NONTRANSITIVE-NONTOTAL CONSUMER THEORY

by

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Abstract

We show the parallel nature of two approaches to nontransitive or nontotal consumers: through "weak" preferences and through "strict" preferences. This yields specific results (e.g., a new equilibrium existence theorem with weak preferences), general results (a Metatheorem translating between the two approaches), and general concepts (a new notion of "rationality"). We give revealed preference axioms to characterize preferences, and prove equivalences among several axiom systems, showing that the apparent weakness of some axiom systems is illusory. We resolve a Weak Axiom conjecture, and we introduce a new axiom. We prove that our nonclassical consumers generalize classical equilibrium theory.

We are indebted to Nicholas Yannelis for many discussions, and for his valuable insights on equilibrium existence theorems. We are also grateful to Donald Campbell and Andreu Mas-Colell for comments on an earlier version.

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1. INTRODUCTION

Does competitive equilibrium exist when some consumers have nontransitive or nontotal preferences? In recent years, several studies have extended neoclassical equilibrium theory to allow such consumers. In this paper we examine the behavior of these consumers, and provide revealed preference conditions to characterize them.

There are two common approaches to preference theory: one through "weak" (i.e., reflexive) preferences, and one through "strict" (i.e., irreflexive) preferences. The distinction becomes important when preferences are not total, so our first aim is to show that the two approaches are equally valid: definitions, axioms, and theorems in one approach correspond to definitions, axioms, and theorems in the other approach.

This correspondence is not at all clear from the crooked historical path that new developments have followed in recent years, finding some results for weak preferences and others for strict preferences. (See Section 2 below.) Indeed, one may be left with the impression that the various results in nontransitive-nontotal consumer theory depend on which approach is used -- weak or strict preference. Our results correct that impression, making it clear that each result under one approach has a corresponding result under the other approach.

In this way, our theorems enlarge the class of known results for nontransitive-nontotal consumer theory. Theorem 1, for example, is a new equilibrium existence result. And this strengthens the evidence that one can drop transitivity and totality assumptions from classical consumer theory, while retaining many of the important results.

Our second aim is to show that different hypotheses on preferences frequently lead to identical demand behavior. In part, we do this by characterizing rationality types through revealed preference axioms (cf. Remark 8); and, in part, we do it by detailed construction (cf. Theorems 9 and 11). Again, an important consequence is that one can frequently drop the totality assumption without much cost. Similarly, one can relax preference continuity somewhat.

Our third aim is to settle a conjecture raised by Kihlstrom, Mas-Colell, Sonnenschein, and Shafer [13] on what kind of preferences the Weak Axiom of Revealed Preferences characterized. We show that the conjecture is false, but under additional hypotheses we show it is true.

After outlining some history in Section 2, we define equilibrium notions for the weak and strong preference approaches in Section 3, illustrating them with two existence theorems. In these sections, the emphasis is on preferences.

In Section 4 we introduce a new notion, motivation, to parallel the notion of rationality introduced in [18]. And we provide some revealed preference links between the two notions. In this section, the emphasis shifts from preferences to behavior.

Section 5 gives revealed preference characterizations of preference types suitable for proving existence of competitive equilibrium.

Section 6 outlines an abstract theory of conjugate preferences as a framework for viewing the earlier sections. A metatheorem clarifies the parallel nature of the approaches through weak and strict preferences.

In Section 7 we specialize our analysis, limiting attention to singleton-valued demands, and to the Weak Axiom of Revealed Preference. We introduce a new axiom, the C-Axiom, to characterize the demand behavior of an important type of "equilibrating" preferences. And we settle a conjecture of Kihlstrom, Mas-Colell, Sonnenschein, and Shafer [13].

2. HISTORY

Neoclassical theories assumed that each consumer maximized a preference relation $\succsim$ over a "commodity set" $X \subseteq R^n$, subject to perfectly competitive budget constraints. Typically, each $\succsim$ was assumed to be

$$\text{reflexive: } x \succsim x \quad (2.1)$$

$$\text{total: } x \neq y \Rightarrow x \succ y \text{ or } y \succ x \quad (2.2)$$

$$\text{transitive: } x \succ y \text{ \& } y \succ z \Rightarrow x \succ z \quad (2.3)$$

$$\text{continuous: } \{w \in X: w \succ y\} \text{ and } \{w \in X: y \succ w\} \text{ are closed in } X \quad (2.4)$$

$$\text{convex: } \{w \in X: w \succ y\} \text{ is convex.} \quad (2.5)$$

In particular, preferences were

$$\text{complete: } x \succsim y \text{ or } y \succsim x. \quad (2.6)$$

An apparent break from that tradition was made by Schmeidler [20], who proved existence of equilibrium with strict preferences, and emphasized the incompleteness (nontotality) of his preferences. He retained transitivity, so he could dispense with convexity conditions in his continuum model.

A different break was made by Sonnenschein [24], who showed that, if we drop transitivity from our hypotheses, then, demand correspondences will still be nonempty-valued and upper-hemicontinuous, and equilibrium existence proofs are still possible. Naturally, the definitions of competitive equilibrium differed between the strict and the weak preference approaches.

Then Mas-Colell returned to the strict preference approach and showed that transitivity could be dropped [15]. Having dropped transitivity, he needed some convexity assumption on strict preferences.

Increasingly general existence results were obtained by Gale and Mas-Colell [8], [9], Shafer and Sonnenschein [22], and Yannelis and Prabhakar [26], in the strict preference framework. Theorem 2, below, is a fruit of this development. Using the definition of equilibrium appropriate for strict preferences, there is no transitivity or totality. The only assumptions are open sections for the upper and lower preference sets, and a mild convexity condition.

While equilibrium theory was zigzagging between the weak and strict approaches, individual consumer theory was also being generalized. Shafer [21] invented a

numerical, utility-like representation for weak preferences, dropping transitivity. Then Kihlstrom, Mas-Colell, and Sonnenschein [13], in a similar context, obtained equivalence of a revealed preference condition, to negative semidefiniteness of the Slutsky matrix.

The connection between the weak preference approach and the strict preference approach has never been made clear. For example, much reference has been made in the strict preference literature to the “incompleteness” of the preferences. But one would never expect strict preference to be complete, or even total. Instead, the appropriate questions would be: Are the strict preferences the asymmetric part¹ of complete preferences? Can the demands also be generated by weak preferences that are complete?

We will answer these questions, and also clarify the relationship between the two approaches. First we return to the neoclassical weak preference approach and prove a new equilibrium existence theorem. Then we compare the weak and strict approaches, and show a precise sense in which they are conjugate approaches: anything one can do, so can the other.

3. EQUILIBRATING PREFERENCES: TWO NOTIONS, TWO THEOREMS

We will state conditions on preferences, sufficient to guarantee existence of competitive equilibrium. Our focus is on *preferences* here. Later, in Sections 5 and 7, we discuss what kinds of *behavior* can be generated by such preferences.

Of course, what we mean by competitive equilibrium will depend on our interpretation of preferences. For weak preferences the natural meaning is different than for strict preferences. In order to relate the two approaches, we must sketch the definitions.

¹ A relation R on X is called **asymmetric** if, for all $x, y \in X$, $xRy \Rightarrow \neg yRx$. A relation R on X is called the **asymmetric part** of a relation S if, for all $x, y \in X$, $xRy \Leftrightarrow xSy \ \& \ \neg ySx$.

3.1. Weakly Equilibrating Preferences

For weak (i.e. reflexive) preferences $\succsim$, it is natural to require that equilibrium consumption $x(t)$ of each trader t satisfy, with respect to trader t 's endowment $\bar{x}(t)$ and the equilibrium price vector $p \in R^n$:

$$px(t) \leq p\bar{x}(t) \quad \text{and} \quad \forall y_{py \leq p\bar{x}(t)} \quad x(t) \succsim y. \quad (3.1.1)$$

Then, of course, $x(\cdot)$ is a **free-disposal weak competitive equilibrium allocation** in a pure exchange economy with m traders if it also satisfies:

$$\sum_{t=1}^m x(t) \leq \sum_{t=1}^m \bar{x}(t), \quad (3.1.2)$$

and it is a **non-free-disposal weak competitive equilibrium allocation** if it also satisfies:

$$\sum_{t=1}^m x(t) = \sum_{t=1}^m \bar{x}(t). \quad (3.1.3)$$

(Since the only equilibria we discuss will be perfectly competitive, we'll drop the word "competitive.")

Any set of preferences (whether reflexive or not) with properties that guarantee existence of a pure exchange weak equilibrium, whenever endowments are in the interior of X_t , will be called **weakly equilibrating preferences**.

Theorem 1. Consider a pure exchange economy with a finite number of traders and commodities. Assume each trader t has a "preference" relation $\succsim_t$ defined on a closed, convex nonempty set X_t that is bounded below, and has an endowment in the interior of X_t .

a) If

$$\begin{aligned} \succsim_t \text{ has closed sections} \\ \{w \in X_t; w \succsim_t x\} \text{ and } \{w \in X_t; x \succsim_t w\} \text{ in } X_t \end{aligned} \quad (3.1.4a)$$

$$\succsim_t \text{ satisfies this convexity condition:}^2 \quad (3.1.4b)$$

for all integers k and all $x, y_i \in X_t$, $\alpha_i \in (0,1)$ with $\sum_{i=1}^k \alpha_i = 1$,

$$x = \sum_{i=1}^k \alpha_i y_i \Rightarrow \exists i \text{ } i = 1, \dots, k \text{ } x \succsim_t y_i,$$

then there exists a weak pure exchange equilibrium.

b) Furthermore, a non-free-disposal pure exchange weak equilibrium exists if each $\succsim_t$ is also locally insatiable: for all $x \in X_t$,

$$\forall \epsilon_t > 0 \exists y \in X_t \text{ \& } \|y - x\| < \epsilon \text{ \& } y \succ_t x, \quad (3.1.4c)$$

where $\succ_t$ is the asymmetric part of $\succsim_t$.

PROOF. For each t , define $\succ_t^0$ by:

$$x \succ_t^0 y \Leftrightarrow \neg y \succsim_t x. \quad (3.1.5)$$

By (3.1.4a), each $\succ_t^0$ has open sections $\{w \in X_t : w \succ_t^0 x\}$, $\{w \in X_t : x \succ_t^0 w\}$, in X_t . From (3.1.4b) it easily follows that, for all $x \in X_t$, $x \notin \text{con}(\{y \in X_t : y \succ_t^0 x\})$. Then by Theorem 2 below, there exists a strict equilibrium with respect to the $\succ_t^0$. And it is easy to show that every strict equilibrium with respect to the $\succ_t^0$ is a weak equilibrium with respect to the $\succsim_t$. QED

This is a new theorem. As we will see in the next subsection, an analogue for strict preferences is known. For additional insight into the proof, see the discussion following our metatheorem in Section 6.

Remark 1. The reader who is familiar with the notion of abstract economy (cf. Debreu [5], Arrow and Debreu [1], Shafer and Sonnenschein [22]) will see that the theorem and proof are valid for such economies.

Remark 2. Note that (3.1.4b) implies that $\succsim_t$ is reflexive.

Remark 3. If we drop the local nonsatiation assumption, a non-free-disposal equilibrium may not exist.

