OPTIMAL CARTEL TRIGGER PRICE STRATEGIES*

by

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ABSTRACT

This paper describes a dynamic model of industry equilibrium in which a cartel deters deviations from collusive output levels by threatening to produce at Cournot quantities for a period of fixed duration whenever the market price falls below some trigger price. In this model firms can observe only their own production level and a common market price. The market demand curve is assumed to have a stochastic component, so that an unexpectedly low price may signal either deviations from collusive output levels or a "downward" demand shock.

This paper characterizes the optimality properties of this model, from the standpoint of the firms which participate in the cartel. In particular, the implications for the equilibrium quantity vector, of setting the trigger price and punishment period length at their optimal values, are assessed. It is demonstrated that, in general, the optimal quantity will exceed that which would maximize expected joint net returns in any single period. The optimal aggregate quantity is shown to be a nondecreasing function of the number of firms, equaling the aggregate Cournot level in the limit, and a nondecreasing function of the variance of the stochastic demand component.

1. INTRODUCTION

Consider an industry trade association which incurs costs in the process of collecting and disseminating member price and quantity data, as well as costs of verifying or monitoring the accuracy of these data. In the absence of such transaction costs, Friedman [1] and Osborne [3] have demonstrated that adherence to a collusive agreement could be monitored and enforced without having to revert to "punishing" behavior.

This paper examines situations in which these costs do exist, and so the firms of the industry are faced with a problem of detecting and deterring cheating on an agreement. In particular, it is assumed that firms observe only their own production level and the market price, but not the quantity produced by any other firm. (Their output is assumed to be of homogeneous quality, and so they face a common market price.) If the market demand curve has a stochastic component, an unexpectedly low price may signal either deviations from collusive output levels or a "downward" demand shock.

Green and Porter [2] have shown that under these circumstances, participating firms can deter deviations by threatening to produce at Cournot levels for a period of fixed duration whenever they observe market price below some trigger price. (They go on to propose econometric techniques employing aggregate industry time series data which are designed to detect such an enforcement mechanism. The equilibrium conditions of the theoretical model determine the estimating equations and stochastic structure of the econometric

model. This model exploits the fact that there will be periodic switches between the Cournot and collusive output levels when such a non-cooperative equilibrium exists.) More generally, they have developed a model in which price wars can be viewed as an equilibrium phenomenon. Their paper is an extension and reinterpretation of Stigler's [4] model of detection of secret price cutting in an oligopolistic market. This extension of Stigler's model offers an explanation that what looks like collusive behavior at a point in time is actually the equilibrium outcome of a regularly repeated non-cooperative market game. A firm which considers a secret expansion of output above the collusive level must trade off immediate profit gains with the increased probability that the market price will fall below the trigger price, thereby increasing the likelihood of lower profits in future periods as the industry reverts to Cournot output levels.

The purpose of this paper is to characterize the optimality properties of the Green and Porter model, from the standpoint of the firms participating in the cartel. In particular, the implications for the equilibrium quantity vector, of setting the trigger price and punishment period length at their optimal values, are assessed.

A version of the Green and Porter model is presented in section 2, and general properties of the equilibrium outlined. Optimal cartel strategies are examined in sections 3 and 4. There is also a discussion of some comparative statics results. Section 5 provides some summary remarks. An appendix examines second order conditions.

2. THE MODEL

Consider an oligopoly of N risk neutral firms. The industry output vector is denoted by $\bar{q}_t = (q_{1t}, \dots, q_{Nt})$, where q_{it} is the output of firm i in period t . (Throughout the paper, a bar over a variable indicates that it is an N -dimensional vector.)

Total industry output is then $Q_t = \sum_{i=1}^N q_{it}$. It is assumed that the firms of the industry produce an output of homogeneous quality. As a result, they face a common market price, which is determined by an inverse demand function depending solely on total industry output, and not the distribution of outputs across firms. It is assumed that the inverse demand function has a multiplicative stochastic component, so that the market price firms observe, $\hat{p}_t$, is determined by

$$(1) \quad \hat{p}_t = p(Q_t)\theta_t,$$

where $p: R_+ \rightarrow R_+$ is the inverse demand function and $\{\theta_t\}$ is an identically and independently distributed sequence of random variables not directly observed by firms, with mean μ , density function f and cumulative distribution F . It is assumed that $F(0) = 0$, $F(\infty) = 1$, and both F and f are continuously differentiable. (Section 4 studies the properties of a model where the inverse demand function has an additive stochastic component, i.e. where $\hat{p}_t = p(Q_t) + \theta_t$.) Throughout this paper it is assumed that the inverse demand function is linear and decreasing in output, so that

$$(2) \quad p(Q_t) = a - bQ_t,$$

where a and b are positive constants.

Firms are hypothesized to be symmetric, in that they each face the cost function

$$(3) \quad C(q_i) = c_0 + c_1 q_i$$

where c_0 and c_1 are constants, and where the time subscripts have been dropped for notational simplicity. It is assumed that

$$(4) \quad 0 < c_1 < \mu a,$$

and that the number of firms, N , is small enough so that

$$(5) \quad 0 < c_0 < \frac{(\mu a - c_1)^2}{\mu b(N + 1)^2}.$$

Given $Q_i = Q - q_i = \sum_{j \neq i} q_j$, the total output of the other firms, firm i faces the single period expected profit function

$$(6) \quad \pi_i(q_i, Q_i) = [A - B(Q_i + q_i)]q_i - c_0,$$

where $A = \mu a - c_1$ and $B = \mu b$. Equation (4) and $b > 0$, imply that A and B are positive constants.

Let $\bar{s} = (s_1, \dots, s_N)$ denote the Cournot output vector. Then, given $S_i = \sum_{j \neq i} s_j$, s_i maximizes $\pi_i(q_i, S_i)$ with respect to q_i for every firm i . It is easy to show that

$$(7) \quad s_i = \frac{A}{B(N + 1)} \quad \text{for } i = 1, \dots, N,$$

$$= s \quad \text{say,}$$

and that

$$(8) \quad \pi_i(\bar{s}) = \frac{A^2}{B(N+1)^2} - c_0 \quad \text{for } i = 1, \dots, N.$$

Equation (5) guarantees that firms earn positive profits at Cournot output levels.

Denote the output vector which maximizes expected joint net returns in a single period by $\bar{r} = (r_1, \dots, r_N)$. Then, given $R_i = \sum_{j \neq i} r_j$, r_i maximizes $\sum_{j=1}^N \pi_j(q_i, R_i)$ with respect to q_i for every firm i . It can be shown that

$$(9) \quad r_i = A/2BN \quad \text{for } i = 1, \dots, N$$

$$= r \quad \text{say,}$$

and that

$$(10) \quad \pi_i(\bar{r}) = \frac{A^2}{4BN} - c_0 \quad \text{for } i = 1, \dots, N.$$

As long as there is more than one firm, single firm expected profits will be higher when $\bar{q} = \bar{r}$ than at Cournot output levels.

Now suppose that the industry behaves in the manner posited by Green and Porter. Consider, therefore, firms which follow a strategy of reversion to Cournot output levels for a finite number of periods, say $T - 1$, whenever market price, $\hat{p}_t$, falls below a predetermined trigger price, say $\bar{p}$. Then cooperative and non-cooperative periods are defined recursively by

- (a) $t = 0$ is a cooperative period
- (b) if t is a cooperative period and $\bar{p} \leq \hat{p}_t$, then $t + 1$ is a cooperative period.

- (c) if t is a cooperative period and $\tilde{p} > \hat{p}_t$, then $t + 1, \dots, t + T - 1$ are noncooperative periods with $\bar{q}_t = \bar{s}$, and $t + T$ is a cooperative period.

Suppose that firms face a common discount factor β , where $0 < \beta < 1$. Then, given an industry output vector $\bar{q}$ in cooperative periods, the expected present discounted value of firm i is defined by

$$(11) \quad V_i(\bar{q}) = \pi_i(\bar{q}) + \Pr\{\tilde{p} \leq \Theta p(Q)\} \beta V_i(\bar{q}) \\ + \Pr\{\tilde{p} > \Theta p(Q)\} \left[\sum_{\tau=1}^{T-1} \beta^\tau \pi_i(\bar{s}) + \beta^T V_i(\bar{q}) \right].$$

Thus the expected discounted value of firm i satisfies a recursive relation. This value equals present period expected profits, plus the expected value next period. If present period market price exceeds the trigger price, next period's value equals that of the present period, discounted once. Otherwise, firms earn expected Cournot profits for $T-1$ periods, at which point the industry returns to cooperative behavior.

