 4 in our companion paper [5]. Proposition D is stated as Proposition 5 in [5].

(i) For $i = 1, \dots, n, E_i^v = \{e_i = (e_{i1}, \dots, e_{iK_i(v)}) : \bar{e}_{ij}^v \leq e_{ij} < \bar{\bar{e}}_{ij}^v, j = 1, \dots, K_i(v)\}, v \geq 1$

(ii) For $i = 1, \dots, n, \bar{e}_{ij}^v, \bar{\bar{e}}_{ij}^v \in \mathbb{R}$ and $\bar{e}_{ij}^v < \bar{\bar{e}}_{ij}^v, j = 1, \dots, K_i(v), v \geq 1$.

(iii) $K_1(v+1) + K_2(v+1) + \dots + K_n(v+1) > K_1(v) + K_2(v) + \dots + K_n(v), v \geq 1$.

Thus E^v has dimension

$$J^v = K_1(v) + K_2(v) + \dots + K_n(v),$$

where

$$J^{v+1} > J^v,$$

and hence

(iv) $\lim_{v \rightarrow \infty} J^v = \infty$.

In Proposition C we shall make the following two further assumptions on the environment-set sequence.

(v) there exist $W, X \in \mathbb{R}$, with $W < X$, such that $\bar{e}_{ij}^v \leq W, X \leq \bar{\bar{e}}_{ij}^v$,

$$i = 1, \dots, n, j = 1, \dots, K_i(v), v \geq 1.$$

(vi) there exist $Y, Z \in \mathbb{R}$, with $Y < Z$, such that $Y \leq \bar{e}_{ij}^v, \bar{\bar{e}}_{ij}^v \leq Z$,

$$i = 1, \dots, n, j = 1, \dots, K_i(v), v \geq 1.$$

Next we consider a sequence of indirect regular mechanisms $\{\Lambda^v = \langle M^v, (g_1^v, \dots, g_n^v), h^v \rangle\}$, where

(vii) for $v \geq 1, \Lambda^v$ is a mechanism on E^v and its action space is a subset of $\mathbb{R}^\alpha, v \geq 1$,

(viii) for $v \geq 1, M^v$ is a subset of $\mathbb{R}^D$,

and

$$(ix) \quad \begin{cases} \text{there exist } a, b, r, s \in \mathbb{R}, \text{ with } a < b, r < s, \text{ such that for} \\ M = \underbrace{[a, b] \times \cdots \times [a, b]}_{D \text{ times}}, M' = \underbrace{[r, s] \times \cdots \times [r, s]}_{D \text{ times}}, \text{ and for all } v \geq 1, \text{ we have} \\ M' \subseteq M^v \subseteq M \subset \mathbb{R}^D. \end{cases}$$

Condition (viii) and the regularity of Λ^v imply that M^v includes an open set of $\mathbb{R}^D$. Thus every mechanism in the indirect sequence has a message space of dimension D and (by condition (ix)) all of the message spaces are subsets of the same D -dimensional cube M and all of them include the D -dimensional cube M' .

Finally we consider a sequence of regular continuum DR mechanisms $\{\Lambda^{*v} = \langle M^{*v}, (g_1^{*v}, \dots, g_n^{*v}), h^{*v} \rangle\}$ where

$$(x) \quad \Lambda^{*v} \text{ is a DR mechanism on } E^v, \text{ i.e. } M^{*v} = E^v, v \geq 1,$$

and

$$(xi) \quad \text{for } v \geq 1, \text{ the action space of } \Lambda^{*v} \text{ is a subset of } \mathbb{R}^\alpha.$$

Note that (vi) implies that for all $v \geq 1$ there is a “universal” upper bound to the mesh of an approximation of Λ^{*v} , namely $\frac{Z-Y}{2}$. For any mesh larger than that, the set E^v contains no lattice points.

In Proposition C, we make the following assumption (xii) concerning exact-outcome approximations of the DR mechanisms Λ^{*v} . It says that if we fix the mesh, then for such approximations there is a uniform lower bound, valid for all v , to the ratio of overall error to mesh, provided the fixed mesh is sufficiently small. One can show that this condition follows if there is a uniform lower bound, valid for all v , to the sensitivity of the goal function realized by Λ^{*v} to some environment component, where sensitivity is measured by the absolute value of the derivative of the goal function with respect to that component.¹³ Our assumption is:

$$(xii) \quad \text{there exists } \tau^* > 0 \text{ such that for all } v \geq 1 \text{ and } 0 < \tilde{\epsilon} \leq \frac{X-W}{2}, \quad \mathbb{E}(\Lambda_{\tilde{\epsilon}}^{*v}) \geq \tau^* \cdot \tilde{\epsilon}.$$

Finally, Propositions C and D use a condition on the derivatives associated with the indirect-mechanism sequence and the DR sequence $\{\Lambda^v\}$. Since we require this property for both the

¹³The required argument is suggested by an argument found in the proof of Part 2 of Lemma 3 of [5]. (The proof is found in the Appendix of the present paper). Denote the α real-valued components of the outcome function h^{*v} as $h_1^{*v}, \dots, h_\alpha^{*v}$. To insure ((xii)), it suffices to assume that there exists $\tau > 0$ such that for every $v \geq 1$, there is some triple (i, j, k) , with $i \in \{1, \dots, n\}, j \in \{1, \dots, K_i(v)\}, k \in \{1, \dots, \alpha\}$ such that for all $e \in E^v$, the absolute value of $\frac{\partial h_k^{*v}}{\partial e_{ij}}$ is not less than τ .

indirect sequence and the DR sequence.¹⁴, we state it for an arbitrary sequence, $\{\bar{\Lambda}^v\}$, where $\bar{\Lambda}^v = \langle \bar{M}^v, (\bar{g}_1^v, \dots, \bar{g}_n^v), \bar{h}^v \rangle$ is a regular continuum mechanism on E^v with actions in $\mathbb{R}^\alpha$. We shall say that the sequence $\{\bar{\Lambda}^v\}$ has *uniformly bounded derivatives* if there exist $\bar{\delta} > 0, \xi > 0, \rho > 0, \lambda > 0$ such that for every $v \geq 1$ the following holds:

- The mechanism Λ^v satisfies Regularity Condition (c), concerning the approximate-solution function of Λ^v , which we denote $\bar{t}^v = (\bar{t}_{11}^v, \dots, \bar{t}_{1, K_1(v)}^v, \dots, \bar{t}_{n1}^v, \dots, \bar{t}_{n, K_n(v)}^v)$, where the domain of each $\bar{t}_{ij}^v$ is $[-\bar{\delta}, \bar{\delta}]$.
- For each e in E^v , and each $(\delta_{11}, \dots, \delta_{1, K_1(v)}, \dots, \delta_{n1}, \dots, \delta_{n, K_n(v)})$ with every component in $[-\bar{\delta}, \bar{\delta}]$, the derivative of each $\bar{t}_{ij}^v$ with respect to each component of e does not exceed ρ in absolute value.
- For each (m, e) in $M^v \times E^v$ the derivative of each real-valued component of each agreement function g_i^v with respect to each component of m does not exceed ξ in absolute value.
- For each m in M^v , the derivative of h^v with respect to each component of m does not exceed λ in absolute value.

We shall consider exact-outcome approximations for both of the continuum-mechanism sequences.¹⁵

We now state Proposition C.

Proposition C

Let the environment sequence $\{E^v\}$ satisfy (i)–(vi). Let the continuum indirect-mechanism sequence $\{\Lambda^v\}$ satisfy (vii)–(ix). Let the continuum DR sequence $\{\Lambda^{*v}\}$ satisfy (x)–(xii). Let both sequences have uniformly bounded derivatives. Then there exists $\hat{\epsilon} > 0$ and a function $\bar{r}$ from $\mathbb{R}^+ \times I^+$ to $\mathbb{R}^+$, such that

¹⁴To interpret the following four parts of the “uniform boundedness of derivatives” condition when we apply them to each continuum DR mechanism Λ^{*v} , recall that for that mechanism and for $i \in \{1, \dots, n\}, j \in \{1, \dots, K_i(v)\}$, the thread is the identity function and for the typical real-valued component of the approximate-solution function $\bar{t}^v$, we have $\bar{t}_{ij}^v(e, \delta_{11}, \dots, \delta_{1, K_1(v)}, \dots, \delta_{n1}, \dots, \delta_{n, K_n(v)}) = e_{ij} - \delta_{ij}$. The derivative of the latter expression with respect to any component of e is either zero or one. Every component of each agreement function is the difference between a message component and an environment component; hence its derivative with respect to any message component is either zero or one. Thus the first three parts of the uniform-boundedness-of-derivatives condition are automatically satisfied. The fourth part requires that there exist $\lambda > 0$ such that for every v , the absolute value of the derivative of every component of the outcome function with respect to any of its arguments does not exceed λ , or (equivalently) the absolute value of the derivative of every component of the goal function realized by Λ^{*v} with respect to any environment component does not exceed λ in absolute value.

¹⁵Uniform boundedness of derivatives guarantees the existence of a number $\bar{G} > 0$ such that for all $v \geq 1$, and for any sufficiently small mesh ϵ , we have $\mathbb{CE}(\bar{\Lambda}_\epsilon^v) \leq \bar{G} \cdot \epsilon$. To show that, one applies an argument found in the proof of Lemma 2 of [5].

(†). for $0 < \epsilon < \hat{\epsilon}$, the approximations $\Lambda_\epsilon^v, \Lambda_{\bar{r}(\epsilon,v)}^{*v}$ exist for all v

and

(††) for $0 < \epsilon < \hat{\epsilon}$, $\mathfrak{C}(\Lambda_{\bar{r}(\epsilon,v)}^{*v}) \leq \mathfrak{C}(\Lambda_\epsilon^v)$.

(The DR approximation is error-equivalent to the indirect approximation). Moreover, for any $\hat{\epsilon}, \bar{r}$ satisfying (†), (††), we have

(†††) there exists $\epsilon^* \leq \hat{\epsilon}$ such that for all ϵ with $0 < \epsilon < \hat{\epsilon}$, $\lim_{v \rightarrow \infty} (C(\Lambda_{\bar{r}(\epsilon,v)}^{*v}) - C(\Lambda_\epsilon^v)) = \infty$.

(The DR-versus-indirect cost differential goes to infinity as v goes to infinity).

To introduce the further assumptions that we shall make on the DR approximations for Proposition D, note first that the approximation $\Lambda_{\tilde{\epsilon}}^{*v}$ has message space $E^{J^v} \cap S_{\tilde{\epsilon}}^{J^v}$. It follows from (i), (ii) that for some mesh $\tilde{\epsilon}$ we have¹⁶

$$E^{J^v} \cap S_{\tilde{\epsilon}}^{J^v} \neq \emptyset.$$

Thus a DR approximation $\Lambda_{\tilde{\epsilon}}^{*v}$ (with at least one message) always exists for an appropriately chosen mesh $\tilde{\epsilon}$.

In Proposition D we shall assume that there is some fixed integer $\Psi > 0$ such that for all $v \geq 1$, a DR approximation costing not more than Ψ can be constructed. That is to say, we shall assume

There exists an integer $\Psi > 0$ such that for all $v \geq 1$ there is some mesh, say $\epsilon(v)$, for which

$$(+) \quad 1 \leq \#(E^{J^v} \cap S_{\epsilon(v)}^{J^v}) \leq \Psi.$$

Condition (+) holds, for example, if for all $v \geq 1$, for $i = 1, \dots, n$, and for $j = 1, \dots, K_i(v)$ we have $\bar{e}_{ij}^v = 0$, and for some $T > 0$ we have

$$\text{for all } v \geq 1, \quad \max_{i=1, \dots, n; j=1, \dots, K_i(v)} \bar{e}_{ij}^v = T.$$

¹⁶Choose a rational J^v -tuple in E^v , say $\tilde{\epsilon} = \left(\frac{a_1}{b_1}, \dots, \frac{a_{J^v}}{b_{J^v}}\right)$, where a_k, b_k are integers, $k = 1, \dots, J^v$. Then for the mesh $\tilde{\epsilon} = \frac{1}{2b_1 \cdot b_2 \cdot \dots \cdot b_{J^v}}$, the point $\tilde{\epsilon}$ is a lattice point, since each of its coordinates is an integer multiple of $2\tilde{\epsilon}$.

For then, when we choose a mesh $\tilde{\epsilon} > T/2$, the *only* lattice point in E^{J^v} is, for all v , the origin in $\mathbb{R}^{J^v}$, and hence for that mesh we have a *one-message DR approximation*.

For Proposition D we shall need to consider meshes $\tilde{\epsilon}$ such that the approximation $\Lambda_{\tilde{\epsilon}}^{*v}$ is *sparse*, defined as follows:

the approximation $\Lambda_{\tilde{\epsilon}}^{*v}$ is *sparse* if its message space has fewer than 2^{J^v} elements.

A particular case of a sparse DR exact-outcome approximation is a one-message approximation. A one-message exact-outcome approximation of Λ^{*v} obtains the constant action $h^{*v}(\hat{e})$ for all environments in E^v , where $\hat{e}$ is the approximation's solitary message. It is easily verified that if E^v is a cube in $\mathbb{R}^{J^v}$ with edges of equal length and one corner of the form $\underbrace{\bar{U}, \dots, \bar{U}}_{J^v \text{ times}}$, then the *only* sparse DR approximation is a one-message approximation.¹⁷

Proposition D considers a continuum DR-mechanism sequence $\{\Lambda^{*v}\}$ with the property that *its sparse approximations are monotone*. That will mean that if $\{\Lambda_{f(v)}^{*v}\}$ is any sequence of sparse approximations such that for some integer $B > 0$ we have

$$\text{for all } v, C(\Lambda_{f(v)}^{*v}) \leq B < 2^{J^v},$$

then

$$v'' > v' \Rightarrow \mathbb{E}(\Lambda_{f(v'')}^{*v''}) \geq \mathbb{E}(\Lambda_{f(v')}^{*v'}).$$

Informally, we consider, for every v in our sequence, a sparse DR approximation that does not “cost” more than some fixed B . As v increases, the overall error of the sparse approximation does not get smaller.

We are now ready to state Proposition D. Its proof uses the fact that if we are given a bound on cost (number of messages), and if the environment set has sufficiently high dimension, then only sparse DR approximations can satisfy that bound.

¹⁷Without loss of generality, let every edge of our cube E^v be the interval $[\bar{U}, \bar{\bar{U}}]$. Consider any mesh $\tilde{\epsilon}$. A point in the cube E^v belongs to the $\tilde{\epsilon}$ -lattice

if and only if each of its J^v real components belongs to $[\bar{U}, \bar{\bar{U}}]$ and is an integer multiple of $2\tilde{\epsilon}$.

If E^v contains two distinct points of the mesh —differing in at least one coordinate— then there are at least two numbers that are integer multiples of $2\tilde{\epsilon}$. We obtain a $\tilde{\epsilon}$ -lattice point belonging to E^v if we let each of the J^v components equal *one or the other* of those two numbers. So if there are two or more $\tilde{\epsilon}$ -lattice points in E^v , then there at least 2^{J^v} of them. Since sparseness requires that there be less than 2^{J^v} such points, a sparse approximation can have only one.

Proposition D

Suppose that the environment sequence $\{E^v\}$, the indirect-mechanism sequence $\{\Lambda^v\}$, and the DR sequence $\{\Lambda^{*v}\}$ satisfy (i) – (iv) and (vii) – (xi). Suppose in addition that the DR sequence $\{\Lambda^{*v}\}$ has the property (+).

Part I

For all sufficiently small ϵ , the following statements I-1 and I-2 hold.

I-1 We have

$$M^v \cap S_\epsilon^D \neq \emptyset \text{ for all } v \geq 1,$$

(and hence the minimal-tolerance exact-outcome approximation Λ_ϵ^v exists for all $v \geq 1$).