Remark 4. If $\succsim_t$ is total, it is easy to check that (3.1.4b) is equivalent to the generally weaker condition: for all $x \in X_t$,

² We consistently use subscripts to denote vectors, not vector components. We denote vector components by superscripts.

$$x \notin \text{con}(\{y \in X_t : y \succ_t x\}). \quad (3.1.4b')$$

3.2. Strictly Equilibrating Preferences

We now define equilibrium in the usual manner for strict preferences. With irreflexive preferences $\succ_t$, it is natural to require that equilibrium consumption $x(t)$ of each trader t satisfy, with respect to trader t 's endowment $\bar{x}(t)$ and the equilibrium price vector $p \in R^n$:

$$px(t) \leq p\bar{x}(t) \quad \text{and} \quad \neg \exists y_{py \leq p\bar{x}(t)} \quad y \succ x(t). \quad (3.2.1)$$

Then, of course, $x(\cdot)$ is a **free-disposal strict competitive equilibrium allocation** in a pure exchange economy with m traders if it also satisfies:

$$\sum_{t=1}^m x(t) \leq \sum_{t=1}^m \bar{x}(t), \quad (3.2.2)$$

and it is a **non-free-disposal strict competitive equilibrium allocation** if it also satisfies:

$$\sum_{t=1}^m x(t) = \sum_{t=1}^m \bar{x}(t). \quad (3.2.3)$$

(Again we drop the word "competitive.")

Any set of preferences (whether irreflexive or not) with properties that guarantee existence of a pure exchange strict equilibrium, whenever endowments $\bar{x}(t)$ are in the interior of X_t , will be called **strictly equilibrating preferences**.

Theorem 2. Consider a pure exchange economy with a finite number of traders and commodities. Assume each trader t has a "preference" relation $\succ_t$ defined on a closed, convex nonempty set X_t that is bounded below, and has an endowment in the interior of X_t .

a) If

$$\begin{aligned} \succ_t \text{ has open sections} \\ \{w \in X_t : w \succ_t x\} \text{ and } \{w \in X_t : x \succ_t w\} \text{ in } X_t \end{aligned} \quad (3.2.4a)$$

$\succ_t$ satisfies this convexity condition: (3.2.4b)

for all integers k and all $x, y_i \in X_t, \alpha_i \in (0,1)$ with $\sum_{i=1}^k \alpha_i = 1$,

$$x = \sum_{i=1}^k \alpha_i y_i \Rightarrow \exists i_{i=1, \dots, k} \neg y_i \succ_t x,$$

then there exists a strict pure exchange equilibrium.

b) Furthermore, a non-free-disposal pure exchange strict equilibrium exists if each $\succ_t$ is also locally insatiable: for all $x \in X_t$,

$$\forall \epsilon_\epsilon > 0 \exists y_{y \in X_t} \& \parallel y - x \parallel < \epsilon \quad y \succ_t x \quad (3.2.4c)$$

PROOF. Part (a) is Yannelis and Prabhakar's generalization of Mas-Colell, Gale and Mas-Colell, Shafer and Sonnenschein, and Shafer, proved for abstract economies in [26]. A proof for part (b) can follow Shafer's proof in [23].

Remark 5. It is easy to check that condition (3.2.4b) is equivalent to (3.1.4b').

3.3. Comparison

In general neither the strict nor the weak equilibrium notion is stronger than the other, and so neither Theorem 1 nor Theorem 2 is stronger. However, when each $\succ_t$ is the asymmetric part of the preference $\succsim_t$, then $\succsim_t$ -greatest elements are clearly $\succ_t$ -maximal elements, so these equilibria are related this way:

$$WCA(\succsim) \subseteq SCA(\succ), \quad (3.3.1)$$

where $WCA(\succsim)$ is the set of weakly competitive allocations for $\succsim \equiv (\succsim_1, \dots, \succsim_m)$, and $SCA(\succ)$ is the set of strictly competitive allocations for $\succ \equiv (\succ_1, \dots, \succ_m)$. Example 1 shows that the opposite inclusion can fail.

Example 1. For $t = 1, 2$ let each $\succsim_t$ be the vectorial ordering $\geq$ on $X \equiv R_{\leq}^2$. Then each asymmetric part $\succ_t$ is the vectorial ordering $\geq$ on $R_{\leq}^2$. If we endow the traders with:

$$\bar{x}(1) = (1,0)$$

$$\bar{x}(2) = (0,1),$$

then in the Edgeworth box it is easy to see that all bundles except those on the “bottom” and “right” edges are strict competitive allocations. But there are no weak competitive allocations.

In this special case, where each $\succ_i$ is the asymmetric part of the corresponding $\succsim_i$, Theorem 1’s *conclusion* is therefore stronger than Theorem 2’s. On the other hand, in this case, Theorem 1’s convexity *hypothesis* (3.1.4b) is also stronger than Theorem 2’s convexity hypothesis (3.2.4b). Again, neither theorem implies the other.

In the even more special case where each $\succsim_i$ is *complete* and $\succ_i$ is its asymmetric part, then $\succsim_i$ -greatest elements and $\succ_i$ -maximal elements clearly coincide, so:

$$WCA(\succ) = SCA(\succ). \quad (3.3.2)$$

Thus the conclusions of Theorems 1 and 2 are equivalent. And it is easy to show that their convexity hypotheses are also equivalent under these conditions. So in this more special case, Theorems 1 and 2 are equivalent.

When is choice behavior generated by equilibrating preferences? It is natural to seek revealed preference answers. And that is the focus of the rest of this paper.

4. CHOICE, RATIONALITY, AND MOTIVATION

We begin with the abstract framework of [18], to which the reader may turn for motivation.

4.1. Choice

We assume there is a fixed universe, a nonempty set X , of alternatives. And we assume there is a distinguished family $\mathcal{B}$ of subsets B of X . By analogy with classical consumer theory, we call each element B of $\mathcal{B}$ a **budget**. We call $(X, \mathcal{B})$ a **choice space**. A **choice** on $(X, \mathcal{B})$ is a function defined on $\mathcal{B}$ that assigns to each $B \in \mathcal{B}$ a (possibly empty) subset $h(B)$ of B . If $h(B) \neq \emptyset$ for all $B \in \mathcal{B}$, then we call h a **decisive choice**. (When $h(B)$ is a singleton for every $B \in \mathcal{B}$, we will simply write $h(B) = x$ instead of $h(B) = \{x\}$.) The intuition, of course, is that $h(B)$ is the set of alternatives chosen by a decision-maker from the alternatives in the constraint set, or "budget," B .

If X is a subset of a real linear space L , then any budget $B \in \mathcal{B}$ of the form $B = \{x \in X: p \cdot x \leq m\}$, for some linear functional p on L and some real m , is called a **competitive budget**, and often denoted $B(p, m)$ for any such (p, m) . We denote by $\mathcal{C}$ the set of competitive budgets $B = \{x \in X: p \cdot x \leq m\}$ with $p > 0$ and $m \geq 0$.

Next we define two notions of revelation. If, for some budget $B \in \mathcal{B}$ and some $x, y \in X$, we have $x \in h(B)$ and $y \in B$, then we say that x is **directly revealed preferred to y** , and we write xVy . If also $x \neq y$, then we write xSy .

With these two notions of revelation, we state three axioms.

The V-Axiom. $\forall x_{\in X} \forall B_{\in \mathcal{B}} [x \in B \ \& \ \forall y_{y \in B} xVy \Rightarrow x \in h(B)]$.

The Weak Axiom of Revealed Preference. S is asymmetric.

The Strong Axiom of Revealed Preference. S is acyclic.³

Remark 6. The Weak Axiom implies the V-Axiom. But, even when h is singleton-valued, the V-Axiom does not imply the Weak Axiom. (Cf. [18], p. 37, footnote 9; and see Example 2 below.) However, for the special case of just two commodities we will see in Theorem 7 below that, under natural conditions, the V-Axiom implies the Weak Axiom, hence the Strong Axiom of Revealed Preference.

³I.e., $y_1Sy_2S \cdots Sy_k \Rightarrow \neg y_kSy_1$.

Remark 7. For competitive budgets in R^n with $X = R_+^n$, it is easy to see that the V-Axiom implies: for all $x \in X$,

$$h^{-1}(x) \text{ is convex} \quad (4.1.1)$$

in the sense that $B(p_1, m_1), B(p_2, m_2) \in h^{-1}(x)$ and $0 < \alpha < 1$ imply $B(\alpha p_1 + (1-\alpha)p_2, \alpha m_1 + (1-\alpha)m_2) \in h^{-1}(x)$. However, the converse is not true.⁴

4.2. Rationality and Revealed Preference

By a **preference** on X we mean simply any binary relation on X . We will use the notion of rationality made precise in [18] as follows. A choice h on $(X, \mathcal{B})$ is called **rational** if there exists a preference $\succsim$ on X such that, for every $B \in \mathcal{B}$:

$$h(B) = \{x \in B: \forall y_{y \in B} x \succsim y\}. \quad (4.2.1)$$

Then $\succsim$ is said to **rationalize** h . If there exists such a preference $\succsim$ that is reflexive, or total, or reflexive and total, etc., then h is said to be **reflexive-** or **total-** or **reflexive-total-rational**, etc.

From [18] we know that:

Proposition 1. A choice h is rational if and only if it satisfies the V-Axiom.

Furthermore, [18] shows that:

Proposition 2. A choice h is reflexive-rational if and only if it satisfies the V-Axiom.

So these two types of rationality are equivalent.

Finding a useful revealed preference characterization of total rationality remains an open problem for the general case, though we shall find one for competitive consumer theory (see Theorems 5, 6, and Remark 8). For the general case, [18] shows that the V-

⁴ We thank Professor Donald Campbell for this observation. Nevertheless, he has shown in [4] that, under some additional hypotheses, convexity of the inverse implies the existence of what we are calling a "motivation." In that case, Theorem 3 below guarantees that the V-Axiom holds.

Axiom does not characterize total rationality:

Proposition 3. There are rational choices that are not total-rational.

However we know that total-rationality is equivalent to reflexive-total-rationality [18].

4.3. Motivation and Revealed Preference

We also need to discuss “strict,” i.e., irreflexive, preferences. Because irreflexive preferences cannot rationalize decisive choices, we define the following analogue. If $\succ$ is a preference on X , and if, for every $B \in \mathcal{B}$,

$$h(B) = \{x \in B: \neg \exists y_{y \in B} \ y \succ x\}, \quad (4.3.1)$$

then we say that $\succ$ is a **motivation** for h . If h has a motivation, we say h is **motivated**. Of course we will call $\succ$ an **irreflexive**, or **asymmetric**, etc. motivation for h , according as $\succ$ is irreflexive, or asymmetric, etc.

The first hint of a relationship between strict and weak preferences is valid in abstract choice spaces:

Theorem 3. A choice h is motivated if and only if it satisfies the V-Axiom. Hence h is motivated if and only if it is rational.

PROOF. By equations (6.19) below, a preference R motivates h if and only if R^* (defined by (6.15) below) rationalizes h , in which case h satisfies the V-Axiom by Proposition 1. QED

While Theorem 3 relates weak and strict preferences through the choices they generate, a second, stronger relationship holds under very special conditions:

Theorem 4. If choice h is rationalized by a total preference $\succsim$, then the asymmetric part of $\succsim$ motivates h . The conclusion may be false if $\succsim$ is not total.

We omit the elementary proof.

A third relationship between strict and weak preferences is also valid in abstract choice spaces:

Theorem 5. A choice h is asymmetric-motivated if and only if it is total-rational; and if and only if it is total-reflexive-rational.

PROOF. First, suppose h is asymmetric-motivated by a preference P . To rationalize h , define R by:

$$xRy \Leftrightarrow xPy \text{ or } \neg yPx. \quad (4.3.2)$$

Clearly R is total. To see that R rationalizes h , we must show (4.2.1) with $\succsim \equiv R$.