Equation (11) can be rewritten as

$$(12) \quad V_i(\bar{q}) = \frac{\pi_i(\bar{q}) + F(\tilde{p}/p(Q))[(\beta - \beta^T)/(1 - \beta)]\pi_i(\bar{s})}{1 - \beta + (\beta - \beta^T)F(\tilde{p}/p(Q))} \\ = \frac{\pi_i(\bar{s})}{1 - \beta} + \frac{\pi_i(\bar{q}) - \pi_i(\bar{s})}{1 - \beta + (\beta - \beta^T)F(\tilde{p}/p(Q))}.$$

Thus expected firm value equals that when firms produce at Cournot levels in every period, plus the gain in profit in cooperative periods. versus Cournot periods, appropriately discounted. Not surprisingly, the appropriate discount factor depends on both the trigger price and the length of noncooperative episodes.

A noncooperative equilibrium is determined by $\bar{s}$, T , $\bar{p}$ and an equilibrium output vector $\bar{q}^* = (q_1^*, \dots, q_N^*)$ if

$$(13) \quad V_i(\bar{q}^*) = \text{Max}\{V_i(\bar{q}) \mid q_j = q_j^* \text{ for } j \neq i, \text{ and } q_i \geq 0\}.$$

The first order necessary conditions are $0 = V_i^1(\bar{q}^*)$, where

$$V_i^1(\bar{q}) = \partial V_i(\bar{q}) / \partial q_i, \text{ or}$$

$$(14) \quad 0 = [1 - \beta + (\beta - \beta^T)F] \pi_i^1(\bar{q}^*)$$

$$+ (\beta - \beta^T)[\bar{p}p'(Q^*)/p(Q^*)^2]f[\pi_i(\bar{q}^*) - \pi_i(\bar{s})]$$

for $i = 1, \dots, N$,

where $\bar{p}/p(Q^*)$, the argument of F and f , has been suppressed, and where $\pi_i^1(\bar{q}) = \partial \pi_i(\bar{q}) / \partial q_i$, $p'(Q) = dp(Q)/dQ$, and $Q^* = \sum_{i=1}^N q_i^*$.

Note that there is no presumption that $\bar{q}^* = \bar{r}$, the single period joint profit maximizing vector. Section 3 shows that in general $\bar{q}^*$ will not equal $\bar{r}$, if $\bar{p}$ and T are chosen to maximize total industry expected value.

One immediate result is that, given the symmetric linear structure of demand and costs, participating firms will produce at equal output levels in equilibrium. If q_k^* and q_l^* are positive for

k and l, then we can rewrite equation (14) as

$$(15) \quad 0 = \alpha \pi_i^1(\bar{q}^*) + \gamma [\pi_i(\bar{q}^*) - \pi_i(\bar{s})] \quad \text{for } i = k, l,$$

where $\alpha = 1 - \beta + (\beta - \beta^T)F$ and $\gamma = (\beta - \beta^T)f\bar{p}p'/p^2$ are common to both firms. Note that $\alpha > 0$ (since $0 < \beta < 1$) and $\gamma < 0$ (since $p' < 0$). But

$$(16) \quad \pi_i^1(\bar{q}^*) = A - BQ^* - Bq_i^* \quad \text{for } i = k, l,$$

and

$$(17) \quad \pi_i(\bar{q}^*) - \pi_i(\bar{s}) = q_i^*(A - BQ^*) - A^2/(B(N+1)^2) \quad \text{for } i = k, l.$$

If we subtract equation (15) for firm k from that of firm l, we obtain

$$(18) \quad 0 = \alpha B(q_k^* - q_l^*) + \gamma(A - BQ^*)(q_l^* - q_k^*).$$

Since $\alpha > 0$, $\gamma < 0$ and q_k^* and q_l^* are positive, equation (18) is satisfied if and only if $q_k^* = q_l^*$. Thus we have:

Proposition 2.1: Given the symmetric linear specification of the demand and cost functions, all firms which have positive equilibrium output levels will produce exactly the same quantity in cooperative periods.

I have chosen to work with a symmetric linear structure of inverse demand and costs precisely because of the resultant symmetric equilibrium, which enables a detailed examination of the nature of the effects of the probability distribution function of the stochastic demand component, $F(\theta)$, on the optimal equilibrium solution.

Also, the fact that the firms are symmetric implies that they will unanimously agree upon the optimal solution, so that we can abstract from problems the cartel may face in trying to reconcile disparate interests.

Denote the equilibrium output in cooperative periods for a single firm by q^* . Then aggregate output $Q^* = Nq^*$. We can then rewrite the first order necessary conditions as

$$(19) \quad [1 - \beta + (\beta - \beta^T)F][A - (N + 1)Bq^*] \\ = (\beta - \beta^T)f[\bar{p}b/(a - Nbq^*)^2][q^*(A - NBq^*) - A^2/(B(N + 1)^2)]$$

where the argument of both F and f is $\bar{p}/(a - Nbq^*)$.

A second result is

Proposition 2.2: For any strictly concave profit function, the single period Cournot output vector, $\bar{s}$, is an equilibrium.

This is easily verified by realizing that $\pi_i^1(\bar{s}) = 0$ for $i = 1, \dots, N$, by the definition of $\bar{s}$, and letting $\bar{q}^* = \bar{s}$ in equation (14) or (19). As a result, the problems posed in this framework are not those of existence of equilibrium, but rather uniqueness. This is the motivating force behind the calculus of the following sections, which try to select the values of $\bar{p}$ and T which maximize total industry expected discounted value, subject to the satisfaction of equation (19). These optimal values in turn imply an optimal value of q^* , given equation (19).

Before turning to these issues, it is useful to derive restrictions on the range of those values of q^* which will be of

interest. Because the Cournot output vector is an equilibrium, we can restrict our attention to values of $\bar{q}^*$ where firms do no worse than they would by producing at $\bar{s}$. From equation (12), we know that $V_i(\bar{q}) \geq V_i(\bar{s})$ if and only if $\pi_i(\bar{q}) \geq \pi_i(\bar{s})$, or, in the symmetric linear structure outlined above,

$$(20) \quad q(A - NBq) - c_0 \geq A^2/(B(N+1)^2) - c_0$$

when $q_i = q$ for $i = 1, \dots, N$. Simple manipulation of equation (20) reveals that it is equivalent to

$$(21) \quad \frac{A}{N(N+1)B} \leq q \leq \frac{A}{(N+1)B}$$

or

$$(22) \quad \bar{s}/N \leq q \leq s,$$

where, as before, s is the Cournot output of a single firm. We can therefore restrict our attention to values of q^* in the closed interval $[s/N, s]$. It can be shown that $\pi_i^i(N^{-1}\bar{s}) > 0$ and $\pi_i(N^{-1}\bar{s}) = \pi_i(\bar{s})$ by construction, so that $V_i^i(N^{-1}\bar{s}) > 0$ for all i . Therefore $N^{-1}\bar{s}$ itself is not an equilibrium.

To summarize, we have

Proposition 2.3: In the symmetric linear structure of this section, a noncooperative equilibrium is characterized by q^* , $\tilde{p}$, T and s satisfying equation (19), where q^* lies within $(s/N, s]$.

It is important to note that in equilibrium, the incentive structure is designed to guarantee that firms do not deviate from q^* , the output level in cooperative periods. If they are rational,

firms will then realize that, in equilibrium, market price will be less than trigger price only when there is a sufficiently small realization of the random demand component, θ , and never because of individual firm deviations. When this event occurs, rational firms will nevertheless revert to Cournot levels for $T-1$ periods, precisely because the enforcement mechanism must be employed on these occasions if firms are to have the correct incentives to produce q^* in cooperative periods. If firms don't believe that the enforcement mechanism will be employed, then they have an incentive to cheat on the agreement. A corollary is that in equilibrium, observed "price wars" will always be preceded by a period with an unexpectedly low realization of θ . In other words, in this framework price wars are precipitated by an unexpected fall in demand, rather than punishment of actual cheating.

A related point is that, if firms know the parameters of the inverse demand and cost functions, this analysis could easily be couched in terms of observed values of θ , where in equilibrium firms assume that $\bar{q} = \bar{q}^*$ and then compare realizations of θ with some trigger value, say $\tilde{\theta}$. This approach will lead to first order necessary conditions which are similar to those derived above, with similar implications.

Finally, note that firms will not deviate from the punishing response of Cournot output levels. When all other firms produce at that level, then it is optimal for any particular firm to adhere to the agreement and behave in a similar manner.

3. OPTIMAL CARTEL STRATEGIES

In general terms, section 2 presented a model in which the expected present discounted value of firm i is represented by $V_i(\bar{q}; \tilde{p}, T)$ for each of the N firms in the industry (as in equation (12)). The solution to the first order necessary conditions (14) is the Nash equilibrium quantity vector $\bar{q}^*(\tilde{p}, T)$, which is a function of the trigger price, $\tilde{p}$, and T , where $T - 1$ is the length of episodes during which firms revert to Cournot output levels. The equilibrium value of firm i will then be $V_i(\bar{q}^*(\tilde{p}, T); \tilde{p}, T)$, or $V_i^*(\tilde{p}, T)$, for $i = 1, \dots, N$, which is a function of $\tilde{p}$ and T . From firm i 's perspective, then, the optimal values of $\tilde{p}$ and T are those which maximize $V_i^*(\tilde{p}, T)$ subject to $\tilde{p} \geq 0$ and $T \geq 1$. (Recall that $T - 1$ is the length of reversionary episodes, rather than T .) Because firms are symmetric, these optimal values will be common to all firms.