I-2 For all $v \geq 1$, there exists a number $r(\epsilon, v) > 0$ such that for all $v \geq 1$

$$E^{J^v} \cap S_{r(\epsilon, v)}^{J^v} \neq \emptyset \text{ (and hence the exact-outcome approximation } \Lambda_{r(\epsilon, v)}^{*v} \text{ exists)}$$

$$\Lambda_{r(\epsilon, v)}^{*v} \text{ is cost-equivalent to } \Lambda_\epsilon^v, \text{ i.e., } C(\Lambda_{r(\epsilon, v)}^{*v}) \leq C(\Lambda_\epsilon^v).$$

Part II Suppose that there exist numbers T, U such that for all v , all $e \in E^v$, and every component $h_k^v : E^v \rightarrow \mathbb{R}$ of the outcome function of Λ^{*v} , we have

$$(\dagger) \quad T \leq h_k^v(e) \leq U.$$

Suppose that the sequence $\{\Lambda^v\}$ of indirect mechanisms has uniformly bounded derivatives, and that the DR sequence's sparse approximations are monotone. Then there exists $H > 0$ such that for all sufficiently small ϵ and for the function r of I-2

$$\lim_{v \rightarrow \infty} \mathbb{E}(\Lambda_{r(\epsilon, v)}^{*v}) = H,$$

and there exists $\hat{v} > 0, L > 0$ such that for any sufficiently small ϵ , we have

$$\text{for all } v \geq \hat{v}, \left(\mathbb{E}(\Lambda_{r(\epsilon, v)}^{*v}) - \mathbb{E}(\Lambda_\epsilon^v) \right) \geq L.$$

IV. THE CONTINUUM WALRASIAN MECHANISM IN AN EXCHANGE ECONOMY WITH QUASI-LINEAR UTILITIES: REGULARITY AND GOAL REALIZATION

The agenda of this section. In this section we consider an exchange economy with n traders, a numeraire commodity, and L other commodities, where n and L are arbitrary. An environment e will identify the n traders' utility functions. We shall construct a version of the continuum Walrasian mechanism, whose typical message is a point in $\mathbb{R}^{nL}$. The message consists of L prices and an L -component trade vector for each of $n - 1$ traders, say Traders $1, \dots, n - 1$. For convenient reference we call this the *incomplete* Walrasian mechanism. In this mechanism trader n 's trade vector is not an explicit part of the typical message, but is understood (by Trader n) to equal the negative of the sum of the other $n - 1$ trader's trade vectors.

It will turn out to be useful to define as well the *complete* Walrasian mechanism, whose typical message has $L(n + 1)$ real components. The message explicitly describes a trade vector for each of the n traders, and those trade vectors sum to zero if the message is an equilibrium message. For every environment e , an equilibrium message of one mechanism uniquely identifies an equilibrium message of the other.

For every e , the incomplete mechanism's equilibrium outcome will be an individually rational and Pareto-optimal vector of trades.¹⁸

Our main task in this section is to show that under appropriate assumptions on the set of environments and on the utility functions, the incomplete Walrasian continuum mechanism meets our regularity conditions. That will allow us to apply Propositions A-C, and to study finite approximations of the Walrasian mechanism as well as finite approximations of other regular continuum mechanisms that also realize Pareto optimality and individual rationality. Those other mechanisms will include a DR mechanism.

While regularity conditions (a),(b),(d) are quickly established for the incomplete mechanism (under the assumptions that we make about the exchange economy), that turns out not to be true for condition (c).

It turns out, in fact, that a straightforward way of establishing condition (c) for the incomplete mechanism is to establish it first for the complete mechanism. We do so by following, for the complete mechanism, the program given in Remark R1. We shall show that the complete mechanism indeed has the four properties presented in that Remark. Recall, in particular, that the fourth of those properties requires the Jacobian associated with the agreement functions to satisfy a suitable Gale/Nikaido condition. The task of verifying such a condition occupies a major part of the argument. Once the four properties are established, Regularity Condition (c) will follow for the

¹⁸The same is true of the complete mechanism.

complete mechanism. It is then easily argued that this implies satisfaction of Regularity Condition (c) by the incomplete mechanism as well.

As for regularity condition (d), the complete mechanism *fails* to satisfy it ¹⁹, but the incomplete mechanism does satisfy it.

We conclude that the incomplete Walrasian continuum mechanism indeed satisfies all four regularity conditions. Hence we may proceed to study its finite approximations by applying Propositions A-C.

The exchange economy to be studied. We consider an economy with traders $1, \dots, i, \dots, n$ and commodities $1, \dots, \ell, \dots, L, L + 1$. Commodity $L + 1$ will be a numeraire.

In the course of the discussion, we shall occasionally make a temporary departure from the n -trader $(L + 1)$ -commodity case in order to visit a special illustration: a two-commodity economy with two traders whose utility function is linear-quadratic.

Trader i 's initial endowment of each of the $L + 1$ commodities is positive and fixed. For commodity $\ell, \ell \in \{1, \dots, L\}$, i 's endowment is $W_\ell^i > 0$; for commodity $L + 1$ it is $V^i > 0$. Let $v^i, w_1^i, \dots, w_L^i \in \mathbb{R}$ denote i 's net trades (additions to endowment or subtractions from endowment) for the numeraire commodity and the L other commodities, respectively. We shall use the notation

$$w^i \equiv (w_1^i, \dots, w_L^i).$$

Trader i has a utility function defined on the commodity bundles that i would hold if the trade (v^i, w^i) lies in the open set

$$(IV-1) \quad \hat{S}_i \equiv \{(v^i, w^i) : -V^i < v^i < \sum_{j \neq i} V^j + \gamma; \text{ for all } \ell \in \{1, \dots, L\}, -W_\ell^i < w_\ell^i < \sum_{j \neq i} W_\ell^j + \gamma\},$$

where $\gamma > 0$.

We shall also be considering a subset of $\hat{S}_i$, namely the closed set

$$(IV-2) \quad S_i \equiv \{(v^i, w^i) : -V^i + \sigma \leq v^i \leq \sum_{j \neq i} V^j; \text{ for all } \ell \in \{1, \dots, L\}, -W_\ell^i + \sigma \leq w_\ell^i \leq \sum_{j \neq i} W_\ell^j\},$$

where σ is a real number satisfying

$$(IV-3) \quad 0 < \sigma < \min(V^i, W_\ell^i), \text{ for all } i \in \{1, \dots, n\}, \ell \in \{1, \dots, L\}.$$

¹⁹The message space of the complete mechanism is a subset of $\mathbb{R}^{L(n+1)}$. Condition (d) requires that for the mechanism's unique thread (exact-solution function), denoted t^c , the image set $t^c(E)$ includes an open set of $\mathbb{R}^{L(n+1)}$. But that is not the case, since at any equilibrium message, say $t^c(e)$, the n trade vectors sum to zero.

Thus for every trade vector in S_i , trader i 's post-trade holding of every commodity is at least $\sigma > 0$ but is less than the upper bound given in the definition of $\hat{S}_i$ in (IV.1).

It will be convenient to define the nonnumeraire projections

$$S_i^\# \equiv \{(w_1^i, \dots, w_L^i) : \text{for some } v^i \text{ we have } (v^i, w_1^i, \dots, w_L^i) \in S_i\}$$

$$\hat{S}_i^\# \equiv \{(w_1^i, \dots, w_L^i) : \text{for some } v^i \text{ we have } (v^i, w_1^i, \dots, w_L^i) \in \hat{S}_i\}.$$

Finally, we fix a real number Δ such that

$$(IV-4) \quad 0 < \Delta < \min(\sigma, \gamma)$$

and we define the following closed set of nonnumeraire ²⁰trade vectors for trader i :

$$(IV-5) \quad S_{i\Delta}^\# \equiv \{w^i : -W_\ell^i + \Delta \leq w_\ell^i \leq \sum_{j \neq i} W_\ell^j + \Delta, \ell = 1, \dots, L\}.$$

Note that since $0 < \Delta < \sigma$, the most that i gives up of nonnumeraire commodity ℓ , among the trade vectors in $S_i^\#$, is his endowment minus σ . But the most that i gives up among the trade vectors in $S_{i\Delta}^\#$ is a larger quantity, namely his endowment minus Δ . We have

$$(IV-6) \quad S_i^\# \text{ is in the interior of } S_{i\Delta}^\#.$$

Moreover, since $\Delta < \gamma$

$$(IV-7) \quad S_{i\Delta}^\# \text{ is in the interior of } \hat{S}_i^\#.$$

Since i 's endowments are assumed fixed, we can write i 's utility as a function of i 's trade vector. The function belongs to a family of real-valued functions, each of them uniquely identified by a vector of parameters $e_i = (e_{i1}, \dots, e_{iJ_i}) \in \mathbb{R}^{J_i}$. The vector e_i is i 's local environment (individual characteristic). We consider two local-environment sets: (i) an open set

$$(IV-8) \quad \hat{E}_i = \{e_i = (e_{i1}, \dots, e_{iJ_i}) : \bar{e}_{ij} - \zeta_i < e_{ij} < \bar{\bar{e}}_{ij} + \zeta_i, j = 1, \dots, J_i\},$$

where $\zeta > 0$ and $\bar{e}_{ij} < \bar{\bar{e}}_{ij}, i = 1, \dots, n; j = 1, \dots, J_i$; and (ii) a subset of $\hat{E}_i$, namely the closed set

$$E_i = \{e_i = (e_{i1}, \dots, e_{iJ_i}) : \bar{e}_{ij} \leq e_{ij} \leq \bar{\bar{e}}_{ij}, j = 1, \dots, J_i\}.$$

²⁰We again use the superscript $\#$ to indicate that the set contains only nonnumeraire trade vectors.

For every local environment $e_i \in \hat{E}_i$ and every trade vector $(v^i, w^i) \in \hat{S}_i$, trader i 's utility is given by the following quasi-linear function:

$$u^i = v^i + \phi^i(w^i; e_i).$$

We shall make the following assumptions about the sets $\hat{E}_i$ and the “valuation” functions ϕ^i :

($\alpha 1$) Every function $\phi^i : \hat{S}_i \times \hat{E}_i \rightarrow \mathbb{R}$ is $\mathcal{C}^3$. Thus, in particular, for $r, s \in \{1, \dots, L\}$, the function

$$\phi_{rs}^i(w^i; e_i) \equiv \frac{\partial \phi^i}{\partial w_r^i \partial w_s^i}$$

is continuously differentiable ($\mathcal{C}^1$) at every (w^i, e_i) in $\hat{S}_i \times \hat{E}_i$.

($\alpha 2$) For every $e_i \in \hat{E}_i$, and every (nonnumeraire) trade vector $w^i = (w_1^i, \dots, w_L^i)$ such that $-W_\ell^i < w_\ell^i < \sum_{j \neq i} W_\ell^j + \gamma, \ell = 1, \dots, L$, the function ϕ^i :

- is strictly increasing in $w_\ell^i, \ell = 1, \dots, L$
- is *totally differentially concave*, i.e., the following Hessian is negative definite:

$$\phi_{ww}^i \equiv \begin{pmatrix} \phi_{11}^i & \dots & \phi_{1L}^i \\ \dots & \ddots & \dots \\ \phi_{L1}^i & \dots & \phi_{LL}^i \end{pmatrix}.$$

($\alpha 3$) In stating this three-part assumption we shall use the following notation:

$$(IV-9) \quad \phi_\ell^i(w^i; e_i) \equiv \frac{\partial \phi^i(w^i; e_i)}{\partial w_\ell^i}.$$

Part a. For every $e \in \hat{E}$, there exists a (nonnumeraire) trade-vector n -tuple $(w^1, \dots, w^n)$ which belongs to $S_1^\# \times \dots \times S_n^\#$ and satisfies

$$\phi_\ell^1(w^1; e_1) = \dots = \phi_\ell^n(w^n; e_n), \ell = 1, \dots, L.$$

and

$$\sum_{i=1}^n w_\ell^i = 0, \ell = 1, \dots, L.$$

Part (b). For every $e \in \hat{E}$, every trade-vector n -tuple satisfying the equalities in Part (a) belongs to $S_1^\# \times \dots \times S_n^\#$.

Part (c).²¹ For $e = (e_1, \dots, e_n) \in E$, define

$$\Phi(e) \equiv \left\{ \left((w^1, \dots, w^{n-1}), p \right) : (w^1, \dots, w^{n-1}, -\sum_{i=1}^{n-1} w^i) \in S_{1\Delta}^\# \times \dots \times S_{n\Delta}^\#; \right. \\ \left. p = (\phi_1^1(w_1^1; e_1), \dots, \phi_L^1(w_L^1; e_1)) \right\}.$$

Define also

$$\Phi(E) \equiv \{\Phi(e) : e \in E\}.$$

We assume that

$$\Phi(E) \text{ includes an open set of } \mathbb{R}^{nL}.$$

Under parts (a) and (b) of Assumption ($\alpha 3$), there is, for every $e \in \hat{E}$, a trade n -tuple, summing to zero for every commodity, for which the marginal rates of substitution for a given commodity pair are the same for all traders; every such n -tuple gives each trader a post-trade holding of at least $\Delta > 0$ for every commodity.

An illustration. To illustrate, consider the case of two traders ($n = 2$), a numeraire commodity, one non-numeraire commodity ($L = 1$), and the following linear-quadratic utility function:

$$Q + \alpha_i R - \frac{1}{2} \beta_i R^2,$$

where Q, R are i 's total holdings of the non-numeraire commodity and the numeraire commodity respectively; $\alpha_i > 0, \beta_i > 0$; and the function gives i 's utility for bundles (Q, R) such that $R < \frac{\alpha}{\beta}$. (At such a bundle the function is increasing in R).

Now for $i = 1, 2$, we shall fix $V_i > 0, W_i > 0$, trader i 's endowments of the numeraire and non-numeraire commodities, respectively. We shall also fix the parameters α_i .

The non-numeraire trade vector w^i has a single component. We have

$$\hat{S}_1 = \{(v^1, w^1) : -V^1 < v^1 < V^2 + \gamma; -W^1 < w^1 < W^2 + \gamma\},$$

$$S_1 = \{(v^1, w^1) : -V^1 + \sigma \leq v^1 \leq V^2; -W^1 + \sigma \leq w^1 \leq W^2\},$$

$$\hat{S}_2 = \{(v^2, w^2) : -V^2 < v^2 < V^1 + \gamma; -W^2 < w^2 < W^1 + \gamma\},$$

²¹We impose this requirement on the environment set E in order that the incomplete Walrasian mechanism on E , which we construct below, obey Regularity Condition (d). The set $\Phi(e)$ will be the set of equilibrium messages for e in the incomplete mechanism, and $\Phi(E)$ will be the mechanism's entire set of equilibrium messages.

$$\begin{aligned}
S_2 &= \{(v^2, w^2) : -V^2 + \sigma \leq v^2 \leq V^1; -W^2 + \sigma \leq w^2 \leq W^1\}, \\
S_{1\Delta}^\# &= \{w^1 : -W^1 + \Delta < w^1 < W^2 + \Delta\}, \\
S_{2\Delta}^\# &= \{w^2 : -W^2 + \Delta < w^2 < W^1 + \Delta\},
\end{aligned}$$

where

$$(IV-10) \quad \gamma > 0, 0 < \sigma < \min\{V_1, W_1, V_2, W_2\}; 0 < \Delta < \min(\sigma, \gamma).$$

Define

$$\theta_i \equiv \alpha_i - \beta_i W^i.$$

Let trader i 's local environment be the pair

$$e_i = (e_{i1}, e_{i2}) = (\theta_i, \beta_i)$$

and, as in (IV-8), let

$$\hat{E}_i = \{e_i = (\theta_i, \beta_i) : \bar{\theta}_i - \zeta_i < \theta_i < \bar{\bar{\theta}}_i + \zeta_i, \bar{\beta}_i - \zeta_i < \beta_i < \bar{\bar{\beta}}_i + \zeta_i\},$$

where

$$(IV-11) \quad 0 < \zeta_i < \bar{\theta}_i, 0 < \zeta_i < \bar{\beta}_i.$$

Then

$$E_i = \{e_i = (\theta_i, \beta_i) : \bar{\theta}_i \leq \theta_i \leq \bar{\bar{\theta}}_i, \bar{\beta}_i \leq \beta_i \leq \bar{\bar{\beta}}_i\}.$$

For the trades $(v^i, w^i) \in \hat{S}_i$, we can write i 's utility as

$$v^i + \phi^i(w^i; e_i),$$

where

$$\phi^i : \hat{S}_i \times \hat{E}_i \rightarrow \mathbb{R}; \phi^i(w^i; e_i) = (\theta_i + \beta_i W^i)(W^i + w^i) - \frac{1}{2}\beta_i(W^i + w^i)^2.$$

The function ϕ^i satisfies Assumption ($\alpha 1$). We have

$$\frac{d\phi^i}{dw^i} = \theta_i + \beta_i W^i - \beta_i(W^i + w^i) = \theta_i - \beta_i w^i.$$

So ϕ^i is increasing as long as

$$w^i < \frac{\theta_i}{\beta_i}.$$

Assumption ($\alpha 2$) requires that ϕ^i be strictly increasing at every w^i for which $-W^i < w^i \leq W^i + \gamma, j \neq i$. That is the case if

$$(IV-12) \quad W^2 + \gamma < \frac{\bar{\theta}_1 - \zeta_1}{\beta_1 + \zeta_1}, W^1 + \gamma < \frac{\bar{\theta}_2 - \zeta_2}{\beta_2 + \zeta_2}.$$

(Even when trader i adds to his endowment of the non-numeraire commodity the upper bound in the set $\hat{S}_i$, trader i 's utility is still increasing with respect to the holding of that commodity at the resulting bundle.)

