ON THE SPECIFICATION OF MULTI - MARKET
DISEQUILIBRIUM ECONOMETRIC MODELS

by

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ABSTRACT

The specifications of multi-market disequilibrium econometric models are clouded with different concepts of effective demand. This paper points out that the specification of such models for econometric analysis can be achieved from the basic concept of fixed price equilibrium and without the use of the concepts of effective demand. The specifications of Ito and Gourieroux, Laffont and Monfort are justified within this framework.

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1. Introduction.

When prices or wages are rigid, markets may be in disequilibrium in the sense that the Walrasian demands and supplies of goods are not equal. There is a relatively large volume of theoretical works on the study of such problems, see, e.g., Clower (1965), Drèze (1975) and Benassy (1982). For empirical analysis, econometric disequilibrium models for single markets have been specified in Fair and Jaffee (1972), Maddala and Nelson (1974), Goldfeld and Quandt (1975) and Bowden (1978). The two survey articles by Maddala (1980) and Quandt (1982) summarize the econometric development and the empirical studies. Single market models do not account for spillover effects from other markets which are in disequilibrium. This motivates the formulation of multi-market disequilibrium models in Quandt (1976), Ito (1980) and Gourieroux, et al (1980). However, the formulations in these articles are different in the specification of effective demand. Effective demands are hypothesized actions or demands of agents expressed on markets when they face quantity rationing; see Benassy (1982, pp. 28-29). However, there are different opinions on the specifications of effective demands, see, e.g., Clower (1965), Drèze (1975), Malinvaud (1977), Benassy (1977) and Svensson (1980). The specification of effective demand in Gourieroux, et al, for the two markets model follows Malinvaud's suggestion. Ito's specification of effective demand differs from Gourieroux, et al. Ito (1980, p. 103) criticizes the effective demand in Gourieroux, et al, as unwarranted behavior.

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On the other hand, Portes (1977, p. 15) criticizes Ito's effective demands. Both the econometric modeling in Ito and Gourieroux, et al, have adopted the Clower demand concept, Clower (1965). Quandt (1982) points out the disadvantage of the Clower demand in that Clower demands need not satisfy the budget constraint. Svensson (1980) argues that the choice theoretic foundations of Clower demand are weak. All these potential conflicts are due to the fact, as pointed out in Portes and Quandt, among others, that so little about behaviors of agents under rationing are known and there does not exist an empirically relevant general theory of rationing. These arguments cloud the validity of the various formulations in the specification of the multi-market disequilibrium market models.

In this article, we suggest a more basic approach on the specification of multi-market disequilibrium models which take into account spillover effects but without a priori hypotheses about how effective demands are determined. Our approach demonstrates that, while the effective demand theories may be interested in theoretical inquiry, the explicit emphasis on the interpretation of effective demands are unnecessary for econometric model formulation and estimation. Our approach utilizes the basic concept of fixed price equilibria defined in the theoretical literature and is built on a well specified microeconomic foundation. Our approach will also justify the compatibility of the approaches of Ito and Gourieroux, et al, from the mathematical point of view and it points out once again its nonnecessity on the interpretation of effective demands as actual behavior of agents for the formulation of multi-market disequilibrium models.

2. Fixed Price Equilibrium, Its Characterization and Spillover Effects.

The study of fixed price equilibria, their existence and efficiency have drawn much attention in the theoretical literature, e.g., Drèze (1975), Laroque (1978), Nayok (1980) and Benassy (1982), among others. The concept of equilibrium when prices are fixed has been established in Drèze (1975). The Drèze article has emphasized the exchange economy but its concept can be straightforward extended to economies with production, see, e.g., Laroque (1978). A fixed price equilibrium is defined as a set of transactions that fulfill the following properties: (i) transactions balance on each market; (ii) transactions are the utility or profit maximizing ones, taking all constraints into account; and (iii) one does not find rationed demanders and rationed suppliers simultaneously in any market.

To leave aside delicate aggregation problems, we follow Ito and Gourieroux, et al, to consider an economy with two represented agents, a consumer and a producer. There are n markets and m ($m > n$) commodities. Without loss of generality, let $m = n+1$. The m^{th} commodity is always in equilibrium without constraint and the n markets are referred to the remaining n goods. Let $U(x_1, \dots, x_m)$ be the utility of the consumer which is assumed to be continuously differentiable. Some of the commodities may be labor supply quantities which have disutility. Without loss of generality, the first ℓ goods are assumed to be such commodities. Therefore, the utility function is strictly decreasing for the first ℓ goods and strictly increasing for the remaining goods. Furthermore, U possesses some concavity properties which guarantee the existence of a unique solution of utility maximization on any convex set. Given the prices p_i , $i=1, \dots, m$ with $p_m = 1$, and endowed income M , the notional demand functions

$$(2.1) \quad x_i^d = D_i(p_1, p_2, \dots, p_n, M) \quad i=1, \dots, m$$

are solutions of the following utility maximization problem of the consumer,

$$(2.2) \quad \max_{x_1, \dots, x_m} \{U(x_1, \dots, x_m) \mid -p_1 x_1 - \dots - p_\ell x_\ell + p_{\ell+1} x_{\ell+1} + \dots + p_{m-1} x_{m-1} + x_m = M\}.$$

On the other hand, the producer possesses the production technology $F(x_1, \dots, x_m) = 0$. In general, F can be a multiple inputs and multiple outputs technology which will be assumed to be continuously differentiable, strictly quasi-concave, increasing function of outputs and decreasing functions of inputs. The notional derived demand and supply functions

$$(2.3) \quad x_i^S = S_i(p_1, \dots, p_n) \quad i=1, \dots, m$$

are solutions of the profit maximization problem of the producer,

$$(2.4) \quad \max_{x_1, \dots, x_m} \{-p_1 x_1 - \dots - p_\ell x_\ell + p_{\ell+1} x_{\ell+1} + \dots + p_m x_m \mid F(x_1, \dots, x_m) = 0\}$$

where the first ℓ goods are inputs purchased from the consumer and the remaining goods are outputs. However, with rigid prices, the notional demands and supplies of (2.1) and (2.3) are not necessarily equal. Given the setting, a fixed price equilibrium is defined as the traded quantity vector Q ,

$Q = (Q_1, \dots, Q_m)'$ such that it satisfies the following two properties:

(I) the traded quantity vector Q is a constrained-optimum for both the consumer and producer in the sense that

$$(2.5) \quad U(Q) = \max_x \{U(x) \mid p_*' x = M, x_i \leq Q_i, \text{ for all } i=1, \dots, n\}$$

and

$$(2.6) \quad p_*' Q = \max_x \{p_*' x \mid F(x) = 0, x_i \leq Q_i \text{ for all } i=1, \dots, n\}$$

where $x = (x_1, \dots, x_m)'$ and $p_* = (-p_1, \dots, -p_\ell, p_{\ell+1}, \dots, 1)'$;

(II) for each market i , $i=1, \dots, n$, the consumer and the producer will not be simultaneously rationed.