For reasons which will become obvious later in the paper, for the purposes of this section I have chosen to regard T as a continuous variable and assume that the optimal value happens to be an integer.

In this case the optimal values of $\tilde{p}$ and T , if they are interior solutions, will satisfy the first order necessary conditions

$$(23) \quad 0 = dV_i^*/d\tilde{p}$$

and

$$(24) \quad 0 = dV_i^*/dT,$$

which will have the same solution for each firm i , because of symmetry. Equations (23) and (24) can be rewritten as

$$(25) \quad 0 = \sum_{j=1}^N \frac{\partial V_i}{\partial q_j} \frac{\partial q_j^*}{\partial \bar{p}} + \frac{\partial V_i}{\partial \bar{p}}$$

$$= \sum_{j \neq i} \frac{\partial V_i}{\partial q_j} \frac{\partial q_j^*}{\partial \bar{p}} + \frac{\partial V_i}{\partial \bar{p}},$$

since V_i is evaluated at $\bar{q}^*$ and $V_i^i(\bar{q}^*) = 0$ by construction, and

$$(26) \quad 0 = \sum_{j \neq i} \frac{\partial V_i}{\partial q_j} \frac{\partial q_j^*}{\partial T} + \frac{\partial V_i}{\partial T}.$$

Because of the symmetry results, these equations can in turn be rewritten as

$$(27) \quad 0 = (N - 1) \frac{\partial V_i}{\partial q_j} \frac{\partial q_j^*}{\partial \bar{p}} + \frac{\partial V_i}{\partial \bar{p}} \quad \text{for } j \neq i,$$

and

$$(28) \quad 0 = (N - 1) \frac{\partial V_i}{\partial q_j} \frac{\partial q_j^*}{\partial T} + \frac{\partial V_i}{\partial T} \quad \text{for } j \neq i.$$

The remainder of this section studies the solutions to these equations given the linear structure of the previous section, and is divided into four parts. The first examines interior solutions of these equations, and the second presents an example. The third subsection then demonstrates that for a wide class of distributions the optimal value of T is infinite, and the fourth presents a further example which illustrates this point. To simplify the algebra of this section, we shall assume that μ , the mean of θ , equals one, so that $b = B$.

3.1 Interior Solutions

This section calculates the implications of choosing optimal values of $\bar{p}$ and T , assuming that they are interior. By interior I mean that the optimal value of $\bar{p}$ is positive and finite (more accurately, that $\bar{p}/p(Q^*)$ lies in the interior of the support of the distribution function, F), and that T is greater than unity and finite. Some comparative statics results are presented near the end of the section.

To solve equations (27) and (28) for the optimal values of $\bar{p}$ and T , first note that, if $p^* = p(Nq^*) = a - Nbq^*$,

$$(29) \quad \frac{\partial V_i(\bar{q}^*)}{\partial q_j} = \frac{-(A - Nbq^*)}{[1 - \beta + (\beta - \beta^T)F]} \quad \text{for } j \neq i$$

$$(30) \quad \frac{\partial V_i(\bar{q}^*)}{\partial T} = \frac{\beta^T \ln \beta F[\pi_i(\bar{q}^*) - \pi_i(\bar{s})]}{[1 - \beta + (\beta - \beta^T)F]^2}$$

$$= \frac{p^{*2} [A - (N+1)Bq^*] \beta^T \ln \beta F}{(\beta - \beta^T) f [1 - \beta + (\beta - \beta^T)F] \bar{p} b}$$

where the second equality follows from equation (19), and

$$(31) \quad \frac{\partial V_i(\bar{q}^*)}{\partial \bar{p}} = \frac{-(\beta - \beta^T) f [\pi_i(\bar{q}^*) - \pi_i(\bar{s})]}{[1 - \beta + (\beta - \beta^T)F]^2 p^*}$$

$$= \frac{-p^* [A - (N+1)Bq^*]}{\bar{p} b [1 - \beta + (\beta - \beta^T)F]}$$

where we have again used equation (19). Note that these derivatives are identical for all firms i . Again, $\tilde{p}/p^*$, the argument of F and f , has been suppressed.

Totally differentiating the first order necessary condition, equation (19), and employing Proposition 2.1, which shows that $q_j^* = q^*$ for all j , yields

$$(32) \quad \frac{\partial q^*}{\partial \tilde{p}} = \frac{[A - (N + 1)Bq^* - b\Delta\eta]p^*}{\tilde{p}bK}$$

and

$$(33) \quad \frac{\partial q^*}{\partial T} = \frac{\beta^T \ln \beta p^* [\tilde{p}b\Delta f - Fp^*(A - (N + 1)Bq^*)]}{\tilde{p}fb(\beta - \beta^T)K}$$

where

$$(34) \quad \Delta = \pi_i(\bar{q}^*) - \pi_i(\bar{s}),$$

which is common to all firms i ,

$$(35) \quad \eta = 1 + \frac{f'(\tilde{p}/p^*)}{f(\tilde{p}/p^*)} \frac{\tilde{p}}{p^*},$$

and

$$(36) \quad K = \frac{B\Delta(N + 1)p^*}{A - (N + 1)Bq^*} + bN\Delta(\eta + 1) - (N - 1)(A - NBq^*).$$

For each firm i , solution of equation (27), the optimality condition for $\tilde{p}$, yields

$$(37) \quad q^* = \frac{A}{2NB} \left(\frac{N + \eta + (N + 1)a/A}{N + 1 + \eta} \right) \\ = r \left(\frac{N + \eta + (N + 1)a/A}{N + 1 + \eta} \right)$$

where r is the value of q_i which maximizes total single period expected profits of the industry, given $Q_i = (N - 1)r$. Solution of equation (28), the optimality condition for T , taking $\tilde{p}$ as the optimal value of $\tilde{p}$, and using equations (19) and (37), yields

$$(38) \quad \frac{f(\tilde{p}^*/p^*)}{F(\tilde{p}^*/p^*)} \frac{\tilde{p}^*}{p^*} - \frac{f'(\tilde{p}^*/p^*)}{f(\tilde{p}^*/p^*)} \frac{\tilde{p}^*}{p^*} = 1$$

or

$$(39) \quad \frac{f(\tilde{p}^*/p^*)}{F(\tilde{p}^*/p^*)} \frac{\tilde{p}^*}{p^*} = \eta^*$$

where

$$\eta^* = 1 + \frac{f'(\tilde{p}^*/p^*)}{f(\tilde{p}^*/p^*)} \frac{\tilde{p}^*}{p^*} .$$

Equation (39) implies that η^* is nonnegative, and so q^* is greater than r . Since $q^* > r$, $p(Nq^*) < p(Nr)$. Thus we have:

Proposition 3.1: If $\tilde{p}$ and T are chosen to maximize expected value of the firms of the industry, and the optimal values are an

interior solution, then the equilibrium output level will exceed the single period joint profit maximizing output. As a corollary, the equilibrium price will be lower.

Another implication of equation (38) is a comparative statics result. Let $\theta^* = \tilde{p}^*/p^*$. Then equation (38) can be rewritten as

$$(40) \quad \frac{f(\theta^*)}{F(\theta^*)} \theta^* - \frac{f'(\theta^*)}{f(\theta^*)} \theta^* = 1.$$

Thus the difference between the elasticity of F with respect to θ and that of f with respect to θ , evaluated at θ^* , is equal to unity. If there is a unique solution to this equation at θ^* , q^* will be determined by equation (37) with $\eta = \eta^*$, and then the optimal trigger price will be uniquely determined as $\tilde{p}^* = \theta^* p(Nq^*)$. (Section 3.3 discusses cases in which there is no solution for equation (40).) Thus we have:

Proposition 3.2: The optimal trigger price $\tilde{p}^*$ will adjust in exactly the same proportion as $p(Nq^*)$ adjusts in response to changes in the parameters a, A, B and N . Neither $\tilde{p}^*$ nor q^* will be affected by small changes in the discount factor β if T^* is unique.