Assumption ($\alpha 2$) also requires that the one-by-one matrix whose single entry is $-\beta_i$ be negative semi-definite. That is the case since $0 < \bar{\beta} \leq \beta_i$.

We turn to the three parts of Assumption ($\alpha 3$). To check Part (a), notice that the equation which specifies equality of marginal rates of substitution, namely

$$(IV-13) \quad \frac{d\phi^1(w^1; (\theta_1, \beta_1))}{dw^1} = \frac{d\phi^2(w^2; (\theta_2, \beta_2))}{dw^2},$$

and the equation

$$w^1 + w^2 = 0,$$

are solved simultaneously by

$$(IV-14) \quad w^1 = \frac{\theta_1 - \theta_2}{\beta_1 + \beta_2}, w^2 = -w^1.$$

Part (b) requires that for that pair (w^1, w^2) we have, for every $e = ((\theta_1, \beta_1), (\theta_2, \beta_2))$ in $\hat{E}$,

$$w^1 + W^1 > \sigma > 0, w^2 + W^2 > \sigma > 0.$$

Using the definition of θ_i , it is readily verified that this is indeed the case if

$$(IV-15) \quad (W^1 + W^2) \cdot \min(\bar{\beta}_1 - \zeta, \bar{\beta}_2 - \zeta) > \sigma + |\alpha_1 - \alpha_2|.$$

As for Part (c), we first verify that for the value of w^1 satisfying (IV-14), i.e., $w^1 = \frac{\theta_1 - \theta_2}{\beta_1 + \beta_2}$ we have

$$\frac{\partial \phi^1(w^1; e_1)}{\partial w^1} = \frac{\theta_1 \beta_2 + \theta_2 \beta_1}{\beta_1 + \beta_2}.$$

So for $e = (\theta_1, \beta_1, \theta_2, \beta_2)$, the set $\Phi(e)$ is a singleton, namely

$$\left\{ \left(\frac{\theta_1 - \theta_2}{\beta_1 + \beta_2}, \frac{\theta_1 \beta_2 + \theta_2 \beta_1}{\beta_1 + \beta_2} \right) \right\}.$$

Part (c) is satisfied if $\Phi(E)$ contains an open set of $\mathbb{R}^2$. If

$$(IV-16), \quad \bar{\beta}_1 = \bar{\beta}_2 = \bar{\beta}, \bar{\bar{\beta}}_1 = \bar{\bar{\beta}}_2 = \bar{\bar{\beta}},$$

then for any β^* satisfying $\bar{\beta} \leq \beta^* \leq \bar{\bar{\beta}}$, the set $\Phi(E)$ includes the set

$$\Phi_{\beta^*}(E) \equiv \left\{ \left(\frac{\theta_1 - \theta_2}{2\beta^*}, \frac{\theta_1 + \theta_2}{2} \right) : \theta_1 \in [\bar{\theta}_1, \bar{\bar{\theta}}_1], \theta_2 \in [\bar{\theta}_2, \bar{\bar{\theta}}_2] \right\}.$$

Part (c) is satisfied if

$$(IV-17) \quad \Phi_{\beta^*}(E) \text{ contains a nondegenerate rectangle.}$$

To summarize: Assumptions $(\alpha 1) - (\alpha 3)$ are satisfied if the fixed endowments, the fixed utility function parameters α_i , and the sets $\hat{S}_i, \hat{E}_i$ satisfy (IV-10)-(IV-17). An example of $\gamma, \sigma, \Delta, \beta^*$ and $(V_i, W_i, \alpha_i, \bar{\beta}_i, \bar{\bar{\beta}}_i, \bar{\theta}_i, \bar{\bar{\theta}}_i, \zeta_i), i = 1, 2$, satisfying (IV-10)-(IV-17) is as follows:

$$\begin{aligned} \gamma &= 1, \sigma = \frac{1}{100}, \Delta = \frac{1}{200}, V_1 = 3, W_1 = 4, \alpha_1 = \frac{54}{4}, \bar{\theta}_1 = \\ 10, \bar{\bar{\theta}}_1 &= \frac{52}{5}, V_2 = 4, W_2 = 3, \alpha_2 = \frac{51}{5}, \bar{\theta}_2 = \frac{48}{5}, \bar{\bar{\theta}}_2 = \frac{99}{10}, \bar{\beta}_1 = \\ \bar{\beta}_2 &= \beta^* = \frac{1}{10}, \bar{\bar{\beta}}_1 = \bar{\bar{\beta}}_2 = \frac{1}{5}. \end{aligned}$$

The complete mechanism. In the complete mechanism a message specifies a nonnumeraire trade vector for every trader $i, i = 1, \dots, n$ and also specifies a price for each of the L nonnumeraire commodities. The message space's possible values for i 's nonnumeraire trade vector are those in the compact set $S_{i\Delta}^\#$, defined in (IV-5).

As remarked in (IV-6) and (IV-7),

$$(IV-18) \quad \begin{cases} \text{(i) the compact set } S_i^\# \text{ lies in the interior of the set } S_{i\Delta}^\#, \text{ and (ii) } S_{i\Delta}^\# \\ \text{lies, in turn, in the interior of the open set } \hat{S}_i^\#, \end{cases}$$

where $\hat{S}_i^\#$ denotes the projection of the open set $\hat{S}_i$ (defined in (IV-1)) with respect to the non-numeraire trade vector w^i .²²

To construct the set of “price” components of the mechanism’s possible messages, we first need to identify, for each non-numeraire commodity ℓ , a smallest and largest marginal utility. We again use the marginal-utility notation introduced in (IV-9). Define

$$p_\ell^* \equiv \min_{w^i \in S_i^\#, e_i \in E_i, i \in \{1, \dots, n\}} \phi_\ell^i(w^i; e_i)$$

and

$$p_\ell^{**} \equiv \max_{w^i \in S_i^\#, e_i \in E_i, i \in \{1, \dots, n\}} \phi_\ell^i(w^i; e_i).$$

These minima and maxima exist since the sets $S_i^\#, E_i$ are compact and, by Assumption ($\alpha 1$) and the fact that $S_i^\# \subset \hat{S}_i^\#$, the function ϕ_ℓ^i is continuous on $S_i^\#$. Moreover, by Assumption ($\alpha 2$), and the definition of the sets S_i , every marginal utility is positive when the trade n -tuple lies in $S_1 \times \dots \times S_n$. We can therefore choose a real number $\bar{\Delta}$ so that

$$(IV-19) \quad 0 < \bar{\Delta} < \min_{\ell \in \{1, \dots, L\}} p_\ell^*.$$

For each non-numeraire commodity ℓ , our set of possible prices will be

$$\{p_\ell : p_\ell^* - \bar{\Delta} \leq p_\ell \leq p_\ell^{**} + \bar{\Delta}\}.$$

All members of that set are positive.

²²The compactness of $S_{i\Delta}^\#$ is needed for Regularity Condition (a) to hold (in both the complete and the incomplete mechanism). Statement (i) of (IV-18) will help us to show that the complete mechanism has the first of the four properties in Remark R1: its compact message space contains in its interior an equilibrium message for every $e \in E$. On the other hand, (ii) of (IV-18) will help us to show that the complete mechanism satisfies the extension conditions required for the second of the four properties in Remark R1.

We are now ready to combine the above statements into a definition of the compact message space of the complete continuum Walrasian mechanism, and to specify the mechanism's remaining elements. The mechanism is the triple

$$\Lambda^c = \langle M^c, (\bar{g}_1, \dots, \bar{g}_n), h^c \rangle.$$

The elements of the mechanism are as follows.

The message space is the compact set

$$M^c = \left\{ m = ((w^1, \dots, w^n), (p_1, \dots, p_L)) : w^i \in S_{i\Delta}^\#; i = 1, \dots, n; \right. \\ \left. p_\ell^* - \bar{\Delta} \leq p_\ell \leq p_\ell^{**} + \bar{\Delta}, \ell \in \{1, \dots, L\} \right\}.$$

For Trader i , with $i \in \{2, \dots, n\}$, the agreement function is

$$\bar{g}_i = (\bar{g}_{i1}, \dots, \bar{g}_{iL}),$$

where, for $\ell \in \{1, \dots, L\}$,

$$\bar{g}_{i\ell}(m, e_i) = \phi_\ell^i(w^i; e_i) - p_\ell.$$

Trader 1 has the additional task of checking whether or not the proposed trades balance and has the agreement function

$$\bar{g}_1 = (\bar{g}_{11}, \dots, \bar{g}_{1L}, g_1^*, \dots, g_L^*),$$

where, for $\ell \in \{1, \dots, L\}$,

$$\bar{g}_{1L}(m, e_1) = \phi_\ell^1(w^1; e_1) - p_\ell,$$

and

$$g_\ell^*(m, e_1) = \sum_{i=1}^n w_\ell^i.$$

For $m = ((w^1, \dots, w^n), (p_1, \dots, p_L)) \in M$, we have $h^c(m) = (w^1, \dots, w^n)$.

The incomplete mechanism. This mechanism is the triple

$$\Lambda = \langle M, (g_1, \dots, g_n), h \rangle.$$

Its elements are as follows.

The message space is the compact set

$$M = \left\{ m = ((w^1, \dots, w^{n-1}), (p_1, \dots, p_L)) : w^i \in S_{i\Delta}^\#; i = 1, \dots, n-1; \right.$$

$$p_\ell^* - \bar{\Delta} \leq p_\ell \leq p_\ell^{**} + \bar{\Delta}, \ell \in \{1, \dots, L\}.$$

For Trader i , with $i \in \{1, \dots, n-1\}$, the agreement function is

$$g_i = (g_{i1}, \dots, g_{iL}),$$

where, for $\ell \in \{1, \dots, L\}$

$$g_{i\ell}(m, e_i) = \phi_\ell^i(w^i; e_i) - p_\ell.$$

For trader n , the agreement function is $g_n = (g_{n1}, \dots, g_{nL})$, where

$$g_{n\ell}(m, e_n) = \phi_\ell^n\left(-\sum_{i=1}^{n-1} w_\ell^i; e_n\right) - p_\ell, \ell = 1, \dots, L.$$

For $m = ((w^1, \dots, w^n), (p_1, \dots, p_L)) \in M$, we have

$$h(m) = (w^1, \dots, w^{n-1}).$$

Results to be established for the two mechanisms. We now show that under Assumptions $(\alpha 1) - (\alpha 3)$:

- Both mechanisms cover E .
- Both mechanisms satisfy regularity conditions (a) and (b).
- The complete mechanism satisfies regularity condition (c), and that implies that the incomplete mechanism also satisfies regularity condition (c).
- While the complete mechanism does not satisfy regularity condition (d), the incomplete mechanism does. Thus the incomplete mechanism satisfies all four regularity conditions.
- For every e in E , the trade-vector $(n-1)$ -tuple which is the incomplete mechanism's unique equilibrium outcome, when combined with the balancing trade vector for trader n , is individually rational and Pareto-optimal in the set of all trade-vector n -tuples that sum to zero.

Coverage. To deal with the coverage issue, recall first that parts (a) and (b) of Assumption $(\alpha 3)$ guarantee the existence of a nonnumeraire trade n -tuple which sums to zero, lies in the closed set $S_1^\# \times \dots \times S_n^\# \subset S_{1\Delta}^\# \times \dots \times S_{n\Delta}^\#$, yields each trader a bundle with a positive amount (not less than σ) of each non-numeraire commodity, and has the property that all traders have the same marginal

utility for each non-numeraire commodity ℓ . Under Assumptions $(\alpha 1), (\alpha 2)$, the common marginal utility for ℓ is a positive number, which we may call p_ℓ and which satisfies

$$p_\ell^* \leq p_\ell \leq p_\ell^{**}.$$

It then follows from the definition of M^c that

$$(\dagger) \quad \left[\begin{array}{l} \text{for every } e \in E, \text{ the equation system} \\ \bar{g}_{i\ell}(m, e_i) = 0, g_\ell^*(m, e_1) = 0, i \in \{1, \dots, n\}, \ell \in \{1, \dots, L\} \\ \text{of the complete mechanism has a solution} \\ m = ((w^1, \dots, w^n), (p_1, \dots, p_L)) \in M^c, \\ \text{with } w^i \in S_i^\# \text{ for all } i \text{ and } p_\ell^* \leq p_\ell \leq p_\ell^{**} \text{ for all } \ell. \end{array} \right.$$

So the complete mechanism covers E . Similarly, using the definition of M :

$$(\dagger\dagger) \quad \left[\begin{array}{l} \text{for every } e \in E, \text{ the equation system} \\ g_{i\ell}(m, e_i) = 0, i \in \{1, \dots, n-1\}, \ell \in \{1, \dots, L\} \\ \text{of the incomplete mechanism has a solution} \\ m = ((w^1, \dots, w^{n-1}), (p_1, \dots, p_L)) \in M, \\ \text{with } w^i \in S_i^\# \text{ for } i = 1, \dots, n-1, \text{ and } p_\ell^* \leq p_\ell \leq p_\ell^{**} \text{ for all } \ell. \end{array} \right.$$

So the incomplete mechanism also covers E

Regularity conditions (a), (b). By construction, the message space M^c of the complete mechanism is compact and, by Assumption $(\alpha 1)$ (which says that every valuation function ϕ^i is C^3), every component of every agreement function $\bar{g}_i$ is C^2 at every (m, e_i) in $M^c \times E_i$. The message space M of the incomplete mechanism is also compact and every component of every agreement function g_i is also C^2 at every (m, e_i) in $M \times E_i$. So Regularity Conditions (a) and (b) are satisfied for both mechanisms.

The complete mechanism satisfies regularity condition (c). We shall establish this by verifying that the four properties in Remark R1 hold for the complete mechanism.

Property (1)

The complete mechanism has this property if M^c includes in its interior a set $\bar{M}^c$, which contains an equilibrium message for every e in E . In view of the above statement ($\dagger$), and the fact that $S_{i\Delta}^\#$ contains $S_i^\#$ in its interior, $i = 1, \dots, n$, that is indeed the case for the set

$$\bar{M}^c = \left\{ m = ((w^1, \dots, w^n), (p_1, \dots, p_L)) : w^i \in S_i^\#; i = 1, \dots, n; p_\ell^* \leq p_\ell \leq p_\ell^{**}, \ell \in \{1, \dots, L\} \right\}.$$

Property (2)

This property requires the agreement functions to obey extension conditions.

The open set $\hat{E}$ — the extended environment set — was defined in (IV-8). We define the open set $\hat{M}^c$ — our extended message space — as follows:

$$\hat{M}^c \equiv \left\{ m = (w^1, \dots, w^n), (p_1, \dots, p_L) : w^i \in \hat{S}_i^\#; i = 1, \dots, n; p_\ell^* - \tau < p_\ell < p_\ell^{**} + \tau, \ell \in \{1, \dots, L\} \right\},$$

where, for the number $\bar{\Delta}$ introduced in (IV-19), we have

$$\bar{\Delta} < \tau < p_\ell^*, \text{ all } \ell.$$

By assumption $\alpha 1$, each utility function $u_i = v^i + \phi^i(\cdot; \cdot)$ is defined on $\hat{S}_i \times \hat{E}_i$ and is $\mathcal{C}^3$. So each agreement function $\bar{g}_i$ is well defined on the extended set $\hat{M}^c \times \hat{E}$. Let $\hat{g}_i$ denote the extension of g_i to $\hat{M}^c \times \hat{E}_i$. To write $\hat{g}_i$, one just repeats the formula for g_i . Each extended function is $\mathcal{C}^2$, as Property (2) requires.

Property (3)

For every $(m, e) \in \hat{M}^c \times \hat{E}$, the Jacobian associated with the equation system

$$\hat{g}(m, e) = 0$$

and, more generally, with the equation system

$$(*) \quad \hat{g}_{i\ell}(m, e_i) = \delta_{i\ell}, \hat{g}_\ell^*(m, e_1) = \delta_\ell^*, i = 1, \dots, n, \ell = 1, \dots, L,$$

is

$$\hat{Q}(m; e) = \begin{pmatrix} \phi_{ww}^1 & 0 & \dots & 0 & -I_L \\ 0 & \phi_{ww}^2 & \dots & 0 & -I_L \\ \vdots & & \ddots & 0 & \vdots \\ 0 & 0 & & \phi_{ww}^n & -I_L \\ I_L & I_L & \dots & I_L & 0 \end{pmatrix}.$$

Here I_L is an L -by- L identity matrix and the zero symbol denotes an L -by- L null matrix. The entries in the bottom row are the partial derivatives of the components of Trader 1's "equilibrating" agreement function g^* . The last column contains partial derivatives with respect to prices. By Assumption ($\alpha 2$), each of the matrices ϕ_{ww}^i has nonzero determinant. Hence the Jacobian $\hat{Q}(m; e)$ has nonzero determinant for all (m, e) in $\hat{M}^C \times \hat{E}$. That establishes Property (3).