If $x \in h(B)$, then, since P motivates h ,

$$\neg \exists y_{y \in B} yPx,$$

i.e.,

$$\forall y_{y \in B} \neg yPx,$$

so by (4.3.2):

$$\forall y_{y \in B} xRy. \quad (4.3.3)$$

Conversely, suppose (4.3.3) holds for some $x \in B$ we must show $x \in h(B)$. If $x \notin h(B)$, then, since P motivates h , we have yPx for some $y \in B$; since P is asymmetric, we also have $\neg xPy$. Then by (4.3.2) we have $\neg xRy$, a contradiction of (4.3.3).

Thus asymmetric-motivation implies total-rationality.

Second, suppose h is total-rational under some preference R . To motivate h , define P by:

$$xPy \Leftrightarrow [xRy \text{ \& } \neg yRx]. \quad (4.3.4)$$

Clearly P is asymmetric. And methods analogous to the first part above show that P motivates h . QED

A fourth relationship is valid in concrete competitive budget spaces:

Theorem 6. Let $X = R^{\geq}_0$ and let $\mathcal{B} = \mathcal{C}$. Suppose h is a decisive choice on $(X, \mathcal{B})$ and suppose the budget equality holds.

Then h is asymmetric-motivated if and only if h satisfies the V-Axiom.

PROOF. The necessity of the V-Axiom is immediate from Theorem 3. To see the sufficiency of the V-Axiom, define R as follows.

For each $i = 0, 1, 2, 3, 4$, define a relation R^i by:

(4.3.5)

$$xR^0y \Leftrightarrow x = 0 \ \& \ y \notin \text{Range}(h);$$

$$xR^1y \Leftrightarrow xR^0y, \text{ or} \\ xVy, \text{ or} \\ x > y \neq 0;$$

$$xR^2y \Leftrightarrow xR^1y, \text{ or} \\ x \notin R^{\geq}_0 \ \& \ y \in R^{\geq}_0 \cap \text{Range}(h) \ \& \ \neg yVx;$$

$$xR^3y \Leftrightarrow xR^2y, \text{ or} \\ y \notin R^{\geq}_0 \ \& \ \exists z_{z \notin R^{\geq}_0} x = \frac{z+y}{2} \notin \text{Range}(h) \ \& \ \neg zVy \ \& \ \neg yVz;$$

$$xR^4y \Leftrightarrow xR^3y, \text{ or}$$

$$x, y \notin R^{\geq}_0 \ \& \ \frac{x+y}{2} Vx \ \& \ \neg xVy \ \& \ \neg yVx;$$

$$xRy \Leftrightarrow xR^4y, \text{ or} \\ \neg xR^4y \ \& \ \neg yR^4x \ \& \ xLy, \text{ or} \\ x = y,$$

where L is the **lexicographic** ordering:

$$xLy \Leftrightarrow \exists i \ i = 1, \dots, n \ \forall j_j < i \ x_j = y_j \ \& \ x_i > y_i.$$

From the last part of the definition (4.3.5), R is clearly total and reflexive.

Let P be the asymmetric part of R . It suffices to show that, for all $B \in \mathcal{B}$,

$$h(B) = \{x \in B: \neg \exists y_{y \in B} yPx\}. \quad (4.3.6)$$

First, let $x \in h(B)$. Then it is immediate that there is no $y \in B$ with yPx : for $x \in h(B)$ implies xVy , hence xRy for all $y \in B$.

Next let $z \in B \equiv (p, m)$ & $z \notin h(B)$. We must show there exists a $w \in B$ with wPz . If $z = 0$, then let $w \in h(B)$; clearly wPz . If $z \neq 0$, we consider two cases. In case $p \cdot z < m$, then there are $w \in B$ with $w > z$; and for any such w we have wR^1z ; and it follows easily from the cases of (4.3.5) that $\neg zRw$; so wPz .

For the second case, suppose $p \cdot z = m$. Since $z \notin h(B)$, the V-axiom implies $\neg zV\tilde{y}$ for some $\tilde{y} \in B$. Since $p \cdot z = m$ & $p > 0$, the line from z through $\tilde{y}$ intersects the boundary of $R^{\geq}$, say at y . Clearly $\neg zVy$.

If $z \notin \text{Range}(h)$, then $0R^0z$; and it follows easily from all the cases of (4.3.5) that $\neg zR0$. So $0Pz$.

If $z \in \text{Range}(h)$, there are several cases to consider:

- if $z \in R^{\geq}$, then yR^2z ;
- if $z \notin R^{\geq}$ & yVz , then yR^1z ;
- if $z \notin R^{\geq}$ & $\neg yVz$, let $w \equiv \frac{z+y}{2}$; then:
 - if $w \notin \text{Range}(h)$, then clearly wR^3z ;
 - if $w \in \text{Range}(h)$, then clearly wVz or wVy ; and
 - wVz implies wR^1z , and
 - wVy implies wR^1z or yR^4z .

And in each of the $wR^i z$ or $yR^i z$ cases, it is easily verified from (4.3.5) that $\neg zRw$ or $\neg zRy$ (respectively). So wPz or yPz . QED

Remark 8. The last three theorems show that, in the classical consumer framework, rationality, total-rationality, reflexive-total-rationality, motivation, and asymmetric-motivation are all "equivalent"; and they are all characterized by the V-Axiom. This suggests that existence theorems based on strict preferences will not be more general than those based on weak total preferences. Of course, that doesn't tell us whether the continuity and convexity properties of the strict preferences used in

existence theorems can be obtained from continuity and convexity properties of weak total preferences. We will later see that this can be done quite generally.

We will sometimes make the following common assumption:

Desirability Hypothesis.

$$\begin{aligned} \text{If } 0 < (p_k, m_k) \xrightarrow{k \rightarrow \infty} (\bar{p}, \bar{m}) \ \& \ \bar{m} > 0 \ \& \ \bar{p}_i = 0 \text{ for some } i = 1, \dots, n, \\ \text{and } x_k \in h(p_k, m_k) \text{ for all } k, \text{ then } \|x_k\| \rightarrow \infty. \end{aligned} \tag{4.3.7}$$

Now for the special case of just two commodities, we can show:

Theorem 7. Let $X = R_+^2$ and let $\mathcal{B} = \mathcal{C}$. Suppose h is a continuous singleton-valued choice on $(X, \mathcal{B})$ and suppose the budget equality and the Desirability Hypothesis hold.

Then h satisfies the Weak Axiom of Revealed Preference if and only if it satisfies the V-Axiom.

And h satisfies the Strong Axiom of Revealed Preference under the same conditions.

PROOF (sketch). The Weak Axiom implies the V-Axiom by Remark 6. To show that the V-Axiom implies the Strong Axiom, it suffices to show it implies the Weak Axiom, since it is well known [19] that the axioms are equivalent when there are only two commodities.

Suppose the Weak Axiom fails; say xSy & ySx , with $x = h(B_1)$ for $B_1 = (p_1, m_1)$, and $y = h(B_2)$ for $B_2 = (p_2, m_2)$. Without loss of generality, we can assume $p_2 \cdot x = m_2$, as in Figure 1.

Figure 1

(For if $p_2 \cdot x < m_2$, then by the budget equality and continuity of h , we can increase m_1 until $p_2 \cdot h(p_1, m_1) = m_2$.) Also without loss of generality we can assume $y^1 < x^1$ & $y^2 > x^2$. And we can clearly assume all budgets are normalized with $p_i^2 = 1$ ($i = 1, 2$). Rotating the budget line for B_2 through x , letting $p_2^1 \rightarrow 0$, the Desirability Hypothesis implies $x^1 \rightarrow \infty$. By the budget equality, and the continuity of h , together with the Intermediate Value Theorem, y must pass through x , say at budget B_2' . But the $B_1, B_2' \in h^{-1}(x)$, and $B_2 \notin h^{-1}(x)$. Since B_2 can clearly be obtained as a convex combination of B_1 and B_2' , this means $h^{-1}(x)$ is not convex, so the V-Axiom is violated, by Remark 7. QED

Remark 9. Since the Strong Axiom is equivalent to reflexive-total-transitive-rationality, this gives us a very strong type of rationality from the very weakest rationality type when $n = 2$.

5. THE V-AXIOM AND EQUILIBRATING PREFERENCES

In Section 3 we gave conditions on preferences, sufficient to guarantee existence of competitive equilibrium. Now we characterize the individual *behavior* arising from such preferences. We will show that the V-Axiom characterizes some equilibrating preferences that are not necessarily either transitive or total.

We will also show that the apparent generality of some very weak assumptions on preferences is a little misleading. For under quite general conditions, the weaker assumptions imply stronger assumptions.

In particular we will show that the recent emphasis on “incomplete” preferences is somewhat misplaced.⁵ Indeed, the same demands generated by “incomplete” preferences, can often be generated by “complete” preferences. We will show how to pass back and forth between these two kinds of preferences. This clarifies the relationship between these two approaches to equilibrium theory.

We will also show that sectional continuity is not really weaker than full continuity, as far as behavioral consequences go. This is true despite the fact that sectional continuity is a weaker assumption on preferences, when preferences are nontotal.⁶

Finally, we will characterize several preference types by revealed preference axioms.

To characterize demands that can be motivated and rationalized by equilibrating preferences, we begin with some sufficient conditions.

For any set $A \subseteq R^n$, we denote by $L(A)$ the smallest linear variety (i.e., translate of a linear space) containing A . By the relative interior, $\text{RInt}(A)$ of A , we mean the interior in the relative topology of $L(A)$.⁷

For equilibrium analysis it is common to drop the assumption that prices are

⁵ A similar point was made by McKenzie ([16], first paragraph), in the context of weak preferences. Our discussion will apply to both weak and strict preferences, and will show that many special properties (monotonicity, continuity, convexity, etc.) can be preserved in passing between the preference types.

⁶ When preferences are total and either transitive or convex, then sectional continuity implies full continuity. (Cf. [3] and the Lemma in [21].)

⁷ Cf. [25], p. 7; [17], p. 22.

positive, and to assume that the consumption sets X_i are compact.⁸ So let

$$P \equiv \{(p, m) \in R^{n+1}: \|p\| = 1\}, \quad (5.1)$$

and

$$P' \equiv \{(p, m) \in P: \text{RIInt}(B(p, m)) \neq \emptyset\}, \quad (5.2)$$

where $B(p, m) \equiv \{x \in X: p \cdot x \leq m\}$.

For any $x, y \in R^n$ we will use the notation $[x, y]$ to denote the closed line interval between x and y . And (x, y) , $(x, y]$, and $[x, y)$ will denote the corresponding intervals, excluding both or one of the endpoints.

Theorem 8. Let X be a compact, convex subset of R^n , and let $\mathcal{B}$ be the family of all competitive budgets determined by P . Suppose h is a decisive choice on $(X, \mathcal{B})$ such that:

$$\begin{aligned} &h \text{ is upper-hemicontinuous on } P' \\ &\text{Range}(h) \supseteq \text{RIInt}(X). \end{aligned}$$

If h satisfies the V-Axiom, then h can be rationalized by a preference $\succsim$ that:

$$\begin{aligned} &\text{is total} \\ &\text{has closed graph in } X \times X \\ &\text{satisfies convexity condition (3.1.4b)} \end{aligned}$$

and whose asymmetric part $\succ$ is a motivation for h that:

$$\begin{aligned} &\text{is asymmetric} \\ &\text{has an open graph in } X \times X \\ &\text{satisfies the convexity condition (3.1.4b')}. \end{aligned}$$

PROOF.

⁸ By standard techniques (cf. Debreu [6], Chapter 5), the original consumption sets can be compactified in such a way that all equilibria of the compactified problem can be proved to be equilibria of the original problem.