Here $dp^*/dq^* = -bN$, and so $d\tilde{p}^*/dq^* = -bN\theta^*$. But

$$\frac{dq^*}{dA} = \frac{N + \eta^*}{2NB(N + 1 + \eta^*)} > 0, \quad \text{so} \quad \frac{d\tilde{p}^*}{dA} = -\frac{(N + \eta^*)\theta^*}{2(N + 1 + \eta^*)} < 0$$

$$\frac{dq^*}{da} = \frac{2N + 1 + \eta^*}{2NB(N + 1 + \eta^*)} > 0, \quad \text{so} \quad \frac{d\tilde{p}^*}{da} = -\frac{(N + 1)\theta^*}{2(N + 1 + \eta^*)} < 0$$

$$\frac{dq^*}{dB} = -\frac{q^*}{B} < 0, \quad \text{so} \quad \frac{d\tilde{p}^*}{dB} = N\theta^*q^* > 0,$$

and

$$\frac{d(Nq^*)}{dN} = \frac{A + a\eta^*}{2B(N + 1 + \eta^*)^2} > 0, \quad \text{and so} \quad \frac{d\tilde{p}^*}{dN} = -\frac{(A + a\eta^*)\theta^*}{2(N + 1 + \eta^*)^2} < 0.$$

Thus the equilibrium quantity (trigger price) is a decreasing (increasing) function of the absolute value of the slope of the inverse demand curve, b , and an increasing (decreasing) function of the intercept of the inverse demand curve, a , and $A = a - c_1$. (Recall equations (2) and (3).) As N , the number of firms, increases, total output increases and so the trigger price will fall. Also, q^* is a decreasing function of N , since

$$\frac{dq^*}{dN} = -\frac{q^*}{N} + \frac{A + a\eta^*}{2BN(N + 1 + \eta^*)^2} < 0 .$$

It is easy to show that as N becomes infinitely large (suppose that fixed costs, c_0 , are zero, so that firms earn positive profits), q^* , r and s all tend to zero. However,

$$Nq^* = \frac{A}{2B} \left(\frac{N + \eta^* + (N + 1)a/A}{N + 1 + \eta^*} \right) , \quad Nr = \frac{A}{2B} \quad \text{and} \quad Ns = \frac{NA}{B(N + 1)}$$

so that

$$\lim_{N \rightarrow \infty} Nq^* = \lim_{N \rightarrow \infty} Ns = \frac{A}{B} > \frac{A}{2B} = \lim_{N \rightarrow \infty} Nr .$$

(For large enough N , the q^* of equation (37) exceeds s , and so firms set $q^* = s$.) Thus, as the number of firms goes to infinity, aggregate output converges to that of Cournot equilibrium. (Again, this implicitly assumes that there are zero fixed costs. If not, then there will be some value of N , say $\hat{N}$, for which $\pi_1((\hat{N}q^*(\hat{N})) \geq 0 > \pi_1((\hat{N} + 1)q^*(\hat{N} + 1))$. Then $\hat{N}$ will be the maximum number of firms. In the concluding section of the paper there is a discussion of the effects of allowing N to be endogenously determined.)

The equilibrium value of η , η^* , is uniquely determined by θ^* according to

$$(41) \quad \eta^* = 1 + f'(\theta^*)\theta^*/f(\theta^*) = f(\theta^*)\theta^*/F(\theta^*) ,$$

so that $\eta^* \geq 0$. But then

$$(42) \quad q^* = \frac{A}{2BN} \left(\frac{N + \eta^* + (N + 1)a/A}{N + 1 + \eta^*} \right) .$$

Since $\eta^* > 0$, $q^* > s/N$ as required by equation (22). Also, $q^* \leq s$ only if $\eta^* \geq (N + 1)((N + 1)(a/A) - N)/(N - 1) = \eta^0$, say, an increasing function of N . If $\eta^* < \eta^0$, $q^* > s$, but the industry can do better by producing at $\bar{s}$, and will do so since it is an equilibrium. Since η^0 is an increasing function of N , the larger the number of firms, the less likely is the event of $\eta^* \geq \eta^0$, and so the less likely is $q^* < s$. Finally, $\lim_{\eta^* \rightarrow \infty} q^* = r$, and

$$(43) \quad \frac{dq^*}{d\eta^*} = \frac{A - (N + 1)a}{2BN(N + 1 + \eta^*)^2} < 0 \quad \text{when } \eta^* > \eta^0$$

$$= 0 \quad \text{otherwise.}$$

Thus we have

Proposition 3.3: As the parameters of the distribution function F change, θ^* and η^* change according to equations (40) and (41), respectively. Equilibrium quantity q^* is a nonincreasing function

of η^* with slope given by equation (43). As η^* goes to infinity, q^* converges to r , the single period joint profit maximizing output.

Recall that μ , the mean of θ , is being held constant. Thus an increase in η^* represents a mean preserving contraction of the density function f about $\theta^* = \bar{p}^*/p^*$. Heuristically, as η^* increases without bound, the density function converges to a mass point at the mean μ , so that θ^* must converge to μ . If μ is unity, then $\bar{p}^* = p(Nq^*) = p(Nr)$. Then we essentially have the certainty world of Friedman [1], in which $\bar{r}$ can be supported as an equilibrium. If $\eta^* < \eta^0$, then in some sense the distribution of θ is too noisy, so that no equilibrium output less than s can be supported.

The optimal value of T , say T^* , can be obtained by substituting the values of $\bar{p}^*$ and q^* obtained from equations (38) and (42) into equation (19), the first order necessary condition. Thus T^* solves

$$(44) \quad [1 - \beta + (\beta - \beta^{T^*})F(\theta^*)][A - (N + 1)Bq^*] \\ = (\beta - \beta^{T^*})f(\theta^*)[b\theta^*/p(Nq^*)][q^*(A - NBq^*) - A^2/B(N + 1)^2].$$

Inversion of equation (44) yields

$$(45) \quad T^* = \left\{ \frac{1}{\ln \beta} \right\} \ln \left\{ \beta - \frac{(1 - \beta)[A - (N + 1)Bq^*]}{f(\theta^*)(b\theta^*/p^*)\Delta - F(\theta^*)[A - (N + 1)Bq^*]} \right\},$$

where Δ is given by equation (34). For interior solutions of $\bar{p}^*$ it can be shown that the expression in $\{ \}$ brackets will be positive and less than β , which in turn is less than unity, so that T^* will be greater than unity, as required. To summarize, we have:

Proposition 3.4: If the optimal value of $\bar{p}^*$ and T^* are interior solutions, then q^* , $\bar{p}^*$ and T^* will be determined by equations (42), (38) and (44), respectively, when $\eta^* > (N+1)[(N+1)a/A - N]/(N-1) = \eta^0$. If $\eta^* \leq \eta^0$, q^* will equal s , the single period Cournot output.

3.2 An Example

Suppose that θ is distributed according to

$$F(\theta) = [\alpha\theta/(\alpha+1)]^\alpha \quad \text{for } 0 \leq \theta \leq (\alpha+1)/\alpha, \text{ and } \alpha > 0,$$

so that

$$f(\theta) = \frac{\alpha F(\theta)}{\theta}, \quad f'(\theta) = \frac{(\alpha-1)f(\theta)}{\theta}, \quad \text{and} \quad E[\theta] = 1.$$

Then decreases in α represent a mean-preserving spread in $f(\theta)$. Note that if $\alpha = 1$, θ is uniformly distributed. Now

$$\frac{f(\theta)\theta}{F(\theta)} = \frac{f'(\theta)\theta}{f(\theta)} + 1 = \alpha \quad \text{for all } \theta \in [0, (\alpha+1)/\alpha].$$

Then equation (40) is satisfied for all possible values of θ , and equilibrium output is given by

$$q^* = \frac{A}{2BN} \left(\frac{N + \alpha + (N+1)a/A}{N+1+\alpha} \right) \quad \text{if } \alpha \geq (N+1)[(N+1)a/A - N]/(N-1) = \alpha^0$$

$$= \frac{A}{B(N+1)} \quad \text{if } \alpha < \alpha^0$$

For finite α , as N goes to infinity, α^0 also goes to infinity, and so Nq^* converges to the same limit as Ns , namely A/B . Since $\lim_{N \rightarrow \infty} Nq^* = A/B$, $\lim_{N \rightarrow \infty} p(Nq^*) = c_1$. Also, as α approaches infinity, q^* converges to r . As α falls, q^* increases towards s , and $q^* = s$ once α falls to α^0 . If there is more than one firm in the industry (i.e. $N > 1$), $\alpha^0 > 1$ because $a \geq A$. As a result, if θ is uniformly distributed, $q^* = s$. This could alternatively be verified by examining the first order necessary conditions (19) for any uniform distribution, and seeing that they can only be satisfied when $q^* = s$, if we restrict q^* to $(s/N, s]$.

The optimal interior values of $\bar{p}^*$ and T^* must be chosen to satisfy the first order necessary conditions, given q^* . There is a degree of indeterminacy, in that if $\bar{p}^*$ adjusts with T^* so that the necessary conditions continue to hold, it doesn't matter what value T^* takes on.

