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DYNAMICS OF HYPERINFLATION

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Introduction

This is a study of some theoretical difficulties and estimation problems that arise in economic models in which current expectations of future values of some of the endogenous variables enter in an essential way.² Such models are common, especially in monetary economics and macroeconomics. An example would be a model in which the public's expectations of future inflation influence aggregate demand, which together with aggregate supply helps determine the current actual rate of inflation. In order to keep the exposition simple and concrete, we shall center our discussion around Phillip Cagan's [2] model of hyperinflation. This permits us to analyze all of the theoretical problems involved in larger models, and to illustrate some empirical methods for determining the validity of alternative methods of modeling the formation of expectations.

We are particularly interested in exploring the possibility of building and estimating a version of Cagan's model in which expectations are "rational." Expectations about a variable are said to be rational if

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²The model studied by Walters [13] bears a superficial resemblance to the one analyzed in this paper, but does not come to grips with the problem we are addressing. In Walters' model, current variables do not depend on currently held expectations about future values of any endogenous variables.

they depend, in the proper way, on the same things that economic theory says actually determine that variable. By contrast, the usual method of modeling expectations involves supposing that they are formed by extrapolating past values of the variable being predicted, a scheme that usually though not always assumes that the people whose expectations count are ignorant of the economic forces governing the variable they are trying to predict.

Section 1 contains our discussion of the alternative ways expectations might be assumed to be formed in Cagan's model, while section 2 contains some empirical results reflecting on the adequacy of various of these alternatives. Our conclusions are contained in section 3.

1. Models of Hyperinflation

Cagan's explanation of hyperinflation relies on a demand function for real balances of the form

$$(1) \quad \log \left(\frac{M}{P} \right)_t = \alpha \pi_t + \gamma Y + \delta + U_t, \quad \alpha < 0, \gamma > 0$$

where M is the demand for nominal balances, which Cagan assumes always equals the supply, P is the commodity price level, π_t is the public's expectation of the future rate of inflation, Y is real income, α , γ , and δ are parameters, and U_t is a stochastic term with central tendency equal to zero. It is assumed that real income Y is constant over time. Cagan posited that the unobservable expectation π_t is a distributed lag of current and past actual rates of inflation, one with geometrically declining lag weights:

$$(2) \quad \pi_t = (1 - \lambda) \sum_{i=0}^{\infty} \lambda^i \log \frac{P_{t-i}}{P_{t-i-1}}, \quad 0 \leq \lambda < 1.$$

Letting X_t denote $\log (P_t/P_{t-1})$ and using the lag operator L , which is defined by the operation $L^n X_t = X_{t-n}$, (2) can be written as

$$(3) \quad \begin{aligned} \pi_t &= (1 - \lambda) \left(\sum_{i=0}^{\infty} (\lambda L)^i \right) X_t \\ &= \frac{1 - \lambda}{1 - \lambda L} X_t \end{aligned}$$

By taking the first difference of equation (1), we find that the rate of inflation is related to the rate of increase of the money supply by

$$(4) \quad \mu_t = X_t + \alpha(\pi_t - \pi_{t-1}) + U_t - U_{t-1},$$

where μ_t equals $\log (M_t/M_{t-1})$. Substituting (3) into (4), assuming that the appropriate lag-generating function is invertible, and solving for the rate of inflation X_t yields

$$(5) \quad X_t = \frac{\frac{1-\lambda L}{1+\alpha(1-\lambda)}}{1 - \left(\frac{\lambda+\alpha(1-\lambda)}{1+\alpha(1-\lambda)}\right) L} \mu_t - \frac{\left(\frac{1-\lambda L}{1+\alpha(1-\lambda)}\right) (1-L)}{1 - \left(\frac{\lambda+\alpha(1-\lambda)}{1+\alpha(1-\lambda)}\right) L} U_t$$

where we are assuming that $\left|\frac{\lambda+\alpha(1-\lambda)}{1+\alpha(1-\lambda)}\right| < 1$.

According to expression (5), the current rate of inflation can be viewed as determined by distributed lags of the change in the money supply and of the disturbance in the demand function for money, U_t .

Two implications of equation (5) are worthy of note. First, unless special some/restrictions are placed on μ_t , the current rate of inflation X_t will be correlated with current and past values of the random disturbance U . Cagan implemented his model by substituting (2) into (1) to arrive at

$$(1') \quad \log \left(\frac{M}{P}\right)_t = \alpha(1-\lambda) \sum_{i=0}^{\infty} \lambda^i X_{t-i} + \gamma Y + \delta + U_t,$$

an equation whose parameters he estimated by the method of least squares.

If there is correlation between X_t and current and past values of U , then this estimator is not statistically consistent.

Second, notice that Cagan's model is one in which expectations are not necessarily assumed to be "rational" in the sense of Muth [8]. That is, the public is assumed to form expectations of inflation according to (2), whereas the actual rate of inflation evolves according to equation (5). The prediction of future inflation generated by equation (5) might be different from the public's expectation of inflation. Unless some restrictions are placed on the stochastic process describing the μ 's, the public's expectation of inflation may differ systematically from the forecast of inflation produced by the model. Thus, it is possible that the public is assumed not to be able to forecast inflation as well as does the model. In this sense, the model permits "irrational" expectations.

In order to explore the possibility of building and estimating a version of Cagan's model that incorporates "rational" expectations, it is necessary to be more specific about the horizon to which π_t corresponds. We make the simplest assumption, namely that the horizon is one period, π_t being the rate of inflation that the public expects, as of time t , to prevail between time t and time $t + 1$. Thus, π_t is the public's forecast of X_{t+1} . (It would clearly be possible to carry through our argument while assuming that the horizon appropriate to π_t is longer than one period.)