For econometric practice, these two requirements need to be characterized by some equivalent but operationally convenient conditions. This can be done with the concept of virtual prices originated in Rothbarth (1941) in the microeconomic literature. The virtual price concept has been shown to be a valuable tool for the analysis of household behavior under quantity rationing, Rothbarth (1941) and Neary and Roberts (1980), and for microeconomic modeling for consumer demand under rationing, Deaton (1981), and with limited dependent variables, Lee and Pitt (1984). Consider the consumer constrained utility maximization problem,

$$(2.7) \quad \max_x \{U(x) | p_*'x = M, x_i \leq Q_i, i=1, \dots, n\}.$$

That the vector Q is the optimal solution of the constrained problem (2.7) can be characterized by the Kuhn-Tucker conditions:

$$(2.8) \quad \begin{aligned} \frac{\partial U(Q)}{\partial x_i} + p_i \lambda - \phi_i &= 0, \quad \phi_i \geq 0, \quad i=1, \dots, \ell; \\ \frac{\partial U(Q)}{\partial x_j} - p_j \lambda - \phi_j &= 0, \quad \phi_j \geq 0, \quad j=\ell+1, \dots, n \\ \frac{\partial U(Q)}{\partial x_m} - \lambda &= 0 \\ p_*'Q &= M \end{aligned}$$

where λ and ϕ 's are Lagrangean multipliers. Define

$$(2.9) \quad \begin{aligned} \xi_i &= - \frac{\partial U(Q)}{\partial x_i} / \frac{\partial U(Q)}{\partial x_m} \quad i=1, \dots, \ell; \\ \xi_j &= \frac{\partial U(Q)}{\partial x_j} / \frac{\partial U(Q)}{\partial x_m} \quad j=\ell+1, \dots, n \end{aligned}$$

which are the prices which support the quantity vector Q as the unconstrained utility maximization solution satisfying the budget constraint, i.e.,

$$(2.10) \quad Q_i = D_i(\xi_1, \dots, \xi_n, C) \quad i=1, \dots, n$$

where $C = M + \sum_{i=1}^{\ell} (p_i - \xi_i)Q_i + \sum_{j=\ell+1}^n (\xi_j - p_j)Q_j$. The prices ξ_i are known as the virtual prices at Q_i and the income C is known as the corresponding virtual income. The Kuhn-Tucker conditions in (2.8) can be written equivalently as

$$(2.11) \quad \begin{aligned} p_i &\geq \xi_i & i=1, \dots, \ell; \\ \xi_j &\geq p_j & j=\ell+1, \dots, n \end{aligned}$$

given that Q has already satisfied the budget constraint.

The corresponding Kuhn-Tucker conditions that characterize the vector Q as the optimal solution of the quantity constrained profit maximization problem of the producer, namely,

$$(2.12) \quad \max_x \{p'_*x \mid F(x) = 0, \quad x_i \leq Q_i, \quad i=1, \dots, n\}$$

are

$$(2.13) \quad \begin{aligned} -p_i - \lambda \frac{\partial F(Q)}{\partial x_i} - \delta_i &= 0, \quad \delta_i \geq 0, \quad i=1, \dots, \ell \\ p_j - \lambda \frac{\partial F(Q)}{\partial x_j} - \omega_j &= 0, \quad \omega_j \geq 0, \quad j=\ell+1, \dots, n \\ 1 - \lambda \frac{\partial F(Q)}{\partial x_m} &= 0 \\ F(Q) &= 0. \end{aligned}$$

Define the virtual prices for inputs at Q as

$$\eta_i = - \frac{\partial F(Q)}{\partial x_i} / \frac{\partial F(Q)}{\partial x_m} \quad i=1, \dots, \ell$$

and the virtual prices for outputs at Q as

$$\eta_j = \frac{\partial F(Q)}{\partial x_j} / \frac{\partial F(Q)}{\partial x_m} \quad j=\ell+1, \dots, n.$$

These virtual prices support the quantity vector Q as the unconstrained profit maximization solution and hence

$$(2.14) \quad Q_i = S_i(\eta_1, \dots, \eta_n) \quad i=1, \dots, n.$$

Given that the vector Q is on the production function, the Kuhn-Tucker conditions (2.13) can be rewritten equivalently as

$$(2.15) \quad \begin{aligned} \eta_i &\geq p_i & i=1, \dots, \ell; \\ p_j &\geq \eta_j & j=\ell+1, \dots, n. \end{aligned}$$

Hence the optimality of Q in (I) is characterized by the inequalities between the virtual prices and the fixed market prices,

$$(2.16) \quad \begin{aligned} \eta_i &\geq p_i \geq \xi_i & i=1, \dots, \ell; \\ \xi_j &\geq p_j \geq \eta_j & j=\ell+1, \dots, n. \end{aligned}$$

Consider the inequality $p_i \geq \xi_i$ in (2.11). If the strict inequality $p_i > \xi_i$ holds, it implies obviously that the corresponding Lagrangean multiplier ϕ_i is strictly positive in (2.8) and the quantity constraint $x_i \leq Q_i$ is strictly binding at Q_i . The consumer is rationed in the market i under this situation. On the other hand, if $p_i = \xi_i$ in (2.11), the traded good Q_i is just binding and the quantity constraint $x_i \leq Q_i$ for the i^{th} good is not effective. The consumer is not rationed in the market i for this case. Similarly, the producer is rationed in the corresponding market i if and only if $\eta_i > p_i$. Hence the property (II) for the fixed price equilibrium simply

means that either $p_i = \eta_i$ or $p_i = \xi_i$ but never $\eta_i > p_i > \xi_i$ for any i , $i=1, \dots, \ell$; and never $\xi_j > p_j > \eta_j$ for any j , $j=\ell+1, \dots, n$. The property (II) has implications on market efficiency. Denote the indirect utility function as $V(p, M)$. Given that Q is the optimal solution of (2.7),

$$\begin{aligned} (2.17) \quad V(\xi_1, \dots, \xi_n, 1, C) &= \max_x \{U(x) | p_\star' x = M, x_i \leq Q_i, i=1, \dots, n\} \\ &= \max_x \{U(x) | p_\star' x = M, x_i = Q_i, i=1, \dots, n\}. \end{aligned}$$