A) First define R^0 by:

$$\begin{aligned} xR^0y &\Leftrightarrow xVy, \text{ or} \\ &[x,y \in \text{RInt}(X) \ \& \ \neg xVy \ \& \ \neg yVx \\ &\ \& \ d(x, W(x,y)) \leq d(y, W(x,y))], \end{aligned} \tag{5.3}$$

where $W(x,y)$ is defined in Lemma 2 below, and $d(\cdot, \cdot)$ is euclidean distance. Then define R by:

$$xRy \Leftrightarrow (x,y) \in \text{Closure}(\{(a,b): aR^0b\}). \tag{5.4}$$

Clearly R is total, reflexive, and has a closed graph in $X \times X$.

B) To show the convexity condition (3.1.4b), let $x = \sum_{i=1}^k \alpha_i y_i$, with $\alpha_i \geq 0$ & $\sum_{i=1}^k \alpha_i = 1$. If $x \in \text{Range}(h)$, then clearly xVy_i for some y_i , so xR^0y_i , so xRy_i .

If $x \notin \text{Range}(h)$, then for each i we can approximate y_i by $y_{i,m} \xrightarrow{m \rightarrow \infty} y_i$, with $y_{i,m} \in \text{RInt}(X)$.⁹ Let $x_m \equiv \sum_{i=1}^k \alpha_i y_{i,m}$. By the preceding paragraph, $x_m R^0 y_{i,m}$ for some i . Then we can choose an i and a subsequence so that $x_m \xrightarrow{m \rightarrow \infty} x$ and $y_{i,m} \xrightarrow{m \rightarrow \infty} y_i$. So xRy_i , and (3.1.4b) holds.

C) Next we verify the rationality condition (4.2.1). First let $x \in h(B)$ & $y \in B$, so xVy , hence xRy . So $h(B)$ is a subset of the righthand side.

In the opposite direction, let $y \notin h(B)$. We must show that $\neg yRv$ for some $v \in B$. If yVz for all $z \in \text{RInt}(X) \cap B$, then yVz for all $z \in B$, by the preceding footnote and Lemma 1 below; then by the V-Axiom, $y \in h(B)$, a contradiction. So there exists a $z \in \text{RInt}(X) \cap B$ with $\neg yVz$. We consider the following cases.

1) Suppose $\neg zVy$. Then by Lemma 2, $W(z,y)$ is closed and nonempty, so we can define v by:

$$v \text{ is the closest element of } W(z,y) \text{ to } y. \tag{5.5}$$

⁹ Because X is a compact convex subset of R^n , it is homeomorphic to the unit ball of some R^m with $m \leq n$. See [17], p. 23, Theorem 2.8; [25], p. 13, Theorem 1.17.

So vVy , hence vRy . It suffices to show that $\neg yRv$. Suppose yRv . Then, since $\neg yVz$, clearly $\neg yVv$. So the only possibility of yRv is that $y_i R^0 v_i$ and $y_i \rightarrow y$ & $v_i \rightarrow v$. We can then assume one of the following two subcases:

1.a) $y_i Vv_i$, for all i . Then by Lemma 1 below, yVv , which contradicts $\neg yVv$.

1.b) $\neg y_i Vv_i$ & $\neg v_i Vy_i$ & $d(y_i, W(y_i, v_i)) \leq d(v_i, W(y_i, v_i))$ for all i . Then for each i , by Lemma 2 below we can let w_i be the closest element to y_i in $W(y_i, v_i)$. Then $w_i Vy_i$ and $w_i Vv_i$ and $w_i \rightarrow$ some $\bar{w} \in (v, y]$. So by Lemma 1 below, $\bar{w}Vy$ and $\bar{w}Vv$ and $\bar{w} \in (v, y]$. This contradicts (5.5). So $\neg yRv$.

2) Suppose zVy , so zRy . Again there are two subcases to consider:

2.a) If $W(z, y) \neq \emptyset$, then by Lemma 2 we can let v be the closest element to y in $W(z, y)$. Then vVy . So vRy , hence it suffices to show that $\neg yRv$. But if yRv , then we obtain a contradiction as in (1).

2.b) If $W(z, y) = \emptyset$, it suffices to show $\neg yRz$. If yRz , then since $\neg yVz$ we must have sequences $y_i \rightarrow y$ and $z_i \rightarrow z$ with $y_i R^0 z_i$. By taking subsequences we can assume one of the following two cases:

2.b.i) $y_i Vz_i$ for all i . Then by Lemma 1, yVz , which contradicts $\neg yVz$.

2.b.ii) $\neg y_i Vz_i$ & $\neg z_i Vy_i$ & $d(y_i, W(y_i, z_i)) \leq d(z_i, W(y_i, z_i))$ for all i . for each i we can let w_i be the closest element of $W(y_i, z_i)$ to y_i , by Lemma 2. By taking another subsequence we can obtain $w_i \rightarrow$ some $v \in [z, y]$. If $v \in (z, y]$, then $W(x, y) \neq \emptyset$, a contradiction. So $w_i \rightarrow z$. But this contradicts the fact that $d(y_i, W(y_i, z_i)) \leq d(z_i, W(y_i, z_i))$ for all i .

D) The remaining assertions concern the asymmetric part $\succ$ of $\succcurlyeq$. According to Theorem 4, $\succ$ motivates h . Asymmetry of $\succ$ is immediate by definition; openness of the graph of $\succ$ follows easily from the totality of $\succcurlyeq$ and closedness of the graph of $\succcurlyeq$; and convexity condition (3.1.4b') for $\succ$ follows from convexity condition (3.1.4b) for $\succcurlyeq$, by Remark 4. QED

Lemma 1. (Continuity of V .) Let X be a compact convex subset of R^n , and let correspondence $h: P \rightarrow R^n$ be upper hemicontinuous on P' . If $x_i, y_i \in X$ & $x_i Vy_i$ & $x_i \xrightarrow{i \rightarrow \infty} x$ & $y_i \xrightarrow{i \rightarrow \infty} y$ and $x \in \text{RInt}(X)$ or $y \in \text{RInt}(X)$, then xVy .

PROOF. Because x or y is in $\text{RInt}(X)$, we can suppose that, for each i there is $(p_i, m_i) \in P'$ such that $x_i \in h(p_i, m_i)$ & $p_i \cdot y_i \leq m_i$. Since each $\|p_i\| = 1$, we can find a subsequence with $p_i \rightarrow p$, for some $p \in R^n$. For the m_i sequence, we can find either a subsequence $m_i \rightarrow m$ for some $m \in R^1$, or a subsequence $m_i \rightarrow \infty$. In the second case, the compactness of X and the fact $\|p_i\| = 1$ imply that, for all large i , the budget set $B(p_i, m_i) = X$. Then there exists $\bar{m} \in R^1$ such that $h(p_i, \bar{m}) = h(p_i, m_i)$, for all large i . So in both cases we can assume $m_i \rightarrow m$ for some $m \in R^1$.

Since $p_i \rightarrow p$ & $m_i \rightarrow m$ & $x_i \rightarrow x$ & $y_i \rightarrow y$ & $m_i \geq p_i \cdot x_i$ and $m_i \geq p_i \cdot y_i$, we therefore have $(p, m) \in P'$. And since h is upper hemicontinuous on P' , we see that $x \in h(p, m)$. Therefore xVy . QED

Lemma 2. Suppose $\text{Range}(h) \supseteq \text{RInt}(X)$, and h is upper hemicontinuous on P' . Let $x \in \text{RInt}(X)$ or $y \in \text{RInt}(X)$, where $x \neq y$ & $\neg xSy$ & $\neg ySx$. Then:

$$\text{i) } \exists w_{w \in (x, y)} wSx \text{ \& } wSy.$$

Let

$$W(x, y) \equiv \{w \in (x, y) : wSx \text{ \& } wSy\}. \quad (5.6)$$

Then

$$\text{ii) } W(x, y) \text{ is closed, and} \\ d(x, W(x, y)) > 0 \text{ \& } d(y, W(x, y)) > 0.$$

PROOF. To prove (i), note that $W_x \equiv \{u \in [x, y] : uSx \text{ or } u = x\}$ and $W_y \equiv \{u \in [x, y] : uSy \text{ or } u = y\}$ are closed by Lemma 1, and they are clearly nonempty. And their union is all of the interval $[x, y]$, by the hypothesis $\text{Range}(h) \supseteq \text{RInt}(X)$ and since X is convex. By connectedness of $[x, y]$, we therefore have $W_x \cap W_y \neq \emptyset$. If $w \in W_x \cap W_y$, then $w \neq x$ because $\neg xVy$; similarly $w \neq y$; so wSx & wSy .

To prove (ii), first observe that $W(x, y) = W_x \cap W_y$, and note again that W_x and W_y are closed.

To see that $d(x, W(x, y)) > 0$, suppose not. Then there exist $v_i \in (x, y)$ with $v_i \rightarrow x$ & v_iSx & v_iSy . Say $v_i \in h(p_i, m_i)$ & $p_i \cdot y \leq m_i$; then there is a subsequence $p_i \rightarrow p$ for some $p \in R^n$, and we can assume (as in the proof of Lemma 1) that $m_i \rightarrow m$ for some $m \in R^1$. Since x or $y \in \text{RInt}(X)$, it is not hard to show that $(p, m) \in P'$, so by

upper hemicontinuity of h on P' , we see $x \in h(p, m)$ & $m \geq p \cdot y$. This implies xSy , contradicting $\neg xSy$. Therefore $d(x, W(x, y)) > 0$. Similarly $d(y, W(x, y)) > 0$. QED

The hypotheses of Theorem 8 are implied by the hypotheses of Sonnenschein [24], Schmeidler [20], Mas-Colell [15], Gale and Mas-Colell [8], and Yannelis and Prabhakar [26]. Indeed, we have this characterization:

Theorem 9. Let X be a compact, convex subset of R^n , and let $\mathcal{B}$ be the family of all competitive budgets determined by P . Suppose h is a (not necessarily decisive) choice on $(X, \mathcal{B})$. Then the following sets of conditions are equivalent:

i) h can be motivated by a preference $\succ$ that:

*has open sections $\{w \in X: w \succ x\}$ and $\{w \in X: x \succ w\}$ in X
satisfies convexity condition (3.1.4b')*

ii) h can be motivated by a preference $\succ$ that:

*has open graph in $X \times X$
satisfies convexity condition (3.1.4b')
is the asymmetric part of a preference $\succsim$ satisfying (iv)*

iii) h can be rationalized by a preference $\succsim$ that:

*has closed sections $\{w \in X: w \succsim x\}$ and $\{w \in X: x \succsim w\}$ in X
satisfies convexity condition (3.1.4b)*

iv) h can be rationalized by a preference $\succsim$ that:

*has closed graph in $X \times X$
satisfies convexity condition (3.1.4b)
is total
has asymmetric part $\succ$ satisfying (ii)*

v) h is:

*upper-hemicontinuous on P'
decisive, and satisfies
 $\text{Range}(h) \supseteq \text{RInt}(X)$
the V-Axiom.*

PROOF. We first prove that (v) is equivalent to (iii) and (iv), and then we prove that (v) is equivalent to (i) and (ii).

A) Note that (v) implies (iv) by Theorem 8.

B) And (iv) implies (iii), a fortiori.

C) To see that (iii) implies (v), we begin by showing that h is decisive on $\mathcal{P}$. Let $B \in \mathcal{B}$, and note that $B_x \equiv \{w \in B: w \succcurlyeq x\}$ is compact, for each x , because $\succcurlyeq$ has closed sections. Furthermore, by convexity condition (3.1.4b) it can be seen that, for any m and any $x_1, \dots, x_m \in B$, we have $\text{con}(\{x_1, \dots, x_m\}) \subseteq \bigcup_{i=1}^m B_{x_i}$. Therefore $\bigcap_{x \in B} B_x \neq \emptyset$, by Ky Fan's generalization ([7], Lemma 1) of the Knaster-Kuratowski-Mazurkiewicz Lemma ([14]; [2], p. 172). So, since every finite collection of the compact sets B_x has a nonempty intersection, we know some $\bar{x} \in \bigcap_{x \in B} B_x$, and then $\bar{x} \succcurlyeq x$ for all $x \in B$. So $h(B) \neq \emptyset$.

