# AN INTEGRABILITY CONDITION, WITH APPLICATIONS TO UTILITY THEORY AND THERMODYNAMICS

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# AN INTEGRABILITY CONDITION, WITH APPLICATIONS TO UTILITY THEORY AND THERMODYNAMICS+

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## I. An integrability condition.

Let $M$ be a $C^r$ n-manifold ($1 \leqq r \leqq \infty$) and let $\omega$ be a $C^k$ 1-form ($1 \leqq k \leqq r$) on $M$ which never vanishes. We seek conditions on $\omega$ that it admit, for a set $N \subseteq M$, a $C^k$ integrating pair $(\varphi,\lambda)$ on $N$, more specifically a never-vanishing function (integrating factor) $\lambda\colon N \to R^1$ and a $C^k$ function $\varphi\colon N \to R^1$ such that $\lambda\omega = d\varphi$ on $N$; it follows that such $\lambda$ would be $C^{k-1}$ and could be chosen either positive or negative.

For any $N \subseteq M$, by a $C^r$ positive $\omega$-cycle in $N$ we mean a $C^r$ function $\gamma\colon [0,T] \to N$, for some real $T > 0$, such that $\gamma(0) = \gamma(T)$ and for all $t\epsilon[0,T]$, $\omega_{\gamma(t)}(\dot{\gamma}) > 0$.[1]

Theorem. Let $\bar{y}\epsilon M$. There is a neighborhood of $\bar{y}$ on which $\omega$ admits a $C^k$ integrating factor if some neighborhood of $\bar{y}$ contains no $C^r$ positive $\omega$-cycle and only if some neighborhood of $\bar{y}$ contains no continuous, piecewise $C^1$ positive $\omega$-cycle.

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+ The approach to integrability here is based on an axiom of J. Ville [18], but the method of proof is a modification of Turner's modification [17] of Carathéodory's accessibility theorem ([3], §4). We have found that there is substantial parallelism between the methods we use in proving our theorem and the method used by Serrin in proving an analogous result (cf. [15], p. 145); however we employ a Ville-like axiom, natural in the economic context, in place of the usual "Second Law." We are also extremely grateful to Professor Serrin for some of his Minnesota classroom notes which suggested a step in our proof.

Proof. The "only if" is trivial. We prove the "if" as follows. Let N be a neighborhood of $\bar{y}$ containing no $C^r$ positive $\omega$-cycle. Clearly we may identify N with a neighborhood $N_0$ of the origin $0_n$ in $R^n$, and $\bar{y}$ with $0_n$. Then for every $y \in N_0$, $\omega_y$ has the form $\omega_y = \sum_{i=1}^{n} X^i(y) d_y x^i : T_y M \to R^1$, for some $C^k$ functions $X^i : M \to R^1$, where the $x^i$ are the standard coordinate functions on $R^n$. Since $\omega$ never vanishes, we may without loss of generality assume that $X(\bar{y}) = (X^1(\bar{y}), \ldots, X^n(\bar{y})) = (\bar{r}, 0, \ldots, 0)$ for some real $\bar{r} > 0$.

Because X is continuous, there is an open neighborhood $N_1$ of $\bar{y}$ such that, for all $y \in N_1$, $X^1(y) > 0$

$$\text{I.1)} \qquad \left| \frac{X^j(y)}{X^1(y)} \right| < 1 \qquad (j = 2, \ldots, n).$$

For all $y \in R^n$ let $\|y\| = \sum_{i=1}^{n} |y_i|$. We choose $\delta > 0$ to be so small that

$$\text{1.2)} \qquad N_2 = \{y \in R^n : \|y\| < \delta\}$$

is an open subneighborhood of $\bar{y}$ in $N_1$.

For any given $C^k$ path $\alpha: [0,T] \to \{y \in N_2 : y_1 = 0\}$, and real $\mu$, we consider solutions to the corresponding ordinary differential equation

$$\text{I.3.}\mu) \qquad \frac{dz^1(t)}{dt} = -\sum_{j=2}^{n} \frac{X^j(z^1(t), \alpha^2(t), \ldots, \alpha^n(t))}{X^1(z^1(t), \alpha^2(t), \ldots, \alpha^n(t))} \dot{\alpha}^j(t) + \mu$$

$$\equiv f^{\alpha}(t, z^1(t), \mu),$$

say.