For the producer, let $\pi(p) = \max_x \{p_\star' x | F(x) = 0\}$ be the profit function. We have

$$\begin{aligned} (2.18) \quad \pi(\eta_1, \dots, \eta_n, 1) &= \max_x \{p_\star' x | F(x) = 0, x_i \leq Q_i, i=1, \dots, n\} \\ &= \max_x \{p_\star' x | F(x) = 0, x_i = Q_i, i=1, \dots, n\}. \end{aligned}$$

Consider for example one of the ℓ goods, say the i^{th} good, $i \in \{1, \dots, \ell\}$. By Theorem 1 in the Appendix, if $p_i > \xi_i$, it implies $\frac{\partial V(\xi_1, \dots, \xi_n, 1, C)}{\partial Q_i} > 0$. On the other hand, for the producer, $\eta_i > p_i$ implies $\partial \pi(\eta_1, \dots, \eta_n, 1) / \partial Q_i > 0$. Hence if $\eta_i > p_i$, $i \in \{1, \dots, \ell\}$, the producer can increase his profit by using more input x_i than the quantity Q_i with adjustment on x_m such that the input and output quantities are on the production function with the other quantities kept at the level $Q_1, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_n$. Meanwhile, if $p_i > \xi_i$, the consumer can increase his welfare by increasing the supply of good i than Q_i with adjustment on x_m such that the budget constraint is satisfied. Therefore if $\eta_i > p_i > \xi_i$, both the welfare of the consumer and the producer can be improved upon and the trade would be inefficient. The property (II) assumes that the agents can always eliminate such inefficiency. In summary, the fixed price equilibrium Q is characterized by the conditions,

$$(2.19) \quad \begin{aligned} \eta_i &\geq p_i \geq \xi_i, \quad p_i = \xi_i \quad \text{or} \quad p_i = \eta_i, \quad i=1, \dots, \ell; \\ \xi_j &\geq p_j \geq \eta_j, \quad p_j = \xi_j \quad \text{or} \quad p_j = \eta_j, \quad j=\ell+1, \dots, n. \end{aligned}$$

When the consumer or the producer are rationed in some markets, there will be spillover effects to the other markets. This is the important nature of the multi-markets model. In our framework, the spillover effects are captured in the differences between the virtual prices and the market prices. For the consumer, if he is not rationed in the market i , the virtual price ξ_i will be equal to the market price p_i ; otherwise, they are not equal. Suppose, for example, the consumer is rationed in the first k markets, $k \leq n$. His demand system will be

$$Q_i = D_i(\xi_1, \dots, \xi_k, p_{k+1}, \dots, p_n, C) \quad i=1, \dots, n$$

where $C = M + \sum_{i=1}^k (\xi_i - p_i) \bar{Q}_i$ where $\bar{Q}_i = Q_i$ for $i=\ell+1, \dots, n$ and $\bar{Q}_i = -Q_i$ for $i=1, \dots, \ell$. The spillover effects of the rationed markets to the remaining markets are captured by the terms E_i^d in

$$Q_i = D_i(p, M) + E_i^d \quad i=k+1, \dots, n$$

where

$$E_i^d = D_i(\xi_1, \dots, \xi_k, p_{k+1}, \dots, p_n, C) - D_i(p, M).$$

For the producer, if he is rationed on the markets $k+1, \dots, n$, the spillover effects to the markets $1, \dots, k$ will be captured by the terms E_i^s in

$$Q_i = S_i(p) + E_i^s \quad i=1, \dots, k$$

where

$$E_i^s = S_i(p_1, \dots, p_k, \eta_{k+1}, \dots, \eta_n) - S_i(p).$$

3. Clower Demands and Minimum Conditions.

The specification of the multi-market disequilibrium models in both Ito (1980) and Gourieroux, et al (1980) has used the concept of Clower demands. The Clower demands have been interpreted as effective demands and have behavior interpretations. Much of the controversy occurs around their behavior interpretations. In this section, we will point out its role in our framework and justify the specification of Ito's and Gourieroux, et al's approaches from the mathematical point of view.

Consider the i^{th} utility optimization with the quantity constraints on all the goods except the i^{th} one,

$$\begin{aligned} & \max_x U(x) \\ (3.1) \quad & \text{subject to } p_{\star}^i x = M \\ & \text{and } x_j = Q_j, \quad j \neq i, \quad j=1, \dots, n. \end{aligned}$$

The optimal solution of (3.1) for the i^{th} good, $\tilde{x}_i^d$, is a conditional demand function,

$$(3.2) \quad \tilde{x}_i^d = D_i(p, M | Q_j, \quad j \neq i, \quad j=1, \dots, n).$$

The Clower demands are obtained by solving optimization problems of the type in (3.1) for all i , $i=1, \dots, n$. The Clower demand vector is formed by the vector which has, as its i^{th} component, the i^{th} component of the i^{th} optimization problem. It is apparent that the Clower demand vector need not satisfy the budget constraint. Such a vector, however, can be used to provide equivalent characterization conditions for the fixed price equilibrium Q .

As we have already shown in the previous section, the traded quantity vector Q as a constrained optimum for the consumer is characterized by conditions

$p_i \geq \xi_i$ for $i=1, \dots, \ell$ and $\xi_j \geq p_j$ for $j=\ell+1, \dots, n$. By Theorem 1 and Theorem 2 in the Appendix, it is apparent that $p_i \geq \xi_i$, $i=1, \dots, \ell$ are equivalent to $D_i(p, M|Q_j, j \neq i, j=1, \dots, n) \geq Q_i$; and $\xi_j \geq p_j$, $j=\ell+1, \dots, n$ are equivalent to $D_j(p, M|Q_k, k \neq j, k=1, \dots, n) \geq Q_j$. Hence the optimality of Q for the consumer can be characterized by the following inequality conditions in terms of the Clower demand vector:

$$(3.3) \quad D_i(p, M|Q_j, j \neq i, j=1, \dots, n) \geq Q_i,$$

for all i , $i=1, \dots, n$. By the same theorems, we show that $p_i = \xi_i$ is equivalent to $Q_i = D_i(p, M|Q_j, j \neq i, j=1, \dots, n)$. Similar analysis is applicable to the producer. Denote the optimal solution for the i^{th} constrained profit maximization problem,

$$(3.4) \quad \begin{aligned} & \max_x p'_* x \\ & \text{subject to } F(x) = 0 \end{aligned}$$

$$\text{and} \quad x_j = Q_j, \quad j \neq i, \quad j=1, \dots, n$$

for the i^{th} good as

$$(3.5) \quad \tilde{x}_i^S = S_i(p|Q_j, j \neq i, j=1, \dots, n).$$

By Theorem 3 and Theorem 4 in the Appendix, the optimality of Q for the producer can be characterized by the following inequalities:

$$(3.6) \quad S_i(p|Q_j, j \neq i, j=1, \dots, n) \geq Q_i, \quad i=1, \dots, n.$$