The assumption that expectations are rational is imposed by requiring that

$$\pi_t = E_t X_{t+1}$$

where $E_t X_{t+1}$ is the mathematical expectation of X_{t+1}

formed using the model and information available as of time t . Then equation (4) becomes

$$(6) \quad X_t = \mu_t - \alpha E_t X_{t+1} + \alpha E_{t-1} X_t - U_t + U_{t-1}.$$

Similarly, X_{t+1} is given by

$$X_{t+1} = \mu_{t+1} - \alpha E_{t+1} X_{t+2} + \alpha E_t X_{t+1} - U_{t+1} + U_t.$$

Taking expectations as of time t , we have

$$E_t X_{t+1} = E_t \mu_{t+1} - \alpha E_t X_{t+2} + \alpha E_t X_{t+1} - E_t U_{t+1} + U_t$$

or

$$(7) \quad E_t X_{t+1} = \frac{1}{1-\alpha} E_t \mu_{t+1} - \frac{\alpha}{1-\alpha} E_t X_{t+2} - \frac{1}{1-\alpha} (E_t U_{t+1} - U_t).$$

More generally, we have that

$$(8) \quad E_t X_{t+j} = \frac{1}{1-\alpha} E_t \mu_{t+j} - \frac{\alpha}{1-\alpha} E_t X_{t+j+1} - \frac{1}{1-\alpha} (E_t U_{t+j} - E_t U_{t+j-1}).$$

By repeatedly substituting (8) into (7) we find that

$$(9) \quad E_t X_{t+1} = \frac{1}{1-\alpha} \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^{j-1} E_t \mu_{t+j} - \frac{1}{1-\alpha} \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^{j-1} (E_t U_{t+j} - E_t U_{t+j-1}).$$

Substituting this expression into equation (6) yields the following expression that describes the evolution of inflation:

$$(10) \quad X_t = \mu_t + \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^j E_t \mu_{t+j} - \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^j E_{t-1} \mu_{t+j-1} \\ - \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^j (E_t U_{t+j} - E_t U_{t+j-1}) \\ + \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^j (E_{t-1} U_{t+j-1} - E_{t-1} U_{t+j-2}) - U_t + U_{t-1}$$

Writing down the corresponding expression for X_{t+1} and taking its expectation as of time t yields an expression identical to equation (9), which verifies that the public's expectation is indeed equivalent with the model's forecast of the rate of inflation one period hence. Notice that since $\alpha < 0$, we are assured that $0 < (-\alpha / (1-\alpha)) < 1$, which makes it possible for the infinite sums in (9) and (10) to converge.

Equation (10) exhibits an important feature that characterizes all models in which expectations of future values of endogenous variables enter and in which those expectations are posited to be equivalent with the model's forecasts: the current values of some of the endogenous variables in such models will depend on the public's expectations of some of the "exogenous" variables from now until forever.³ The reason for this outcome can be seen

³See Wallace [12] for a discussion of how this property of rational expectations is dealt with in a standard macroeconomic model.

by inspecting equation (6). The current rate of inflation is influenced by the current forecast of inflation for next period. But next period's rate of inflation depends on next period's expectation of inflation two periods hence. An optimal forecasting scheme necessarily takes into account this dependence of current values on expectations of future values, which in turn depend on expectations of values even farther into the future, and so on, leading to a progression of infinite extent into the future. The current rate of inflation is then governed by the public's forecasts of the variables determining the subsequent rates of inflation, which in this case are the future rates of growth of the money supply and future disturbances in the demand function for money.

In order to complete our model of hyperinflation, it is necessary to supplement equation (10) with a description of how the public forms its expectations of subsequent rates of growth of the money supply. To accomplish this while remaining true to the objective of incorporating rational expectations in the model, it is necessary to write down an expression assumed to govern the actual evolution of the rate of increase in the money supply. The public can then be assumed to utilize this expression in forming its expectations of subsequent rates of growth in the money supply, thus guaranteeing that its expectations are consistent with the actual process governing the growth in money.

One tractable way of modeling the rate of growth in the money supply is to assume that it follows a purely autoregressive process, i.e.

$$(11) \mu_t = \sum_{i=1}^{\infty} w_i \mu_{t-i} + \tilde{\varepsilon}_t,$$

where the w_i 's are constants and $\tilde{\varepsilon}_t$ is a serially uncorrelated random term with finite mean, which we take to be zero, and finite variance.

The random variable $\tilde{\varepsilon}_t$ is assumed to be distributed independently of the U_t 's that appear in the demand function for real balances. This specification

implies that the rate of growth of the money supply is exogenous with respect to the rate of inflation.⁴

On the basis of equation (11), the public's expectation of future values of μ_t can be written down as the appropriate mathematical expectations as of time t . These expectations will have the forms

$$(12) \quad E_t \mu_{t+j} = \sum_{i=0}^{\infty} v_{ij} \mu_{t-i}$$

where the v_{ij} 's are functions of the w_i 's that appear in (11). Substituting (12) into equation (10) yields

$$\begin{aligned} X_t = & \mu_t + \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^j \sum_{i=0}^{\infty} v_{ij} \mu_{t-i} - \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^j \sum_{i=0}^{\infty} v_{ij} \mu_{t-i-1} \\ & - \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^j (E_t U_{t+j} - E_t U_{t+j-1}) \\ & + \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^j (E_{t-1} U_{t+j-1} - E_{t-1} U_{t+j-2}) - U_t + U_{t-1} \end{aligned}$$

This can be rewritten as

$$(13) \quad X_t = \sum_{i=0}^{\infty} Z_i \mu_{t-i} + \tilde{U}_t$$

$$\text{where } Z_0 = 1 + \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^j v_{0j}$$

$$Z_i = \sum_{j=1}^{\infty} \left[\left(\frac{-\alpha}{1-\alpha}\right)^j (v_{ij} - v_{i+1,j}) \right], \quad i \geq 1,$$

$$\begin{aligned} \tilde{U}_t = & - \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^j (E_t U_{t+j} - E_t U_{t+j-1}) \\ & + \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^j (E_{t-1} U_{t+j-1} - E_{t-1} U_{t+j-2}) - U_t + U_{t-1}. \end{aligned}$$

Equation (13) is a "reduced form" expressing the rate of inflation as a distributed lag of current and past rates of money creation. Since the $\tilde{\epsilon}$'s in (11) were assumed to be uncorrelated with the U 's, it follows that the

⁴Lucas [7] closes his model by assuming that government policy is governed by a purely autoregressive process.