First let $y \in N_2$ and let $\alpha(t) = (0, (1-t)y_2, \ldots, (1-t)y_n)$ for all t. Then $f^{\alpha}$ is $C^k$, and the equation (I.3.$\mu$) with $\mu = 0$ has a unique solution $z^1_y(\cdot;\mu)$ with $z^1_y(0;\mu) = y_1$, defined on a maximal interval J about 0. By the choice of $N_2$, $z^1_y(\cdot;\mu)$ remains bounded on $[0,1] \cap J$, and so (by [8], p. 12 (Thm. 3.1)) the maximal interval includes [0,1], hence $z^1_y(1;\mu)$ is defined and is thus locally a $C^k$ function of y and $\mu$ ([8], p. 100 (Thm. 4.1)). We may therefore define a $C^k$ function $\varphi: N_2 \to R^1$ by

I.4) $\varphi(y) = z_y^1(1;0)$ for all $y \in N_2$.

Note that $(z_y^1(\cdot;0), \alpha^2(\cdot),\ldots,\alpha^n(\cdot))$ lies in $N_2$, and so $(z_y^1(\cdot;\mu), \alpha^2(\cdot),\ldots,\alpha^n(\cdot))$ also lies in $N_2$ for all small $\mu$.

We now show that, for any real $t_2 > t_1$ and any $C^k$ path $\beta: [t_1, t_2] \to N_2$ such that $\omega(\dot{\beta}(t)) = 0$ for all $t \in [t_1, t_2]$, we have $\varphi(\beta(t_1)) = \varphi(\beta(t_2))$. Without loss of generality we take $t_1 = 1$ and $t_2 = 2$. First, define $\beta^1(\cdot;\mu)$ to be the unique solution $z^1_{\beta(1)}(\cdot;\mu)$ of (I.3.$\mu$) through $\beta^1(1)$, where $\alpha^j = \beta^j$ for $j = 2,\ldots,n$; and define $\beta^j(\cdot;\mu) = \beta^j(\cdot)$ for $j = 2,\ldots,n$. Thus $\beta(\cdot;0) = \beta(\cdot)$, and $\beta(\cdot;\mu)$ lies in $N_2$ for all small $\mu$. Next define a real valued function $q_{\beta(1)}(\cdot;\mu)$ as the solution of (I.3.$\mu$) going through $\beta^1(1)$ at time $t = 1$, for $\alpha^j(t) \equiv t\beta^j(1)$ $(j = 2,\ldots,n)$. (That this solution exists on all of $[0,1]$ follows from the fact that $q_{\beta(1)}(t;\mu) = z^1_{\beta(1)}(1-t;-\mu)$ for all $t$, and for all small $\mu$ the latter exists for all $t \in [0,1]$ by the previous paragraph.) Now suppose $\varphi(\beta(1)) > \varphi(\beta(2))$, and consider the continuous, piecewise $C^k$ path $\eta(\cdot;\bar{\mu})$ in $R^n$, for some small $\bar{\mu} > 0$, obtained by following the solution of (I.3.$\bar{\mu}$), first radially from $L = \{y \in R^n: y_2 = \ldots = y_n = 0\}$ to $\beta(1)$, then from $\beta(1)$ to $\beta(2;\bar{\mu})$ and finally from $\beta(2;\bar{\mu})$ radially back to $L$; equivalently, define:

$$\text{I.5.}\bar{\mu})\quad \eta(t;\bar{\mu}) = \begin{cases} (q_{\beta(1)}(t;\bar{\mu}), t\beta^2(1),\ldots,t\beta^n(1)) & \text{for all } t \in [0,1] \\ \beta((t-1)2 + (2-t)1;\bar{\mu}) & \text{for all } t \in [1,2] \\ (z^1_{\beta(2;\bar{\mu})}(t;\bar{\mu}), (1-t)\beta^2(2;\bar{\mu}),\ldots,(1-t)\beta^n(2;\bar{\mu})) & \text{for all } t \in [2,3]. \end{cases}$$

Just as with $(z_y^1(\cdot;\mu),\alpha^2(\cdot),\ldots,\alpha^n(\cdot))$ earlier, $\eta(\cdot;\bar{\mu})$ lies in $N_2$ for all small $\bar{\mu}$. Note that $\eta(0;0)$ lies above $\eta(3;0)$ on L, since $\eta^1(0;0) = \varphi(\beta(1)) > \varphi(\beta(2)) = \eta^1(3;0)$. Then, because $\eta$ is continuous, $\eta^1(0;\bar{\mu}) > \eta^1(3;\bar{\mu})$ for all small $\bar{\mu}$. Let us extend $\eta(\cdot;\bar{\mu})$ from $[0,3]$ to $[0,4]$ by going straight up on L:

I.6) $$\eta(t;\bar{\mu}) = (4-t)\eta(3;\bar{\mu}) + (t-3)\eta(0;\bar{\mu}) \quad \text{for all } t\in[3,4]$$

so $\eta(0;\bar{\mu}) = \eta(4;\bar{\mu})$. It is readily verified that:

I.7) $$\omega(\frac{d}{dt}\eta(t;\bar{\mu})) = \begin{cases} X^1(\eta(t;\bar{\mu}))\bar{\mu} > 0 & \text{for all } t\in[0,3] \\ \eta^1(0;\bar{\mu}) - \eta^1(3;\bar{\mu}) > 0 & \text{for all } t\in[3,4] \end{cases}$$

for all small $\bar{\mu} > 0$. Finally, it is straightforward to show, using the linearity of $\omega$ and the Weierstrass approximation theorem, that the continuous but only piecewise $C^k$ path $\eta(\cdot;\bar{\mu})$ may be smoothed to give a $C^r$ path $\eta(\cdot)$ with $\eta(0) = \eta(4)$, still lying in $N_2$, and on which $\omega$ is still positive. Thus we have a positive $C^r$ $\omega$-cycle in $N_2$, a contradiction. Therefore $\varphi(\beta(1)) \leqq \varphi(\beta(2))$. A similar argument shows $\varphi(\beta(1)) \geqq \varphi(\beta(2))$, and we conclude $\varphi(\beta(1)) = \varphi(\beta(2))$.

For any $y\in(L\cap N_2)$ we have $\varphi(y) = y_1$, so[1] $\varphi_1(y) = 1$. Thus $\varphi_1 > 0$ on some subneighborhood $N_3$ of $\bar{y}$ in $N_2$. We next define an integrating pair $(\varphi,\lambda)$ on $N_3$. Let $y\in N_3$ and $1 \leqq j \leqq n$, and define $\alpha$: $[0,1] \to R^n$ by:

$$\alpha^i(t) = \begin{cases} y_i, & i \neq j \\ y_i + t, & i = j \end{cases} \quad \text{for all } t\in[0,1].$$

Let $z^1(\cdot)$ be the solution of (I.3.$\mu$) through $y^1$, for $\mu = 0$. Then by the constancy property proved for $\varphi$, $\frac{d}{dt}\varphi(z^1(t),\alpha^2(t),\ldots,\alpha^n(t))\big|_{t=0} = 0$, so (since $y\in N_1$)

I.8) $$-\varphi_1(y)\frac{X^j(y)}{X^1(y)} + \varphi_j(y) = 0,$$

and thus

$$\frac{\varphi_1(y)}{X_1(y)}\,\omega_y = \sum_{i=1}^{n}\varphi_i(y)\,d_y x^i$$

I.9) $$= d_y\varphi,$$

where $\varphi$ is $C^k$. Now for $y\epsilon N_3$, $\frac{\varphi_1(y)}{X^1(y)}$ is positive and $C^{k-1}$, and so $(\varphi,\lambda)$, with $\lambda = \frac{\varphi_1}{X^1}$, is a positive $C^k$ integrating pair for $\omega$ on $N_3$. This completes the proof of the Theorem.

II. An application to thermodynamics.

Let the equilibrium states of a "simple" ([3], §3; [17], p. 785) or "standard" ([2], §10) thermodynamic system form a $C^r$ (n+1)-dimensional manifold M with global coordinates $U, x^1, \ldots, x^n$ representing the internal energy (U) and such relevant "deformation" variables as volumes, etc. Let $\alpha$ be a nonvanishing $C^k$ 1-form on M representing heat inflow, and let $\beta = \sum_{i=1}^{n} p_i\,dx^i$ (for some functions $p_i$ of $U, x_1^1, \ldots, x^n$) be a 1-form on M representing the flow of work (performed by the system). Then a differential version of the First Law of Thermodynamics is often stated[1] for quasi-static[2] process as:

II.1) $$\alpha = dU + \beta.$$

One part[3] of what is often called the Second Law of Thermodynamics concerns the existence of a $C^k$ function $S: M \to R^1$ called entropy, with the property: for quasi-static processes,

II.2) $$\alpha = \frac{1}{\lambda}\,dS,$$

for some $C^{k-1}$ integrating factor $\lambda$ on M. To establish the existence of such a function, we state as an axiom:

II.3) There do not exist any positive $C^r$ $\alpha$-cycles.

The intuition of axiom (II.3) is clear: there exist no quasi-static paths on which heat is continually flowing into the system, but on which the state variables $U, x^1, \ldots, x^n$ all return to their initial values. Such a "perpetual" perpetuum mobile of the second kind is *a fortiori* prohibited by Landsberg's version ([11], pp.176-177) of the Kelvin formulation of the Second Law.