Furthermore, $p_i = \eta_i$ is equivalent to $Q_i = S_i(p|Q_j, j \neq i, j=1, \dots, n)$. Hence the fixed price equilibrium can be characterized by the conditions:

$$(3.7) \quad Q_i = \min_{i=1, \dots, n} \{D_i(p, M|Q_j, j \neq i, j=1, \dots, n), S_i(p|Q_j, j \neq i, j=1, \dots, n)\}$$

This analysis justifies Ito's specification and the use of Clower's demand vector without behavior interpretations on effective demands.

The specification in Gourieroux, et al (1980) differs slightly from Ito's specification. In their article, only a two market model has been explicitly specified, but that can be extended to the n market case with appropriate care. It differs from Ito's specification in the construction of conditional demand and supply functions in some regimes. The demand system for the consumer is (2.10):

$$(3.8) \quad Q_i = D_i(\xi_1, \dots, \xi_n, M + \sum_{j=1}^n (\xi_j - p_j) \bar{Q}_j), \quad i=1, \dots, n.$$

The virtual prices ξ can be solved from (3.8) and are functions of p , M and Q , i.e.,

$$(3.9) \quad \xi_i = \xi_i(p, M, Q_1, \dots, Q_n) \quad i=1, \dots, n.$$

Without loss of generality, consider the regime where the consumer is rationed in the first k markets, $k \leq n$, and the producer is rationed in the remaining markets. For this regime, we have $\xi_i = p_i$ for $i=k+1, \dots, n$ and the remaining ξ 's can be solved from the following equations:

$$Q_i = D_i(\xi_1, \dots, \xi_k, p_{k+1}, \dots, p_n, M + \sum_{j=1}^k (\xi_j - p_j) \bar{Q}_j), \quad i=1, \dots, k.$$

The solutions of ξ_i , $i=1, \dots, k$, are effectively only functions of p , M and $Q_1, \dots, Q_k$, i.e.,

$$\xi_i = \xi_i(p, M, Q_1, \dots, Q_k) \quad i=1, \dots, k.$$

Similarly, for the producer, $n_i = p_i$ for $i=1, \dots, k$ and

$$n_i = n_i(p, Q_{k+1}, \dots, Q_n) \quad i=k+1, \dots, n$$

for this regime. From the theorems in the Appendix, $p_i \geq \xi_i$ for i in $\{1, \dots, \ell\}$, or $p_i \leq \xi_i$ for i in $\{\ell+1, \dots, n\}$ is equivalent to $D_i(p, M|Q_j, j \neq i, j=1, \dots, k) \geq Q_i$. Furthermore, $p_i \leq \eta_i$ for i in $\{1, \dots, \ell\}$, or $p_i \geq \eta_i$ for i in $\{\ell+1, \dots, n\}$ is equivalent to $S_i(p|Q_j, j=k+1, \dots, n) \geq Q_i$. Hence this regime can be equivalently characterized by the conditions that

$$Q_i = S_i(p|Q_j, j=k+1, \dots, n)$$

$$Q_i \leq D_i(p, M|Q_j, j \neq i, j=1, \dots, k)$$

for $i=1, \dots, k$; and

$$Q_i = D_i(p, M|Q_j, j=1, \dots, k)$$

$$Q_i \leq S_i(p|Q_j, j=k+1, \dots, n, j \neq i)$$

for $i=k+1, \dots, n$. These conditions are equivalent to the following conditions from (3.7) for the characterization of the occurrence of this regime:

$$Q_i = S_i(p|Q_j, j \neq i, j=1, \dots, n)$$

$$Q_i \leq D_i(p, M|Q_j, j \neq i, j=1, \dots, n)$$

for $i=1, \dots, k$; and

$$Q_i = D_i(p, M|Q_j, j \neq i, j=1, \dots, n)$$

$$Q_i \leq S_i(p|Q_j, j \neq i, j=1, \dots, n)$$

for $i=k+1, \dots, n$. Thus both the specifications by Ito and Gourieroux, et al, are consistent. They are equivalent ways to characterize the fixed price equilibrium Q .

4. Econometric Specification.

For econometric implementation, stochastic elements will be introduced. The basic structures for the consumer are the demand functions

$$(4.1) \quad x_i^d = D_i(p_1, p_2, \dots, p_n, M; \theta, u) \quad i=1, \dots, n$$

where θ is a finite dimensional parameter vector and u is an n -dimensional vector of random variables. For the producer, the derived demand and supply equations are

$$(4.2) \quad x_i^s = S_i(p_1, p_2, \dots, p_n; \delta, v) \quad i=1, \dots, n$$

where δ is a finite vector of parameters and v is an n -dimensional vector of random variables. Given the exogenous price vector p , the observed dependent variables are the traded quantities Q_i , $i=1, \dots, n$ in the n markets. For econometric implementation, the vector Q needs to be the unique equilibrium for given p . The traded quantities satisfy the following system of equations:

$$(4.3) \quad Q_i = D_i(\xi_1, \xi_2, \dots, \xi_n, M + \sum_{j=1}^n (\xi_j - p_j) \bar{Q}_j; \theta, u) \quad i=1, \dots, n$$

and

$$(4.4) \quad Q_j = S_j(\eta_1, \eta_2, \dots, \eta_n; \delta, v) \quad j=1, \dots, n$$

where the virtual prices ξ 's and η 's satisfy the conditions (2.19). Each regime corresponds to a pattern of market rationing on the consumer and the producer. For each market, either the producer or the consumer but not both is rationed. With n markets, there are totally 2^n number of regimes. To simplify notations, define $\xi_{*i} = -\xi_i$, $\eta_{*i} = -\eta_i$ for $i=1, \dots, \ell$, and $\xi_{*j} = \xi_j$, $\eta_{*j} = \eta_j$ for $j=\ell+1, \dots, n$. It follows that (2.16) can be rewritten as $\xi_{*j} \geq p_{*j} \geq \eta_{*j}$ for all j , $j=1, \dots, n$. For each regime, the system of

equations (4.3), (4.4) and (2.19) determine a set of $2n$ endogenous variables, namely, the n traded quantities and the n virtual prices. Consider the regime where the first L markets are rationed for the consumer and the remaining $n - L$ markets are rationed for the producer. For this regime,

$$(4.5) \quad Q_i = D_i(\xi_1, \dots, \xi_L, p_{L+1}, \dots, p_n, M + \sum_{j=1}^L (\xi_j - p_j) \bar{Q}_j; \theta, u) \quad i=1, \dots, n$$

and

$$(4.6) \quad Q_j = S_j(p_1, \dots, p_L, n_{L+1}, \dots, n_n; \delta, v) \quad j=1, \dots, n.$$