Property(4)

To establish the final property, we have to show that $\hat{Q}(m; e)$ satisfies a univalence condition. We shall use Corollary (ii), stated by Nikaido and found on page 379 of [10].²³

That Corollary tells us that for any fixed e , and for any m in the open rectangular region $\hat{M}$, the Jacobian $\hat{Q}(m, e)$ has the univalence property if it meets the following two conditions (a),(b). Hence, if those conditions are met, any message in $\hat{M}^c$ which solves the equation system (*) is the only such message to be found in all of $\hat{M}^c$.

For brevity we denote the Jacobian $\hat{Q}(m; e)$ by $\hat{Q}$.

The two conditions are

- (a) The determinant of $\hat{Q}$ is positive if it has an even number of rows (and columns) and negative if it has an odd number of rows (and columns).
- (b) The symmetric matrix $\frac{1}{2}(\hat{Q} + \hat{Q}')$, where $\hat{Q}'$ is the transpose of $\hat{Q}$, is negative semi-definite.

We find that $\frac{1}{2}(\hat{Q} + \hat{Q}')$ is the following (bordered) diagonal matrix:

$$\begin{pmatrix} \phi_{ww}^1 & & & & 0 \\ & \phi_{ww}^2 & & & 0 \\ & & \ddots & & 0 \\ & & & \phi_{ww}^n & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

This is negative semi-definite because we assumed (in ($\alpha 2$)) that each ϕ_{ww}^i is negative definite. So the above condition (b) is satisfied.

To verify condition (a), first define

$$\psi_i \equiv (\phi_{ww}^i)^{-1}.$$

The matrix ψ_i is negative definite (if a matrix is negative definite then so is its inverse). Next premultiply $\hat{Q}$ by the following matrix product:

²³The discussion in [10] makes use of results found in [1].

$$A \cdot A^{-1} = \begin{pmatrix} \phi_{ww}^1 & & 0 \\ & \ddots & \\ & & \phi_{ww}^n \\ 0 & & & I_L \end{pmatrix} \cdot \begin{pmatrix} \psi_{ww}^1 & & 0 \\ & \ddots & \\ & & \psi_{ww}^n \\ 0 & & & I_L \end{pmatrix}.$$

Since this product is the $(nL + L)$ -by- $(nL + L)$ identity matrix, it follows that

$$A \cdot A^{-1} \cdot \hat{Q} = \hat{Q}.$$

Note that

$$A^{-1} \cdot \hat{Q} = T,$$

where

$$T = \begin{pmatrix} I_L & & 0 & -\psi_1 \\ & \ddots & & \vdots \\ 0 & & I_L & -\psi_n \\ \bullet\bullet & \bullet\bullet & \bullet\bullet & \bullet\bullet \\ I_L & \dots & I_L & 0 \end{pmatrix}.$$

But we can modify T , by “elementary” operations, to obtain a matrix K whose determinant equals the determinant of T . To do so, successively subtract from the bottom row (the row below the line that is composed of bullets) each of the rows above the bottom row. We are left with

$$K = \begin{pmatrix} I_L & & 0 & -\psi_1 \\ & \ddots & & \vdots \\ 0 & & I_L & -\psi_n \\ \bullet\bullet & \bullet\bullet & \bullet\bullet & \bullet\bullet \\ 0 & \dots & 0 & \psi_1 + \dots + \psi_n \end{pmatrix}.$$

The matrix K is block-triangular, so its determinant equals the product of the determinants of the diagonal blocks, i.e.,

$$\begin{aligned} \det K &= \underbrace{(\det I_L) \cdot (\det I_L) \cdots (\det I_L)}_{n \text{ times}} \cdot \left(\det(\psi_1 + \dots + \psi_n) \right) \\ &= \det(\psi_1 + \dots + \psi_n). \end{aligned}$$

Note that $\psi_1 + \dots + \psi_n$ is a sum of negative-definite L -by- L matrices, so it is negative-definite and L -by- L . Hence

$$(+) \quad \det K \begin{cases} > 0 \text{ if } L \text{ is even} \\ < 0 \text{ if } L \text{ is odd} \end{cases}$$

Finally, we have

$$(++) \quad \det \hat{Q} = (\det A) \cdot (\det A^{-1}) \cdot (\det Q) = (\det A) \cdot (\det T) = (\det A) \cdot (\det K).$$

But

$$\det A = (\det \phi_{ww}^1) \cdot (\det \phi_{ww}^2) \cdots (\det \phi_{ww}^n).$$

So, since each ϕ_{ww}^i is negative definite and L -by- L ,

$$\text{sign of } \det A \begin{cases} \text{is positive if } L \text{ is even} \\ = \text{sign of } (-1)^n \text{ if } L \text{ is odd} \end{cases},$$

or more compactly

$$\text{sign of } \det A = \text{sign of } (-1)^{Ln}.$$

That, together with $(+), (++)$ imply

$$\text{sign of } \det \hat{Q} = \text{sign of } (-1)^{Ln+L}.$$

That means that the Jacobian $\hat{Q}$ satisfies the condition (a).

So both requirements of Nikaido's Corollary are met, Property (4) of Remark R1 is established, and the complete mechanism indeed obeys Regularity Condition (c).

The satisfaction of condition (c) by the complete mechanism implies its satisfaction by the incomplete mechanism. First note that for all $e \in E$,

$$(\S) \quad \left\{ \begin{array}{l} \text{if } ((w^1, \dots, w^n), (p_1, \dots, p_L)) \text{ satisfies} \\ \quad \bar{g}_{i\ell}((w^1, \dots, w^n), (p_1, \dots, p_L), e_i) = \delta_{i\ell}, i = 1, \dots, n, \ell = 1, \dots, L \\ \text{and} \\ \quad g_{\ell}^*((w^1, \dots, w^n), (p_1, \dots, p_L), e_1) = 0, \ell = 1, \dots, L, \\ \text{then } ((w^1, \dots, w^{n-1}), (p_1, \dots, p_L)) \text{ satisfies} \\ \quad g_{i\ell}((w^1, \dots, w^{n-1}), (p_1, \dots, p_L), e_i) = \delta_{i\ell}, i = 1, \dots, n, \ell = 1, \dots, L \end{array} \right.$$

and

$$(\S\S) \left\{ \begin{array}{l} \text{if } ((w^1, \dots, w^{n-1}), (p_1, \dots, p_L)) \text{ satisfies} \\ \quad g_{i\ell}((w^1, \dots, w^{n-1}), (p_1, \dots, p_L), e_i) = \delta_{i\ell}, i = 1, \dots, n, \ell = 1, \dots, L, \\ \text{then } ((w^1, \dots, w^{n-1}, -\sum_{i=1}^{n-1} w^i), (p_1, \dots, p_L)) \text{ satisfies} \\ \quad \bar{g}_{i\ell}((w^1, \dots, w^{n-1}, -\sum_{i=1}^{n-1} w^i), (p_1, \dots, p_L)) = \delta_{i\ell}, i = 1, \dots, n, \ell = 1, \dots, L \\ \text{and} \\ \quad g_\ell^*((w^1, \dots, w^{n-1}, -\sum_{i=1}^{n-1} w^i), (p_1, \dots, p_L)), e_1) = 0, \ell = 1, \dots, L. \end{array} \right.$$

Now the satisfaction of regularity condition (c) by the complete mechanism means that there exists a number $\bar{\delta} > 0$ and a C^2 approximate-solution function

$$\bar{t}^c : E \times \underbrace{[-\bar{\delta}, \bar{\delta}] \times \dots \times [-\bar{\delta}, \bar{\delta}]}_{n(L+1) \text{ times}} \rightarrow M^c,$$

with trade components $\bar{t}_{i\ell}^c$ and price components $\bar{t}_\ell^c$, $i = 1, \dots, n, \ell = 1, \dots, L$, such that

$$(\S\S\S) \left\{ \begin{array}{l} \text{for all } \delta_{i\ell}, \delta_\ell^* \in [-\bar{\delta}, \bar{\delta}], i = 1, \dots, n, \ell = 1, \dots, L \text{ and all } e \in E, \\ \quad \bar{t}^C(e, \delta_{11}, \dots, \delta_{1L}, \dots, \delta_{n1}, \dots, \delta_{nL}, \delta_1^*, \dots, \delta_L^*) \\ \text{is the unique message } m \text{ in } M^C \text{ satisfying the complete-mechanism} \\ \text{equation system} \\ \quad \bar{g}_{i\ell}(m, e_i) = \delta_{i\ell}, g_\ell^*(m, e_1) = \delta_\ell^*, i = 1, \dots, n, \ell = 1, \dots, L. \end{array} \right.$$

We obtain the incomplete mechanism's approximate-solution function from the complete mechanism's function $\bar{t}^c$ by setting the quantities $\delta_1^*, \dots, \delta_L^*$ equal to zero. Thus we consider the function

$$\bar{t} : E \times \underbrace{[-\bar{\delta}, \bar{\delta}] \times \dots \times [-\bar{\delta}, \bar{\delta}]}_{nL \text{ times}} \rightarrow M,$$

with trade components $\bar{t}^{i\ell}$ and price components $\bar{t}_\ell$, $i = 1, \dots, n-1, \ell = 1, \dots, L$, defined by:

$$\bar{t}_{i\ell}(e, \delta_{11}, \dots, \delta_{1L}, \dots, \delta_{n1}, \dots, \delta_{nL}) = \bar{t}_{i\ell}^c(e, \delta_{11}, \dots, \delta_{1L}, \dots, \delta_{n1}, \dots, \delta_{nL}, \underbrace{0, \dots, 0}_{L \text{ times}})$$

and

$$\bar{t}_\ell(e, \delta_{11}, \dots, \delta_{1L}, \dots, \delta_{n1}, \dots, \delta_{nL}) = \bar{t}_\ell^c(e, \delta_{11}, \dots, \delta_{1L}, \dots, \delta_{n1}, \dots, \delta_{nL}, \underbrace{0, \dots, 0}_{L \text{ times}}).$$

It follows from (§), (§§), (§§§) that this $\mathcal{C}^2$ function $\bar{t}$ is indeed the approximate-solution function for which the incomplete mechanism satisfies regularity condition (c). That is to say, for all $\delta_{i\ell} \in [-\bar{\delta}, \bar{\delta}]$, $i = 1, \dots, n$, $\ell = 1, \dots, L$ and all $e \in E$,

$$\bar{t}(e, \delta_{11}, \dots, \delta_{1L}, \dots, \delta_{n1}, \dots, \delta_{nL})$$

is the unique message m in M satisfying the incomplete-mechanism equation system

$$g_{i\ell}(m, e_i) = \delta_{i\ell}, i = 1, \dots, n, \ell = 1, \dots, L.$$

So the incomplete mechanism satisfies regularity condition (c).

Regularity condition (d). Let t^c, t denote, respectively, the threads of the complete mechanism and the incomplete mechanism. (Recall that the thread is the exact-solution function and one obtains it from the approximate-solution function by setting all the “ δ ” variables equal to zero). The complete mechanism has $n(L + 1)$ message variables. It fails to satisfy condition (d) since at every equilibrium message the n trade vectors sum to zero. Hence the image set $t^c(E) \subset \mathbb{R}^{n(L+1)}$ of equilibrium messages has dimension nL and does not contain an open set of $\mathbb{R}^{n(L+1)}$.

On the other hand, the incomplete mechanism has nL message variables: L -component trade vectors for each of $n - 1$ traders and L prices. Part (c) of Assumption $\alpha 3$ guarantees that the incomplete-mechanism’s equilibrium-message set $t(E) \subset \mathbb{R}^{nL}$ contains an open set of $\mathbb{R}^{nL}$.

We conclude that the incomplete mechanism Λ obeys all four of the regularity conditions.

Pareto – optimality and individual rationality of the incomplete mechanism’s equilibrium outcomes. Suppose $\bar{m} = ((\bar{w}^1, \dots, \bar{w}^{n-1}), (\bar{p}_1, \dots, \bar{p}_L)) \in M$ is an equilibrium message of the incomplete mechanism for the environment $\bar{e} \in E$. We first argue that the trade n -tuple $(\bar{w}^1, \dots, \bar{w}^{n-1})$ is an interior Pareto optimum.

Since the incomplete mechanism obeys regularity condition (c), $\bar{m}$ is the unique equilibrium message for $\bar{e}$. In view of statement ($\dagger\dagger$), and the definition of the incomplete mechanism’s message space M , the pair $((\bar{w}^1, \dots, \bar{w}^{n-1}, -\sum_{i=1}^{n-1} \bar{w}^i), (\bar{p}_1, \dots, \bar{p}_L))$ belongs to the set

$$\left\{ m = ((w^1, \dots, w^n), (p_1, \dots, p_L)) : w^i \in S_i^\#, i = 1, \dots, n; p_\ell^* \leq p_\ell \leq p_\ell^{**}, \ell \in \{1, \dots, L\} \right\}.$$

Recall that

$$S_i^\# \equiv \{w^i : \text{for all } \ell \in \{1, \dots, L\}, -W_\ell^i + \sigma \leq w_\ell^i \leq \sum_{j \neq i} W_\ell^j\},$$

and (from (IV-2))

$$0 < \sigma < W_\ell^i, \text{ for all } i \in \{1, \dots, n\}, \ell \in \{1, \dots, L\}.$$

Thus after the trades specified in $\bar{m}$ have taken place, and trader n has received the negative of their sum, each trader holds a positive amount of every non-numeraire commodity.

The univalence argument that we gave in checking Property (4) of Remark R1 for the complete mechanism implies that $(\bar{w}^1, \dots, \bar{w}^{n-1}, -\sum_{i=1}^{n-1} \bar{w}^i)$ is the only solution to the equation system²⁴

$$\phi_\ell^1(w^1, \bar{e}_1) = \phi_\ell^2(w^2, \bar{e}_2) = \dots = \phi_\ell^n(w^n, \bar{e}_n), \ell = 1, \dots, L$$

$$\sum_{i=1}^n w_\ell^i = 0, \ell = 1, \dots, L$$

to be found in all of the open set

$$\hat{S}^\# \equiv \hat{S}_1^\# \times \dots \times \hat{S}_n^\#,$$

where (recalling the definition)

$$\hat{S}_i^\# \equiv \{(w_1^i, \dots, w_L^i) : \text{for all } \ell \in \{1, \dots, L\}, -W_\ell^i < w_\ell^i < \sum_{j \neq i} W_\ell^j + \gamma\},$$

and $\gamma > 0$.

That implies (applying the First Welfare Theorem) that there is a numeraire trade vector $(\bar{v}^1, \dots, \bar{v}^n)$ in the set

$$V^* \equiv \{(v^1, \dots, v^n) : \sum_{i=1}^n v^i = 0, v^i \geq V^i, \text{ all } i\}$$

such that $((\bar{v}^1, \dots, \bar{v}^n), (\bar{w}^1, \dots, \bar{w}^{n-1}, -\sum_{i=1}^{n-1} \bar{w}^i))$ is Pareto-optimal for $\bar{e}$ in the set of all trade n -tuple pairs $((v^1, \dots, v^n), (w^1, \dots, w^n))$ for which $(v^1, \dots, v^n) \in V^*$, $(w^1, \dots, w^n) \in S^\#$, and $\sum_{i=1}^n w^i = 0$.

Next we turn to individual rationality. For every trader i , utility at the bundle held after the trades $\bar{w}^1, \dots, \bar{w}^{n-1}, -\sum_{i=1}^{n-1} \bar{w}^i, v^1, \dots, v^n$ take place is not less than utility at the endowment bundle. That is the case since the equality $g^i(\bar{m}, \bar{e}_i) = 0$, and the conditions that we imposed on ϕ^i in Assumption ($\alpha 2$), imply that for traders $1, \dots, n-1$, the pair $(\bar{v}^i, \bar{w}^i)$, maximizes the utility $v^i + \phi^i(w^i; \bar{e}_i)$ under the budget constraint

$$\sum_{\ell=1}^L \bar{p}_\ell \cdot (W_\ell^i + w_\ell^i) + (V^i + v^i) \leq \sum_{\ell=1}^L \bar{p}_\ell \cdot W_\ell^i + V^i,$$

²⁴We again use the notation of (IV- 9).

and the analogous statement holds for trader n and the pair $(v^n, -\sum_{i=1}^{n-1} \bar{w}^i)$.

So, at an equilibrium message of the incomplete mechanism, each trader's trade is individually rational.

The complete and incomplete continuum Walrasian mechanisms in the illustrative two-person linear-quadratic case.