For example, consider the case of $a = A = 2$, $B = 1$ and $N = 2$. Then $r = \frac{1}{2}$, $p(\bar{r}) = 1$, $s = \frac{2}{3}$, $p(\bar{s}) = \frac{2}{3}$, and

$$q^* = \frac{1}{2} \frac{(5 + \alpha)}{(3 + \alpha)} \quad \text{and} \quad p(\bar{q}^*) = \frac{1 + \alpha}{3 + \alpha} \quad \text{if } \alpha \geq 3 ,$$

and

$$q^* = \frac{1}{3} \quad \text{and} \quad p(\bar{q}^*) = \frac{2}{3} \quad \text{if } \alpha < 3 .$$

Simple manipulation of the first order conditions reveals that $\bar{p}^*$ and T^* must be chosen so that

$$\bar{p}^* = p(\bar{q}^*) \left(\frac{\alpha+1}{\alpha} \right) \left(\frac{(1 - \beta)}{(\beta - \beta^{T^*})} \frac{9(1 + \alpha)}{(\alpha^2 - 9)} \right)^{\frac{1}{\alpha}} \quad \text{if } \alpha > 3$$

In this case the single period joint profit maximizing output, r , will be supportable as an equilibrium if, using the first order conditions,

$$1 - \beta = (\beta - \beta^{T^*}) F(\bar{p}) (\alpha - 9) / 9 .$$

But this is only possible for interior values of $\bar{p}$ and T if $\alpha > 9$, which is more restrictive than $\alpha > 3$, the condition under which a q^* less than s can be supported.

3.3 "Corner" Solutions

We now turn to cases in which the optimal values of $\bar{p}$ or T , or both, may not be interior solutions.

For some distributions of θ , no symmetric quantity vector less than the Cournot output vector is supportable as an equilibrium

e.g. the uniform distribution, as shown in section 3.2. (Recall that the linear structure of the inverse demand and cost functions guarantees that all equilibrium quantity vectors be symmetric.)

When this is not the case, we can show that $\bar{p}^*$ must be an interior solution, and that $T^* > 1$. (We shall demonstrate that $\bar{p}^*$ is positive and finite. More precisely, $\bar{p}^*/p(Nq^*)$ must lie in the interior of the support of F .) In order to demonstrate this point, suppose that some symmetric $\bar{q}^0 = (q^0, \dots, q^0)$, where q^0 lies in the open interval $(s/N, s)$, is supportable as an equilibrium by some $(\bar{p}, T)$ combination, say $(\bar{p}^0, T^0)$. Then

$$\pi_i(\bar{q}^0) > \pi_i(\bar{s}) \text{ for all } i, \text{ and so } V_i(\bar{q}^0, \bar{p}^0, T^0) > V_i(\bar{s}, \bar{p}, T)$$

for all i , $\bar{p}$, and T . Then we know that $\bar{q}^*$ cannot equal $\bar{s}$, since $\bar{q}^0$ is preferable to $\bar{s}$ for all firms, and is supported by $(\bar{p}^0, T^0)$.

In this case, $\bar{p}^*$ must be greater than zero. Suppose not, i.e. suppose that $\bar{p}^* = 0$. Then

$$(46) \quad V_i(\bar{q}, 0, T) = \frac{\pi_i(\bar{s})}{1 - \beta} + \frac{\pi_i(\bar{q}) - \pi_i(\bar{s})}{1 - \beta + (\beta - \beta^T)F(0)}$$

$$= \frac{\pi_i(\bar{q})}{1 - \beta}$$

since $F(0) = 0$, and so $\bar{q}^* = \bar{s}$, which contradicts the fact that $\bar{q}^*$ cannot equal $\bar{s}$. Hence $\bar{p}^* > 0$.

Also $\tilde{p}^*$ must be finite. If not, since $F(\infty) = 1$,

$$(47) \quad V_1(\bar{q}, \infty, T) = \frac{(\beta - \beta^T)}{(1 - \beta^T)} \frac{\pi_1(\bar{s})}{(1 - \beta)} + \frac{\pi_1(\bar{q})}{1 - \beta^T},$$

and so $\bar{q}^* = \bar{s}$, a contradiction.

Further, T^* must be greater than one. If $T = 1$, then

$$(48) \quad V_1(\bar{q}, \tilde{p}, 1) = \frac{\pi_1(\bar{q})}{1 - \beta},$$

and so $\bar{q}^* = \bar{s}$, which is again a contradiction.

Since $\tilde{p}^*$ must be an interior solution, the optimal quantity q^* will continue to be determined by equation (37), as in section 3.1. The problem, however, is that for many distributions the optimal value of T^* is infinite. For these distributions there is no value θ^* of θ in the support of F which will satisfy equation (40). Included in this class are the exponential, Rayleigh and lognormal distributions. To see that T^* must be infinite in these cases, note that

$$\begin{aligned} \frac{dV_1^*}{dT} &= (N - 1) \frac{\partial V_1}{\partial q_j} \frac{\partial q_1^*}{\partial T} + \frac{\partial V_1}{\partial T} \quad \text{for } j \neq 1 \\ &= \frac{\beta^T \ln \beta p^{*2}}{\tilde{p} b f (\beta - \beta^T) [1 - \beta + (\beta - \beta^T) F]} \left\{ \frac{-(N - 1) [\tilde{p} b f \Delta - F p^* (A - (N + 1) B q^*)]}{K} \right. \\ &\quad \left. + [A - (N + 1) B q^*] F \right\}. \end{aligned}$$

Equation (38) is derived by setting the expression in { } brackets equal to zero, and making use of equations (19) and (37), with the implicit assumption that T is finite. But if there is no value of $\theta^* = \bar{p}^*/p^*$ which allows equation (38) to be satisfied, as will be the case for the aforementioned distributions, then $T^* = \infty$ guarantees that $dV_1^*/dT = 0$, since $0 < \beta < 1$. Thus we have the desired result.

For each distribution where T^* is infinite, one can construct examples in which some symmetric vector $\bar{q}^0 = (q^0, \dots, q^0)$, where q^0 lies in the open interval $(s/N, s)$, is supportable as an equilibrium by some $(\bar{p}^0, T^0)$ combination where T^0 is finite. In other words, for each of these distributions, even though T^* is infinite, finite values of T can support quantities less than Cournot output levels. The cartel can do better than just following single period Cournot strategies by producing $\bar{q}^*$ or $\bar{q}^0$.

If we set $T = T^*$ when T^* is infinite, then q^* and $\bar{p}^*$ must be chosen to simultaneously satisfy equation (37) and

$$(49) \quad [1 - \beta + \beta F^*](A - (N + 1)Bq^*)$$

$$= \frac{\beta b f^* \bar{p}^*}{p^{*2}} [q^*(A - NBq^*) - A^2/B(N + 1)^2],$$

where $p^* = p(Nq^*)$, $F^* = F(\bar{p}^*/p^*)$, and $f^* = f(\bar{p}^*/p^*)$. By substituting equation (37) into (49), and by letting $\theta^* = \bar{p}^*/p^*$ as before, one obtains, for example when $A = a$,

$$(50) \quad (1 + \eta^*)(N + 1)^2(1 - \beta + \beta F^*) \\ = \beta f^* \theta^* [2N^2 + (N - 1)(1 + \eta^*)]$$

where $f^* = F(\theta^*)$, $f^* = f(\theta^*)$ and, as before,

$$(51) \quad \eta^* = 1 + f'(\theta^*)\theta^*/f^* .$$

Thus θ^* is chosen to satisfy equation (50), which yields η^* via equation (51). Then q^* is given by equation (42), and $\tilde{p}^*$ will again vary as the parameters A and B vary to exactly offset any proportional changes in p^* , since A and B don't enter equation (50) and so do not affect θ^* . Now the derivatives of q^* and $\tilde{p}^*$ with respect to a , A and B are exactly the same as those given on page 17. Thus we again have that the equilibrium quantity (trigger price) is an increasing (decreasing) function of the intercept of the demand curve, a , and $A = a - c_1$, where c_1 is the constant portion of marginal costs, and a decreasing (increasing) function of the absolute value of the slope of the inverse demand curve, b .

Now, however, changes in the number of firms, N , as well as variations in the discount factor β and the parameters of the distribution function F , will alter θ^* in a nontrivial manner, so

that the comparative statics results of section 3.1 with respect to changes in N will no longer hold. It is also important to note that we continue to have

$$(52) \quad q^* = \frac{A}{2NB} \left(\frac{N + \eta^* + (N+1)a/A}{N+1+\eta^*} \right) \quad \text{for } \eta^* \geq \eta^0$$

$$= \frac{A}{B(N+1)} = s \quad \text{otherwise .}$$

As shown in the appendix, if η^* exceeds η^0 then the second order equilibrium conditions will be satisfied. Now q^* will approach $r = A/2NB$ as η^* approaches infinity. When η^* is greater than η^0 , q^* is greater than r and is a nonincreasing function of η^* . We summarize in

Proposition 3.5: If the optimal value of T^* is infinite, then q^* will be determined by equation (52). As the parameters of the distribution function F and the number of firms changes, θ^* and η^* will vary according to equations (50) and (51), respectively. Then the optimal trigger price is given by $\tilde{p}^* = \theta^*p(Nq^*)$. Equilibrium quantity q^* is a nonincreasing function of η^* . As η^* goes to infinity, q^* converges to r , the single period joint profit maximizing output.