$\tilde{U}$'s are uncorrelated with the μ 's in (13), thus confirming that it is legitimate to treat the rates of growth in the money supply as exogenous in (13). Hence, equation (13) can properly be estimated using least squares, the lack of correlation between the $\tilde{U}$'s and the μ 's guaranteeing that the estimates will at least be consistent. Whether it would be possible to estimate the structural parameter α from the estimate of the reduced form coefficients would depend on precisely what form the autoregression in (11) was assumed to take, and on how much prior information about the autoregression coefficients was available. Unless the form of (11) were severely restricted, it would not be possible to estimate the structural parameter on the basis of the estimated reduced form parameters alone. However, the following two-stage estimator for α would be available: by least squares estimate some versions of

$$\hat{X}_t = \sum_{i=0}^{\infty} \hat{Z}_i \mu_{t-i}$$

$$\hat{X}_{t+1} = \sum_{i=0}^{\infty} \hat{h}_i \mu_{t-i} ;$$

then substitute $\hat{X}_{t+1}$ for π_t and $\hat{X}_t$ for π_{t-1} in equation (4) and estimate α by least squares. Since the μ_t 's are valid instruments for π_t , being uncorrelated with the $\tilde{U}$'s, the estimator of α will be statistically consistent.

Assuming that the rate of increase in the money supply is governed by (11) thus enables us to complete a version of Cagan's model in which expectations are posited to be "rational." But there is reason to believe that (11) is not an adequate description of growth in the money supply in the context of this model. The model is designed to explain the behavior of inflation during periods of hyperinflation, periods in which the government is resorting to creation of money as the principal means of financing its expenditures.

In order to keep real expenditures at the level it desires, the government is likely to respond to a decline in the purchasing power of money by increasing the rate at which it is adding to the stock of money. Such behavior makes the rate of increase in the money supply depend partly on the price level, thus setting up feedback from the public's expected rate of inflation, which helps determine the price level, to the rate of money creation. Such feedback must not occur if equation (11) is to be an adequate description of the money creation process. Using equation (11) to model the formation of expectations thus amounts to assuming that the public never really catches on to what the government is doing in financing its expenditures mainly by money creation.

Under certain conditions, it is possible to complete a version of Cagan's model in which the public understands that the government is printing money in order to finance most of its expenditures. This is well known for the case of a continuous-time, nonstochastic version of Cagan's model.⁵ Thus, suppose that the demand for money is governed by

$$(14) \quad \frac{M(t)}{P(t)} = f(\pi(t)), \quad f' < 0$$

where M , P , and π are now continuous functions of time. Suppose that there are no private bank deposits and that all money is non-interest-bearing debt of the government. Also suppose that the government finances all of its expenditures, which in real terms equal $G(t)$ at time t , by the creation of new money. Suppose the government wants to keep $G(t)$ constant over time at the rate G . The rate of money creation can be determined from the government's budget constraint,

$$(15) \quad \frac{\dot{M}(t)}{P(t)} = G,$$

⁵For example, see Friedman [3].

where a dot above a variable denotes its derivative with respect to time.

Equation (15) can be rewritten as

$$\frac{\dot{M}}{M} = \frac{P}{M} \cdot G.$$

Substituting (14) into the above equation gives

$$(16) \quad \frac{\dot{M}}{M} = \frac{G}{f(\pi)},$$

which shows that the rate of money creation varies directly with the public's anticipated rate of inflation. Differentiating (14) with respect to time and rearranging gives

$$(17) \quad \frac{\dot{P}}{P} = \frac{\dot{M}}{M} - \frac{f'(\pi)}{f(\pi)} \pi.$$

Now suppose we add the requirement that expectations be "rational", i.e.

$$(18) \quad \pi = \frac{\dot{P}}{P}.$$

Equations (16), (17), and (18) form a system that governs the evolution of M , P , and π over time. The equilibrium of the system, if it exists, is easily found. Notice that if $\dot{\pi} = 0$ for all t , then

$$\frac{\dot{P}}{P} = \frac{\dot{M}}{M} = \frac{G}{f(\pi)}$$

or

$$\pi = \frac{G}{f(\pi)},$$

which is one equation in the unknown π . If this equation has a unique solution, then it determines π , which also equals $\dot{P}/P$ and $\dot{M}/M$. The equilibrium of the system is one in which $\dot{P}/P$, π , and $\dot{M}/M$ are all unchanging over time.