The environment sets E_i were given above, following (IV-11). We assume that conditions (IV-11), (IV-12), (IV-15) are satisfied. Recall that

$$\frac{d\phi^i}{dw^i} = \theta_i - \beta_i w^i.$$

Using the definition of S_i , and defining $\phi^{i'} \equiv \frac{d\phi^i}{dw^i}$, we obtain

$$(IV - 20) \quad p^* = \min_{w^i \in S_i^\#, e_i \in E_i, i \in \{1,2\}} \phi^{i'}(w^i; e_i) = \min(\bar{\theta} - \bar{\beta} W^1, \bar{\theta} - \bar{\beta} W^2),$$

which is positive by (IV-12); and

$$(IV - 21) \quad p^{**} = \max_{w^i \in S_i^\#, e_i \in E_i, i \in \{1,2\}} \phi^{i'}(w^i; e_i) = \max(\bar{\theta} + \bar{\beta}(W^1 - \sigma), \bar{\theta} + \bar{\beta}(W^2 - \sigma)).$$

To define the message space of the complete mechanism, we again need the compact sets

$$S_{1\Delta}^\# = \{-W^1 + \Delta \leq w^1 \leq W^2 + \Delta\}, S_{2\Delta}^\# = \{-W^2 + \Delta \leq w^2 \leq W^1 + \Delta\},$$

where (as in (IV-10)), Δ satisfies

$$(IV - 22) \quad 0 < \Delta < \min(W^1, W^2), \Delta < \sigma.$$

The message space of the complete mechanism is the compact set

$$(IV - 23) \quad M^c = \{m = ((w^1, w^2), p) : w^i \in S_{i\Delta}^\#; i = 1, 2; p^* - \bar{\Delta} \leq p \leq p^{**} + \bar{\Delta}\}.$$

Trader 1's agreement function $\bar{g}_1$ is a pair, which we denote $(\bar{g}_{11}, g^*)$. We have

$$\begin{aligned} \bar{g}_{11}(((w^1, w^2), p); e_1) &= \theta_1 - \beta_1^1 - p \\ g^*(((w^1, w^2), p); e_1) &= w^1 - w^2. \end{aligned}$$

For trader 2, we have

$$\bar{g}_2\left((w^1, w^2), p; e_2\right) = \theta_2 - \beta_2^1 - p.$$

The solution to the equation system $\bar{g}_1\left((w^1, w^2), p, e_1\right) = \bar{g}_2\left((w^1, w^2), p, e_2\right) = g^*\left((w^1, w^2), p; e_1\right) = 0$ is

$$(IV - 24) \quad w^1 = \frac{\theta_1 - \theta_2}{\beta_1 + \beta_2}, w^2 = \frac{\theta_2 - \theta_1}{\beta_1 + \beta_2}, p = \frac{\theta_1\beta_2 + \theta_2\beta_1}{\beta_1 + \beta_2}.$$

We argued above that our assumed condition (IV-15) implies that for this solution we have

$$w^1 \in S_1^\#, w^2 \in S_2^\#.$$

By the definition of p^*, p^{**} in (IV-20) and (IV-21), we also have

$$\frac{\theta_1\beta_2 + \theta_2\beta_1}{\beta_1 + \beta_2} \in [p^*, p^{**}],$$

since $\bar{g}_{11} = g^* = \bar{g}_2 = 0$ implies that for $i = 1, 2$, we have $p = \phi^{i'}(w^i; e_i)$, where $w^i \in S_i$.

So the complete mechanism indeed covers E .

The incomplete mechanism. The message space of the incomplete mechanism is the compact set

$$M = \left\{ m = (w^1, p) : w^1 \in S_{1\Delta}^\#, -w^1 \in S_{2\Delta}^\#; p^* - \bar{\Delta} \leq p \leq p^{**} + \bar{\Delta} \right\}.$$

The agreement functions are

$$\begin{aligned} g_1(w^1, p, e_1) &= \theta_1 - \beta_1^1 - p \\ g_2(w^1, p, e_1) &= \theta_1 + \beta_1^1 - p \end{aligned}$$

For the outcome function h we have

$$h(w^1, p) = w^1.$$

The solution to the equation system $g_1(w^1, p, e_1) = g_2(w^1, p, e_2) = 0$ is

$$w^1 = \frac{\theta_1 - \theta_2}{\beta_1 + \beta_2}, p = \frac{\theta_1\beta_2 + \theta_2\beta_1}{\beta_1 + \beta_2}.$$

Note that $((w^1, w^2), p)$ is an equilibrium mechanism for e in the complete mechanism if and only if (w^1, p) is an equilibrium mechanism for e in the incomplete mechanism. Hence the fact that (as just argued) the complete mechanism covers E implies that the incomplete mechanism covers E .

Regularity. Regularity Conditions (a) and (b) are immediately verified.

We turn to Regularity Condition (c). We shall explicitly identify the constant $\bar{\delta}$ and the approximate-solution function $\bar{t}^c$ which are required if the complete function is to obey Condition (c). required by Condition (c). But first we check the Properties (1) - (4) of Remark R1, which have to be verified, in the general case to ensure the existence of δ and $\bar{t}^c$.

Property (1)

The message space M^c was defined in (IV-23). We now define

$$\bar{M}^c \equiv \{m = ((w^1, w^2), p) : w^i \in S_i, i = 1, 2; p^* \leq p \leq p^{**}\}.$$

As we remarked in the preceding discussion of the coverage property, $\bar{M}^c$ contains an equilibrium message for every e in E . As Property (1) requires, the set $\bar{M}^c$ is in the interior of M^c .

Property (2)

The set $\hat{E}$ was defined just prior to (IV-11). We define the extended message space

$$\hat{M}^c \equiv \{m = ((w^1, w^2), p) : w^i \in \hat{S}_i^\#; i = 1, 2; p^* - \tau < p < p^{**} + \tau\},$$

where $\bar{\Delta} < \tau < p^*$. Defining each of the two extended functions $\hat{g}_i$ by the same formula used to define g_i , we have:

$$\hat{g}_1 = (\hat{g}_{11}, \hat{g}^*), \hat{g}_2 = \theta_1 - \beta_1 w^1, \hat{g}_2 = \theta_2 - \beta_2 w^2, \hat{g}^* = w^1 - w^2.$$

Each of the extended functions is C^2 on $\hat{M}^c \times \hat{E}$.

Property (3)

The Jacobian $\hat{Q}(m, e)$ for the complete mechanism (denoted $\hat{Q}$ for brevity) is as follows (with rows and columns appropriately indexed):

$$\begin{matrix} & w^1 & w^2 & p \\ \begin{matrix} \bar{g}_{11} \\ \bar{g}_2 \\ g^* \end{matrix} & \begin{pmatrix} -\beta_1 & 0 & -1 \\ 0 & -\beta_2 & -1 \\ 1 & 1 & 0 \end{pmatrix} \end{matrix}$$

This has determinant $-\beta_1 - \beta_2$, which is nonzero, since $\beta_i > 0, i = 1, 2$.

Property (4)

$\hat{Q}$ has an odd number of rows and is negative, so the first of Nikaido's conditions is met.

The symmetric matrix $\frac{1}{2}(Q + Q')$ is

$$\begin{pmatrix} -\beta_1 & 0 & 0 \\ 0 & -\beta_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is negative semi-definite. So the second of Nikaido's conditions is met.

All four properties of Remark R1 have been verified. That assures the existence of the number $\bar{\delta}$ and the C^2 function $\bar{t}^c$ required by Regularity Condition (c).

We now explicitly identify $\bar{\delta}$ and $\bar{t}^c$.

For $e = (\theta_1, \beta_1, \theta_2, \beta_2) \in \hat{E}$, the equation system

$$\begin{aligned} \hat{g}_1((w^1, w^2), p) &= \delta_1 \\ (*) \quad \hat{g}_2((w^1, w^2), p) &= \delta_2 \\ \hat{g}^*((w^1, w^2), p) &= \delta^* \end{aligned}$$

has the solution

$$(IV - 25) \quad w^1 = \frac{\theta_1 - \theta_2}{\beta_1 + \beta_2} + \frac{\delta_2 - \delta_1 + \beta_2 \delta^*}{\beta_1 + \beta_2}$$

$$(IV - 26) \quad w^2 = \frac{\theta_2 - \theta_1}{\beta_1 + \beta_2} + \frac{\delta_1 - \delta_2 + \beta_1 \delta^*}{\beta_1 + \beta_2}$$

$$(IV - 27) \quad p = \frac{\theta_1 \beta_2 + \theta_2 \beta_1}{\beta_1 + \beta_2} - \frac{\delta_1 \beta_2 + \delta_2 \beta_1 + \beta_1 \beta_2 \delta^*}{\beta_1 + \beta_2}.$$

The triple composed of the first terms in these three equations is the solution to the system $g(m, e) = 0$; that solution lies in the set $\bar{M}^c$ defined above. Recall that $\bar{M}^c$ is in the interior of M^c , which we may write (foregoing the symbols $S_{i\Delta}^\#$)

$$M^c = \left\{ m = ((w^1, w^2), p) : -W^1 + \Delta \leq w^1 \leq W^2 + \Delta, -W^2 + \Delta \leq w^2 \leq W^1 + \Delta, \right. \\ \left. p^* - \bar{\Delta} \leq p \leq p^{**} + \bar{\Delta} \right\},$$

where

$$p^* = \min(\bar{\theta} - \bar{\beta} W^1, \bar{\theta} - \bar{\beta} W^2), p^{**} = \max(\bar{\theta} + \bar{\beta} (W^1 - \sigma), \bar{\theta} + \bar{\beta} (W^2 - \sigma)).$$

(Recall that for $e = ((\theta_1, \beta_1), (\theta_2, \beta_2))$ an equilibrium message (w^1, w^2, p) satisfies $p = \theta_i - \beta_i w^i, i = 1, 2$, so that the p^*, p^{**} just defined are indeed, respectively, the minimum and the maximum of the equilibrium values of p .) For an arbitrary $\bar{\delta} > 0$, consider all $(\delta_1, \delta_2, \delta^*)$ such that $|\delta_1| \leq \bar{\delta}, |\delta_2| \leq \bar{\delta}, |\delta^*| \leq \bar{\delta}$. Over all such $(\delta_1, \delta_2, \delta^*)$ and all $(\theta_1, \beta_1, \theta_2, \beta_2) \in E$:

- the largest value of the second term in (IV-25) is $\bar{\delta} Z_1$ and the smallest value is $-\bar{\delta} Z_1$,
- the largest value of the second term in (IV-26) is $\bar{\delta} Z_2$ and the smallest value is $-\bar{\delta} Z_2$,
- the largest value of the second term in (IV-27) is $\bar{\delta} Z^*$ and the smallest term is $-\bar{\delta} Z^*$,

where

$$Z_1 \equiv \frac{\bar{\delta} \cdot (\bar{\beta}_2 + 2)}{\bar{\beta}_1 + \bar{\beta}_2}, \\ Z_2 \equiv \frac{\bar{\delta} \cdot (\bar{\beta}_1 + 2)}{\bar{\beta}_1 + \bar{\beta}_2}, \\ Z^* \equiv \frac{\bar{\delta} \cdot (\bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_1 \bar{\beta}_2)}{\bar{\beta}_1 + \bar{\beta}_2}.$$

Now suppose we are given

$$W^1, W^2, \bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_1, \bar{\beta}_2, \bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_1, \bar{\theta}_2, \Delta, \bar{\Delta} \text{ satisfying (IV-11), (IV-12), (IV-22), (IV-23).}$$

Then it can be verified that we can choose $\bar{\delta} > 0$ so that

$$-W^i + \Delta < -\bar{\delta} Z_i, W^i + \Delta > \bar{\delta} Z_i, i = 1, 2$$

and

$$p^* - \bar{\Delta} < -\bar{\delta}Z^*, < p^{**} + \bar{\Delta} > \bar{\delta}Z^*.$$

Suppose we have chosen such a $\bar{\delta}$. Now define

$$\bar{t}^c(e, \delta_1, \delta_2) = (w^1, w^2, p) \text{ satisfying (IV-25), (IV-26), (IV-27).}$$

The function $\bar{t}^c$ is C^2 . We can conclude, moreover, that for $|\delta_1| \leq \bar{\delta}, |\delta_2| \leq \bar{\delta}, |\delta_2| \leq \bar{\delta}$, and for $e \in E$, the message space M^c contains one and only one message satisfying the above equation system (*), namely the message $\bar{t}^c(e, \delta_1, \delta_2)$.

The first and third components of $\bar{t}^c$ (corresponding to trader 1's nonnumeraire trade and the price) give us the approximate solution function $\bar{t}$ of the incomplete mechanism.

It remains to consider Regularity Condition (d). We argued above, in our initial discussion of the linear-quadratic illustration, that the incomplete mechanism satisfies Part (c) of Assumption ($\alpha 3$). That implies that the mechanism satisfies Regularity Condition (d).

So all four regularity conditions are met.

Pareto optimality and individual rationality in the illustrative

linear-quadratic case. The incomplete mechanism's unique equilibrium message (w^1, p) for the environment $e = (\theta_1, \beta_1, \theta_2, \beta_2)$ in E is the solution given in (IV-24), namely

$$w^1 = \frac{\theta_1 - \theta_2}{\beta_1 + \beta_2}, p = \frac{\theta_1 \beta_2 + \theta_2 \beta_1}{\beta_1 + \beta_2}.$$

The trades $w^1, -w^1$ are the unique trades at which the marginal utilities for the non-numeraire commodity are equal and the two trades sum to zero. They are therefore Pareto-optimal in the set of trades that sum to zero. Individual rationality is readily checked directly by computing utility at the endowments and comparing it with utility at the equilibrium values of $(w^1, -w^1)$.

V. APPLYING PROPOSITIONS A-D TO THE WALRASIAN MECHANISM

We can now apply Propositions A-D.

We start with Proposition A.

We note that one can construct a continuum DR mechanism on our environment set E which, like our incomplete Walrasian mechanism, has the property that an equilibrium outcome is a Pareto-optimal and individually rational trade vector. Recall that in a DR mechanism a message is a point e of E . We may, in particular, let the DR mechanism's outcomes be the Walrasian trade vectors themselves. Then the DR mechanism's outcome function assigns to any message e the Walrasian trade vector for the environment e . The goal function realized by both mechanisms assigns to any environment e the Walrasian trade vector for e . Under our assumptions $(\alpha 1) - (\alpha 3)$, that goal function is well-behaved in the sense of Proposition A.

Whatever outcome function we choose, our DR mechanism has a larger message-space dimension than the incomplete Walrasian mechanism as long as $J > nL$.

As long as the action space of the (incomplete) Walrasian mechanism and the action space of the DR mechanism are both a subsets of the same finite-dimensional space, namely $\mathbb{R}^{nL}$, we can apply Proposition A without specifying that the two mechanisms realize the same goal function. Since the "inferior" mechanism is a DR mechanism, we obtain, in fact, the strongest of the superiority statements in proposition A. Even if we choose rounded outcomes for our approximation of the Walrasian mechanism, and exact outcomes for our DR approximation, the former has lower error than the latter, if the former has a sufficiently fine mesh and the latter is not more costly than the former.

Formally:

If $J > nL$, then a complete class of rounded-outcome approximations of the continuum Walrasian mechanism on E is superior to a complete class of exact-outcome approximations of any continuum DR mechanism on E which realizes a well-behaved goal function whose range is $\mathbb{R}^{nL}$.

Next we apply Proposition B.

Using techniques developed in earlier papers²⁵ one can show that that there is no regular continuum mechanism on our set E of n -trader $(L + 1)$ -commodity exchange economies which realizes a Pareto-optimal individually rational allocation for each environment e in E and has fewer than nL message variables. Proposition A tells us that the rounded-outcome approximations of a regular

²⁵See [3], [9], [11]

continuum mechanism which has the same realization property but has more than nL message variables, are inferior, in our sense, to the rounded-outcome approximations of the Walrasian mechanism.

It remains, however, to consider the (possibly empty) class of regular continuum mechanisms that realize Pareto optimality and individual rationality, are exactly *tied* with the Walrasian mechanism with regard to number of message variables, and are *distinct* from the Walrasian mechanism. Proposition B assures us that nothing is to be gained by considering that class. Finite approximations of members of that class are indifferent, in the sense of the Proposition, to finite approximations of the Walrasian mechanism.

Formally:

Consider our set E of n -trader $(L + 1)$ -commodity exchange economies meeting assumptions $(\alpha 1) - (\alpha 3)$. Any complete class of finite controlled rounded-outcome approximations of the (incomplete) Walrasian mechanism on E is either superior or indifferent to a complete class of controlled rounded-outcome approximations of any other regular continuum mechanism on E , where the other mechanism also has nL action variables.²⁶

Next we turn to Propositions C and D. We consider a sequence $\{E^v\}$ of successively higher-dimensional exchange-economy environment sets E^v . With each member of the sequence we associate a continuum DR mechanism that realizes the same goal function as our (incomplete) Walrasian mechanism does. We shall suppose that endowments are fixed and stay the same for all v . Every environment $e = (e_1, \dots, e_n)$ in E^v specifies a vector e_i of utility-function parameters for every trader i . As v grows, so does the dimension of E^v .