The upper hemicontinuity of h on $\mathcal{P}'$ follows from the closedness of the $\succcurlyeq$ sections, by standard arguments.¹⁰

To show that $\text{Range}(h) \supseteq \text{RInt}(X)$, let $x \in \text{RInt}(X)$; we must show $x \in \text{Range}(h)$. Define relation T by:

$$vTw \Leftrightarrow \neg w \succcurlyeq v. \quad (5.7)$$

Then if yTx hence $\neg x \succcurlyeq y$, we have $y \notin \{w \in X: x \succcurlyeq w\}$. Now this last set is closed in X , by the assumption of closed sections; since X itself is closed in R^n , that set is also closed in R^n , hence in $L(X)$. So there is a neighborhood $N(y)$ in $L(X)$, of y with $N(y) \cap \{w \in X: x \succcurlyeq w\} = \emptyset$. Let $K(x) \equiv \bigcup \{N(y): yTx\}$, so $K(x)$ is open in $L(X)$, hence $\text{con}(K(x))$ is also open in $L(X)$.

Now we show that $x \notin \text{con}(K(x))$ for each x . For otherwise, if $x = \sum_{i=1}^k \alpha_i y_i$ & $\sum_{i=1}^k \alpha_i = 1$ & $\alpha_i \geq 0$ ($i = 1, \dots, k$) & $\neg x \succcurlyeq y_i$, then from convexity condition (3.1.4b) we have $x \succcurlyeq y_j$ for some j , which is a contradiction. So $x \notin \text{con}(K(x))$, for each x .

Since $x \notin \text{con}(K(x))$, which is open in $L(X)$, a separating hyperplane theorem ensures the existence of a $p \in R^n$ with $p \neq 0$ and $p \cdot x < p \cdot z$ for all $z \in \text{con}(K(x))$.

Now we show that $x \succcurlyeq y$ for all $y \in X$ with $p \cdot y \leq p \cdot x$. If not, say $\neg x \succcurlyeq y$ & $p \cdot y \leq p \cdot x$, so yTx , hence $y \in K(x) \subseteq \text{con}(K(x))$. But then $p \cdot x < p \cdot y$, a

¹⁰ Cf. The proof of [11], p. 151 (Proposition 6.1 (3)), in p. 152 (second paragraph).

The result generalizes [24], Theorem 6, because we only assume closed sections for $\succcurlyeq$, rather than a closed graph. (We know from [3] that assuming closed sections is weaker than assuming a closed graph.)

contradiction. Since $\succsim$ rationalizes h , we therefore have $x \in h(p, m)$. So $x \in \text{Range}(h)$.

The last part of (v) is the V-Axiom, which follows immediately by Proposition 1.

D) Parts (A), (B), and (C) show the equivalence of (iii), (iv), and (v).

E) Note that (v) implies (ii) by Theorem 8.

F) And (ii) implies (i) a fortiori.

G) Finally, we prove that (i) implies (v). Define $\succsim$ by:

$$x \succsim y \iff \neg y \succ x.$$

In Lemma 5 below, we note that $\succsim$ rationalizes the same demand h that is motivated by $\succ$. Since the $\succ$ - sections are open, clearly the $\succsim$ - sections are closed. It is easily checked that convexity condition (3.1.4b') for $\succ$ implies convexity condition (3.1.4b) for $\succsim$. By our proof (C), then, h satisfies (v).

H) Parts (E), (F), and (G) show the equivalence of (i), (ii), and (v). So (i) through (v) are all equivalent. QED

We can drop the X -compactness assumption and insist on strictly positive prices, if we assume the budget equality and Desirability Hypothesis. We will also get the following monotonicity conclusion:¹¹ for all $x, y \in X \subseteq R^n$,

$$x > 0 \text{ \& } x \geq y \Rightarrow x \succ y. \quad (5.8)$$

Define

$$P'' \equiv \{(p, m) \in R^n \times R^1: \|p\| = 1 \text{ \& } p > 0 \text{ \& } m \geq 0\}. \quad (5.9)$$

Theorem 10. Let $X = R^n_+$ and let $\mathcal{B}$ be the family of all competitive budgets determined by P'' . Suppose h is a decisive choice on $(X, \mathcal{B})$ such that:

h is upper-hemicontinuous on P''
the budget equality holds
the Desirability Hypothesis holds
 $\text{Range}(h) \supseteq R^n_+$.

¹¹ We use the following definitions. For all $x, y \in R^n$, $x \geq y$ means that each component of x is at least as great as the corresponding component of y ; $x \geq y$ means that $x \geq y$ & $x \neq y$; and $x > y$ means that each component of x is strictly greater than the corresponding component of y .

If h satisfies the V-Axiom, then h can be rationalized by a preference $\succsim$ that:

is total
has a closed graph in $X \times X$
satisfies convexity condition (3.1.4b)
satisfies the monotonicity condition (5.8)

and whose asymmetric part $\succ$ is a motivation for h that:

is asymmetric
has an open graph in $X \times X$
satisfies the convexity condition (3.1.4b')
satisfies the monotonicity condition (5.8)

PROOF. Use the proof for Theorem 8, replacing references to $R\text{Int}(X)$ by $\text{Int}(X)$, and references to Lemmas 1 and 2 by Lemmas 3 and 4, below, respectively.

Lemma 3. Let $X = R_{\geq}^n$, and let correspondence $h: P'' \rightarrow R^n$ be upper hemicontinuous on P'' and satisfy the budget equality and Desirability Hypothesis. If $x_i, y_i \in X$ & $x_i \rightarrow x$ & $y_i \rightarrow y$ & $x_i V y_i$ and $x \in \text{Int}(X)$ or $y \in \text{Int}(X)$, then $x V y$.

PROOF. Since $x_i V y_i$, say $x_i \in h(p_i, m_i)$ & $p_i \cdot y_i \leq m_i$ for some $(p_i, m_i) \in P''$. Since $\|p_i\| = 1$, we can take a subsequence with $p_i \rightarrow p$, for some $p \in R_{\geq}^n$. By the Desirability Hypothesis, $p > 0$. By the budget equality, $m_i \rightarrow p \cdot x \equiv m$. So $m \geq p \cdot y$. By upper hemicontinuity of h , $x \in h(p, m)$. So $x V y$. QED

Lemma 4. Suppose $\text{Range}(h) \supseteq \text{Int}(X)$, and let correspondence $h: P'' \rightarrow R^n$ be upper hemicontinuous on P'' and satisfy the budget equality and Desirability Hypothesis. Let $x \in \text{Int}(X)$ or $y \in \text{Int}(X)$, where $x \neq y$ & $\neg x S y$ & $\neg y S x$. Then:¹²

- i) $\exists w_{w \in (x, y)} w S x$ & $w S y$.
- ii) $W(x, y)$ is closed, and
 $d(x, W(x, y)) > 0$ & $d(y, W(x, y)) > 0$.

PROOF. Use Lemma 3 in the same way that Lemma 1 was used in the proof of Lemma 2. QED

¹² $W(x, y)$ is defined in (5.6).

And for a characterization we have:

Theorem 11. Let $X = R^n$ and let $\mathcal{B}$ be the family of all competitive budgets determined by P'' . Suppose h is a (not necessarily decisive) choice on $(X, \mathcal{B})$ satisfying the Desirability Hypothesis. Then the following sets of conditions are equivalent:

i) h can be motivated by a preference $\succ$ that:

has open sections $\{w \in X: w \succ x\}$ and $\{w \in X: x \succ w\}$ in X
satisfies convexity condition (3.1.4b')
satisfies monotonicity condition (5.8)

ii) h can be motivated by a preference $\succ$ that:

has open graph in $X \times X$
satisfies convexity condition (3.1.4b')
satisfies monotonicity condition (5.8)
has a C^∞ Shafer representation¹³
is the asymmetric part of a preference $\succsim$ satisfying (iv)

iii) h can be rationalized by a preference $\succsim$ that:

has closed sections $\{w \in X: w \succsim x\}$ and $\{w \in X: x \succsim w\}$ in X
satisfies convexity condition (3.1.4b)
satisfies monotonicity condition (5.8):

iv) h can be rationalized by a preference $\succsim$ that:

has closed graph in $X \times X$
satisfies convexity condition (3.1.4b)
satisfies monotonicity condition (5.8)
is total
has asymmetric part $\succ$ satisfying (ii)

v) h is:

upper-hemicontinuous on P'' ,
decisive, and satisfies the
budget equality,
 $\text{Range}(h) \supseteq \text{Int}(X)$, and
 V -Axiom.

PROOF. This can parallel the proof for Theorem 9.

¹³ Cf. [21]. The fact that such a representation can be made C^∞ follows from the fact that closed sets are the zero sets of C^∞ functions (cf. [10], p. 17, Proposition 4.8).

Remark 10. This extends Theorems 3, 5, and 6 by showing that we can not only pass between strict and weak preferences, but we can carry continuity and convexity properties between the two approaches as well.

Remark 11. One implication of this theorem is that, even if we start with a strict preference $\succ^+$ that is not known to be the asymmetric part of a nice weak preference, there exists some nice weak preference $\succsim$ whose asymmetric part $\succ$ motivates the same demand as $\succ^+$.

That is typical of Revealed Preference Theory, and of Consumer Integrability Theory as well. For they are concerned with *equivalence classes* of consumers -- those whose preferences (and endowments, if given) generate the same demand correspondences.

For purposes of equilibrium existence theory, for example, this allows us to replace “difficult” preferences with ones that are easier to work with. This theorem and the remaining ones give a useful catalogue for such applications.

In the next section we formalize these ideas.

6. CONJUGATE PREFERENCES

It should now be clear that the approach through “weak” preferences is closely related to the approach through “strict” preferences. It may be helpful to formalize this notion.

Let $(X, \mathcal{B})$ be a budget space. For any (binary) relation R on X , let $\rho(R)$ be the choice on $(X, \mathcal{B})$ rationalized by R , and let $\mu(R)$ be the choice motivated by R . Let $\mathcal{U}$ be the set of all relations on X .

We can partition the space $\mathcal{U}$ into equivalence classes in two natural ways:

We will say that $R, S \in \mathcal{U}$ are **rationally equivalent** (with respect to $(X, \mathcal{B})$) if they rationalize the same choice:

$$\rho(R) = \rho(S). \tag{6.1}$$

Let $\mathcal{R}$ be the quotient space of $\mathcal{U}$ under this equivalence relation; and for any $R \in \mathcal{U}$ let R/ρ be the ρ -equivalence class of R

We will say that $R, S \in \mathcal{U}$ are **motivationally equivalent** (with respect to $X, \mathcal{B}$) if they motivate the same choice:

$$\mu(R) = \mu(S). \quad (6.2)$$

Let $\mathcal{M}$ be the quotient space of $\mathcal{U}$ under this equivalence relation; and for any $R \in \mathcal{U}$ let R/μ be the μ -equivalence class of R .

The equivalence classes R/ρ and R/μ are clearly the appropriate subjects of some important economics topics. For example, in proving existence of competitive allocations, any rationalizing preference can be replaced by any rationally equivalent one, and any motivating preference can be replaced by any motivationally equivalent one, and the competitive properties will be preserved.

Even more is true: not only can we replace preferences by any others in their equivalence classes, but there is a natural correspondence between ρ -equivalence classes and μ -equivalence classes.