When T is infinite, the cartel will behave cooperatively until the market price is less than the trigger price in some period,

and then the firms will revert to Cournot behavior permanently beginning with the next period. We can then talk about the expected life of the cartel, which will be denoted by L . Then L is the expected number of cooperative periods before the "break-down" of the cartel.

The probability that the cartel has a life of j periods is equal to $[1 - F]^{j-1}F$, where the argument of F is $\tilde{p}/p(\bar{q})$. In this case the market price has exceeded the trigger price in the previous $(j-1)$ periods, which occurs with probability $[1 - F]^{j-1}$, and then market price falls below $\tilde{p}$ in the j^{th} period, which occurs with probability F . (Note that we are making use of the assumption that the sequence $\{\theta_t\}$ is independently distributed.) Then the expected life of the cartel will be

$$L = \sum_{j=1}^{\infty} F[1 - F]^{j-1}j$$

$$= 1/F .$$

If $\tilde{p}$ and q equal $\tilde{p}^*$ and q^* , respectively, then

$$(53) \quad L = 1/F(\theta^*)$$

where $\theta^* = \tilde{p}^*/p(Nq^*)$. But then L depends only on the number of firms, the discount factor β , and the parameters of the distribution function F , which affect θ^* , and not on the parameters

of the inverse demand and cost functions, which don't affect θ^* .

Thus we have

Proposition 3.6: If T is infinite and $\bar{p}$ is chosen optimally, then the expected life of the cartel will be $F(\theta^*)^{-1}$, which is a decreasing function of $\theta^* = \bar{p}^*/p(Nq^*)$.

3.4 Another Example

Suppose now that $\ln \theta$ is normally distributed with mean $-\sigma^2/2$ and variance σ^2 . Then θ will follow a lognormal distribution with mean one and variance $\exp(\sigma^2) - 1$, so that increases in σ^2 result in a mean-preserving spread in the distribution of θ . If we denote the density function of a standard normal distribution by ϕ and its cumulative by Φ , then $\theta f(\theta)/F(\theta) = \phi(z)/\Phi(z)\sigma$ and $\eta = z/\sigma$ where $z = (\ln \theta + \sigma^2/2)/\sigma$. Since there is no value of z for which $\phi(z) = z\phi(z)$, there will be no value of θ satisfying equation (40), which requires that η equal $\theta f(\theta)/F(\theta)$. By the reasoning of section 3.3, then, the optimal value of T is infinite. Equation (53) then tells us that the expected life of the cartel will be given by

$$L = \Phi^{-1}((\ln \theta^* + \sigma^2/2)/\sigma),$$

when the trigger price is chosen optimally.

For example, suppose that $N = 2$, $a = A = 2$ and $B = 1$. Then $s = \frac{2}{3}$, $p(\bar{s}) = \frac{2}{3}$, $\pi_1(\bar{s}) = \frac{4}{9}$, $r = \frac{1}{2}$, $p(\bar{r}) = 1$, and $\pi_1(\bar{r}) = \frac{1}{2}$. Also,

$s/N = \frac{1}{3}$. If the trigger price is chosen optimally, q^* will be given by

$$q^* = \frac{1}{2} \left(\frac{9\sigma^2 - 2\ln \theta^*}{7\sigma^2 - 2\ln \theta^*} \right).$$

As shown in the appendix, the second order conditions for a Nash equilibrium, when q^* is less than s , will be satisfied when η exceeds -1 . This is equivalent to $\theta^* < \exp(\sigma^2/2)$. But examination of the expression for q^* reveals that this is exactly the restriction on θ^* which will ensure that q^* be less than s . Thus any (θ^*, q^*) combination satisfying both the expression for the optimal quantity (such that $q^* < s$) and the first order necessary conditions will also satisfy the second order conditions.

If $z^* = (\ln \theta^* + \sigma^2/2)/\sigma$, then we can write the first order conditions, taking T as infinite, as

$$1 - \beta + \beta\phi(z^*) = \frac{\beta\phi(z^*)(13\sigma^2 - 2\ln \theta^*)}{9\sigma(5\sigma^2 - 2\ln \theta^*)},$$

where we have employed the expression for q^* . As σ^2 and β vary, we obtain the following values:

σ^2	β	q^*	p^*	θ^*	$\tilde{p}^*$	L
0.13	0.72	0.65	0.69	1.03	0.72	1.85
0.13	0.75	0.64	0.71	1.00	0.71	1.90
0.13	0.76	0.63	0.73	0.97	0.71	2.00
0.10	0.63	0.63	0.73	0.98	0.72	2.00

Thus increases in the discount factor, keeping σ^2 constant, result in decreases in the optimal quantity q^* towards r , and decreases in $\tilde{p}^*$, as the decreases in θ^* more than offset the increases in p^* . The decreases in θ^* lead to increases in L . Decreases in σ^2 , with β falling to ensure that q^* remain constant, result in increases in θ^* and $\tilde{p}^*$, but no change in L . (These examples were calculated by setting $\ln \theta^*$ equal to $k\sigma^2/2$, and letting k equal $\frac{1}{2}$, 0 and $-\frac{1}{2}$. Here q^* lies between s/N and s as long as k is less than 1 and greater than -4 .)

4. ADDITIVE UNCERTAINTY

This section is divided into three subsections. In the first the model of section 2 is reformulated to allow demand to have an additive, as opposed to multiplicative, stochastic component, and necessary conditions for a Nash equilibrium are derived. In the second section, interior solutions of the optimal choice of $\bar{p}$ and T are characterized. The final section discusses cases where the optimal T is infinite.

4.1 The Model

Suppose that the industry inverse demand curve has an additive stochastic component, so that firms observe market price $\hat{p}_t$ in period t according to

$$(54) \quad \hat{p}_t = p(Q_t) + \theta_t$$

where Q_t is aggregate output and $\{\theta_t\}$ is an identically and independently distributed sequence of random variables with zero mean, density function f , and cumulative distribution F . In this case $F(-\infty) = 0$ and $F(\infty) = 1$. It is assumed that the inverse demand function is given by equation (2) and that firms face the cost function described by equation (3), where we now assume that

$$(55) \quad 0 < c_1 < a ,$$

and that the number of firms N is small enough to ensure that

$$(56) \quad 0 < c_0 < (a - c_1)^2 / b(N + 1)^2.$$

Then if $A = a - c_1$ and $B = b$, both of which will be positive because of assumption (55) and $b > 0$, Cournot output and the corresponding value of expected profits are given by equations (7) and (8), respectively. Equation (56) guarantees that expected profits are positive at Cournot output levels. Similarly, single period expected joint net returns are maximized if output is given by equation (9), implying expected profits as shown in equation (10).

If the industry employs the Green-Porter enforcement mechanism, the expected present discounted value of firm i , given an industry output vector $\bar{q}$ in cooperative periods, can be expressed as

$$(57) \quad v_i(\bar{q}) = \frac{\pi_i(\bar{s})}{1 - \beta} + \frac{\pi_i(\bar{q}) - \pi_i(\bar{s})}{1 - \beta + (\beta - \beta^T)F(\bar{p} - p(Q))} \quad \text{for } i = 1, \dots, N,$$

where time subscripts have been suppressed. A non-cooperative equilibrium at $\bar{q}^*$, $\bar{p}$, T and $\bar{s}$ must then satisfy the first order necessary condition

$$(58) \quad [1 - \beta + (\beta - \beta^T)F][A - (N + 1)Bq^*] \\ = (\beta - \beta^T)f_b[q^*(A - NBq^*) - A^2/B(N + 1)^2],$$

where $\bar{p} = p(Nq^*)$, the argument of F and f , has been suppressed. Note that this formulation of the first order conditions has implicitly assumed that the equilibrium output vector $\bar{q}^*$ is symmetric. Algebraic manipulation analogous to that underlying Proposition 2.1 demonstrates that this will be the case. It is also clear that the single period Cournot output vector will continue to be an equilibrium. To ensure that $V_i(\bar{q}^*) \geq V_i(\bar{s})$, any other symmetric equilibrium output must lie in the interval $(s/N, s]$. Thus Proposition 2.3 continues to hold, where equation (58) is now the first order equilibrium condition.

4.2 Interior Optima

In this case the optimal values of $\bar{p}$ and T , if they are interior solutions (so that $\bar{p}$ is finite and T is finite and greater than unity), will again satisfy the first order necessary conditions given by equations (27) and (28).