The virtual prices satisfy the following inequalities:

$$\xi_{*i} \geq p_{*i} \quad i=1, \dots, L$$

and

$$p_{*j} \geq n_{*j} \quad j=L+1, \dots, n$$

Given the joint distribution of (u, v) , it implies through the transformations (4.5) and (4.6) a joint density for the $2n$ endogenous variables which will be denoted as $f(Q_1, \dots, Q_n, \xi_{*1}, \dots, \xi_{*L}, n_{*L+1}, \dots, n_{*n})$. The contribution of an observation from this regime to the likelihood function is

$$\begin{aligned} g(Q_1, \dots, Q_n) \\ = \int_{-\infty}^{p_{*n}} \dots \int_{-\infty}^{p_{*L+1}} \left(\int_{p_{*L}}^{\infty} \dots \int_{p_{*1}}^{\infty} f(Q_1, \dots, Q_n, \xi_{*1}, \dots, \xi_{*L}, n_{*L+1}, \dots, n_{*n}) d\xi_{*1}, \dots, d\xi_{*L} \right) dn_{*L+1}, \dots, dn_{*n}. \end{aligned}$$

Without sample separation information, the likelihood function for one observation Q_t is

$$h(Q_{1t}, \dots, Q_{nt}) = \sum_c g_c(Q_{1t}, \dots, Q_{nt})$$

where g_c denotes the likelihood function for Q_t in regime c , and the summation has taken over all the possible regimes. The likelihood function for the whole sample without serial correlation is L :

$$L = \prod_{t=1}^T h(Q_{1t}, \dots, Q_{nt}).$$

The uniqueness of the fixed price equilibrium will guarantee that the likelihood function L is well defined. Convexity properties for the utility and the production functions are not enough to guarantee that the equilibrium is unique. This has been clearly demonstrated in Ito and Gourieroux, et al. Sufficient conditions for uniqueness have been derived in Ito (see also Gourieroux, et al (1980b)) for the linear expenditure system. Unfortunately, there are no general analyses on the uniqueness of the fixed price equilibrium in the literature, except Schulz (1983). The theoretical literature on this topic has established so far the existence of the fixed price equilibrium. However, if the traded quantity vector Q_t is in the neighborhood of the unique competitive equilibrium for each t , the analysis with contraction mapping theorem in Ito (1980, pp. 122-123) or the analysis in Grouieroux, et al, can be extended to provide sufficient conditions to establish the uniqueness equilibrium in the neighborhood of the competitive equilibrium. A rigorous analysis on the local uniqueness of the fixed price equilibria can be found in Laroque (1981). Conditions for the global uniqueness of fixed price equilibria are derived in Schulz (1983). However, both the articles of Laroque (1981) and Schulz (1983) have considered only the pure exchange economy and have left out the production activities. The main condition for global uniqueness of Schulz is closely related to the macroeconomic condition that the marginal propensity to consume is less than one.

5. Conclusions.

In this article, we have considered fixed price disequilibrium market models. The specifications of such models in the econometric literature have been clouded by the different notions of effective demands and their behavioral interpretations. We point out that for econometric analysis and estimation it is unnecessary to emphasize the validity of effective demands as behavior responses to disequilibrium in the markets. The basic notion is on the concept of fixed price equilibrium. We consider the fixed price equilibrium concept introduced by Drèze. Econometric analysis can be proceeded with the equilibrium concept. The technique used is based on the concept of virtual prices introduced in the microeconomic literature of quantity rationing. Microeconomic foundation can therefore be provided for the specification of multi-market disequilibrium models. We have also demonstrated that the specifications in Ito and Gouriéroux, et al, are compatible and consistent with our approach. Their differences are in the different but mathematically equivalent ways to characterize the occurrence of different regimes of disequilibrium.

The models can be extended to incorporate price adjustment equations in some ways. Such generalizations can be rather straightforward to be incorporated into the econometric specification and will not be considered in detail here.

Appendix

Indirect Utility Function and Profit Function Under Rationing

Some useful theorems on indirect utility function and profit function under quantity rationing will be derived in this appendix. Consider the following utility maximization of the consumer with quantity rationing:

$$\begin{aligned}
 & \max_x U(x_1, \dots, x_m) \\
 (A.1) \quad & \text{subject to } -p_1 x_1 - \dots - p_\ell x_\ell + p_{\ell+1} x_{\ell+1} + \dots + p_m x_m = M \\
 & \text{and } x_i = x_i^0, \quad i=1, \dots, m-1.
 \end{aligned}$$

We note that the analysis in the appendix is applicable to x_m being a vector. Let $x^* = (x_1^0, \dots, x_{m-1}^0, x_m^*)'$ be the optimal solution of (A.1); ξ_i , $i=1, \dots, m-1$ and C , $C = M + \sum_{i=1}^{\ell} (p_i - \xi_i) x_i^0 + \sum_{i=\ell+1}^{m-1} (\xi_i - p_i) x_i^0$, be the corresponding virtual prices and virtual income. To simplify notations, denote $\bar{x}^0 = (x_1^0, \dots, x_{m-1}^0)$ and $\bar{\xi} = (\xi_1, \dots, \xi_{m-1})$. We note that $\bar{\xi} = \bar{\xi}(p, M, \bar{x}^0)$ and $C = C(p, M, \bar{x}^0)$ are functions of p , M and $\bar{x}^0$. Let

$$V(p, M) = \max_x \{U(x) \mid p_*' x = M\},$$

where $p_* = (-p_1, \dots, -p_\ell, p_{\ell+1}, \dots, p_m)$, be the indirect utility function. It follows by the definitions of virtual prices and virtual income that

$$(A.2) \quad V(\bar{\xi}(p, M, \bar{x}^0), p_m, C(p, M, \bar{x}^0)) = U(x^*).$$

For the subsequent analysis, the price vector p and income M are fixed parameters and will be suppressed in the function $\bar{\xi}$ and C to simplify the notations.

