INSTANTANEOUS AND NON-INSTANTANEOUS
ADJUSTMENT TO EQUILIBRIUM
IN TWO-SECTOR GROWTH MODELS

by

Antonio Bosch
Andreu Mas-Colell
Assaf Razin

Discussion Paper No. 8, July 1971

Center for Economic Research
Department of Economics
University of Minnesota
Minneapolis, Minnesota 55455

INSTANTANEOUS AND NON-INSTANTANEOUS ADJUSTMENT
TO EQUILIBRIUM IN TWO-SECTOR GROWTH MODELS

by

Antonio Bosch, Andreu Mas-Colell and Assaf Razin

The purpose of this paper is to test some of the basic simplifying assumptions of the two-sector growth model studied by Uzawa [7], [8], Inada [3] and Drandakis [1] and, in this way, to assess its usefulness as a convenient idealization of reality. In the model one distinguishes between a set of variables which is always adjusted to fulfill some "momentary equilibrium" conditions and the set of variables in the dynamic paths of which one is interested. Short-run disequilibrium behavior is ruled out, partly to simplify the analysis, partly for lack of an adequate theory. However, if we have a model where not all variables are instantaneously adjusted to their partial equilibria the fact that the equilibrium of the first model is stable does not necessarily imply the stability of equilibrium of the latter. Furthermore, even if speeds of adjustment of some variables are very high relative to the rest then with respect to dynamic properties the conclusion that those variables can be treated as instantaneously adjusted is not warranted.¹

1. Consider, for example, in the standard supply and (downward sloping) demand model, the case where equilibrium is "Walras stable" but "Marshall unstable". In the "Walrasian" treatment the quantity variable is always in equilibrium. However, if it is momentarily given and moves according to the difference between demand and supply price the equilibrium is never stable regardless of the speed of adjustment of the quantity variable.

We will study a two-sector growth model in the four cases:² first, labor does not shift instantaneously between sectors (Part I); second, capital does not shift instantaneously between sectors (Part II); third, neither labor nor capital shifts instantaneously between sectors (Part III); fourth, wage-rental ratio is rigid in the short run (Part IV). We will study those cases under two alternatives savings assumptions: a uniform saving-income ratio and the classical saving hypothesis (all wages consumed, all profits invested).

We will show that local stability of an equilibrium of the instantaneous adjustment model (I.M.) will guarantee the local stability of this equilibrium in the non-instantaneous model (N.I.M.) if speeds of adjustment are sufficiently high.

The ideas discussed in this paper are of the same nature as the subject of the "Hicks stability" versus "true dynamic stability" in the multimarket model. Directly relevant to our purpose is McFadden's [5] analysis of the multimarket equilibrium. We will apply his results to prove some global stability properties.

A great deal may be concluded about the stability of the N.I.M. without any reference to speeds of adjustment. In some cases, namely in the second model, the stability in the I.M. implies the stability in the N.I.M. This is not generally true, as was already observed by Uzawa [8] with reference to the fourth model. However, in this case we derive stability conditions which are broad enough.

2. Recently, Shell et al [6] have analyzed a two-sector growth model where the change over time in the price of investment good is related to the excess demand in this market. Our analysis can be straightforwardly applied to this case. However this model poses the basic problem that momentary equilibrium is not unique if factor-intensity reversal is admitted. We observe, however, that even if the momentary equilibrium is determined it is possible for the equilibrium to be stable for an instantaneous adjustment of the price of investment and unstable for a low speed of adjustment of this variable.