To solve these equations, note that,

$$(59) \quad \frac{\partial V_i(\bar{q}^*)}{\partial q_j} = - \frac{A - NBq^*}{1 - \beta + (\beta - \beta^T)F} \quad \text{for } j \neq i,$$

$$(60) \quad \frac{\partial V_i(\bar{q}^*)}{\partial T} = \frac{\beta^T \ln \beta F[\pi_i(\bar{q}^*) - \pi_i(\bar{s})]}{[1 - \beta + (\beta - \beta^T)F]^2} \\ = \frac{\beta^T \ln \beta F[A - (N+1)Bq^*]}{(\beta - \beta^T)f\beta[1 - \beta + (\beta - \beta^T)F]},$$

where the second equality follows from equation (58), and

$$\begin{aligned}
 (61) \quad \frac{\partial V_i(\bar{q}^*)}{\partial \bar{p}} &= - \frac{(\beta - \beta^T) f[\pi_i(\bar{q}^*) - \pi_i(\bar{s})]}{[1 - \beta + (\beta - \beta^T)F]^2} \\
 &= - \frac{[A - (N+1)Bq^*]}{b[1 - \beta + (\beta - \beta^T)F]} ,
 \end{aligned}$$

where equation (58) has been employed again. These derivatives will be the same for all firms.

Totally differentiating the first order necessary condition, equation (58), and exploiting the symmetry properties of the equilibrium, yields

$$(62) \quad \frac{\partial q^*}{\partial \bar{p}} = \frac{f(A - (N+1)Bq^*) - f'b\Delta}{bK}$$

and

$$(63) \quad \frac{\partial q^*}{\partial T} = \frac{\beta^T \ln \beta [fb\Delta - F(A - (N+1)Bq^*)]}{b(\beta - \beta^T)K}$$

where Δ is given by equation (34), as before, but now

$$(64) \quad K = \frac{fB(N+1)\Delta}{(A - (N+1)Bq^*)} - (N-1)f(A - NBq^*) + f'bN\Delta .$$

For each firm i , solution of equation (27), the optimality condition for $\bar{p}$, gives us

$$\begin{aligned}
 (65) \quad q^* &= \frac{A + (N+1)(f/f')}{2BN} \\
 &= r + (N+1)f/(2BNf'),
 \end{aligned}$$

where r is the value of q_i which maximizes total single period expected profits of the industry, given that all other firms produce at a level r . Solution of equation (28), the optimality condition for T , taking $\tilde{p}^*$ as the optimal value of the trigger price and using equations (58) and (65), yields

$$(66) \quad \frac{f(\theta^*)}{F(\theta^*)} = \frac{f'(\theta^*)}{f(\theta^*)}$$

where $\theta^* = \tilde{p}^* - p(Nq^*)$. Equation (66) implies that $f'(\theta^*)$ is nonnegative, and so q^* is greater than r . Simple manipulation reveals that q^* will be less than the Cournot output level if $f(\theta^*)/F(\theta^*)$ exceeds $(N+1)^2/(N-1)A$, which is analogous to the condition in section 3. If $f(\theta^*)/F(\theta^*) < (N+1)^2/(N-1)A$, then $q^* = s$.

If equation (66) uniquely determines θ^* , then q^* is given by equation (65) (as long as it doesn't exceed s), and $\tilde{p}^* = \theta^* + p(Nq^*)$. Since $dp^*/dq^* = -bN$, $d\tilde{p}^*/dq^* = -bN$. When $q^* < s$,

$$\frac{dq^*}{dA} = \frac{1}{2bN} > 0, \quad \text{so} \quad \frac{d\tilde{p}^*}{dA} = -\frac{1}{2} < 0,$$

$$\frac{dq^*}{dB} = -\frac{A}{2b^2N} - \frac{(N+1)f}{2b^2Nf'} < 0, \quad \text{so} \quad \frac{d\tilde{p}^*}{dB} = \frac{A}{2b} + \frac{(N+1)f}{2bf'} > 0,$$

$$\frac{dq^*}{db} = \frac{dq^*}{dB} < 0, \quad \text{so} \quad \frac{d\tilde{p}^*}{db} = \frac{d\tilde{p}^*}{dB} > 0,$$

$$\frac{dq^*}{dN} = -\frac{(A + f/f')}{2bN^2} < 0,$$

and

$$\frac{d(Nq^*)}{dN} = \frac{f}{2f'b} > 0, \quad \text{and so} \quad \frac{d\tilde{p}^*}{dN} = -\frac{f}{2f'} < 0.$$

Thus we have the same comparative statics results as those derived in section 3.2.

If f , the density function of θ , is symmetric about zero with $f'(\theta) > 0$ when $\theta < 0$ and $f'(\theta) < 0$ when $\theta \geq 0$, then the nonnegativity of $f'(\theta^*)$ implies that θ^* is negative. But then $\bar{p}^* < p(Nq^*)$. Thus in this special case, the trigger price will be strictly less than the expected price in cooperative periods.

Suppose instead that θ is distributed according to

$$F(\theta) = \exp(\alpha\theta - 1) \quad \text{for } \theta \leq 1/\alpha, \quad \text{where } \alpha > 0,$$

so that

$$f(\theta) = \alpha F(\theta), \quad f'(\theta) = \alpha f(\theta), \quad \text{and} \quad E[\theta] = 0.$$

Then decreases in α represent a mean-preserving spread in $f(\theta)$. Now equation (66) is satisfied for all possible values of θ , and optimal output is given by

$$q^* = \frac{A + (N+1)/\alpha}{2BN} \quad \text{when } \alpha \geq (N+1)^2/(N-1)A$$

$$= A/B(N+1) \quad \text{otherwise.}$$

Thus q^* is a nonincreasing function of α and converges to r , the single period collusive output level, as α increases to infinity.

We summarize the results of this subsection with

Proposition 4.1: If there is an additive stochastic demand component, and the optimal values of $\bar{p}$ and T are interior solutions, then the optimal output of an individual firm is given by equation (65) when $f(\theta^*)/F(\theta^*)$ exceeds $(N+1)^2/(N-1)A$, and equals the single period Cournot quantity otherwise. Here θ^* satisfies equation (66). In either event q^* will be strictly greater than r when $f(\theta^*)$ is finite.

4.3 "Corner" Solutions

The problem with the derivations of the previous subsection is that for most potential distributions of θ , there is no possible value of θ^* that satisfies equation (66).

Again, the uniform distribution is an example of a distribution of θ for which no symmetric quantity vector less than the Cournot output vector is supportable as an equilibrium. For distributions such as the normal, however, some such quantity vector can be supported, but because the optimal value of T is infinite, no θ^* satisfying equation (66) exists. (Reasoning exactly analogous to that of section 3.3 demonstrates that if equation (66) can't be satisfied, then $dV_1^*/dT = 0$ only if T is infinite.) In cases such as these, q^* will now be determined by equation (65) when $f'(\theta)/f(\theta)$ exceeds $(N+1)^2/(N-1)A$, (this ensures that $q^* < s$), and $\bar{p}^*$ will be chosen to satisfy the first order conditions, given

q^* , so that

$$(67) \quad [1 - \beta + \beta F^*](A - (N + 1)Bq^*) = \beta f^*b[q^*(A - NBq^*) - A^2/B(N + 1)^2] ,$$

where $F^* = F(\tilde{p}^* - p(Nq^*))$ and $f^* = f(\tilde{p}^* - p(Nq^*))$. Substitution of equation (65) into (67) yields,

$$(68) \quad 1 - \beta + \beta F(\theta^*) = \beta f(\theta^*)[(N - 1)A/(N + 1)^2 + f(\theta^*)/f'(\theta^*)] .$$

Then θ^* will be chosen to satisfy this equation, and the optimal trigger price is given by $\tilde{p}^* = p(Nq^*) + \theta^*$. Now θ^* will depend on the parameters of the distribution function F , the discount factor β , the number of firms N , and on $A = a - c_1$, the difference between the intercept of the inverse demand function and the constant portion of marginal costs. The comparative statics results of the previous subsection for movements in $\tilde{p}^*$ and q^* with respect to changes in A and N will no longer be valid, as θ^* will now vary as they change. The expected life of the cartel, L , will again be given by equation (53), where now $\theta^* = p^* - p(Nq^*)$.

Suppose, for example, that θ is distributed according to

$$F(\theta) = (\theta + \alpha/(\alpha + 1))^\alpha \quad \text{for } -\alpha/(\alpha + 1) < \theta < 1/(\alpha + 1) ,$$

where $\alpha > 0$, so that

$$f(\theta) = \alpha(\theta + \alpha/(\alpha + 1))^{\alpha-1} , \quad f'(\theta) = \alpha(\alpha - 1)(\theta + \alpha/(\alpha + 1))^{\alpha-2} ,$$

$$\text{and } E[\theta] = 0 .$$

Then

$$\frac{f'(\theta)}{f(\theta)} = \frac{\alpha - 1}{\theta + \alpha/(\alpha + 1)} \quad \text{and} \quad \frac{f(\theta)}{F(\theta)} = \frac{\alpha}{\theta + \alpha/(\alpha + 1)} ,$$

so that there is no possible realization of θ which will allow equation (66) to be satisfied. Then T^* is infinite.