The possibility that the public's expected rate of inflation influences the rate of money creation is an interesting one, for it turns out that such a mechanism is capable of providing a way of rationalizing the method of forming expectations about inflation that Cagan assumed. We have seen above that if the public's expectation of inflation is governed by Cagan's adaptive scheme, equation (2), then the actual inflation rate is described by equation (5), which can be rewritten as

$$(19) \quad (1 + \alpha(1-\lambda) - (\lambda + \alpha(1-\lambda))L) X_t = (1-\lambda L) \mu_t - (1-\lambda L) (I-L) U_t.$$

To provide a rationalization of Cagan's model, we begin by supposing that the rate of money creation is governed by

$$(20) \quad \mu_t = \left(\frac{1-\lambda}{1-\lambda L}\right) X_t + \varepsilon_t$$

where ε_t is a serially uncorrelated random term. Substituting (20) into (19) yields

$$(\lambda + \alpha(1-\lambda)) (I-L) X_t = (1-\lambda L) (\varepsilon_t - (U_t - U_{t-1})).$$

This can be rewritten as

$$(I - \frac{(1-\lambda)L}{1-\lambda L}) X_t = (\lambda + \alpha(1-\lambda))^{-1} (\varepsilon_t - (U_t - U_{t-1}))$$

or

$$(21) \quad X_t = \left(\frac{1-\lambda}{1-\lambda L}\right) X_{t-1} + (\lambda + \alpha(1-\lambda))^{-1} (\varepsilon_t - (U_t - U_{t-1})).$$

Now suppose that U_t follows the Markov process⁶

⁶ The residuals in Cagan's regressions, which are estimates of the U 's, are highly serially correlated (see Cagan [2] and Barro [1]). Barro reports Durbin-Watson statistics for Cagan's regressions ranging from .25 to .53. Thus, the assumption in the text appears to be an interesting one to investigate the implications of.

$$U_t = U_{t-1} + \eta_t$$

where η_t is a serially uncorrelated random term with mean zero and finite variance. On this assumption, equation (21) shows that the model's prediction of X_t as of time $t-1$ is given by

$$E_{t-1} X_t = \left(\frac{1-\lambda}{1-\lambda L} \right) X_{t-1},$$

which is Cagan's adaptive expectations scheme. Then on the hypothesis that expectations are rational, (20) is equivalent with

$$(22) \quad \mu_t = E_t X_{t+1} + \varepsilon_t,$$

which can be regarded as a linear, discrete-time approximation to equation (16), one that captures the "feedback" from expected inflation to money creation that will occur if the government is financing a roughly fixed rate of real expenditures by money creation.

Thus, we have been able to produce a set of restrictions on the U 's and on the stochastic process governing the μ 's that make Cagan's adaptive expectations scheme one that produces expectations that are identical to the model's predictions. The system that emerges is one in which at any given time the public expects a constant rate of inflation and a constant rate of money creation to prevail over the entire future. By equation (22), we have

$$E_t \mu_{t+1} = E_t X_{t+2},$$

or more generally

$$E_t \mu_{t+j} = E_t X_{t+j+1}.$$

But it is known that Cagan's adaptive expectation scheme has the property⁷

$$E_t X_{t+j} = E_t X_{t+1} \quad \text{all } j > 1.$$

Thus, it follows that

⁷See Muth [9].

$$E_t \mu_{t+j} = E_t X_{t+j} = E_t X_{t+1}, \text{ for all } j > 1.$$

Notice that on our assumptions equation (9) becomes

$$\begin{aligned} E_t X_{t+1} &= \frac{1}{1-\alpha} \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^{j-1} E_t \mu_{t+j} \\ &= E_t \mu_{t+1} \frac{1}{1-\alpha} \sum_{j=1}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^{j-1} \\ &= E_t \mu_{t+1} \end{aligned}$$

The system that we have set out resembles the continuous-time, nonstochastic system above in two ways. First, expectations of inflation influence the rate of money creation. Second, the equilibrium is one in which at each moment the public expects a single rate of inflation and money creation to prevail over the indefinite future. Moreover, the rate of money creation is expected to equal the rate of inflation.

This system is one in which expectations of money creation could equally well be formed as a distributed lag of past rates of money creation. Substituting (21) into (20) and rearranging gives

$$\begin{aligned} (1-L) \mu_t &= [(\lambda + \alpha (1-\lambda))^{-1} + 1] \varepsilon_t - \varepsilon_{t-1} \\ &\quad - (\lambda + \alpha (1-\lambda))^{-1} (1-\lambda) (U_t - U_{t-1}). \end{aligned}$$

This is a mixed moving average, autoregressive process which can be rewritten as

$$\mu_t = \left(\frac{1-\delta}{1-\delta L}\right) \mu_{t-1} + (\lambda + \alpha (1-\lambda))^{-1} (1-\lambda) (\varepsilon_t - (U_t - U_{t-1})) + \varepsilon_t,$$

where δ is a parameter that depends on the ratio of the variance of ε_t to the variance of $(U_t - U_{t-1})$. On our assumptions, the least-squares forecast of μ_t is

$$E_{t-1} \mu_t = \left(\frac{1-\delta}{1-\delta L} \right) \mu_{t-1}.$$

The expected rate of inflation can also be written as the same function of past μ 's.

Now suppose that we do not observe μ_t directly, but instead have data m_t which are polluted by errors of measurement, i.e.

$$(23) \quad m_t = \mu_t + S_t$$

where S_t is a serially uncorrelated random term that is distributed independently of ϵ_t and U_t . In this case, since μ is not observed while X is, the best way to forecast is to make use of equation (20) and to predict both money creation and inflation by distributed lags of past rates of inflation.

In summary, we have described two ways to build a model of hyperinflation in which expectations are rational. The first model consists of equations (11), (13), and perhaps (23). In this system, μ_t and m_t are exogenous with respect to X , being uncorrelated with the random U 's in the demand function for money that help determine X . In this system, money creation influences current and subsequent rates of inflation; but given lagged rates of money creation, past rates of inflation exert no influence on money creation. The system is one in which money creation "causes" inflation, in the sense of Granger [4], while inflation does not "cause" money creation. In such a system, adaptive expectations schemes like Cagan's are not rational.