$$\text{Theorem 1: } \frac{\partial V(\bar{\xi}(\bar{x}^0), p_m, C(\bar{x}^0))}{\partial x_i} = \frac{\partial V(\bar{\xi}(\bar{x}^0), p_m, C(\bar{x}^0))}{\partial C} \cdot (p_i - \xi_i(\bar{x}^0)),$$

$$i=1, \dots, \ell;$$

and

$$\partial V(\bar{\xi}(\bar{x}^0), p_m, C(\bar{x}^0)) / \partial x_j = \frac{\partial V(\bar{\xi}(\bar{x}^0), p_m, C(\bar{x}^0))}{\partial C} \cdot (\xi_j(\bar{x}^0) - p_j),$$

$$j=\ell+1, \dots, m-1.$$

$$\text{Proof: } \frac{\partial V(\bar{\xi}, p_m, C)}{\partial x_i} = \sum_{j=1}^{m-1} \frac{\partial V(\bar{\xi}, p_m, C)}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i} + \frac{\partial V(\bar{\xi}, p_m, C)}{\partial C} \frac{\partial C}{\partial x_i}, \quad i=1, \dots, m-1.$$

$$\text{Roy's identity implies that } \frac{\partial V(\bar{\xi}, p_m, C)}{\partial \xi_i} = x_i^0 \frac{\partial V(\bar{\xi}, p_m, C)}{\partial C} \text{ for } i=1, \dots, \ell$$

and $\frac{\partial V(\bar{\xi}, p_m, C)}{\partial \xi_j} = -x_j^0 \frac{\partial V(\bar{\xi}, p_m, C)}{\partial C}$ for all $j=\ell+1, \dots, m-1$. Hence, it follows that

$$(A.3) \quad \frac{\partial V(\bar{\xi}, p_m, C)}{\partial x_i} = \frac{\partial V(\bar{\xi}, p_m, C)}{\partial C} \left(\sum_{j=1}^{\ell} x_j^0 \frac{\partial \xi_j}{\partial x_i} - \sum_{j=\ell+1}^{m-1} x_j^0 \frac{\partial \xi_j}{\partial x_i} + \frac{\partial C}{\partial x_i} \right).$$

Since $C = M + \sum_{i=1}^{\ell} (p_i - \xi_i) x_i^0 + \sum_{i=\ell+1}^{m-1} (\xi_i - p_i) x_i^0$, it follows that

$$(A.4) \quad \frac{\partial C}{\partial x_i} = p_i - \xi_i - \sum_{j=1}^{\ell} x_j^0 \frac{\partial \xi_j}{\partial x_i} + \sum_{j=\ell+1}^{m-1} x_j^0 \frac{\partial \xi_j}{\partial x_i}, \quad i=1, \dots, \ell;$$

and

$$(A.5) \quad \frac{\partial C}{\partial x_i} = \xi_i - p_i - \sum_{j=1}^{\ell} x_j^0 \frac{\partial \xi_j}{\partial x_i} + \sum_{j=\ell+1}^{m-1} x_j^0 \frac{\partial \xi_j}{\partial x_i}, \quad i=\ell+1, \dots, m-1.$$

The results follow immediately from (A.4) and (A.5), which are substituted into (A.3). Q.E.D.

As the marginal indirect utility of income is always positive, i.e.,

$\partial V(\bar{\xi}, p_m, C) / \partial C > 0$, the marginal change in the value of the indirect utility as the level of rationing is changed depends on the difference between its virtual price and its market price.

For the commodities with disutility, if we define $z_i = -x_i$, $i=1, \dots, \ell$ and denote

$$(A.6) \quad U^*(z_1, \dots, z_\ell, x_{\ell+1}, \dots, x_m) = U(x_1, \dots, x_\ell, x_{\ell+1}, \dots, x_m),$$

U^* will be an increasing function of all its arguments. The indirect utility function V in (A.2) as a function of the rationed goods will preserve some convexity properties of the direct utility function. This property will be useful for the proof of a subsequent theorem.

Lemma 1: The indirect utility function $V(\bar{\xi}(\bar{x}^0), p_m, C(\bar{x}^0))$ is (strictly) quasi-concave with respect to any rationed good x_i , $i=1, \dots, m-1$ if the direct utility function $U^*(z_1, \dots, z_\ell, x_{\ell+1}, \dots, x_m)$ is (strictly) quasi-concave.

Proof: Without loss of generality, consider the rationed good 1. Consider $V(\bar{\xi}(\bar{x}^0), p_m, C(\bar{x}^0))$ as a function of x_1^0 with other variables, namely, p_m and x_i^0 , $i=2, \dots, m-1$, as fixed parameters, and denote it as $\bar{V}(x_1^0)$. Let $x_1^0(0)$ and $x_1^0(1)$ be two different quantities of good 1, and suppose that $\bar{V}(x_1^0(1)) \geq \bar{V}(x_1^0(0))$. Consider any $\lambda \in (0, 1)$ and $x_1^0(\lambda) = \lambda x_1^0(1) + (1-\lambda)x_1^0(0)$. As in (A.2), let $x_m^*(0)$ and $x_m^*(1)$ be defined from the following relations

$$\bar{V}(x_1^0(0)) = U(x_1^0(0), x_2^0, \dots, x_{m-1}^0, x_m^*(0))$$

$$\bar{V}(x_1^0(1)) = U(x_1^0(1), x_2^0, \dots, x_{m-1}^0, x_m^*(1)).$$

In terms of U^* defined in (A.6), we have

$$\bar{V}(x_1^0(0)) = U^*(z_1^0(0), z_2^0, \dots, z_\ell^0, x_{\ell+1}^0, \dots, x_{m-1}^0, x_m^*(0))$$

and

$$\bar{V}(x_1^o(1)) = U^*(z_1^o(1), z_2^o, \dots, z_\ell^o, x_{\ell+1}^o, \dots, x_{m-1}^o, x_m^*(1)).$$

Define the convex combination $x_m(\lambda) = \lambda x_m^*(1) + (1-\lambda)x_m^*(0)$. By the quasi-concavity of U^* , $U^*(z_1^o(\lambda), z_2^o, \dots, z_\ell^o, x_{\ell+1}^o, \dots, x_{m-1}^o, x_m(\lambda)) \geq \bar{V}(x_1^o(0))$. By the optimality of the indirect utility function,

$$\bar{V}(x_1^o(\lambda)) \geq U^*(z_1^o(\lambda), z_2^o, \dots, z_\ell^o, x_{\ell+1}^o, \dots, x_{m-1}^o, x_m(\lambda))$$

and therefore $\bar{V}(x_1^o(\lambda)) \geq \bar{V}(x_1^o(0))$, i.e., $\bar{V}$ is a quasi-concave function of x_1^o . If U^* is strictly quasi-concave, we have a strict inequality. Q.E.D.