Theorem 12. There is a unique operator $\mathbf{C}: \mathcal{R} \cup \mathcal{M} \rightarrow \mathcal{R} \cup \mathcal{M}$ such that:

$$\begin{aligned} \mathbf{C}|_{\mathcal{R}}: \mathcal{R} &\rightarrow \mathcal{M} \\ \mathbf{C}|_{\mathcal{M}}: \mathcal{M} &\rightarrow \mathcal{R}, \end{aligned} \quad (6.3)$$

and, for any $R \in \mathcal{U}$:

$$\begin{aligned} \mathbf{C}(R/\rho) &= \{S \in \mathcal{U}: \mu(S) = \rho(R)\} \\ \mathbf{C}(R/\mu) &= \{S \in \mathcal{U}: \rho(S) = \mu(R)\}. \end{aligned} \quad (6.4)$$

And this operator is idempotent: for all $K \in \mathcal{R} \cup \mathcal{M}$,

$$\mathbf{C}(\mathbf{C}(K)) = K. \quad (6.5)$$

PROOF. We will use (6.4) to define $\mathbf{C}$. Of course that is not a proper definition as it stands. First we must verify that for any $R', R'' \in R/\rho$ we have $\rho(R') = \rho(R'')$. But that is immediate from (6.1). Second, we must verify that both parts of (6.4) define the same set whenever they both apply. I.e., we must show that, if $R/\rho = R/\mu$, then

$$\{S \in \mathcal{U} : \mu(S) = \rho(R)\} = \{S \in \mathcal{U} : \rho(S) = \mu(R)\}. \quad (6.6)$$

If we can show (6.6), then not only is $\mathbf{C}$ well defined, but it clearly satisfies (6.3) and (6.4), hence it is the unique such operation.

To verify (6.6), suppose

$$R/\rho = R/\mu. \quad (6.7)$$

First let

$$\mu(S) = \rho(R); \quad (6.8)$$

we will show that $\rho(S) = \mu(R)$.

Applying the operator $(\)^*: \mathcal{U} \rightarrow \mathcal{U}$ defined below in (6.15), we see from Lemma 5 (6.19a) below that

$$\rho(S^*) = \mu(S). \quad (6.9)$$

Then

$$\rho(S^*) = \rho(R) \quad (6.10)$$

by (6.8) and (6.9). So

$$S^* \in R/\rho = R/\mu, \quad (6.11)$$

hence

$$\mu(S^*) = \mu(R). \quad (6.12)$$

Now we see from Lemma 5 (6.19b) below that

$$\rho(S) = \mu(S^*), \quad (6.13)$$

so

$$\rho(S) = \mu(R) \quad (6.14)$$

by (6.12) and (6.13), as was to be shown.

Second, let $\rho(S) = \mu(R)$. Then, a parallel proof shows $\mu(S) = \rho(R)$. QED

We call $\mathbf{C}(K)$ the **conjugate** class of K . If $R \in K$ and $S \in \mathbf{C}(K)$, we say that R and S are **conjugate** relations.

Although Theorem 12 defines $\mathbf{C}$ as an operation on classes of relations, it is of interest that it can be induced by a simple operation on relations themselves. Indeed, our proof of Theorem 12 made use of the operator $()^*$ defined on $\mathcal{U}$ by:¹⁴ for all $R \in \mathcal{U}$, and all $x, y \in X$,

$$xR^*y \Leftrightarrow \neg yRx. \quad (6.15)$$

We have already used this operation in our proofs of Theorem 1 (cf. (3.1.5)) and Theorem 9 (cf. (G)).

The following properties of $()^*$ are easily shown. Here we define upper and lower sections, R^U and R^L , by:

$$\begin{aligned} R^U(y) &= \{x \in X: xRy\} \\ R^L(y) &= \{x \in X: yRx\}. \end{aligned} \quad (6.16)$$

Lemma 5. $()^*: \mathcal{U} \rightarrow \mathcal{U}$ is

$$\text{one-to-one}, \quad (6.17)$$

and, for all $R \in \mathcal{U}$:

$$R^{**} = R \quad (6.18)$$

$$\rho(R^*) = \mu(R) \quad (6.19a)$$

$$\mu(R^*) = \rho(R) \quad (6.19b)$$

$$(6.20)$$

$$R^* \text{ is irreflexive (resp. reflexive)} \Leftrightarrow R \text{ is reflexive (resp. irreflexive)}$$

$$R^* \text{ is antisymmetric (resp. total)} \Leftrightarrow R \text{ is total (resp. antisymmetric)}$$

$$R^* \text{ is asymmetric (resp. complete)} \Leftrightarrow R \text{ is complete (resp. asymmetric)}$$

$$R^* \text{ is complete} \Rightarrow R \text{ is the asymmetric part of } R^*$$

$$R \text{ is complete} \Rightarrow R^* \text{ is the asymmetric part of } R$$

$$R^* \text{ is negatively transitive}^{15} \text{ (resp. transitive)} \Leftrightarrow R \text{ is transitive (resp. negatively transitive)}$$

¹⁴ This operation is somewhat similar to the operation McKenzie applied to extend weak preferences to total weak preferences ([16], p. 830).

$$R^{*U}(y) \text{ is open (resp. closed)} \Leftrightarrow R^L(y) \text{ is closed (resp. open)}.$$

PROOF. We omit the elementary proof.

Now $()^*: \mathcal{U} \rightarrow \mathcal{U}$ induces $\mathbf{C}: \mathcal{R} \cup \mathcal{M} \rightarrow \mathcal{R} \cup \mathcal{M}$. That is the content of

Theorem 13. For all $R \in \mathcal{U}$,

$$\begin{aligned} \mathbf{C}(R/\rho) &= \{S^* \in \mathcal{U} : S \in R/\rho\} \\ \mathbf{C}(R/\mu) &= \{S^* \in \mathcal{U} : S \in R/\mu\}, \end{aligned} \tag{6.21}$$

so, for all $K \in \mathcal{R} \cup \mathcal{M}$:

$$\mathbf{C}(K) = \{S^* \in \mathcal{U} : S \in K\}. \tag{6.22}$$

PROOF. If $K = R/\rho$, then

$$\mathbf{C}(K) = \mathbf{C}(R/\rho) \tag{6.23a}$$

$$= \{S \in \mathcal{U} : \mu(S) = \rho(R)\} \tag{6.23b}$$

$$= \{S^{**} \in \mathcal{U} : \rho(S^*) = \rho(R)\} \quad (\text{by (6.19a)}) \tag{6.23c}$$

$$= \{T^* \in \mathcal{U} : \rho(T) = \rho(R)\} \tag{6.23d}$$

(defining $T \equiv S^*$)

$$= \{T^* \in \mathcal{U} : T \in R/\rho\} \tag{6.23e}$$

$$= \{T^* \in \mathcal{U} : T \in K\}. \tag{6.23f}$$

And a parallel proof works for the case $K = R/\mu$. QED

In view of Lemma 5 and Theorem 13, we call R^* the **canonical** conjugate relation of R .

The significance of the equivalence notions and conjugacy operation are reflected in

Theorem 14. Any weak competitive equilibrium, with preferences $R_1, \dots, R_m$ remains a weak equilibrium if the R_i are replaced by $S_i \in R_i/\rho$ and becomes a strict equilibrium if the R_i are replaced by $S_i \in \mathbf{C}(R_i/\rho)$.

Any strict competitive equilibrium, with preferences $R_1, \dots, R_m$ remains a strict

¹⁶ I.e., for any $x, y, z \in X$, $\neg xR^*y$ & $\neg yR^*z \Rightarrow \neg xR^*z$.

equilibrium if the R_i are replaced by $S_i \in R_i / \mu$ and becomes a weak equilibrium if the R_i are replaced by $S_i \in \mathbf{C}(R_i / \mu)$.

From this section's point of view, Theorem 14 and Sections 3, 4, and 5 highlight the freedom offered by general equilibrium theory to modify preferences, both within equivalence classes, and between conjugate classes.

By using the canonical conjugacy operation, we can be even more specific about how theorems change when preference are modified between conjugate classes. That is what the following metatheorem shows.

We will not formalize its statement, since the meaning is intuitively clear, and the proof is immediate. We only mention here that we are supposing sentences of mathematics made up in the usual way from $\&$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), $\Rightarrow$ (implication), $\Leftrightarrow$ (bimplication), $\forall$ (universal quantifier), $\exists$ (existential quantifier), $\in$ (set membership), and binary relation symbols R_i and R_i^* .¹⁶ And we suppose a conventional axiom system for mathematics, together with these axioms:

$$\forall x \forall y [x R_i^* y \Leftrightarrow \neg y R_i x].$$

We can simplify its statement with the following definition. If T is a sentence in our restricted language, and if T^* is obtained from T by replacing each formula in T of the form $x R_i y$ by $\neg y R_i^* x$, every quantifier $\forall R_i$ by $\forall R_i^*$, and every quantifier $\exists R_i$ by $\exists R_i^*$, then we will call T^* the **canonical conjugate** of T .

Canonical Conjugacy Metatheorem. Suppose T is a theorem with relation symbols $R_1, \dots, R_k$, but not symbols $R_1^*, \dots, R_k^*$. Then T^* is a theorem.

Furthermore, in this translation, statements equivalent to rationality, motivation, reflexivity, irreflexivity, asymmetry, antisymmetry, totality, completeness, openness, and closedness translate according to Lemma 5. So statements equivalent to weak competitive equilibrium translate into statements equivalent to strict competitive equilibrium, and vice versa.

Although the proof is trivial, this metatheorem provides a useful link between the weak and strong preference approaches in consumer and equilibrium theory. As one

¹⁶ It is well known that all of mathematics can be constructed on this basis. Note, however, that it is a restricted language. It does not allow, for example, a sentence T of the form: ' R is reflexive'. (Indeed, if that is a true sentence, then ' R^* is reflexive' is *false*.) But it does allow expression of the same idea through

application, it provides an immediate proof of either Theorem 1 or Theorem 2 from the other, since it is easy to check they are canonical conjugates of each other. And it alerts us to look for theorems conjugate to every known one.

7. THE WEAK AXIOM AND EQUILIBRATING PREFERENCES

In Section 5 we found revealed preference axioms to characterize some equilibrating preferences. Kihlstrom, Mas-Colell, and Sonnenschein have already sought to do this through the Weak Axiom. So let us see how much we can characterize with the Weak Axiom.

7.1. *Characterization and Conjecture*

The Weak Axiom implies more than motivation and rationality:

Theorem 15. If h satisfies the Weak Axiom of Revealed Preference, then h is total-asymmetric-motivated, hence total-reflexive-rational.

PROOF. Suppose h satisfies the Weak Axiom. Then, since S is asymmetric, it can be extended to a total asymmetric relation T , using the Axiom of Choice. (In the simple case when X is a subset of R^n , or any linear space with a well ordered basis, we can avoid the Axiom of Choice, and extend S lexicographically.) Clearly, T is a (total asymmetric) motivation, and $T_{\cup=}$ is a (total-reflexive-) rationalization. QED

However, the converse is not true. In Remark 6 we noted that the V-Axiom, which characterizes rationality, does not imply the Weak Axiom. We now see that, even for singleton-valued competitive demands in $R_{\geq}^2$, total-reflexive-rationality need not imply the Weak Axiom.

' $\forall x[xRx]$ '.

Example 2. Let $X = R_+^2$, and let $M = \{x, y, z, w\}$ as in Figure 2, and let $N = X \setminus M$. Define $\succsim$ on X by:

$a \succsim b$ means

$a \in M$ & $b \in N$, or

$a, b \in N$ & $a L b$, or

$a = z$ & $b = x$, or

$a = w$ & $b = y$, or

$a = x$ & [$b = w$ or $b = y$], or

$a = y$ & [$b = z$ or $b = x$], or

$a = z$ & $b = w$, or,

$a = b$.