The second model is one in which Cagan's adaptive expectation mechanism is a rational one to employ. It consists of equations (10), (20), and (23). In this system, the best way to forecast the subsequent rates of money creation that appear in equation (10) is to extrapolate lagged rates of inflation. This in turn implies that inflation itself is best predicted by

extrapolating lagged rates of inflation. This is a system in which both money creation and inflation are best predicted by extrapolating current and lagged rates of inflation and in which lagged rates of money creation add nothing to the predictions formed in this way. In this system, lagged inflation influences money creation, but lagged money creation does not appear to influence inflation once lagged rates of inflation are taken into account. A critical element in this system is the hypothesized feedback that occurs from expected inflation to money creation, a feedback that emerges because the government is attempting to finance a roughly constant rate of real expenditures principally by money creation.

2. Empirical Results

Our two models have very different implications about the structure of feedback between X and m . The model formed by (11), (13), and (23), implies that m influences X with no feedback occurring from X to m . The model consisting of (10), (20), and (23) implies that X influences m but that there is no reverse feedback from m to X . More general models can be imagined in which mutual feedback from X to m and m to X occurs.⁸ In this section, we apply statistical methods that are capable of determining which of these models best describes the data.

A test of the null hypothesis that there is no feedback from X_t to subsequent values of m can be carried out as follows.⁹ Consider the following quite general representation of m_t :

$$(24) \quad m_t = \sum_{i=1}^{\infty} w_i m_{t-i} + \sum_{i=0}^{\infty} c_i X_{t-i} + e_t$$

Here e_t is again a serially uncorrelated random variable assumed to have a finite variance and to be distributed independently of U in equation (1). Equation (11) is obviously a special case of (24), one with the c_i 's all taken to be zero. Equation (20) is another special case of (24), one with the w_i 's all taken to be zero. Equation (24) incorporates the possibility of feedback from the current rate of inflation to subsequent rates of money creation. To test the hypothesis that there is no feedback, we have to test the null hypothesis that all the c_i 's are zero. This can be done by

⁸ For example, if to the system formed by (10), (20), and (23), we add the specification that we observe only $\tilde{X}$ which equals X plus measurement error, mutual feedback will in general characterize the relationship between $\tilde{X}$ and m .

⁹ The test was proposed by Christopher Sims [10].

noticing the implications of the presence of nonzero c_i 's in equation (20). Through equation (13), say, an increment in $\tilde{U}_t$ leads to a decrease in X_t . But through equation (24), the decrease in X_t causes changes in subsequent values of m_t , since the c_i 's are not all zero. Thus nonzero c 's imply nonzero correlations between the $\tilde{U}$'s and subsequent values of m . A test of the null hypothesis that the c 's are zero can be carried out by estimating equation (13) by least squares, which on the null hypothesis produces consistent estimates of the parameters. Then correlations can be calculated between the residuals, which on the null hypothesis are consistent estimates of the true disturbances, and future values of m . Nonzero correlations lead to rejection of the hypothesis and imply that some of the c 's are best taken not to be zero. As a practical matter, it is not necessary to carry out the test in the two separate steps of estimating (13) and then correlating the residuals with future m 's. Instead, the same thing is accomplished by adding future values of m directly to (13) and calculating a two-sided distributed lag regression

$$(25) \quad X_t = \sum_{i=-n}^n z_i m_{t-i} + e_t'$$

where n is a parameter and e_t' is a statistical residual. Sizable coefficients on future values of m imply that some of the c 's in (24) are not zero. Similarly, sizable coefficients on future values of X_t in the reverse distributed lag

$$(26) \quad m_t = \sum_{i=-n}^n h_i X_{t-i} + e_t'',$$

where e_t'' is a statistical residual, permit the inference that there is feedback from the current rate of money creation to subsequent rates of inflation.

Estimating (25) and (26) by least squares unfortunately requires time series longer than most of those examined by Cagan. Only in the case of the German hyperinflation are data available over a long enough

period.¹⁰ For the German monthly data over the period March 1921 through May 1923, we have estimated (25) and (26) by least squares. The data have been "quasi-differenced", i.e. multiplied by $(1-\hat{p}L)$ where $\hat{p}$ is the least-squares estimate of the first-order autoregression parameter of the residuals from (25) or (26) estimated by least squares for the levels of m_t and X_t . Each regression includes a trend term. Our estimates are recorded in Tables 1 and 2. In each table we report the F-statistic pertinent for testing the null hypothesis that the coefficients on future values of the variable on the right side of the equation are zero. The F-statistic in Table 1 is 38.4, well above the critical value of 5.56 at the 1 percent level of significance. This means that at the one percent level of significance we must reject the hypothesis that there is no feedback from current inflation to future rates of money creation.

On the other hand, the F-statistic in Table 2, which reports the regression of money creation on inflation, is 4.8, a value that causes us to reject at the 95 percent confidence level the null hypothesis that there is no feedback from the rate of money creation to subsequent rates of inflation, but that fails to cause us to reject the null hypothesis at the 99 percent confidence level. The absolute values of the coefficients on future rates of inflation are only one-third to one-fourth of those on lagged rates of inflation. This is consistent with the presence of relatively weak feedback from money creation to inflation. The results imply that inflation strongly influences subsequent rates of money creation, but that the influence of money creation on subsequent rates of inflation is harder to detect.

For the remaining six countries studied by Cagan, there are insufficient data to estimate (25) and (26) by ordinary least squares. There

¹⁰ The data, which are monthly, are to be found in Cagan's article [2].

are available, however, alternative techniques that economize on data at the expense of introducing restrictions on the forms of distributed lag. It is convenient to adopt Klein's [6] method of estimating distributed lags in which the coefficients (in positive lags) trail off or oscillate geometrically as the lag increases. Thus, we have estimated the following particular forms of (25) and (26):

$$(25') \quad X_t = \gamma_0 + \gamma_1 \sum_{i=0}^{t-1} \delta^i m_{t-i} + \gamma_2 \delta^t + \gamma_3 m_{t+1} + \gamma_4 m_{t+2} + \gamma_5 t + e_t'$$

$$(26') \quad m_t = \alpha_0 + \alpha_1 \sum_{i=0}^{t-1} \phi^i X_{t-i} + \alpha_2 \phi^t + \alpha_3 X_{t+1} + \alpha_4 X_{t+2} + \alpha_5 t + e_t'$$

Quasi-differenced versions of equations (25') and (26') were estimated using the search procedure recommended by Hildreth and Lu [5]. The search over δ and ϕ was carried down to intervals of .01 over the interval $[-.99, .99]$.