To apply Propositions C and D, we have to restrict the sequence $\{E^v\}$ of exchange-economy environment sets. For Proposition C, they have to obey conditions (i) – (vi) in the statement of that proposition. That is achieved, in particular, if

²⁶In the continuum literature the typical technique for establishing message-space minimality of the Walrasian mechanism is as follows. One considers a “test class” of exchange economies (for example, economies with Cobb-Douglas utility functions). One shows that a regular mechanism with lower message-space dimension than the Walrasian mechanism cannot realize Pareto optimality and individual rationality on the test class. Then, *a fortiori*, no such mechanism can realize that goal on any wider class that contains the test class. Unfortunately a parallel technique will not work in our problem. We cannot, for example, use the quasi-linear class as a “test class” in a parallel way. Given a wider class of exchange economies that includes the quasi-linear class, we cannot rule out the possibility that (i) for an approximation of the Walrasian mechanism, maximum error occurs *outside* the quasi-linear class, (ii) a cost equivalent (not more costly) approximation of some non-Walrasian mechanism has a smaller maximum error outside the quasi-linear class even though its maximum error within the quasi-linear class exceeds that of the Walrasian approximation, (iii) as a consequence, the overall error of the non-Walrasian approximation on the entire wide class is less than the overall error of the Walrasian approximation.

(#) there exist W, Z , with $W < Z$ such that for all v , all i in $\{1, \dots, n\}$ and all j in $\{1, \dots, K_i(v)\}$ we have $\bar{e}_{ij}^v = W, \bar{\bar{e}}_{ij}^v = Z$.

Proposition C also assumes a condition (condition (xii)) on the DR sequence $\{\Lambda^{*v}\}$, namely the existence of a number τ^*0 which is a lower bound to the ratio of overall error to mesh for every approximation of every Λ^{*v} . But as we remarked in Part III, that is, in turn guaranteed if for some person, the sensitivity of some outcome-function component (or, equivalently, some real-valued component of the goal function) to some component of that person's local environment has a common lower bound for all v , i.e., if

(##) $\left\{ \begin{array}{l} \text{for some person } i \text{ and some action component } k, \text{ there} \\ \text{exists } \tau > 0 \text{ such that for every } v \text{ and some } j \text{ in} \\ \{1, \dots, K_i(v)\} \text{ we have } |\partial h_k^{*v} / \partial e_{ij}| \geq \tau. \end{array} \right.$

We have the following application of Proposition C.

For n traders and L nonnumeraire commodities, consider a sequence $\{E^v\}$ of exchange-economy environment sets. Let the sequence $\{E^v\}$ meet the above condition (#). For each $v \geq 1$, let every member e of E^v define an exchange economy that meets our assumptions $(\alpha 1) - (\alpha 3)$, where the endowments $\{(V^i, W_\ell^i)\}_{i=1, \dots, n, \ell=1, \dots, L}$ are the same for every v and for all the exchange economies defined by members of E^v . Let Λ^v be the incomplete Walrasian mechanism on the exchange-economy set E^v . For every $v \geq 1$, let Λ^{*v} be a continuum DR mechanism on the exchange-economy set E^v with an action space the same as that of Λ^v . Let the sequences $\{\Lambda^v\}, \{\Lambda^{*v}\}$ have uniformly bounded derivatives and let the DR sequence $\{\Lambda^{*v}\}$ obey condition (##). For all v , let the DR mechanism Λ^{*v} realize the same goal function as the Walrasian mechanism Λ^v .

For every $v \geq 1$, and for any fixed and sufficiently small ϵ , consider any exact-outcome approximation $\Lambda_{\bar{r}(\epsilon, v)}^{*v}$ of Λ^{*v} which is error-equivalent to the exact-outcome approximation Λ_ϵ^v . Then

$$\lim_{v \rightarrow \infty} [C(\Lambda_{\bar{r}(\epsilon, v)}^{*v}) - C(\Lambda^v)] = \infty.$$

Finally, consider Proposition D. It requires condition (+), which says that there is some fixed “budget” $\Psi \geq 1$ such that for all $v \geq 1$ one can find a mesh having no more than Ψ points in E^v . That is the case, in particular, if for every $v \geq 1$, one can find a mesh having exactly *one* point in E^v . Then we can take Ψ to be one.

But if the environment sequences obey the above condition (#), then, as we showed in Part III, for all v , every sparse approximation Λ^{*v} must be a one-message approximation. Now assume further that the number W in (#) is zero, i.e. that for every v , the origin in $\mathbb{R}^{J^v}$ belongs to E^v . Then it is readily shown that *for any fixed v , all one-message approximations of Λ^{*v} have the same message, namely the origin, which is a lattice point for every mesh.*

Proposition D assumes that the DR sequence $\{\Lambda^{*v} = \langle M^{*v}, (g_1^{*v}, \dots, g_n^{*v}), h^{*v} \rangle\}$ has a further property: its sparse approximations are monotone.

To establish that monotonicity, continue to assume that $W = 0$ and add a further restriction on the environment-set sequence, namely

$$(\#\#\#) \quad \text{for } v \geq 1, i = 1, \dots, n, K_i(v+1) > K_i(v).$$

(That is a stronger requirement than $J^{v+1} = \sum_{i=1}^n K_i(v+1) > \sum_{i=1}^n K_i(v) = J^v$, which is all we have required so far). Next suppose that for all $v \geq 1$, every exchange economy identified by a point e in E^v has, for each trader i , the “valuation” function

$$(\dagger) \quad \phi^i(w^i; e_i) = \mu^i(w^i) + \sum_{\ell=1}^L \xi_{\ell}^{iv}(w_{\ell}^i; e_i),$$

where ξ_{ℓ}^{iv} is a polynomial of degree $K_i(v)$ and e_i is the ordered $K_i(v)$ -tuple of the polynomial's coefficients, any of which may be zero. When all of them are zero, i 's valuation function is just μ^i . Then

$$(\dagger\dagger) \quad \begin{array}{l} \text{an exchange economy identified by a point in } E^v \\ \text{is also an exchange economy identified by a point} \\ \text{in } E^{v+1}, \end{array}$$

the latter point having its last $K(v+1) - K(v)$ components equal to zero.

In applying Proposition D to our exchange economies, we compare the Walrasian sequence $\{\Lambda^v\}$ with the DR sequence $\{\Lambda^{*v}\}$, where both Λ^v and Λ^{*v} realize the Walrasian allocation for every economy in E^v . Consider a sparse approximation of the DR mechanism Λ^v on the exchange-economy set E^v . Under our assumptions, that sparse approximation is a one-message approximation with the origin as its sole message. The sparse approximation therefore yields the same allocation for all economies in E^v and for all $v \geq 1$, namely the Walrasian allocation when every trader i has the valuation function μ^i . That means, in view of ($\dagger\dagger$), that every error²⁷ that can occur for a sparse approximation of Λ^{*v} can also occur for a sparse approximation of $\Lambda^{*,v+1}$. Hence

the overall error of a sparse approximation of $\Lambda^{*,v+1}$ is at least as great as the overall error of a sparse approximation of Λ^{*v} .

²⁷Here “error” means a value of *error at e* for some e in E^v .

Thus the requirement that sparse approximations be monotone is met: for any integer $B \geq 1$, the only sparse approximation costing not more than B is a one-message approximation, and the overall error of one-message approximations is nondecreasing in v .

Formally, we have the following illustrative application of Proposition D.

For n traders and L nonnumeraire commodities, consider a sequence $\{E^v\}$ of exchange-economy environment sets. Let the sequence $\{E^v\}$ meet the above condition (#), with $W = 0$ (i.e., the origin belongs to every environment set), as well as condition (###). For each $v \geq 1$, let every member e of E^v define an exchange economy that meets our assumptions ($\alpha 1$) – ($\alpha 3$), where the endowments $\{(V^i, W_\ell^i)\}_{i=1, \dots, n, \ell=1, \dots, L}$ are the same for every v and, for all the exchange economies defined by members of E^v , each trader's valuation function takes the form ($\dagger$). Let Λ^v be the incomplete Walrasian mechanism on the exchange-economy set E^v and let the sequence $\{\Lambda^v\}$ have uniformly bounded derivatives. For every $v \geq 1$, let Λ^{*v} be a continuum DR mechanism on the exchange-economy set E^v with an action space the same as that of Λ^v . Let the DR mechanism Λ^{*v} realize the same goal function as the Walrasian mechanism Λ^v .

For every $v \geq 1$, and for any fixed and sufficiently small ϵ , let $r(\epsilon, v)$ be a mesh such that the exact-outcome approximation $\Lambda_{r(\epsilon, v)}^{*v}$ of Λ^{*v} is cost-equivalent to the exact-outcome approximation Λ_ϵ^v . Then there exist $H > 0, L > 0, \hat{v} > 0$ such that

- $\lim_{v \rightarrow \infty} \mathbb{E}(\Lambda_{r(\epsilon, v)}^{*v}) = H$.
- for sufficiently small ϵ we have

$$\text{for all } v \geq \hat{v}, \left(\mathbb{E}(\Lambda_{r(\epsilon, v)}^{*v}) - \mathbb{E}(\Lambda_\epsilon^v) \right) \geq L.$$

VI. CONCLUDING REMARKS

In the exchange economy we have studied, traders have concave quasi-linear utility functions. That property permitted us to establish the uniqueness of the continuum Walrasian mechanism's solution function and its approximate-solution functions. Uniqueness is one of our regularity conditions and is not easy to establish.²⁸ It remains open whether the superiority of finite approximations of the Walrasian mechanism extends to a wider class of economies.

²⁸We note that most results about the uniqueness of Walrasian equilibrium, are achieved by imposing conditions on the excess demand functions, not on individual utility functions. (See [8], Section 5.7). But it is conditions of the latter sort that we need, in order to define the individual environment sets E_i on which our mechanism operates,

There are, moreover, other styles of approximation, in addition to the one we have studied.²⁹ . It remains open whether we can obtain analogues of our results for those alternative styles while imposing a regularity condition on the continuum mechanisms which is weaker than the one we have imposed. If so, it may be possible to obtain stronger results about the informational superiority of approximations to the Walrasian mechanism in a wide class of exchange economies.

While the subject is certainly not closed, our finite counterparts of the previous continuum statements substantially extend the claims one can make about the informational merit of mechanisms that use prices.

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²⁹In one alternative style of approximation, we assign to each e_i in E_i one of a finite collection of "surrogate environments". If $s(e_i)$ is the surrogate for e_i then in the finite approximation, with message set M_e , person i agrees to a message $m \in M_e$ if and only if i would agree to m in the continuum mechanism at the environment $s(e_i)$.

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APPENDIX: PROOFS OF LEMMA 3, LEMMA 4, AND PROPOSITION 5 IN THE PAPER
 "Comparing Finite Mechanisms"

The Lemmata and the Propositions in "Comparing Finite Mechanisms" are stated for the case of *two* persons.

Lemma 3

Part 1 (concerning exact-outcome approximations of a regular continuum projection mechanism)

Consider a regular continuum mechanism $\Lambda = \langle M, g_1, g_2, h \rangle$ on the compact set $E = E_1 \times E_2$, with action set $A \subset \mathbb{R}$. Suppose

- $M = A \times Y$, where $Y \subset \mathbb{R}^{D-1}$
- $g_i : M \times E_i \rightarrow \mathbb{R}^{D_i}, i = 1, 2$, where $D_1 + D_2 = D$
- for every $m = (a, y) \in M$, we have $h(m) = a$.

Then any complete class of (minimal-tolerance) exact-outcome approximations $\Lambda_\epsilon = \langle M_\epsilon, g_1^\epsilon, g_2^\epsilon, h^0 \rangle$ with $0 < \epsilon \leq \epsilon^\#$, is linearly error-bounded from below.

Part 2 (concerning exact-outcome approximations of a continuum DR mechanism)

Consider the regular continuum Direct Revelation mechanism $\Lambda^* = \langle M^*, g_1^*, g_2^*, h^* \rangle$ on the set $E = E_1 \times E_2$ with action set $A \subseteq \mathbb{R}$, where $M^* = E$. For $i = 1, 2$, let

$$E_i = \{e_i = (e_{i1}, \dots, e_{iJ_i}) : \bar{e}_{ij} \leq e_{ij} \leq \bar{\bar{e}}_{ij}, j = 1, \dots, J_i\},$$

where

$$\bar{e}_{ij} < \bar{\bar{e}}_{ij}, j = 1, \dots, J_i.$$

Suppose that

(*) the set $h^*(E) \subseteq \mathbb{R}$ contains a nondegenerate interval .

Suppose that

(**) $h^* : E \rightarrow A$ is C^1 .

Then any complete class of exact-outcome approximations $\Lambda_\epsilon^* = \langle M_\epsilon^*, g_1^{*\epsilon}, g_2^{*\epsilon}, h^0 \rangle$ is linearly error-bounded from below.

Part 3 (concerning rounded-outcome approximations of any regular continuum mechanism)

Consider a regular continuum mechanism $\Lambda = \langle M, g_1, g_2, h \rangle$ on the compact set $E = E_1 \times E_2$, with action set $A \subseteq \mathbb{R}$. Suppose that

(*) the set $h(t(E)) \subseteq \mathbb{R}$ contains a nondegenerate interval ,

where t denotes the unique thread of Λ . Let the function $\nu : \mathbb{R}^+ \rightarrow \mathbb{R}$ have the following property:

there exist $Q_1 > 0, Q_2 > 0$ such that for all sufficiently small positive x , we have

$$(***) \quad Q_1 \cdot x \geq \nu(x) \geq Q_2 \cdot x.$$

Then any complete class of (minimal-tolerance) rounded-outcome approximations $\Lambda_\epsilon^{\nu(\epsilon)} = \langle M_\epsilon, g_1^\epsilon, g_2^\epsilon, h^{\nu(\epsilon)} \rangle$ is linearly error-bounded from below.

Proof:

Part 1

Since Λ realizes γ and satisfies Regularity Condition (c), it has a unique thread $t = (t_a, t_y)$, where $t_a : E \rightarrow A$ and $t_y : E \rightarrow \mathbb{R}^{D-1}$. Consider the set

$$\{a = h(m) : m \text{ is an equilibrium message for } \epsilon; e \in E\} = h(t_a(E)) = t_a(E).$$

Since Λ satisfies Regularity Condition (d),

(+) the set $t_a(E)$ contains a nondegenerate interval in $\mathbb{R}$.

Since $m = A \times Y$ and $M_\epsilon = M \cap S_\epsilon^D$, it follows that M_ϵ is a Cartesian product $A_\epsilon \times Y_\epsilon$, where $A_\epsilon = A \cap S_\epsilon$ and $Y_\epsilon = Y \cap S_\epsilon^{D-1}$, and that for $m = (a, y) \in A_\epsilon \times Y_\epsilon$, we have

$$(++) \quad h^0(m) = a \in A_\epsilon.$$

It follows from (+) and (++) that for all sufficiently small ϵ there is an integer K , an environment $\hat{e} \in E$, and messages $(a^*, y^*), (a^{**}, y^{**}) \in M_\epsilon$ such that

$$(+++) \quad a^* = 2K \cdot \epsilon, \quad a^{**} = (2K + 2) \cdot \epsilon, \quad t_a(\hat{e}) = (2P + 1) \cdot \epsilon$$

and

$$(++++) \quad \text{for all } m = (a, y) \in M_\epsilon \text{ we have } |a - (2K + 1) \cdot \epsilon| \geq \max(|a - a^*|, |a - a^{**}|).$$

Note that no message in M_ϵ has an action component that is closer to $t_a(\hat{e})$ (the equilibrium action for $\hat{e}$ in the continuum-mechanism Λ) than a^* is, or than a^{**} is. Yet neither (a^*, y^*) nor (a^{**}, y^{**}) are necessarily equilibrium messages for $\hat{e}$ in the mechanism Λ_ϵ . Consequently, while a^*, a^{**} are actions that are available in the mechanism Λ_ϵ , neither of them need be an equilibrium action for the environment $\hat{e}$. Whether they are or not, it follows from (+++) and (++++) that

$$\text{error at } \hat{e} \text{ of } \Lambda_\epsilon \geq \epsilon.$$

That implies that for all sufficiently small ϵ

$$\mathbb{E}(\Lambda_\epsilon) \geq \epsilon.$$

So any complete class of mechanisms Λ_ϵ is indeed linearly error-bounded from below.

Part 2.