INTRODUCTION

The momentary equilibrium of the two-sector model with instantaneous adjustment is characterized by:

$$F_i (K_i, L_i) \quad i = 1, 2$$

$$L_1 + L_2 = L$$

$$K_1 + K_2 = K$$

$$\frac{\partial F_1}{\partial K_1} = p \frac{\partial F_2}{\partial K_2} \quad ; \quad \frac{\partial F_1}{\partial L_1} = p \frac{\partial F_2}{\partial L_2}$$

$$\text{and either } 1. \quad s(F_1 + pF_2) = pF_2 \quad 0 < s < 1$$

$$\text{or } 2. \quad \frac{\partial F_1}{\partial K_1} K = pF_2 \quad (1)$$

Its dynamic path is given by:

$$\frac{\dot{L}}{L} = n$$

$$\dot{K} = F_2$$

We assume the F_i 's are linear homogeneous, and

$$\frac{\partial F_1}{\partial K_1} > 0 ; \quad \frac{\partial F_1}{\partial L_1} > 0 ; \quad \frac{\partial^2 F_1}{\partial K_1^2} < 0 ; \quad \frac{\partial^2 F_1}{\partial L_1^2} < 0 ; \quad i = 1, 2 .$$

When it is convenient we also assume that the Inada Conditions hold.

We state as lemmas the specific stability results we will use.

Consider the dynamic system:

$$[A] \quad \begin{cases} \dot{X}_1 = F_1 (X_1, X_2) \\ \dot{X}_2 = c F_2 (X_1, X_2) \quad , \quad c > 0 \end{cases}$$

with F_1, F_2 continuously differentiable. Let $\hat{X} = (\hat{X}_1, \hat{X}_2)$ be a

stationary point of [A]. If $\begin{bmatrix} \partial \dot{X}_2 \\ \partial X_2 \end{bmatrix}_{\hat{X}} \neq 0$, $\begin{bmatrix} d\dot{X}_1 \\ dX_1 \end{bmatrix}_{\hat{X}} \dot{X}_2=0$ is well defined.

Lemma 1: If $\left[\frac{d\dot{X}_1}{dX_1} \right]_{\dot{X}_2=0} < 0$ and $\left[\frac{\partial \dot{X}_2}{\partial X_2} \right]_{\hat{x}} < 0$ there is a $\bar{c} \geq 0$ such that if $c > \bar{c}$ the system $[A]$ is locally stable. Moreover, if $\left[\frac{\partial \dot{X}_1}{\partial X_1} \right]_{\hat{x}} < 0$ we can take $\bar{c} = 0$.

Proof: This is a well-known result. A sufficient condition for the stability of $[A]$ is that the determinant of the matrix of the linear approximation at $\hat{X}$ is positive and its trace is negative.

$$\text{Since } \left[\frac{d\dot{X}_1}{dX_1} \right]_{\dot{X}_2=0} = \left[\frac{\partial \dot{X}_1}{\partial X_1} \right]_{\hat{x}} + \left[\frac{\partial \dot{X}_1}{\partial X_2} \right]_{\hat{x}} \left[\frac{dX_2}{dX_1} \right]_{\dot{X}_2=0},$$

$$\left[\frac{dX_2}{dX_1} \right]_{\dot{X}_2=0} \cdot \left[\frac{\partial \dot{X}_2}{\partial X_2} \right]_{\hat{x}} = - \left[\frac{\partial \dot{X}_2}{\partial X_1} \right]_{\hat{x}} \quad \text{and} \quad \left[\frac{\partial \dot{X}_2}{\partial X_2} \right]_{\hat{x}} < 0,$$

the determinant is positive and the result follows.

Consider the system:

$$[B] \quad \begin{cases} \dot{X}_1 = F_1(X_1, X_2, X_3) \\ \dot{X}_2 = c_2 F_2(X_1, X_2, X_3) & c_2 > 0 \\ \dot{X}_3 = c_3 F_3(X_1, X_2, X_3) & c_3 > 0 \end{cases}$$

where $(X_1, X_2, X_3) \in \Delta \subset \mathbb{R}^3$. Δ is a rectangle and the F_i 's are continuously differentiable. The following lemma is an immediate application of McFadden's Theorem II ([5], p. 338). It should be clear that the fulfillment of the conditions of the lemma does not depend on c_2, c_3 .