The estimates of (25') and (26') are reported in Tables 4 and 5, while Table 3 reports F-statistics pertinent for testing the null hypotheses: $\gamma_3 = \gamma_4 = 0$ in (25'), and $\alpha_3 = \alpha_4 = 0$ in (26'). High values of the F-statistic lead to rejection of the null hypothesis.

For Germany, Austria, and Hungary I the F-statistics imply that at the 99 percent confidence level the null hypothesis that there is no feedback from X to m must be rejected. Only in the case of Greece are we forced to reject (at the 95 percent confidence level) the null hypothesis that there is no feedback from m to X. The regression for Austria yields very sizable coefficients on future value of X_t , indicating the likely presence of feedback from m to X despite the insignificance of the F-statistic at the 95 percent confidence level. In the cases of Russia, Poland, and Hungary II, the formal statistical tests permit us to reject

neither of our two hypotheses at the 95 percent confidence level. Overall, the picture that emerges from Tables 3, 4, and 5 is one in which the evidence for influence extending from X to m is rather stronger than for influence going the other direction.

As shown in Section 1, such a pattern of feedback could occur if expectations of inflation were formed by extrapolating past rates of inflation while the government was expected to use money creation to finance a roughly constant rate of real government expenditures. In this regard, it is interesting to inspect Table 6 and Graph 1, which record for each country the statistic $(M_t - M_{t-1}) / (1/2 (P_t + P_{t-1}))$, which approximately equals the real resources commanded by creators of money in each period. With the exception of the last several observations, these data are generally without noticeable trends.¹¹ The assumption that the public expected the government to keep its rate of real purchases roughly constant may thus not be a bad approximation.

Our explanation for the feedback from X to m tends to confirm the wisdom of Cagan's decision to model expectations by an extrapolation of lagged rates of inflation. Such a method of forming expectations seems to have been rational, since the data indicate that lagged rates of money creation exerted little influence on inflation beyond that already accounted for by lagged rates of inflation.

In Section I we mentioned that since Cagan's model implies that the disturbances in the demand function for money, equation (1), cause changes in the current price level, the current rate of inflation is likely not

¹¹We do not take into account the changing importance of the private banking system in creating money over the courses of the various hyperinflations. A complete explanation of the apparent feedback from X to m during the various hyperinflations would probably assign a role to the behavior of the private banks.

to be uncorrelated with the current disturbance U_t . The correlation between U and X implies that least squares estimates of equation (1'), Cagan's equation, are not statistically consistent. It seems useful to investigate the nature of the inconsistency, if only for a special case. We can do this for our "rational" version of Cagan's model that incorporates feedback from X to m , and that leads to the rate of inflation being governed by equation (21). Equation (21) can be rewritten as

$$(\lambda + \alpha(1-\lambda)) X_t = (1-L)^{-1} (1-\lambda L) \varepsilon_t - (1-\lambda L) U_t.$$

Inverting the above equation and solving for U_t gives

$$(27) \quad U_t = - \frac{\lambda + \alpha(1-\lambda)}{1-\lambda L} X_t + (1-L)^{-1} \varepsilon_t,$$

which summarizes the correlation between U_t and current and lagged X 's. Now where (1') is the correct model and where data are as abundant as necessary, consider using least squares to estimate the parameters of the equation

$$(28) \quad \left(\log_e \frac{M}{P}\right)_t = \sum_{i=0}^{\infty} h_i X_{t-i} + \delta + U_t$$

where $h_i = \alpha(1-\lambda)\lambda^i$. Let us assume that the ε 's in (27) are so small that they can be neglected, which amounts to ruling out "exogenous" changes in money creation, i.e., changes not governed by expected inflation. On this assumption, application of Theil's [11] specification theorem shows that¹²

¹²Substituting (27) into (1') and using the assumption that ε_t is zero for all t yields the following exact relationship between M/P and X :

$$\log \left(\frac{M}{P}\right)_t = \frac{\alpha(1-\lambda) - (\lambda + \alpha(1-\lambda))}{1-\lambda L} X_t + \gamma Y + \delta$$

or

$$\log \left(\frac{M}{P}\right)_t = -\lambda \sum_{i=0}^{\infty} \lambda^i X_{t-i} + \gamma Y + \delta.$$

$$\begin{aligned}
 (29) \quad \text{plim } \hat{h}_i &= \alpha(1-\lambda)\lambda^i - (\lambda + \alpha(1-\lambda))\lambda^i \\
 &= -\lambda\lambda^i
 \end{aligned}$$

where $\hat{h}_i$ is the least squares estimate of h_i in (28), and where we have used (27) to obtain the regression of U_t on current and lagged X 's. Equation (29) shows that least squares produces a consistent estimate of λ , but an inconsistent estimate of α , one whose probability limit does not even depend on α . The parameter α is usually estimated by taking note of the fact that $h_0 = \alpha(1-\lambda)$ and estimating α by $\hat{\alpha} = \hat{h}_0 / (1-\hat{\lambda})$ where hats denote least-squares estimates of the indicated parameters. Then (29) implies that

$$(30) \quad \text{plim } \hat{\alpha} = \frac{-\lambda}{1-\lambda}.$$

The implication is that, on the assumptions maintained here, λ can be reliably estimated by least squares, while the least squares estimate of α literally conveys no information about α .