Suppose now that $N = 2$, $A = a = 2$ and $B = b = 1$, as in section 3.4. If we let $\gamma^* = \theta^* + \alpha/(1 + \alpha)$, then q^* satisfies

$$q^* = \frac{1}{2} + 3\gamma^*/4(\alpha - 1),$$

where θ^* is chosen to satisfy

$$[1 - \beta + \beta\gamma^{*\alpha}][\frac{1}{2} - 9\gamma^*/4(\alpha - 1)] = \alpha\beta\gamma^{*\alpha-1}[\frac{1}{18} - 9\gamma^{*2}/8(\alpha - 1)^2] .$$

Here $L = F[\theta^*]^{-1} = (\gamma^*)^{-\alpha}$. As α and β vary, we obtain the following values:

α	β	q^*	p^*	θ^*	$\tilde{p}^*$	L
3	$\frac{36}{37}$	$\frac{5}{8}$	$\frac{3}{4}$	$-\frac{5}{12}$	$\frac{1}{3}$	27
3	$\frac{768}{781}$	$\frac{19}{32}$	$\frac{13}{16}$	$-\frac{1}{2}$	$\frac{5}{16}$	64
3	$\frac{864}{871}$	$\frac{9}{16}$	$\frac{7}{8}$	$-\frac{7}{12}$	$\frac{7}{24}$	216
4	$\frac{2304}{2317}$	$\frac{9}{16}$	$\frac{7}{8}$	$-\frac{11}{20}$	$\frac{13}{40}$	256

Thus increases in the discount factor β , with α kept constant, result in decreases in the optimal quantity towards the single period collusive level, and decreases in both θ^* and the optimal trigger price. As a result, L will increase. Also, increases in α , with β increasing to keep the optimal quantity constant, result in decreases in θ^* and therefore decreases in $\tilde{p}^*$ and increases in L . Note that in all these cases q^* exceeds r and $\tilde{p}^*$ is less than p^* .

5. SUMMARY

As the results of the previous sections have demonstrated, the optimal cartel trigger price strategy depends in a critical way on the nature and distribution of the stochastic demand component. If $\tilde{p}^*$ and T^* are interior solutions and the stochastic component doesn't have a degenerate distribution, then the optimal quantity will be greater than the single period collusive level, and less than the single period Cournot level if the distribution isn't too "noisy". More typically, however, the optimal punishment period length $(T^* - 1)$ is infinite. Then the optimal quantity will again be less than Cournot levels if there isn't too much "noise" in the distribution of θ , and again it will be greater than the collusive rate. Unfortunately, the mathematics is so complicated that the distributional of θ must be specified if one is to make qualitative comparisons.

In the examples of this paper I have employed a number of specific distributions, all of which are one parameter distributions whose means are independent of the variable parameter. All are modifications of more familiar distributions (in particular, the exponential, lognormal, negative exponential and gamma). The examples above could have been stated in terms of these more familiar distributions, but only at the cost of a greater algebraic burden, since the mean of θ would now vary with the distributional parameter.

A natural extension of this model would examine cases in which the number of firms is endogenously determined. One possible model would specify fixed costs to be greater than net variable returns for Cournot firms, and then N could be determined by requiring that expected discounted value be nonnegative for all firms which operate at positive output levels, and negative at all positive quantities for all potential entrants, so that their optimal output is zero. Then $\bar{q}^*$ would be a Nash equilibrium quantity vector in cooperative periods given the Green-Porter enforcement mechanism, with some components equal to zero. If one wanted to maintain the symmetric specification of this paper and yet allow the entry decisions of firms to be well-defined problems, it could be assumed that firms make their entry decisions sequentially before time zero given $\bar{p}^*$, T^* and the decisions of the firms choosing before. Also, there would have to be a "no exit" constraint, so that firms which decided to enter would have to operate at positive output levels during noncooperative episodes. For the purposes of the present paper, however, models of this sort must be categorized as the subject of future research.

APPENDIX

This appendix contains a discussion of the second order conditions of the maximization of the value function with respect to output choices by individual firms. The cases of multiplicative and additive demand shocks are dealt with in turn. Note that, in both cases, the concavity of the profit functions ensures that the Cournot output vector $\bar{s}$ satisfies both the necessary and sufficient equilibrium conditions.

Case 1: Multiplicative Uncertainty

In this case the value function is given by equation (12). Then the first order conditions are given by equation (14). When these are satisfied, i.e. when $V_i^1(\bar{q}) = 0$ for $i = 1, \dots, N$, it can be shown that the second derivatives of the value function will be, if we assume $B = b$ as in section 3,

$$(A1) \quad \left. \frac{\partial^2 V_i(\bar{q})}{\partial q_i^2} \right|_{V_i^1(\bar{q}) = 0} = \frac{-B}{[1 - \beta + (\beta - \beta^T)F]} \left\{ 2 + \frac{(A - (N+1)Bq)}{(a - Nbq)}(1 + \eta) \right\}$$

for $i = 1, \dots, N$, where η is given by equation (35). Because of Proposition 2.1, only symmetric quantity vectors are considered here. A sufficient condition for $\bar{q}$ to be a maximum is then that $f'((\bar{p}/p(Nq))$ be positive. But note that if $s/N < q < s$, $A - (N+1)Bq$ and $a - Nbq$ will be positive. Then the condition that $\eta \geq (N+1)[(N+1)a/A - N]/(N-1) = \eta^0$, which guarantees that $q^* < s$, will ensure that the second order conditions hold. (It is easily verified that $1 + \eta$ is positive if $\eta \geq \eta^0$.)

If the trigger price is chosen optimally, so that q satisfies equation (37), and $a = A$ as in the examples of sections 3.2 and 3.4, then equation (A1) reduces to

$$(A2) \quad \left. \frac{\partial^2 V_i(\bar{q}^*)}{\partial q_i^2} \right|_{V_i^1(q^*) = 0} = \frac{-B(N-1)(1+\eta^*)}{[1-\beta + (\beta - \beta^T)F]} \quad \text{for } i = 1, \dots, N,$$

which will be negative as required if $\eta^* \geq -1$. Alternatively, we require that the elasticity of the density function of θ , evaluated at $\bar{p}^*/p(Nq^*)$, exceed -2 . It can be readily verified that the second order conditions are satisfied at the optima of the examples of sections 3.2 and 3.4.

Case 2: Additive Uncertainty

In this instance, the value function is given by equation (57), and the first order conditions for a Nash equilibrium by equation (58). Then the second derivatives of $V_i(\bar{q})$, evaluated when $V_i^1(\bar{q}) = 0$, are given by

$$(A3) \quad \left. \frac{\partial^2 V_i(\bar{q})}{\partial q_i^2} \right|_{V_i^1(\bar{q}) = 0} = \frac{-[2B + (A - (N+1)Bq)(f'b/f)]}{[1-\beta + (\beta - \beta^T)f]}$$

for $i = 1, \dots, N$. Again, only symmetric quantity vectors are considered. Once again, the positivity of $f'(\bar{p} - p(Nq))$ is sufficient for $\bar{q}$ to be a maximum when $V_i^1(\bar{q}) = 0$. (If $s/N < q < s$, $A - (N+1)Bq$ will be positive.)

If the trigger price $\bar{p}$ is chosen optimally, q will be given by equation (65). Substitution of this equation into (A3) yields

$$(A4) \quad \left. \frac{\partial^2 V_i(\bar{q}^*)}{\partial q_i^2} \right|_{V_i^1(\bar{q}^*) = 0} = \frac{-[2B - (N+1)^2 + A(N-1)(f'_b/f)]}{[1 - \beta + (\beta - \beta^T)F]}$$

for $i = 1, \dots, N$. Then q^* will be a Nash equilibrium if

$$(A5) \quad \frac{f'(\theta^*)}{f(\theta^*)} > \frac{[(N+1)^2 - 2]}{A(N-1)},$$

where $\theta^* = \bar{p}^* - p(Nq^*)$. If q^* is given by equation (65), however, then q^* will be less than s , the single period Cournot output, if

$$(A6) \quad \frac{f'(\theta^*)}{f(\theta^*)} > \frac{(N+1)^2}{A(N-1)}.$$

Equation (A6) is more restrictive than (A5). Therefore, if q^* is chosen according to equation (65) and is less than s , the second order conditions for q^* to be an equilibrium will be satisfied. Thus the second order conditions are not violated in the examples of sections 4.2 and 4.3.

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