Lemma 2: Suppose that $\hat{x} = (\hat{X}_1, \hat{X}_2, \hat{X}_3)$ is the unique stationary point of $[B]$.

For every X_1 , the equations $\dot{X}_2 = 0$ and $\dot{X}_3 = 0$ give a unique solution for X_2, X_3 .

For every X_1 and X_2 the equation $\dot{X}_3 = 0$ gives a unique solution for X_3 .

$$\left[\frac{d\dot{X}_1}{dX_1} \right]_{\substack{\dot{X}_2 = \dot{X}_3 = 0 \\ \hat{X}_1}} < 0 ; \quad \left[\frac{\partial \dot{X}_2}{\partial X_2} \right]_{\substack{\dot{X}_3 = 0 \\ \hat{X}_2}} < 0 \quad \text{for every } X_1 ;$$

$$\left[\frac{\partial \dot{X}_3}{\partial X_3} \right]_{\hat{X}_3} < 0 \quad \text{for every } X_1, X_2 .$$

Then for every compact rectangle $\bar{\Delta} \subset \Delta$ there is a $M > 0$ such that if $c_2 > M$ and $c_3 > c_2 M$ the system [B] is globally stable in $\bar{\Delta}$.

The validity of the next lemma is indicated by McFadden ([5], p. 345).

In the model [B] suppose $\left[\frac{\partial \dot{X}_3}{\partial X_3} \right]_{\hat{X}} \neq 0$. Let $X_3 = \phi(X_1, X_2)$ be the solution of $\dot{X}_3 = 0$. Consider:

$$[B^1] = \begin{cases} \dot{X}_1 = F_1(X_1, X_2, \phi(X_1, X_2)) \equiv G_1(X_1, X_2) \\ \dot{X}_2 = c_2 F_2(X_1, X_2, \phi(X_1, X_2)) \equiv G_2(X_1, X_2) \end{cases}$$

Lemma 3: If $[B^1]$ is locally stable at $(\hat{X}_1, \hat{X}_2)$ and $\frac{\partial F_3}{\partial X_3} < 0$ at $\hat{X} = (\hat{X}_1, \hat{X}_2, \phi(\hat{X}_1, \hat{X}_2))$, there is a $\bar{c}_3 \geq 0$ such that if $c_3 > \bar{c}_3$ the system [B] is locally stable at $\hat{X}$.

Sketch of proof: Let $[d_{1j}]$ be the matrix of the coefficients of the linear approximation of B at $\hat{X}$, and D_{11} , D designate the principal minors and the Jacobian determinant. Let $\bar{d}_{11}$, $\bar{D}_{11}$, $\bar{D}$ correspond to $c_3 = 1$.

As in Lemma 1 it is elementary to show that the stability of $[B^1]$ and $d_{33} < 0$ implies:

$$c_2 \bar{D}_{11} = D_{11} > 0, \quad c_3 \bar{D}_{22} = D_{22} > 0, \quad c_3 \bar{D} = D < 0$$

3. Obviously, $\phi(X_1, X_2)$ does not depend on c_3 .

To fulfill the Routh-Hurwitz conditions for stability (see Gantmacher [2], Ch. 15) we additionally need i) $k_1 \equiv \sum_{i=1}^3 d_{1i} < 0$, ii) $k_2 \equiv \sum_{i=1}^3 D_{1i} > 0$ iii) $k_1 k_2 - D > 0$. Obviously i) holds for c_3 high enough ($d_{33} = \bar{d}_{33} \cdot c_3$) the same is true with respect to ii) since $k_2 = c_3 (\bar{D}_{11} + \bar{D}_{22}) + D_{33}$. Condition iii) follows in a similar way. Its left hand side can be expressed, after some manipulation, as a quadratic expression in c_3 , the coefficient of c_3^2 being positive.