Figure 2

It is easily checked that $\succsim$ is reflexive and total, and it generates a singleton-valued demand h . But for the budgets B and B' in Figure 2, we have $h(B) = \{x\}$ & $h(B') = \{y\}$, violating the Weak Axiom of Revealed Preference.

So the Weak Axiom implies more than total-reflexive-rationality. Exactly how much more is shown by:

Theorem 16. Let $X = R^2$, and let $\mathcal{B} = \mathcal{C}$. Suppose h is a decisive choice on $(X, \mathcal{B})$ satisfying the budget equality. Then h can be rationalized by a preference $\succsim$ satisfying:

reflexivity (2.1),

totality (2.2),

and this convexity condition (for asymmetric part $\succ$):

$$\forall x, y, \alpha \quad x, y \in X \text{ \& } \alpha \in (0, 1) \quad [x \succsim y \text{ \& } x \neq y \Rightarrow \alpha x + (1-\alpha)y \succ y], \quad (7.1.1)$$

if and only if h satisfies the Weak Axiom of Revealed Preference.

PROOF. It is well known and easy to prove that rationality and (7.1.1) imply the Weak Axiom. To show the converse, define R by:

$$(7.1.2) \quad xRy \Leftrightarrow [x \neq y \text{ \& } \exists v \in X \exists \alpha \quad 0 \leq \alpha < 1 \quad vSy \text{ \& } x = \alpha y + (1-\alpha)v].$$

It is easy to show that the Weak Axiom implies that R is asymmetric.

Define $x \succsim y$ by:

$$(7.1.3) \quad \begin{aligned} x \succsim y &\Leftrightarrow xRy, \text{ or} \\ &\neg xRy \text{ \& } \neg yRx \text{ \& } xLy, \text{ or} \\ &x = y. \end{aligned}$$

Then $\succsim$ is clearly reflexive and total.

To see that $\succsim$ rationalizes h , suppose $x = h(p, m)$ and $p \cdot y \leq m$ and $y \neq x$. It suffices to show that $x \succsim y$ and $\neg y \succsim x$. Since xSy , clearly xRy , hence $x \succsim y$. By asymmetry of R , $\neg yRx$. Hence $\neg y \succsim x$, as was to be shown.

Finally, we verify convexity condition (7.1.1). Let $x \succsim y$ \& $x \neq y$ and $z \equiv \alpha x + (1-\alpha)y$ for some $\alpha \in (0, 1)$. Then either:

- i) xRy , or
- ii) $\neg xRy \text{ \& } \neg yRx \text{ \& } xLy$.

In case (i) there exist a $v \in X$ and an $s \in (0, 1)$ such that vSy \& $x = sy + (1-s)v$. That means zRy , so $z \succsim y$. By asymmetry of R , we have $\neg yRz$, hence $\neg y \succsim z$. Therefore $z \succ y$, as in (7.1.1).

In case (ii) we have xLy , so $\neg yLx$ by asymmetry of L . That also implies zLy . To show that $z \succ y$, suppose not. Then $y \succcurlyeq z$; and since zLy , we conclude yRz by (7.1.3). So there exist $w \in X$ and $s \in [0,1]$ such that wSz and $y = sz + (1-s)w$. Say $w = h(p', m')$ and (by the budget equality) $m' = p' \cdot w \geq p' \cdot z = p'[ux + (1-u)w]$ for some $u \in (0,1)$. (Recall that y is a convex combination of w and z , and z is a convex combination of x and y .) So $up' \cdot w \geq up' \cdot x$, hence $p' \cdot w \geq p' \cdot x$. Therefore wSx , so yRx , contradicting (ii). Hence $\neg y \succcurlyeq z$, so $z \succ y$ as in (7.1.1). QED

Of course we do not claim these are equilibrating preferences, since we do not claim the continuity property used in most existence proofs. To obtain continuity of demand, Kihlstrom, Mas-Colell, Sonnenschein, and Shafer first studied a preference that was:

$$\text{reflexive} \quad (7.1.4a)$$

$$\text{total} \quad (7.1.4b)$$

$$\text{continuous} \quad (7.1.4c)$$

$$\text{convex in this sense:} \quad (7.1.4d)$$

$$x \succcurlyeq z \ \& \ y \succcurlyeq z \Rightarrow \forall \alpha \ 0 < \alpha < 1 \ \alpha x + (1-\alpha)y \succ z .$$

They noted that the singleton-valued demand h that was generated would satisfy the Weak Axiom and be continuous:

Proposition 4. Suppose that $X = R^2_+$, and $\mathcal{B} = \mathcal{C}$, and h is a choice on $(X, \mathcal{B})$. If h is rationalized by a reflexive, total preference $\succcurlyeq$ satisfying the continuity (7.1.4c) and convexity (7.1.4d) conditions, then h is continuous and satisfies the Weak Axiom of Revealed Preference.

And then they conjectured a converse:

Conjecture. A continuous singleton-valued h on the competitive budgets of $X = R^2_+$, satisfying the budget equality

$$p \cdot h(p, m) = m \quad \text{and} \quad (7.1.5)$$

$$\text{the Weak Axiom of Revealed Preference,} \quad (7.1.6)$$

can be rationalized by a preference $\succcurlyeq$ satisfying (7.1.4).

We will show that the Conjecture is not quite true, even in $R_{\neq}^2$. But if we introduce a slight strengthening of the Weak Axiom, something quite similar becomes true.

7.2. Counterexamples

Example 3. The demand function h generated by the utility function

$$u(x_1, x_2) = x_1 x_2 \quad (7.2.1)$$

in $R_{\neq}^2$ clearly satisfies (7.1.5) and (7.1.6). But no preference $\succsim$ satisfying (7.1.4) can rationalize h . For if we pick $x = (0, 3)$ and $y = (0, 1)$, as in Figure 3,

Figure 3

then (7.1.4b) implies $x \succsim y$ or $y \succsim x$. Without loss of generality, suppose $x \succsim y$. Then (7.1.4d) implies $z \equiv (0, 2) \succ y$. So by (7.1.4c), there is a neighborhood $N(y)$ of y , with $z \succ y'$ for all $y' \in N(y)$. But this contradicts the fact that there are y' arbitrarily close to y with $y' S z$.

The violation of the preference conditions occurs at the boundary in Figure 3. One might hope to avoid such violations by restricting the consumption set to $\text{Range}(h)$.

But the Conjecture would still not be true. For the next example shows violations can also occur in the interior of $R^{\mathfrak{L}}$.

Example 4. Consider the utility function u defined on $R^{\mathfrak{L}}$ by:

$$u(x^1, x^2) \equiv \begin{cases} \text{a) } (x^1+1)x^2, & \text{if } x^1 \geq 0 \text{ \& } x^2 \geq 1 \\ \text{b) } \frac{x^1}{1+x^1+x^2-2(x^2)^{\frac{1}{2}}}, & \text{if } x^1 \geq 0 \text{ \& } x^2 < 1. \end{cases}$$

Indifference curves for this utility function are illustrated in Figure 4.

Figure 4

It is easy to check that maximizing that utility yields this demand function:

$$(h^1(p, m), h^2(p, m)) = \begin{cases} \text{i) } (\frac{m-p}{2p}, \frac{m+p}{2}), & \text{if } m \geq 1 \text{ \& } m-p > 0 \text{ \& } m+p > 2 \\ \text{ii) } (\frac{m-1}{p}, 1), & \text{if } m \geq 1 \text{ \& } m-p > 0 \text{ \& } m+p \leq 2 \\ \text{iii) } (0, m), & \text{if } m \geq 1 \text{ \& } m-p \leq 0 \\ \text{iv) } (\frac{m-m^2}{p}, m^2), & \text{if } m < 1, \end{cases}$$

where $p \equiv p_1$, and we have normalized prices so that $p_2 \equiv 1$. And it is also easily verified that this h is continuous in (p, m) , and satisfies the budget equality.

But, even though this h is continuous and satisfies the Weak Axiom of Revealed Preference (as a singleton-valued maximizer of a utility) -- and, even though $\bar{x}, y$, and z are in $\text{Range}(h)$ -- it cannot be rationalized by any preference $\succsim$ satisfying (7.1.4). For clearly $y \bar{S} \bar{x}$, so such a rationalization would require $y \succsim \bar{x}$, hence (by convexity (7.1.4d)) $z \succ \bar{x}$. Then (by continuity (7.1.4c)) all bundles z' in some neighborhood $N(z)$ of z have $z' \succ \bar{x}$. But clearly, for very "flat" budget lines through $\bar{x}$ (i.e., with p_1 small), $\bar{x} S z'$ for many $z' \in N(z)$. This contradiction of the asymmetry of $\succ$ means that no Kihlstrom-Mas-Colell-Sonnenschein-Shafer type rationalization exists -- even if we restrict X to $\text{Range}(h)$.

So the Weak Axiom and continuous demand are not enough to guarantee the desired type of rationality.

7.3. The C-Axiom and Continuous Nontransitive Consumers

Despite the failure of the Conjecture, we will now show that if $\text{Range}(h) = X$, and if we restrict the convexity condition (7.1.4d) to $\text{Range}(h)$, then a slight strengthening of the Weak Axiom characterizes some equilibrating preferences very similar to (7.1.4). This is the C-Axiom, which is a continuous version of the Weak Axiom of Revealed Preference.

C-Axiom.

$$\begin{aligned} & \text{If } x_k S y_k \text{ \& } x_k \xrightarrow[k \rightarrow \infty]{} x \text{ \& } y_k \xrightarrow[k \rightarrow \infty]{} y \text{ \& } x \neq y \\ & \text{then } \neg y S x. \end{aligned} \tag{7.3.1}$$

It is clear from Figure 4 that Example 4 does violate the C-Axiom. For example, if we rotate budget lines through $\bar{x}$, we can find y_k with $\bar{x}Sy_k$, $y_k \rightarrow \bar{y}$, and $\bar{y}S\bar{x}$.

Remark 12. The C-Axiom clearly implies the Weak Axiom of Revealed Preference. It is also easy to see that the C-Axiom implies continuity of demand h . The converse is false: the Weak Axiom and continuity of h do not imply the C-Axiom -- even assuming the budget equality. (See the preceding paragraph.) So the C-Axiom is stronger than the Weak Axiom and continuity of demand. Later, in Theorem 19 below, we will show it is exactly the strength required by rationality of type (7.3.3) below.

Theorem 17. Let $X = \text{Range}(h) \subseteq R_+^n$, and let $\mathcal{B} = \mathcal{C}$. Assume that h is a singleton-valued demand function on $(X, \mathcal{B})$ whose range is a convex set. If h satisfies

$$\text{the } C\text{-Axiom, and} \quad (7.3.2a)$$

$$\text{the budget equality} \quad (7.3.2b)$$

then h can be rationalized by a preference $\succsim$ on X such that, for all $x, y, y_i \in X$ and $\alpha, \alpha_i \in (0, 1)$ with $\sum \alpha_i = 1$, and positive integers k :

$$x \succsim x \quad (\text{reflexivity}) \quad (7.3.3a)$$

$$x \neq y \Rightarrow x \succsim y \text{ or } y \succsim x \quad (\text{totality}) \quad (7.3.3b)$$

$$\succsim \text{ has a closed graph in } X \times X \quad (\text{continuity}) \quad (7.3.3c)$$

$$[x \succsim y \text{ \& } x \neq y] \Rightarrow \alpha x + (1-\alpha)y \succ y \quad (7.3.3d)$$

$$x = \sum_{i=1}^k \alpha_i y_i \Rightarrow \exists i_{i=1, \dots, k} \ x \succsim y_i \quad (7.3.3e)$$

$$x \geq y \Rightarrow x \succ y \quad (\text{monotonicity}), \quad (7.3.3f)$$

where $\succ$ is the asymmetric part of $\succsim$.