By assumption (*), the set $h^*(E)$ (the set of equilibrium actions for Λ^*) contains a nondegenerate closed real interval. By Assumption (**), all partial derivatives of h^* are continuous. That implies that for some pair, say $(\bar{i}, \bar{k})$, in $\{(i, k) : i \in \{1, 2\}; k \in \{1, \dots, J_i\}\}$, and some $e^* \in E$, we have

$$(\dagger) \quad \left. \frac{\partial h^*}{\partial e_{\bar{i}\bar{k}}} \right|_{e^*} \neq 0.$$

For notational simplicity, suppose, without loss of generality, that $(\bar{i}, \bar{k}) = (1, 1)$. Since the partial derivatives of h^* are assumed continuous, we have:

$$(\dagger\dagger) \quad \left. \frac{\partial h^*}{\partial e_{11}} \right|_e \neq 0 \text{ for all } e \text{ in some neighborhood, say } N_{e^*}, \text{ of } e^*.$$

The neighborhood N_{e^*} contains a closed nondegenerate rectangle, say $E^* \subset E$ such that $\left| \frac{\partial h^*}{\partial e_{11}} \right|_e$ has a nonzero minimum over all $e \in E^*$, i.e.,

$$(\dagger\dagger\dagger) \quad \text{there exists } \tau > 0 \text{ such that for all } e \in E^*, \left(\text{absolute value of } \left. \frac{\partial h^*}{\partial e_{11}} \right|_e \right) \geq \tau.$$

Moreover, for ϵ sufficiently small, there exists a message $\bar{m} = (\bar{m}_{11}, \dots, \bar{m}_{1J_1}, \bar{m}_{21}, \dots, \bar{m}_{2J_2}) \in M_\epsilon^*$ such that the rectangle E^* contains both the environment $\bar{m}$ and the environment

$$\bar{e} = (\bar{m}_{11} + \epsilon, \bar{m}_{12}, \dots, \bar{m}_{1J_1}, \bar{m}_{21}, \dots, \bar{m}_{2J_2}),$$

and (in view of (‡‡))

$$(‡‡‡‡) \quad h^*(\bar{e}) \neq h^*(\bar{m}).$$

Note that since $\bar{m}$ belongs to M_ϵ^* , each of its components is an integer multiple of 2ϵ . On the other hand M_ϵ^* does not contain a message equal to $\bar{e}$, since the first component of $\bar{e}$ is not an integer multiple of 2ϵ .

The message $\bar{m}$ is one of two equilibrium messages for the environment $\bar{e}$. The other is

$$\hat{m} = (\bar{m}_{11} + 2\epsilon, \bar{m}_{12}, \dots, \bar{m}_{1J_1}, \bar{m}_{21}, \dots, \bar{m}_{2J_2}).$$

Consider first the message $\bar{m}$. In view of assumption (**), and the fact that $\bar{m}$ and $\bar{e}$ differ only in their first components, we can use the Mean Value Theorem to obtain

$$(‡‡‡‡‡‡) \quad h^*(\bar{e}) - h^*(\bar{m}) = \left. \frac{\partial h^*}{\partial e_{11}} \right|_z \cdot \epsilon,$$

where $z \in \mathcal{R}$ and

$$\bar{m}_{11} < z < \bar{m}_{11} + \epsilon.$$

Next consider the message $\hat{m}$. The Mean Value Theorem now gives us

$$(‡‡‡‡‡‡‡‡) \quad h^*(\bar{e}) - h^*(\hat{m}) = \left. \frac{\partial h^*}{\partial e_{11}} \right|_{\hat{z}} \cdot \epsilon,$$

where

$$\bar{m}_{11} + \epsilon \leq \hat{z} \leq \bar{m}_{11} + 2\epsilon.$$

Hence

$$(\diamond) \quad \text{error at } \bar{e} \text{ of } \Lambda_\epsilon^* = \max(|h^*(\bar{e}) - h^*(\bar{m})|, |h^*(\bar{e}) - h^*(\hat{m})|).$$

In view of (‡‡‡‡), (‡‡‡‡‡), (‡‡‡‡‡‡‡), (‡‡‡‡‡‡‡‡), we have, for ϵ sufficiently small,

$$(\diamond\diamond) \quad |h^*(\bar{e}) - h^*(\bar{m})| \geq \tau \cdot \epsilon, |h^*(\bar{e}) - h^*(\hat{m})| \geq \tau \cdot \epsilon.$$

We conclude, using $(\diamond)$, $(\diamond, \diamond)$, that for ϵ sufficiently small,

$$\mathbb{E}(\Lambda_\epsilon^*) \geq \tau \cdot \epsilon.$$

So any complete class of approximations Λ_ϵ^* is indeed linearly error-bounded from below.

Part 3

By assumption (**), there is a nondegenerate real interval, say I , all of whose points are equilibrium actions of the continuum mechanism Λ . The interval I is a subset of the action set A . For every point P in I , there is an environment e in E such that $P = h(t(e))$, where t is the unique thread of Λ .

On the other hand, the actions available in the finite approximation $\Lambda_\epsilon^{\nu(\epsilon)}$, i.e. the values taken by $h^{\nu(\epsilon)}$, belong to the set $A \cup S_{\nu(\epsilon)} \subset \mathbb{R}$.

Note that

if $\nu(\epsilon)$ is sufficiently small, then the interval I contains two adjacent points of $A \cup S_{\nu(\epsilon)}$.

But it follows from assumption (****) that

$$\lim_{\epsilon \rightarrow 0} \nu(\epsilon) = 0.$$

Hence

(Δ) if ϵ is sufficiently small, then the interval I contains two adjacent points of $A \cup S_{\nu(\epsilon)}$.

Call the two adjacent points P' and P'' , with $P' < P''$. Each is an integer multiple of $2\nu(\epsilon)$. Consider the point midway between them, i.e.,

$$P^* = \frac{1}{2}(P' + P'').$$

That point is *not* an available action in $\Lambda_\epsilon^{\nu(\epsilon)}$. Let e^* satisfy

$$h(t(e^*)) = P^*.$$

The points P', P'' may or may not be equilibrium actions for e^* in the mechanism $\Lambda_\epsilon^{\nu(\epsilon)}$, but no action available in $\Lambda_\epsilon^{\nu(\epsilon)}$ is closer to $h(t(e^*))$ than P' is and than P'' is. Hence we have

$$(\Delta\Delta) \quad \text{error at } e^* \text{ of } \Lambda_\epsilon^{\nu(\epsilon)} \geq P'' - P^* = P^* - P' = \nu(\epsilon).$$

It now follows from (Δ), ($\Delta\Delta$), and assumption (****), that

$$\mathbb{E}(\Lambda_\epsilon^{\nu(\epsilon)}) \geq Q_2 \cdot \epsilon.$$

So any complete class of finite approximations $\Lambda_\epsilon^{\nu(\epsilon)}$ is indeed linearly error-bounded from below. ■

We now turn to Lemma 4, which concerns *mesh-boundedness*. We first recall the following Definition:

A class $\mathcal{B}$ of finite approximations of a continuum mechanism Λ is *linearly mesh-bounded for small errors* (abbreviated as *mesh-bounded*) if there exist $H > 0, \zeta > 0$ such that

$$\text{if } \mathbf{L} \in \mathcal{B} \text{ has mesh } \epsilon \text{ and } \mathbb{E}(\mathbf{L}) \leq H, \text{ then } \epsilon \leq \zeta \cdot \mathbb{E}(\mathbf{L}).$$

Lemma 4

Part 1 (concerning exact-outcome approximations of a regular continuum projection mechanism)

Consider a regular continuum mechanism $\Lambda = \langle M, g_1, g_2, h \rangle$ on the compact set $E = E_1 \times E_2$, with action set $A \subset \mathbb{R}$. Suppose

- $M = A \times Y$, where $Y \subset \mathbb{R}^{D-1}$
- $g_i : M \times E_i \rightarrow \mathbb{R}^{D_i}, i = 1, 2$, where $D_1 + D_2 = D$
- for every $m = (a, y) \in M$, we have $h(m) = a$.

Then any complete class of (minimal-tolerance) exact-outcome approximations $\Lambda_\epsilon = \langle M_\epsilon, g_1^\epsilon, g_2^\epsilon, h^0 \rangle$ is linearly mesh-bounded for small errors.

Part 2 (concerning exact-outcome approximations of a continuum DR mechanism)

Consider the regular continuum Direct Revelation mechanism $\Lambda^* = \langle M^*, g_1^*, g_2^*, h^* \rangle$ on the set $E = E_1 \times E_2$ with action set $A \subseteq \mathbb{R}$, where $M^* = E$. For $i = 1, 2$, let

$$E_i = \{e_i = (e_{i1}, \dots, e_{iJ_i}) : \bar{e}_{ij} \leq e_{ij} \leq \bar{\bar{e}}_{ij}, j = 1, \dots, J_i\},$$

where

$$\bar{e}_{ij} < \bar{\bar{e}}_{ij}, j = 1, \dots, J_i.$$

Suppose that

$$(\dagger) \left\{ \begin{array}{l} \text{for some} \\ (\bar{i}, \bar{k}) \in \{(i, k) : i = 1, k = 1, \dots, J_1 \text{ or } i = 2, j = 1, \dots, J_2\} \\ \text{and for some fixed } (J-1)\text{-tuple} \\ \{e_{ik}^*\}_{i \neq \bar{i}, k \neq \bar{k}} \\ \text{with} \\ \bar{e}_{ik} \leq e_{ik}^* \leq \bar{\bar{e}}_{ik} \text{ for } i \neq \bar{i}, k \neq \bar{k}, \\ \text{the function } h \text{ is nonconstant with respect to } e_{\bar{i}\bar{k}}. \end{array} \right.$$

Then any complete class of exact-outcome approximations $\Lambda_\epsilon^* = \langle M_\epsilon^*, g_1^{*\epsilon}, g_2^{*\epsilon}, h^0 \rangle$ is linearly mesh-bounded for small errors.

Part 3 (concerning rounded-outcome approximations of any regular continuum mechanism)

Consider a regular continuum mechanism $\Lambda = \langle M, g_1, g_2, h \rangle$ on the compact set $E = E_1 \times E_2$, with action set $A \subseteq \mathbb{R}$. Suppose that

(††) the set $h(t(E)) \subseteq \mathbb{R}$ contains a nondegenerate interval ,

where t denotes the unique thread of Λ . Let the function $\nu : \mathbb{R}^+ \rightarrow \mathbb{R}$ have the following property:

there exist $Q_1 > 0, Q_2 > 0$ such that for all sufficiently small positive x , we have

$$(\dagger \dagger \dagger) \quad Q_1 \cdot x \geq \nu(x) \geq Q_2 \cdot x.$$

Then any complete class of (minimal-tolerance) rounded-outcome approximations $\Lambda_\epsilon^{\nu(\epsilon)} = \langle M_\epsilon, g_1^\epsilon, g_2^\epsilon, h^{\nu(\epsilon)} \rangle$ is linearly mesh-bounded for small errors.

Proof:

Part 1

We refer to the proof of Part 1 of Lemma 3. As argued there, the thread of Λ is a pair (t_a, t_y) and the set $t_a(E)$ contains an interval, say $[V, W]$, with $V < W$. The set of equilibrium actions of the approximation Λ_ϵ is $t_a(E) \cap S_\epsilon$. Suppose that for Λ_ϵ we have

$$(1-1) \quad \mathbb{E}(\Lambda_\epsilon) < H^* \equiv \frac{W - V}{2}.$$

Then

$$(1-2) \quad \epsilon \leq \frac{W - V}{2}.$$

For suppose not. Then $[V, W] \cap S_\epsilon$ contains at most one equilibrium action, say a^* , of Λ_ϵ . It then follows that if e^*, e^{**} are environments for which $t_a(e^*) = V, t_a(e^{**}) = W$, we have

$$\mathbb{E}(\Lambda_\epsilon) \geq \max(\text{error at } V \text{ of } \Lambda_\epsilon, \text{error at } W \text{ of } \Lambda_\epsilon) = \max(W - a^*, a^* - V) \geq \frac{W - V}{2} = H^*.$$

That contradicts (1-1). So (1-1) indeed implies (1-2).

Hence, using (1-1) and (1-2), we indeed have the claimed linear mesh-boundedness for small errors, if we choose $H < H^*$ and $\zeta = 1$. ■

Part 2

Without loss of generality we may let the pair $(\bar{i}, \bar{k})$ in Condition $(\dagger)$ be the pair $(1, 1)$.

Recall that $e = (e_1, e_2)$, $e_1 = (e_{11}, \dots, e_{1J_1})$, $e_2 = (e_{21}, \dots, e_{2J_2})$, and $J = J_1 + J_2$.

Condition $(\dagger)$ assures us that for some $(J - 1)$ -tuple

$$e_{-(11)}^* = (e_{12}^*, \dots, e_{1J_1}^*, e_{21}^*, \dots, e_{2J_2}^*)$$

(with $\bar{e}_{ij} \leq e_{ij}^* \leq \bar{\bar{e}}_{ij}$ for $i = 1, j = 2, \dots, J_1$ and for $i = 2, j = 1, \dots, J_2$), the function

$$\hat{h} : [\bar{e}_{11}, \bar{\bar{e}}_{11}] \rightarrow \mathbb{R}$$

defined by:

$$\hat{h}(e_{11}) = h(e_{11}, e_{-(11)}^*)$$

is nonconstant and is also $\mathcal{C}^1$ (since Λ^* is regular).

It will sometimes be convenient to use the abbreviation

$$E_{-(11)} \equiv \{(e_{12}, \dots, e_{1J_1}, e_{21}, \dots, e_{2J_2}) : \bar{e}_{ij} \leq e_{ij} \leq \bar{\bar{e}}_{ij} \text{ for } i = 1, j = 2, \dots, J_1, \text{ and for } i = 2, j = 1, \dots, J_2\}$$

Since $\hat{h}$ is non-constant and $\mathcal{C}^1$, it follows that

there exists A, B with $\bar{e}_{11} \leq A < B \leq \bar{\bar{e}}_{11}$, such that $\tilde{h}$ is strictly monotonic on $[A, B]$.

Define

$$\Gamma \equiv \min \left(\left| \tilde{h}(A) - \tilde{h} \left(\frac{A+B}{2} \right) \right|, \left| \tilde{h}(B) - \tilde{h} \left(\frac{A+B}{2} \right) \right| \right).$$

Define

$$(2-1) \quad \xi \equiv \min \{ |\tilde{h}'(e_{11})| : e_{11} \in [A, B] \}.$$

The minimum exists, **and it is positive**, since $\tilde{h}$ is strictly monotonic on $[A, B]$. Finally, define

$$(2-2) \quad H^* \equiv \frac{1}{2} \Gamma$$

and

$$(2-3) \quad \zeta \equiv \frac{2}{\xi}.$$

To prove Part 2 of the Lemma, we shall show:

$$(*) \quad \left\{ \begin{array}{l} \text{If, for the approximation } \Lambda_\epsilon^*, \text{ we have } \mathbb{E}(\Lambda_\epsilon^*) \leq H < H^*, \text{ then} \\ \epsilon \leq \zeta \cdot \mathbb{E}(\Lambda_\epsilon^*). \end{array} \right.$$

In showing (*), we shall use the following general fact about exact-outcome approximations of any regular continuum mechanism, whether or not the continuum mechanism has the DR property.

$$(**) \quad \left\{ \begin{array}{l} \text{Suppose } \Lambda, \text{ with outcome function } h, \text{ is a regular continuum mechanism on } E, \\ \text{realizing the goal function } \gamma : E \rightarrow \mathbb{R}. \text{ Consider the exact-outcome approx-} \\ \text{imation } \Lambda_\epsilon. \text{ Suppose that in that approximation, two distinct environments} \\ \text{share an equilibrium message, i.e., some message } m \text{ is an equilibrium message} \\ \text{for } e \in E \text{ and also for } e' \in E, \text{ where } e \neq e'. \text{ Then a lower bound to the overall} \\ \text{error of } \Lambda_\epsilon \text{ is half the “}\gamma\text{-distance” between } e \text{ and } e', \text{ i.e.,} \\ \mathbb{E}(\Lambda_\epsilon) \geq \frac{1}{2} \cdot |\gamma(e) - \gamma(e')|. \end{array} \right.$$

To see that this is true, recall that realization of the goal function γ means that for any e in E we have $\gamma(e) = h(t(e))$, where t is the unique thread of Λ (i.e., $m = t(e)$ if m is an equilibrium message for e). It follows from the definition of “error at e of Λ_ϵ ” that

$$\text{error at } e \text{ of } \Lambda_\epsilon \geq |\gamma(e) - h(m)|$$

and

$$\text{error at } e' \text{ of } \Lambda_\epsilon \geq |\gamma(e') - h(m)|.$$

But the triangle inequality tells us that

$$|\gamma(e) - h(m)| + |\gamma(e') - h(m)| \geq |\gamma(e) - \gamma(e')|.$$

That implies that

$$\max(|\gamma(e) - h(m)|, |\gamma(e') - h(m)|) \geq \frac{1}{2} \cdot |\gamma(e) - \gamma(e')|.$$

Then (**) follows, since

$$\mathbb{E}(\Lambda_\epsilon) \geq \max(\text{error at } e \text{ of } \Lambda_\epsilon, \text{error at } e' \text{ of } \Lambda_\epsilon).$$

In our case, the continuum mechanism being approximated is the DR mechanism Λ^* , the thread t is the identity function, and hence $\gamma = h$.