In the light of these calculations, it is interesting to review Cagan's and Barro's estimates of equation (1'), which we record in Tables 7 and 8. We have recorded values of $-\hat{\lambda}/(1-\hat{\lambda})$, so that the reader can compare them with the estimates of α . Both Cagan's and Barro's estimates reveal a tendency for the absolute values of $\hat{\lambda}$ and $\hat{\alpha}$ to vary directly with one another. That is a pattern predicted by equation (30). In fact, $-\hat{\lambda}/(1-\hat{\lambda})$ is often contained in the confidence interval around $\hat{\alpha}$ reported by Cagan and Barro. These results seem largely compatible with the view that the least squares estimates of α convey little or no information about the population parameter α .

3. Conclusions

Cagan's adaptive mechanism for explaining expectations of inflation has sometimes been criticized as an ad hoc formulation that is inconsistent with the hypothesis that expectations are rational (e.g., Walters [13]). In this paper, we have showed that conditions exist under which adaptive expectations are fully rational. One essential condition is the presence of feedback from inflation to subsequent rates of money creation. Such feedback appears to have been present in at least several of the hyperinflations that we have studied. This might be explained by the government's resorting to money creation in order to finance its expenditures. Our empirical results indicate that to explain the hyperinflations it is not adequate to regard money creation as exogenous with respect to inflation. Instead, the monetary authorities seemed to make money creation respond directly and systematically to inflation, which was probably an important reason that the hyperinflations developed.

Table 1 Inflation Regressed on Money Creation
(Germany, April 1921 - June 1923)

$$(1-\hat{p}L) X_t = \sum_{i=-4}^6 w_i (1-\hat{p}L) m_{t-i}$$

i	Coefficients on future rates of money creation	Coefficients on lagged rates of money creation
0		
1	1.5321(.2494)	1.2634(.2583)
2	-.3920(.2397)	-2.2126(.2630)
3	-.4558(.1611)	.5362(.2828)
4	.6625(.0942)	-.2454(.2911)
5		2.0478(.4346)
6		-1.7855(.6974)
		-3.3095(.6005)
Constant	.0143(.0285)	
Time	.0231(.0040)	
R_A^2	= .9770	
d.w.	= 2.4123	
F	= 38.47	
$F_{4,14}(.05)$	= 3.34	
$F_{4,14}(.01)$	= 5.56	

Estimated standard errors are in parentheses.

The estimate $\hat{p}$ was obtained as the first-order serial regression coefficient for the residuals of equation (25) estimated by least squares; $\hat{p} = -.605$.

Table 2 Money Creation Regressed on Inflation
(Germany, March 1921 through May 1923)

$$(1-\hat{p}L) m_t = \sum_{i=-4}^6 z_i (1-\hat{p}L) X_{t-i}$$

i	Coefficients on future rates of inflation	Coefficients on lagged rates of inflation
0		.1926(.0345)
1	-.0866(.0296)	.2120(.0524)
2	.0420(.0251)	.2234(.0548)
3	-.0398(.0220)	.1104(.0585)
4	.0379(.0217)	.2500(.0600)
5		.0291(.0632)
6		.0701(.0628)
Constant	-.0086(.0163)	
Time	-.0009(.0028)	
R_A^2	= .8786	
d.w.	= 1.9811	
F	= 4.77	
$F_{4,14}(.05)$	= 3.34	
$F_{4,14}(.01)$	= 5.56	

Estimated standard errors are in parentheses.

The estimate $\hat{p}$ was obtained as the first-order serial regression coefficient for the residuals of equation (26) estimated by least squares; $\hat{p} = .648$.

Table 3

F-statistics (X versus m)

<u>Country</u>	<u>m regressed on X</u>	<u>X regressed on m</u>	<u>degrees of freedom</u>	Critical values of F	
				<u>.05</u>	<u>.01</u>
Germany	3.30	17.64**	24	3.40	5.61
Russia	< 1	< 1	15	3.68	6.36
Austria	2.89	15.53**	9	4.26	8.02
Hungary I	1.17	11.01**	9	4.26	8.02
Greece	7.68**	5.02*	12	3.88	6.93
Poland	< 1	2.57	11	3.98	7.20
Hungary II	< 1	< 1	2	19.00	99.01

* Significant at .95 level of significance

** Significant at .99 level of significance

Table 4 Inflation Regressed on Money Creation

$$(1-\rho L) \text{ applied to: } X_t = \gamma_0 + \gamma_1 \sum_{i=0}^{t-1} \delta^i m_{t-i} + \gamma_2 \delta^t + \gamma_3 m_{t+1} + \gamma_4 m_{t+2} + \gamma_5 t$$

Country	γ_0	γ_1	γ_2	δ	γ_3	γ_4	γ_5	d.w.	R_A^2	$\hat{\rho}$
Germany	-.0712 (.1203)	-.5107 (.1118)	.0073 (.1905)	.68	2.1012 (.3060)	-.2633 (.2521)	.0142 (.0102)	2.44	.8037	-.4585
Russia	-.0906 (.1011)	-.1110 (.1587)	.5552 (.1683)	.58	.1639 (.2084)	.2160 (.2142)	.0199 (.0066)	1.89	.5308	.3920
Austria	175.1205 (48.7372)	.6694 (.2164)	-173.4718 (48.2640)	.99	1.4603 (.3106)	.6223 (.4176)	-1.8178 (.5063)	2.74	.9198	-.5278
Hungary I	.2053 (.0920)	-.3092 (.1597)	-.1714 (.1529)	.55	1.8218 (.3581)	-.6498 (.3279)	-.0002 (.0120)	2.87	.8338	-.3984
Greece	.0090 (.0834)	-.4184 (.3238)	.0575 (.0471)	-.99	.2765 (.3219)	.1356 (.1034)	.0462 (.0119)	2.26	.9565	-.2145
Poland	.0276 (.0902)	1.0107 (.5618)	-.0206 (.0738)	-.99	.2431 (.5913)	.6045 (.4807)	-.0122 (.0161)	1.99	.6715	.0223
Hungary II	3470.5169 (610.6772)	1.1459 (.3285)	-3435.7382 (604.9208)	.99	1.6519 (.3336)	-.6487 (.1426)	-35.4153 (6.0521)	3.09	.9862	-.6612

The estimate $\hat{\rho}$ was obtained as the first-order serial regression coefficient for the residuals of equation (25') estimated by least squares.