PART I

We will assume that labor does not move instantaneously and that all markets are cleared. This means that in the short run the wage rate may differ between sectors. We will further assume that the relative labor force in every sector changes over time in the same direction as the wage differential. Formally, let

$$m = \frac{L_1}{L} \quad (2)$$

$$k_1 = \frac{K_1}{L_1}$$

$$f_1(k_1) = F_1(k_1, 1) \quad (3)$$

$$w_1 = f_1 - k_1 f_1$$

$$\omega_1 = \frac{f_1}{F_1} - k_1 \quad (4)$$

A) Uniform Saving Ratio

For $0 < m < 1$, $k > 0$ given, the momentary equilibrium of the economy is determined by:

$$mk_1 + (1-m)k_2 = k \quad (5)$$

$$f'_1(k_1) = p f'_2(k_2) \quad (6)$$

$$s(mf_1(k_1) + p(1-m)f_2(k_2)) = p(1-m)f_2(k_2) \quad (7)$$

Equations (6) and (7) reduce to

$$sm(\omega_1 + k_1) = (1-s)(1-m)(\omega_2 + k_2) \quad (8)$$

Under the assumptions in (1), (5) and (8) have a unique solution in k_1 and k_2 .⁴

The path of $m(t)$, $k(t)$ is described by the following dynamic equations

$$\dot{m} = cH(m, k) \quad , \quad \text{sign } \dot{m} = \text{sign } [w_1 - w_2] \quad , \quad c > 0 \quad (9)$$

$$\dot{k} = (1-m)f_2 - nk \quad (10)$$

We assume that H is continuously differentiable, and the system (9) - (10) has a unique and continuous solution for every initial conditions ($0 < m_0 < 1$, $k_0 > 0$). Moreover Inada conditions insure that the solution stays in the nonspecialization region ($k(t) > 0$, $1 > m(t) > 0$).

Expressions for $\frac{\partial k_1}{\partial m}$, $\frac{\partial k_1}{\partial k}$ are derived in A-1 of the Appendix.

Define $\sigma_1(m, k)$ as $\sigma_1(m, k) = \frac{dk_1}{d\omega_1} \frac{\omega_1}{k_1}$, elasticities of substitution.

It is clear that the steady states of the model (5) - (10) coincide with the steady states of the model (1). Suppose $(\hat{m}, \hat{k})$ is a steady state.

For short let $\hat{x} \equiv (\hat{m}, \hat{k})$. We will study now the stability properties of this model. If at $\bar{x} = (\bar{m}, \bar{k})$, $\dot{m} = 0$, then:

$$a) \quad \left[\frac{\partial \dot{m}}{\partial m} \right]_{\bar{x}} < 0$$

Proof:

In $\bar{x}$, $\omega_1 = \omega_2 = \omega$

4. In the (k_1, k_2) space (8) gives a monotonic increasing relation starting at the origin and (5) a decreasing one.

From (8), if $\omega_1 = \omega_2$:

$$m = \frac{(1-s)(\omega + k_2(\omega))}{s(\omega + k_1(\omega)) + (1-s)(\omega + k_2(\omega))} \quad (11)$$

From Uzawa [7] we know that, if $\omega_1 = \omega_2 = \omega$, k is a function of ω and

$$\frac{dk}{d\omega} > 0 \quad (12)$$

Combining (11) and (12) and manipulating:

$$\text{sign} \left[\frac{dm}{dk} \right]_{\dot{m}=0} = \text{sign} \left[\frac{\omega + \sigma_2 k_2}{\omega + k_2} - \frac{\omega + \sigma_1 k_1}{\omega + k_1} \right]_{\dot{x}} \quad (13)$$

where all the terms are functions of (m, k) .