Consider a modified utility optimization with one of the quantity constraints deleted, say x_i^o ;

$$(A.7) \quad \begin{aligned} & \max_{x_i, x_m} U(x_1, \dots, x_m) \\ & \text{subject to } -p_1 x_1 - \dots - p_\ell x_\ell + p_{\ell+1} x_{\ell+1} + \dots + p_m x_m = M \end{aligned}$$

$$\text{and } x_j = x_j^o, \quad j \neq i, \quad j=1, \dots, m-1.$$

The optimal solution of $\tilde{x}_i$ in (A.7) is a conditional demand function, i.e.,

$$(A.8) \quad \tilde{x}_i = D_i(p, M | x_j^o, \quad j \neq i, \quad j=1, \dots, m-1).$$

The following theorem provides information about the location of the desirable demand quantity for good i with quantity rationing on the other $m-2$ goods and the location of the rationed quantity x_i^o .

Theorem 2: (i) $\partial V(\bar{\xi}(\bar{x}^o), p_m, C(\bar{x}^o)) / \partial x_i \leq 0$ if and only if

$$D_i(p, M | x_j^o, \quad j \neq i, \quad j=1, \dots, m-1) \leq x_i^o,$$

and

(ii) $\partial V(\bar{\xi}(\bar{x}^0), p_m, C(\bar{x}^0)) / \partial x_i \geq 0$ if and only if

$$D_i(p, M | x_j^0, j \neq i, j=1, \dots, m-1) \geq x_i^0,$$

for all $i=1, \dots, m-1$.

Proof: To prove this theorem, we will prove the following equivalent properties:

(a) $\partial V(\bar{\xi}(\bar{x}^0), p_m, C(\bar{x}^0)) / \partial x_i = 0$ implies $\tilde{x}_i = x_i^0$;

(b) $\partial V(\bar{\xi}(\bar{x}^0), p_m, C(\bar{x}^0)) / \partial x_i > 0$ implies $\tilde{x}_i > x_i^0$;

and (c) $\partial V(\bar{\xi}(\bar{x}^0), p_m, C(\bar{x}^0)) / \partial x_i < 0$ implies $\tilde{x}_i < x_i^0$,

where $\tilde{x}_i$ is defined in (A.8). Under the condition in (a), Theorem 1 implies that $\xi_i(\bar{x}^0) = p_i$ and hence $\tilde{x}_i = x_i^0$. To prove (b), we use counter-arguments and suppose $\tilde{x}_i \leq x_i^0$. Under the condition in (b), there exists quantity x_i^1 for good i such that $x_i^1 > x_i^0$ and $\bar{V}(x_i^1) > \bar{V}(x_i^0)$ where $\bar{V}$ was similarly defined in the proof of Lemma 1. Since

$$\bar{V}(\tilde{x}_i) = \max_x \{U(x) | p_*^1 x = M, x_j = x_j^0, j \neq i, j=1, \dots, m\}$$

and

$$\bar{V}(x_i^1) = \max_x \{U(x) | p_*^1 x = M, x_j = x_j^0, j \neq i, j=1, \dots, m \text{ and } x_i = x_i^1\},$$

it is necessary that $\bar{V}(\tilde{x}_i) \geq \bar{V}(x_i^1)$. As $\tilde{x}_i \leq x_i^0 \leq x_i^1$, $\bar{V}(\tilde{x}_i) \geq \bar{V}(x_i^1)$ and $\bar{V}(x_i^1)$ is quasi-concave with respect to x_i from Lemma 1, it implies that $\bar{V}(x_i^0) \geq \bar{V}(x_i^1)$, which is a contradiction. Property (c) can similarly be proved. Q.E.D.

For the producer, consider the following profit maximization problem with quantity rationing:

$$\begin{aligned}
 & \max_x p'_* x \\
 (A.9) \quad & \text{subject to } F(x_1, \dots, x_m) = 0, \\
 & \text{and } x_i = x_i^0, \quad i=1, \dots, m-1.
 \end{aligned}$$

Suppose that the optimal output m in (A.9) exists and is denoted as x_m^* , the optimal value of profit π_R as a function of input and output prices and the fixed quantities is a restricted profit function:

$$\pi_R = \Pi(p | \bar{x}^0)$$

where $\bar{x}^0 = (x_1^0, \dots, x_{m-1}^0)'$. Without restrictions on input and output quantities, the profit maximization problem, $\max_x \{p'_* x | F(x) = 0\}$ will imply an unrestricted profit function π_U :

$$\pi_U = \Pi(p).$$

Let $\bar{n} = (\eta_1, \dots, \eta_{m-1})'$ be the virtual price vector at the rationed quantities $\bar{x}^0$. The restricted profit function is related to the unrestricted profit function via the virtual price vector:

$$(A.10) \quad \Pi(p | \bar{x}^0) = \Pi(\bar{n}, p_m) - \sum_{i=1}^{\ell} (p_i - \eta_i) x_i^0 + \sum_{i=\ell+1}^{m-1} (p_i - \eta_i) x_i^0.$$

The following result is an analogy to Theorem 1, which relates the marginal restricted profit function with respect to the rationed quantities to the differences between the virtual prices and their market prices.

Theorem 3: $\frac{\partial \Pi(p | \bar{x}^0)}{\partial x_i} = \eta_i(\bar{x}^0) - p_i$ for the input i , $i=1, \dots, \ell$;

and $\frac{\partial \Pi(p | \bar{x}^0)}{\partial x_j} = p_j - \eta_j(\bar{x}^0)$ for the output j , $j=\ell+1, \dots, m-1$.