Table 5 Money Creation Regressed on Inflation

$$(1-\hat{p}L) \text{ applied to: } m_t = \alpha_0 + \alpha_1 \sum_{i=0}^{t-1} \phi^2 X_{t-i} + \alpha_2 \phi^t + \alpha_3 X_{t+1} + \alpha_4 X_{t+2} + \alpha_5 t$$

Country	α_0	α_1	α_2	ϕ	α_3	α_4	α_5	d.w.	R_A^2	$\hat{p}$
Germany	.2037 (.0774)	.2717 (.0318)	-.1693 (.0839)	.88	-.0779 (.0335)	.0448 (.0339)	-.0176 (.0056)	1.91	.9349	.1133
Russia	-30.5212 (40.0497)	.2083 (.0846)	30.6696 (39.9155)	.99	-.1051 (.1935)	.2591 (.1911)	.1939 (.3917)	2.39	.7082	-.3033
Austria	60.6528 (15.8772)	.3031 (.0597)	-59.9510 (15.7376)	.99	.2676 (.0797)	.1586 (.0926)	-.6379 (.1562)	2.12	.8306	-.1010
Hungary I	-.3974 (.0978)	.5816 (.0457)	.5169 (.1107)	.80	.0198 (.0642)	.1269 (-.0783)	-.0282 (.0078)	2.67	.9572	-.4125
Greece	92.3591 (74.8968)	.3257 (.1872)	-91.4102 (74.2067)	.99	.2513 (.0681)	-.0010 (.0490)	-.9369 (.7435)	1.45	.9629	.3367
Poland	13.0791 (32.2762)	.2688 (.0746)	-12.8325 (32.0157)	.99	.0039 (.0881)	.0346 (.0880)	-.1721 (.3180)	2.02	.8617	.0601
Hungary II	-1.6880 (.6663)	.6186 (.1673)	1.2515 (.6138)	.34	.1366 (.1066)	-.0307 (.0378)	.2545 (.0735)	3.30	.9934	-.4339

The estimate $\hat{p}$ was obtained as the first-order serial regression coefficient for the residuals of equation (26') estimated by least squares.

Table 6

Index of $(M_t - M_{t-1}) / (1/2(p_t + p_{t-1}))$

Germany	Russia	Austria	Hungary I
.26(Oct 1920)	.11(Feb 1922)	1.07(Feb 1921)	1.04(Aug 1922)
.10	.10	.66	1.36
.23	.07	.97	1.02
.08	.05	.06	.16
-.09	.06	.97	.29
.10	.08	.92	-.14
.08	.10	1.00	.09
.09	.16	1.70	.42
.23	.10	2.34	.56
.29	.12	1.77	.55
.22	.10	1.72	.94
.31	.10	1.32	1.03
.42	.11	.57	1.08
.43	.07	.78	.83
.47	.15	.78	.63
.36	.14	.74	.48
.13	.06	1.41	.31
.30	.08	1.70	.46
.28	.09	1.70	.44
.26	.07		
.38	.05		
.46	.13		
.49	.12		
.38	.09		
.53	.07		
.43			
.38			
.41			
.76			
.27			
.58			
.41			
.66			
.72			
1.46			
1.27			
.41			
20.56			

Table 6 (continued)

Greece	Poland	Hungary II
.36(Feb 1943)	.10(May 1922)	.31(Aug 1945)
.75	.13	.36
.90	.18	.48
.41	.20	.13
.62	.22	.05
.95	.33	.05
.79	.16	.20
.93	.18	.11
1.12	.14	.64
.95	.17	2.09
.67	.27	56.06
.55	.22	
.23	.16	
.44	.32	
.71	.17	
.55	.27	
.37	.27	
.84	.62	
.61	.21	
1.99	.28	
2.45	.39	
520.26

Graph 1
(graphs of $\Delta M_t / (.5(P_t + P_{t-1}))$)

Graph 1.
(graphs of $\Delta M_t / (.5(P_t + P_{t-1}))$)

Table 7
Cagan's estimates of α and λ

Country	Time Period	$\hat{\lambda}$	$\frac{-\hat{\lambda}}{1-\hat{\lambda}}$	$\hat{\alpha}$
Austria	Jan 1921 - Aug 1922	.95	-19.5	-8.55
Germany	Sept 1920 - July 1923	.82	-4.5	-5.46
Greece	Jan 1943 - Aug 1944	.86	-6.2	-4.09
Hungary	July 1922 - Feb 1924	.90	-9.5	-8.70
Hungary	July 1945 - Feb 1946	.86	-6.2	-3.63
Poland	Apr 1922 - Nov 1923	.74	-2.9	-2.30
Russia	Dec 1921 - Jan 1924	.70	-2.4	-3.06

Source: Table 3 of Cagan [2].

Table 8

Barro's estimates of α and λ

Country	Time Period	$\hat{\lambda}$	$\frac{-\hat{\lambda}}{1-\hat{\lambda}}$	$\hat{\alpha}$
Austria	Jan 1921 - Dec 1922	.829	-4.85	-4.09
Germany	Jan 1921 - Aug 1923	.824	-4.68	-3.79
Hungary	Oct 1921 - Feb 1924	.861	-6.19	-5.53
Poland	Jan 1922 - Jan 1924	.709	-2.44	-2.56

Source: Table 3 of Barro [1].

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