By our Theorem, it follows immediately from axiom (II.3) that in a neighborhood $N_z$ of every point $z \epsilon M$ there exists a $C^k$ function $S: N_z \to R^1$, and a positive $C^{k-1}$ function $\frac{1}{\lambda}: N_z \to R^1$ such that, for all $y \epsilon N_z$, $\alpha_y = \frac{1}{\lambda(y)} d_y S$. At least when the coefficients $p_i$ of $\beta$ are nonnegative, the extension to a global integrating pair is straightforward (cf. [6], pp. 608-610).

From the usual assumptions about the way that the parts of a thermodynamic system relate to the whole system, one can deduce that the integrating factor $\lambda$ can be chosen in a special way, usually as a function of just temperature.

## III. An application to utility theory.

Let the possible consumption bundles of a consumer be the set $M = \{y \epsilon R^n : y > 0\}$. Suppose that, for each $x \epsilon M$, there exists a $p(x) \epsilon \{y \epsilon M: \sum_{i=1}^{n} y_i = 1\} \equiv Z$, representing the unique price vector in $Z$ at which bundle $x$ is purchased. Assume that $p(\cdot)$ is a $C^k$ function, for some $k \geqq 1$.

If we let $x^1, \ldots, x^n$ be the standard coordinate functions on $M$, and if we let $\alpha$ be the nonvanishing 1-form $\sum_{i=1}^{n} p_i \, dx^i$ on $M$, then we may consider:[1]

The Ville Axiom. There do not exist any positive $C^\infty$ $\alpha$-cycles.

The intuition of the Ville Axiom is clear: if $\gamma: [0,1] \to M$ is a path on which

$$\text{III.1)} \qquad p(\gamma(t)) \cdot \frac{dx(\gamma(t))}{dt} > 0,$$

then intuitively $\dot{\gamma}(t)$ is revealed a "preferred" direction from $\gamma(t)$. (Cf. [1], pp. 203-5; [7], p. 552; [10], p. 118.) So the Ville Axiom simply states that no path exists which moves always in a revealed preferred direction but ends at its starting point.

By our Theorem, it follows immediately from the Ville Axiom that in a neighborhood $N_x$ of every point $z \in M$ there exists a $C^k$ "utility" function $S: N_z \to R^1$ and a positive $C^{k-1}$ function $\lambda: N_z \to R^1$ such that, for all $y \in N_z$, $\lambda(y)\alpha_y = \lambda(y) \sum_{i=1}^{n} p_i(y) d_y x^i = d_y S$.

The extension to a global integrating factor is again straightforward (cf. [6], pp. 608-10). For more details, and a discussion of second order conditions and utility maximization, cf. [9].

Note that if there corresponds to each positive vector $x \in R^n$ not only a price vector $p(x)$ at which $x$ is purchased by the consumer, but also a positive "income" $m \in R^1$, then the budget identity $p(x) \cdot x = m$ frequently assumed by economists becomes, in differential form, very like the differential version of the First Law of Thermodynamics (II.1):

$$\text{III.2)} \qquad p(x) \cdot dx = dm - x \cdot d_x p .$$

Clearly there are close formal similarities between thermodynamics and utility theory.[1]

## FOOTNOTES

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Page 1, n. 1. We mean right- and left-hand derivatives where derivatives are not defined. Note that a negative $\omega$-cycle is obtained by traversing a positive $\omega$-cycle in the opposite direction.

Page 4, n. 1. Subscripts on $\varphi$ denote partial derivatives.

Page 5, n. 1. Cf. [19], p. 81.

Page 5, n. 2. Cf. [2], §7; [11], p. 35.

Page 5, n. 3. The other part deals with entropy increase on irreversible paths.

Page 6, n. 1. This is essentially the condition used by Ville in [18] to obtain the integrability of the differential form $\alpha$. His proof of integrability was based on Darboux's theorem ([4], Première Partie, V; [16], p. 141, Theorem 6.2) on canonical forms for 1-forms. In thermodynamics this same approach, through Darboux's theorem, was later used by Landsberg [11], pp. 51-3, 392-8). The approach through our Theorem seems much more direct, and also imposes weaker differentiability conditions on the 1-form $\alpha$.

Page 7, n. 1. This was observed already by V. Pareto ([13, p. 543) and P. A. Samuelson ([14], p. 70) in a very general fashion. The similarity of the First Law of Thermodynamics and the consumer's budget identity was noted by H. T. Davis ([5], Chapter 8, Section 5), J. Lisman [12], and N. Georgescu-Roegen ([7], p. 17).

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