Proof: From (A.10), we have

$$\begin{aligned} \partial \Pi(p|\bar{x}^0)/\partial x_i &= \sum_{j=1}^{m-1} \frac{\partial \Pi(\bar{n}, p_m)}{\partial \eta_j} \frac{\partial \eta_j(\bar{x}^0)}{\partial x_i} - (p_i - \eta_i(\bar{x}^0)) + \sum_{j=1}^{\ell} \bar{x}_j^0 \frac{\partial \eta_j(\bar{x}^0)}{\partial x_i} \\ &\quad - \sum_{j=\ell+1}^{m-1} x_j^0 \frac{\partial \eta_j(\bar{x}^0)}{\partial x_i} \end{aligned}$$

for $i=1, \dots, \ell$. By the Hotelling-McFadden duality theorem, $\partial \Pi(\bar{n}, p_m)/\partial \eta_j = -x_j^0$ for input j ; $j=1, \dots, \ell$ and $\partial \Pi(\bar{n}, p_m)/\partial \eta_j = x_j^0$ for output j ; $j=\ell+1, \dots, m-1$. Hence it follows that

$$\partial \Pi(p|\bar{x}^0)/\partial x_i = \eta_i(\bar{x}^0) - p_i, \quad i=1, \dots, \ell.$$

Similarly, we can prove that $\frac{\partial \Pi(p|\bar{x}^0)}{\partial x_j} = p_j - \eta_j(\bar{x}^0)$ holds for the outputs $j=\ell+1, \dots, m-1$. Q.E.D.

Analogously to Lemma 1, we have

Lemma 2: The restricted profit function $\Pi(p|\bar{x}^0)$ is (strictly) quasi-concave with respect to any one of the rationed input or output i , $i=1, \dots, m-1$ if the production possibility set $\{x|F(x) \leq 0\}$ is (strictly) convex.

Consider the modified profit maximization problem (A.9) with the i quantity rationing constraint deleted:

$$\max_x \{p'_* x | F(x) = 0, x_j = x_j^0, j \neq i, j=1, \dots, m-1\}$$

and denote its optimal solution $\tilde{x}_i$ as $S_i(p|x_j^0, j \neq i, j=1, \dots, m-1)$. An analogous result to Theorem 2 is the following theorem.

Theorem 4: (i) $\frac{\partial \Pi(p|\bar{x}^0)}{\partial x_i} \leq 0$ if and only if $S_i(p|x_j^0, j \neq i, j=1, \dots, m-1) \leq x_i^0$

and

$$(ii) \quad \frac{\partial \Pi(p|x^o)}{\partial x_i} \geq 0 \quad \text{if and only if} \quad S_i(p|x_j^o, j \neq i, j=1, \dots, m-1) \geq x_i^o$$

for all $i=1, \dots, m-1$.

The proofs of this theorem and Lemma 2 are similar to the consumer analysis and are omitted.

References

- Benassy, J.P. (1977), "On Quantity Signals and the Foundations of Effective Demand Theory", Scandinavian Journal of Economics 79:147-168.
- Benassy, J.P. (1982), The Economics of Market Disequilibrium (New York: Academic Press).
- Bowden, R.J. (1978), The Econometrics of Disequilibrium (Amsterdam: North-Holland).
- Clower, R.W. (1965), "The Keynesian Counter-Revolution: A Theoretical Appraisal", in The Theory of Interest Rates, edited by F.M. Hahn and F.P.R. Brechling (London: MacMillan).
- Deaton, A.S. (1981), "Theoretical and Empirical Approaches to Consumer Demand Under Rationing", in Essays in the Theory and Measurement of Consumer Behavior, edited by A.S. Deaton (Cambridge: Cambridge University Press).
- Drèze, J.H. (1975), "Existence of an Exchange Equilibrium Under Price Rigidities", International Economic Review 16:301-320.
- Fair, R.C., and D.M. Jaffee (1972), "Methods of Estimation for Markets in Disequilibrium", Econometrica 40:497-514.
- Goldfeld, S.M., and R.E. Quandt (1975), "Estimation of a Disequilibrium Model and the Value of Information", Journal of Econometrics 3:325-348.
- Gourieroux, C., J.J. Laffont and A. Monfort (1980a), "Disequilibrium Econometrics in Simultaneous Equations Systems", Econometrica 48:75-96.
- Gourieroux, C., J.J. Laffont and A. Monfort (1980b), "Coherency Conditions in Simultaneous Linear Equations Models With Endogenous Switching Regimes", Econometrica 48:675-696.
- Ito, T. (1980), "Methods of Estimation for Multi-Market Disequilibrium Models", Econometrica 48:97-126.
- Laroque, G. (1978), "The Fixed Price Equilibria: Some Results in Local Comparative Statics", Econometrica 46:1127-1154.

- Laroque, G. (1981), "On the Local Uniqueness of the Fixed Price Equilibria", Review of Economic Studies 48:113-129.
- Lee, L.F., and M.M. Pitt (1984), "Microeconomic Models of Consumer and Producer Demand With Limited Dependent Variables", manuscript, Department of Economics, University of Minnesota.
- Maddala, G.S. (1980), "Disequilibrium, Self-Selection and Switching Models", forthcoming in Handbook of Econometrics, edited by Z. Griliches and M.D. Intriligator (Amsterdam: North-Holland).
- Maddala, G.S., and F.D. Nelson (1974), "Maximum Likelihood Methods for Models of Markets in Disequilibrium", Econometrica 42:1013-1030.
- Malinvaud, E. (1977), The Theory of Unemployment Reconsidered (Oxford: Basil Blackwell).
- Nayak, P.R. (1980), "Efficiency of Non-Walrasian Equilibria", Econometrica 48: 127-134.
- Neary, J.P., and K.W.S. Roberts (1980), "The Theory of Household Behavior Under Rationing", European Economic Review 13:25-42.
- Portes, R. (1977), "Effective Demand and Spillovers in Empirical Two-Market Disequilibrium Models", Harvard Institute of Economic Research, Discussion Paper #595.
- Quandt, R.E. (1978), "Maximum Likelihood Estimation in Disequilibrium Models", in Pioneering Economics, edited by T. Bagioti and G. Franco (Padova: Edizioni Cedam).
- Quandt, R.E. (1982), "Econometric Disequilibrium Models", Econometric Reviews 1:1-63.
- Rothbarth, E. (1941), "The Measurement of Changes in Real Income Under Conditions of Rationing", Review of Economic Studies 8:100-107.

Schulz, N. (1983), "On the Global Uniqueness of Fix-Price Equilibria",
Econometrica 51:47-68.

Svensson, L.E.O. (1980), "Effective Demand and Stochastic Rationing", Review
of Economic Studies LXVII:339-356.