We now show

$$(2-4) \quad \left\{ \begin{array}{l} \text{if } \text{OE}(\Lambda_\epsilon^*) < H^*, \text{ then the lattice } S_\epsilon \text{ contains two adjacent points, say } r \text{ and } r + 2\epsilon, \text{ such that} \\ \text{(a)} \quad A \leq r < r + \epsilon \leq B \\ \text{or} \\ \text{(b)} \quad A \leq r + \epsilon < r + 2\epsilon \leq B. \end{array} \right.$$

Note first that if $[A, B]$ contains two or more points of S_ϵ , then we can immediately claim the existence of $r, r + 2\epsilon$ in S_ϵ such that both (a) and (b) are satisfied. Consider, therefore, the case where $[A, B]$ contains at most one member of S_ϵ . Consider the possible ways in which both (a) and (b) can then fail.

There are the following three possibilities.

Case (i)

For $r \in S_\epsilon$ we have

$$r < A < \frac{A+B}{2} \leq r + \epsilon \leq B.$$

Then for some $m_{-(11)} \in S_\epsilon^{J-1} \cap E_{-(11)}$, the message $(r, m_{-(11)})$ is an equilibrium message of Λ_ϵ^* for the environment $(A, e_{-(11)}^*)$ and also for the environment $(\frac{A+B}{2}, e_{-(11)}^*)$. Using our general fact (**), and the definition of H^* in (2-2), we then have

$$\text{OE}(\Lambda_\epsilon^*) \geq \frac{1}{2} \cdot \left| \tilde{h}(A) - \tilde{h}\left(\frac{A+B}{2}\right) \right| \geq H^*.$$

Case (ii)

For $r \in S_\epsilon$ we have

$$A \leq r \leq \frac{A+B}{2} < B < r + \epsilon.$$

Then, for some $m_{-(11)} \in S_\epsilon^{J-1} \cap E_{-(11)}$, the message $(r, m_{-(11)})$ is an equilibrium message of Λ_ϵ^* for the environment $(\frac{A+B}{2}, e_{11}^*)$ and also for the environment (B, e_{11}^*) . Using our general fact (**), we then have

$$\text{OE}(\Lambda_\epsilon^*) \geq \frac{1}{2} \cdot \left| \tilde{h}(B) - \tilde{h}\left(\frac{A+B}{2}\right) \right| \geq H^*.$$

Case (iii)

For $r \in S_\epsilon$ we have

$$r < A < B < r + \epsilon.$$

Then, just as in Case (i), for some $m_{-(11)} \in S_\epsilon^{J-1} \cap E_{-(11)}$, the message $(r, m_{-(11)})$ is an equilibrium message of Λ_ϵ^* for the environment $(A, e_{-(11)}^*)$ and also for the environment $(\frac{A+B}{2}, e_{-(11)}^*)$. Just as in Case (i), we have

$$\text{OE}(\Lambda_\epsilon^*) \geq H^*.$$

So (2-4) is established.

Now suppose (a) of (2-4) holds. Consider the environments $(r, e_{-(11)}^*)$ and $(r + \epsilon, e_{-(11)}^*)$. For some $m_{-(11)} \in S_\epsilon^{J-1} \cap E_{-(11)}$, the message $(r, m_{-(11)})$ is an equilibrium message of the mechanism Λ_ϵ^* for both environments. Using our general fact (**), we have

$$(2-5) \quad \text{OE}(\Lambda_\epsilon^*) \geq \frac{1}{2} \cdot |\tilde{h}(r) - \tilde{h}(r + \epsilon)|.$$

Using the Mean Value Theorem and the fact that $[r, r + \epsilon] \subseteq [A, B]$, we obtain, for some $z \in [r, r + \epsilon]$ and for the positive number ξ of (2-1):

$$(2-6) \quad |\tilde{h}(r) - \tilde{h}(r + \epsilon)| = |\tilde{h}'(z)| \cdot \epsilon \geq \xi \epsilon.$$

On the other hand, suppose (b) of (2-4) holds. Consider the environments $(r + \epsilon, e_{-(11)}^*)$ and $(r + 2\epsilon, e_{-(11)}^*)$. For some $m_{-(11)} \in S_\epsilon^{J-1} \cap E_{-(11)}$, the message $(r + 2\epsilon, m_{-(11)})$ is an equilibrium message of the mechanism Λ_ϵ^* for both environments. Using our general fact (**), we have

$$(2-7) \quad \text{OE}(\Lambda_\epsilon^*) \geq \frac{1}{2} \cdot |\tilde{h}(r + \epsilon) - \tilde{h}(r + 2\epsilon)|.$$

Again using the Mean Value Theorem and the fact that $[r, r + \epsilon] \subseteq [A, B]$, we obtain, for some $z \in [r + \epsilon, r + 2\epsilon]$,

$$(2-8) \quad |\tilde{h}(r + \epsilon) - \tilde{h}(r + 2\epsilon)| = |\tilde{h}'(z)| \cdot \epsilon \geq \xi \epsilon.$$

Thus (2-5)-(2-8) imply (whether ((a)) holds or ((b)) holds) that for $\zeta = \frac{2}{\xi}$ (as in (2-3)):

$$\text{if } \text{OE}(\Lambda_\epsilon^*) \leq H < H^*, \text{ then } \text{OE}(\Lambda_\epsilon^*) \geq \left(\frac{\xi}{2}\right) \cdot \epsilon \text{ and hence } \epsilon \leq \zeta \cdot \text{OE}(\Lambda_\epsilon^*).$$

That completes the proof. ■

Part 3

By assumption, the continuum mechanism Λ obeys condition $(\dagger\dagger)$. That implies that the set of equilibrium actions of Λ include some nondegenerate real interval $I = [K, L]$. The set of equilibrium actions of the approximation Λ_ϵ includes the set $[K, L] \cap S_{\nu(\epsilon)}$.

The remaining argument is analogous to that used in the above proof of Part 1 of the present Lemma.

Suppose that for $\Lambda_\epsilon^{\nu(\epsilon)}$ we have

$$(3-1) \quad \mathbb{E}(\Lambda_\epsilon^{\nu(\epsilon)}) < \frac{L - K}{2}.$$

Then

$$(3-2) \quad \nu(\epsilon) \leq \frac{L - K}{2}.$$

For suppose not. Then $[K, L] \cap S_{\nu(\epsilon)}$ contains at most one equilibrium action, say a^* , of $\Lambda_\epsilon^{\nu(\epsilon)}$. It then follows that if e^*, e^{**} are environments for which $t(e^*) = L, t(e^{**}) = K$, where t is the unique thread of Λ , then we have

$$\mathbb{E}(\Lambda_\epsilon^{\nu(\epsilon)}) \geq \max \left(\text{error at } K \text{ of } \Lambda_\epsilon^{\nu(\epsilon)}, \text{error at } L \text{ of } \Lambda_\epsilon^{\nu(\epsilon)} \right) = \max(L - a^*, a^* - K) \geq \frac{L - K}{2}.$$

That contradicts (3-1). So (3-1) indeed implies (3-2).

By assumption, the approximation $\Lambda_\epsilon^{\nu(\epsilon)}$ obeys condition $(\dagger \dagger \dagger)$, i.e.,

$$\epsilon \leq \frac{1}{Q_2} \cdot \nu(\epsilon).$$

Hence, in view of (3-1) and (3-2):

$$\text{if } \mathbb{E}(\Lambda_\epsilon^{\nu(\epsilon)}) < \frac{L - K}{2}, \text{ then } \epsilon \leq \frac{L - K}{2Q_2}.$$

Thus for $H < \frac{L-K}{2}$, we have

$$\text{if } \mathbb{E}(\Lambda_\epsilon^{\nu(\epsilon)}) < H, \text{ then } \epsilon \leq \frac{1}{Q_2} \cdot \mathbb{E}(\Lambda_\epsilon^{\nu(\epsilon)}).$$

So a complete class of approximations $\Lambda_\epsilon^{\nu(\epsilon)}$ meeting the conditions $(\dagger\dagger), (\dagger \dagger \dagger)$ is indeed linearly mesh-bounded for small errors. ■

We now turn to the proof of Proposition 5 in “Comparing Finite Mechanisms”. That proposition is the two-person version of Proposition D of the present paper. The Proposition uses the conditions

(i) – (iv) and (vii) – (ix), which are given in the present paper, just before the statement of Proposition D.

Proposition 5 (Concerning a limit for DR error and a lower bound on DR-versus-indirect error differential when there is an unchanging action cube for the continuum DR mechanisms).

Suppose that the environment sequence $\{E^v\}$, the indirect-mechanism sequence $\{\Lambda^v\}$, and the DR sequence $\{\Lambda^{*v}\}$ satisfy (i) – (iv) and (vii) – (ix). Suppose in addition that the DR sequence $\{\Lambda^{*v}\}$ has the property (+).

Part A

For all sufficiently small ϵ , the following statements A1 and A2 hold.

A1 We have

$$(I) \quad M^v \cap S_\epsilon^D \neq \emptyset \text{ for all } v \geq 1,$$

(and hence the minimal-tolerance exact-outcome approximation Λ_ϵ^v exists for all $v \geq 1$).

A2 For all $v \geq 1$, there exists a number $r(\epsilon, v) > 0$ such that for all $v \geq 1$

$$(II) \quad E^{J^v} \cap S_{r(\epsilon, v)}^{J^v} \neq \emptyset \text{ (and hence the exact-outcome approximation } \Lambda_{r(\epsilon, v)}^{*v} \text{ exists)}$$

$$(III) \quad \Lambda_{r(\epsilon, v)}^{*v} \text{ is cost-equivalent to } \Lambda_\epsilon^v, \text{ i.e., } C(\Lambda_{r(\epsilon, v)}^{*v}) \leq C(\Lambda_\epsilon^v).$$

Part B Suppose that there exist numbers T, U such that for all v , all $e \in E^v$, and every component $h_k^v : E^v \rightarrow \mathbb{R}$ of the outcome function of Λ^{*v} , we have

$$(\dagger) \quad T \leq h_k^v(e) \leq U.$$

Suppose that the sequence $\{\Lambda^v\}$ of indirect mechanisms has uniformly bounded derivatives, and that the DR sequence's sparse approximations are monotone. Then there exists $H > 0$ such that for all sufficiently small ϵ and for the function r of A2

$$(IV) \quad \lim_{v \rightarrow \infty} \mathbb{E}(\Lambda_{r(\epsilon, v)}^{*v}) = H,$$

and there exists $\hat{v} > 0, L > 0$ such that for any sufficiently small ϵ , we have

$$(V). \quad \text{for all } v \geq \hat{v}, \left(\mathbb{E}(\Lambda_{r(\epsilon, v)}^{*v}) - \mathbb{E}(\Lambda_\epsilon^v) \right) \geq L.$$

Proof:

Part A

First we note that for sufficiently small ϵ , condition (I) holds. That is the case since, by assumption, for all $v \geq 1$, the mechanism Λ^v obeys condition (vii) and hence M^v contains the nondegenerate cube $M' = \underbrace{[r, s] \times \cdots \times [r, s]}_{D \text{ times}}$, where $r < s$. For a sufficiently small ϵ , say $\check{\epsilon}$, there

exists an integer ℓ such that the nondegenerate interval $[r, s]$ contains the points $(\ell) \cdot (2\check{\epsilon})$ and $(\ell + 1) \cdot (2\check{\epsilon})$. That implies that for all positive ϵ smaller than $\check{\epsilon}$, there exists an integer $\bar{\ell}$ such that the interval $[r, s]$ contains the points $(\bar{\ell}) \cdot (2\epsilon)$ and $(\bar{\ell} + 1) \cdot (2\epsilon)$. That implies, in turn, that for all positive ϵ smaller than $\check{\epsilon}$, condition (I) holds. That establishes **A1**.

Using condition (ix), we also have

$$C(\Lambda_\epsilon^v) \geq \#(M' \cap S_\epsilon^D).$$

Hence we can make $C(\Lambda_\epsilon^v)$ as large as desired by taking the mesh ϵ sufficiently small. In particular, we can take ϵ small enough so that $C(\Lambda_\epsilon^v)$ is not less than the integer Ψ of Condition (+). Condition (+) assures us that for such an ϵ , there is, for every $v \geq 1$, a mesh $r(\epsilon, v)$ so that (II) and (III) are satisfied. That establishes **A2**.

Part B

We repeat an argument found in the proof of Lemma 2. In view of the uniform boundedness of $\{\Lambda^{*v}\}$, that argument tells us that for all $v \geq 1$, for $\epsilon^* = \bar{\delta}/\xi$ and for $G = D^3\xi\rho\lambda$:

$$\text{(VI)} \quad \mathbb{E}(\Lambda_\epsilon^{*v}) \leq G \cdot \epsilon \text{ if } 0 < \epsilon \leq \epsilon^*.$$

Now by (ix), for all v , we have $M^v \subseteq M \subset \mathbb{R}^D$. Hence for any approximation Λ_ϵ^v , the number of equilibrium messages does not exceed $H_\epsilon \equiv \#(M \cap S_\epsilon^D)$ (where $\#$ denotes number of elements), i.e.,

$$C(\Lambda_\epsilon^v) \leq H_\epsilon.$$

Therefore, by (III),

$$\text{(VII)} \quad \text{for all } v, C(\Lambda_{r(\epsilon, v)}^{*v}) \leq H_\epsilon.$$

Now for any $\epsilon > 0$ such that the approximation $\Lambda_{r(\epsilon, v)}^{*v}$ exists, we have, using (VII) and (iv),

$$\text{(VIII)} \quad \text{for all sufficiently large } v, 2^{J^v} > H_\epsilon \text{ and the approximation } \Lambda_{r(\epsilon, v)}^{*v} \text{ is sparse.}$$

Assumption (†) implies that there exists $Q > 0$ such that

(IX) for any $v \geq 1$ and for any $\epsilon > 0$ such that the approximation $\Lambda_{r(\epsilon, v)}^{*v}$ exists, we have $\mathbb{E}(\Lambda_{r(\epsilon, v)}^{*v}) \leq Q$.

Consider an ϵ small enough so that the approximation $\Lambda_{r(\epsilon, v)}^{*v}$ exists. By (VII and (VIII), there exists an integer $\bar{v}$ such that

(X) for $v \geq \bar{v}$, $\Lambda_{r(\epsilon, v)}^{*v}$ is sparse and $C(\Lambda_{r(\epsilon, v)}^{*v}) \leq H_\epsilon$.

By assumption, the sparse approximations of the mechanisms in the sequence $\{\Lambda_{r(\epsilon, v)}^{*v}\}$ are monotone. Let H_ϵ play the role of the number B in our definition of “monotone”. Then, using (X), we have

$$v'' > v' \geq \bar{v} \Rightarrow \mathbb{E}(\Lambda_{r(\epsilon, v'')}^{*v''}) \geq \mathbb{E}(\Lambda_{r(\epsilon, v')}^{*v'}).$$

Thus the overall errors of the approximations $\Lambda_{r(\epsilon, v)}^{*v}$ form a nondecreasing sequence for $v \geq \bar{v}$. That fact, together with (IX), implies that the sequence goes to a limit, i.e.,

$$\text{there exists } N > 0 \text{ such that } \lim_{v \rightarrow \infty} \mathbb{E}(\Lambda_{r(\epsilon, v)}^{*v}) = N.$$

That establishes statement (IV).

By (VI) and (VIII), there exists $\hat{v} > 0$ and $\hat{\epsilon} > 0$ such that for $0 < \epsilon \leq \hat{\epsilon}$, the mechanism $\Lambda_{r(\epsilon, \hat{v})}^{*\hat{v}}$ is sparse and has an overall error that exceeds $G\epsilon$. Since, by assumption, the sparse DR approximations are monotone, we have, for any $v \geq \hat{v}$, that $\Lambda_{r(\epsilon, v)}^{*v}$ is sparse and has an overall error at least as large as that of $\Lambda_{r(\epsilon, \hat{v})}^{*\hat{v}}$. Hence for $0 < \epsilon \leq \hat{\epsilon}$ and for all $v \geq \hat{v}$, the difference

$$\mathbb{E}(\Lambda_{r(\epsilon, v)}^{*v}) - \mathbb{E}(\Lambda_\epsilon^v)$$

is bounded from below by the positive number

$$L = \mathbb{E}(\Lambda_{r(\epsilon, \hat{v})}^{*\hat{v}}) - G\epsilon.$$

That establishes statement (V) and completes the proof of Part B. ■