PROOF. Define R as follows: for $x, y \in X$,

$$\begin{aligned} xRy &\Leftrightarrow xVy, \text{ or} \\ &\neg xSy \text{ \& } \neg ySx \text{ \& } d(w, x) \leq \frac{1}{2} d(y, x), \end{aligned} \quad (7.3.4)$$

where w is defined in Lemma 6 below.

Then define $\succsim$ by:¹⁷ for $x, y \in X$,

¹⁷ By the closure, we mean the closure in $X \times X$.

$$x \succcurlyeq y \Leftrightarrow (x, y) \in \text{closure}(\{(a, b): aRb\}) \quad (7.3.5)$$

Clearly $\succcurlyeq$ is reflexive, total, and has a closed graph in $X \times X$.

To verify rationality, let $x = h(p, m)$ and $y \in X$ & $p \cdot y \leq m$ & $x \neq y$. We will show $x \succ y$. Since xVy , clearly xRy , hence $x \succcurlyeq y$. Suppose, by way of contradiction, that $y \succcurlyeq x$ for some $y \in B$, where B is the budget determined by (p, m) . Since xSy , the Weak Axiom implies $\neg ySx$. So the only possibility of $y \succcurlyeq x$ is that y_iRx_i for all i , and $y_i \rightarrow y$ and $x_i \rightarrow x$. By taking subsequences, we can assume one of the following two cases:

i) y_iSx_i for all i . Then by the C-Axiom $\neg xSy$, a contradiction.

ii) $\neg y_iSx_i$ & $\neg x_iSy_i$ & $d(y_i, w_i) \leq \frac{1}{2} d(y_i, x_i)$ for all i , where w_i is defined as in Lemma 6 below, for (x_i, y_i) . Then $w_i \rightarrow$ some $w \in (x, y]$, by choosing a subsequence if necessary. Since $w_i \in (x_i, y_i)$ we can choose $\bar{w}_i \in (x_i, w_i)$ with $\bar{w}_i \rightarrow w$. Then, by definition of w_i , we have $\bar{w}_iSx_i$, so the C-Axiom require $\neg xSw$, a contradiction.

To verify convexity condition (7.3.3e), suppose $x \in X$, $x = \sum_{i=1}^k \alpha_i y_i$, $\sum_{i=1}^k \alpha_i = 1$, and $\alpha_i \geq 0$. Then, since $x \in \text{Range}(h)$, clearly xVy_i for some y_i . So xRy_i , hence $x \succcurlyeq y_i$, and (7.3.3e) holds.

To verify the convexity condition (7.3.3d), suppose $x, y \in X$, $x \neq y$ & $x \succcurlyeq y$, and $\alpha \in (0, 1)$. It is enough to show that $z \equiv \alpha x + (1-\alpha)y \succ y$. There are two possible instances of $x \succcurlyeq y$:

A) xSy . Then by the Weak Axiom (cf. Remark 12) and convexity of X , we have zSy . So $z \succ y$ by the rationality proof above.

B) $\neg xSy$ & $\neg ySx$ & $d(w, x) \leq \frac{1}{2} d(y, x)$, where w is defined in Lemma 6 below.

We know that either

B.1) $z \in (w, y)$, or

B.2) $z \in (x, w)$, or

B.3) $z = w$.

In case (B.1), zSy , by the definition of w . So $z \succ y$ by (A).

In case (B.2), $\neg zSy$ and $\neg ySz$, by our hypothesis (B) and by definition of w . Since $d(w, x) \leq \frac{1}{2} d(y, x)$, we have

$$d(w, z) < \frac{1}{2} d(y, z), \quad (7.3.6)$$

so zRy , hence $z \succcurlyeq y$. See Figure 5.

Figure 5

We have to show that $\neg y \succcurlyeq z$. The only possibility of $y \succcurlyeq z$ is that $y_i R z_i$ and $y_i \rightarrow y$ and $z_i \rightarrow z$. By taking subsequences, we can assume one of the following two subcases:

B.2.a) $y_i S z_i$ for all i . We can pick $v_i \in (y_i, z_i)$ such that $v_i \rightarrow$ some $v \in (w, y)$. Then $y_i S v_i$ for all i , and $y_i \rightarrow y$ & $v_i \rightarrow v$. By the C-Axiom, $\neg v S y$, which contradicts the definition of w .

B.2.b) $\neg y_i S z_i$ & $\neg z_i S y_i$ & $d(w_i, y_i) \leq \frac{1}{2} d(z_i, y_i)$ for all i , where w_i is defined as in Lemma 6, for (y_i, z_i) . Then $w_i \rightarrow v$ by choosing a subsequence if necessary, and

$$v \in (z, y] \text{ and } d(v, y) \leq \frac{1}{2} d(z, y). \quad (7.3.7)$$

Since (7.3.6) and (7.3.7) imply $v \neq w$, we can pick $\bar{w} \in (w, v)$. And we can pick $\bar{w}_i, w_i^* \in (w_i, z_i)$ with $\bar{w}_i \rightarrow \bar{w}$ & $w_i^* \rightarrow v$ & $w_i^* S \bar{w}_i$. Then by the C-Axiom, $\neg \bar{w} S v$, hence $\neg \bar{w} S y$, contradicting $\bar{w} \in (w, y)$. Therefore $z \succ y$.

In case (B.3), $z = w$. Then either $z S y$ or $\neg z S y$. If $z S y$, then by (B.1), $z \succ y$. If $\neg z S y$, then by (B.2), $z \succ y$.

So in all three cases -- (B.1), (B.2), and (B.3), we have $z \succ y$, which proves convexity condition (7.3.3d). QED

Lemma 6. Let $x, y \in X = \text{Range}(h)$ & $x \neq y$, where X is a convex subset of R^n . If choice h on $(X, \mathcal{B})$ satisfies the budget equality and the Weak Axiom of Revealed Preference, then there exists a unique $w \in [x, y]$ such that zVx for all $z \in [x, w]$, and zVy for all $z \in (w, y]$.

PROOF. Define $W_1 \equiv \{z \in [x, y]: zVx\}$, and $W_2 \equiv \{z \in [x, y]: zVy\}$. By the Weak Axiom, W_1 and W_2 are intervals. These intervals are either closed or half open, but by the Weak Axiom they do not intersect in more than one point. Since $[x, y] \subseteq \text{Range}(h)$, we have $[x, y] = W_1 \cup W_2$. By connectedness of $[x, y]$, at least one of these intervals must be closed; so $W_1 = [x, w]$ or $W_2 = [w, y]$, for some w . Clearly such a w satisfies the conclusion of the Lemma. QED

Remark 13. It is the fact $\text{Range}(h) = X$ that avoids the counterexample of Example 1.

Remark 14. These preferences are weakly equilibrating, even for non-free-disposability, by Theorem 1.

Remark 15. These preferences (7.3.3) satisfy all of Shafer's conditions (7.1.4), except they have a slightly weaker convexity condition.

Remark 16. The budget equality and C-Axiom hypotheses are not too strong in Theorem 17. Clearly the budget equality follows from the monotonicity condition (7.3.3f). And the C-Axiom and singleton-valuedness of h follow from (7.3.3c & d):

Theorem 18. Let consumption set $X \subseteq R^n$, and let $\mathcal{B} = \mathcal{C}$.

If choice h on $(X, \mathcal{B})$ is rationalized by a relation $\succsim$ that is continuous (7.3.3c), and satisfies the convexity condition (7.3.3d), then h is singleton-valued, and satisfies the C-Axiom.

PROOF. That h is singleton-valued follows immediately from convexity (7.3.3d). To verify the C-Axiom, suppose $x_i \rightarrow x$ & $y_i \rightarrow y$ & $x_i Sy_i$ & $x \neq y$. We must show

$\neg ySx$.

Since x_iSy_i , any rationalization $\succsim$ satisfies $x_i \succsim y_i$. So if $\succsim$ has a closed graph, $x \succsim y$, so $(\frac{1}{2}x + \frac{1}{2}y) \succ y$ by convexity (7.3.3d).

Suppose, by way of contradicting the C-Axiom, ySx ; say $y = h(B)$ & $y \neq x \in B$. Then $yS(\frac{1}{2}x + \frac{1}{2}y)$, so $y \succsim (\frac{1}{2}x + \frac{1}{2}y)$, a contradiction. QED

Finally we have this characterization of some weakly equilibrating preferences through the C-Axiom:

Theorem 19. Let X be a closed, convex subset of R^n , and let $\mathcal{B} = \mathcal{C}$. Suppose that $\text{Range}(h) = X$.

Then any choice h on $(X, \mathcal{B})$ can be rationalized by a preference satisfying the

reflexivity,
totality,
continuity,
convexity, and
monotonicity

conditions (7.3.3) if and only if h satisfies

the C Axiom (hence is singleton-valued),
continuity on P'' ,
the budget equality.

PROOF. Most of this is immediate from Theorem 17, Remark 16, and Theorem 18. To prove the continuity of h on P'' from the preference hypotheses, we first use an argument much like that in the proof of Theorem 9 (Part C) (cf. footnote 10) to prove the upperhemicontinuity of h . Then, since h is singleton-valued, it is continuous. QED

8. GENERALITY

In Section 6 we emphasized the notion of equivalent preferences. Section 5 illustrated its importance by showing that some very general, nonclassical preferences were actually equivalent to "nicer" preferences (Theorems 9 and 11).

How far can we carry this reduction of very general preferences to more special ones? For example, can the nonclassical preferences in Theorems 1 and 2 be replaced by classical ones? Or do those theorems allow some preferences that are not equivalent to classical ones? In other words, are Theorems 1 and 2 **inessential** or **essential** generalizations of the classical theorems?

One answer is given by Mas-Colell's paper [15], in which he proved an equilibrium existence result¹⁸ for convex preferences that were incomplete, and thus allowed a demand correspondence that was not convex-valued. Since convex, complete preferences could not generate such a demand, his existence theorem was an essential generalization of the classical theorems.

Our theorems allow a different proof of essentiality, based on nontransitivity rather than incompleteness. They enable us to exhibit a continuous *nontransitive*, consumer -- even a convex, monotone one -- with no continuous transitive, total equivalent.¹⁹

Example 5. Suppose that each bundle w in Figure 6 is chosen under the budget whose line goes through w and is tangent at w to the curve on which w is located.

¹⁸ Now a special case of Theorem 2 above.

¹⁹ Of course our Example 2 is a simple illustration of a competitive consumer that is rational yet not transitive rational. But can we find such examples if we require the continuity and convexity conditions of Theorems 1 and 2? A very strong affirmative answer is provided by Example 5.

Figure 6

A similar example was used by Hurwicz and Richter [12], p. 62, who showed that no continuous, transitive, and total preference could rationalize the demand.

But now we can see that continuous, total, convex, monotone, *nontransitive* rationalizations do exist. A first result is provided by Theorem 11, since the demand function is clearly continuous and satisfies the budget equality and the V-Axiom (indeed, even the Strong Axiom²⁰). Of course the convexity conclusions are only those of the hypotheses of Theorems 1 and 2. The full convexity result follows from Theorem 19, since the demand function of Figure 6 also clearly satisfies the C-Axiom.

So our nonclassical preferences yield essential generalizations of classical equilibrium theory.

²⁰ So there exists a classical (noncontinuous) preference.

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