However

$$\left[\frac{\partial \dot{m}}{\partial m} \right]_{\dot{x}} = - \frac{\left[\frac{\partial \dot{m}}{\partial k} \right]_{\dot{x}}}{\left[\frac{dm}{dk} \right]_{\dot{x}}} \dot{m} = 0 \quad (14)$$

and $\text{sign} \left[\frac{\partial \dot{m}}{\partial k} \right]_{\dot{x}} = \text{sign} \left[\frac{d\omega_1}{dk_1} \frac{\partial k_1}{\partial k} - \frac{d\omega_2}{dk_2} \frac{\partial k_2}{\partial k} \right]_{\dot{x}} \quad (15)^5$

Referring to Lemma 1 in the Introduction we can already conclude that if $\hat{x}$ is a locally stable steady state in the I.M. (1) it will be locally stable in the N.I.M. (5) - (10) for c high enough. Since $\left[\frac{dk}{d\omega} \right]_{\dot{m}=0} > 0$ conditions for Lemma 2 are fulfilled. Therefore, in an appropriate compact rectangle, if $\hat{x}$ is globally stable in the I.M. (1) it will be globally stable in the N.I.M. (5) - (10) for c high enough.

b) If $\sigma_1(\hat{m}, \hat{k}) \geq 1$ or $\sigma_1(\hat{m}, \hat{k}) \cdot k_1 \geq \sigma_2(\hat{m}, \hat{k}) \cdot k_2$,⁶ then

$$\left[\frac{\partial \dot{k}}{\partial k} \right]_{\hat{x}} < 0 \quad \text{where } \hat{x} = (\hat{m}, \hat{k})$$

is a steady state.

5. Since, when $\dot{m} = 0$, $\text{sign} \frac{\partial \dot{m}}{\partial \alpha} = \text{sign} \frac{\partial (\omega_1 - \omega_2)}{\partial \alpha}$.

6. For example: $\sigma_1(\hat{m}, \hat{k}) \geq \sigma_2(\hat{m}, \hat{k})$ and $k_1(\hat{m}, \hat{k}) \geq k_2(\hat{m}, \hat{k})$.

Proof: From (10),

$$\left[\frac{\partial \dot{k}}{\partial k} \right]_{\hat{k}} = (1 - m) f_2'(k_2) \left[\frac{\partial k_2}{\partial k} - \frac{\omega + k_2}{k} \right]_{\hat{k}} \quad (16)$$

From A-3 in the Appendix,

$$\left[\frac{\partial k_2}{\partial k} \right]_{\hat{k}} = \left[\frac{\omega + k_2}{B(m, k) m k_1 + (1-m) k_2 + (1-m) \omega} \right]_{\hat{k}} \quad (17)$$

$$\text{where } B(m, k) = \frac{\frac{\omega}{\sigma_2 k_2} + 1}{\frac{\omega}{\sigma_1} + k_1} \quad (\omega + k_1) \quad (18)$$

and $B(m, k) \geq 1$ if $\sigma_1 \geq 1$ or if $\sigma_1 k_1 \geq \sigma_2 k_2$.

Therefore, under these conditions, and taking (5) into account,

$$\left[\frac{\partial k_2}{\partial k} \right]_{\hat{k}} < \left[\frac{\omega + k_2}{k} \right]_{\hat{k}} \quad \text{or} \quad \left[\frac{\partial \dot{k}}{\partial k} \right]_{\hat{k}} < 0$$

Combining Lemma 1, a) and b) we can state the following proposition.

Proposition 1:

If the steady state $\hat{k}$ corresponding to the model (1) with uniform saving rate and instantaneous adjustment of labor is locally stable and either 1) $\sigma_1(\hat{k}) \geq 1$ or 2) $\sigma_1(\hat{k}) \geq \sigma_2(\hat{k})$ and $k_1(\hat{k}) \geq k_2(\hat{k})$, the steady state $(m(\hat{k}), \hat{k})$ corresponding to the model (5) - (10) will be locally stable.

In particular, using the results of Uzawa [8] and Drandakis [1], we can say that the steady state $(\hat{m}, \hat{k})$ in the model (5) - (10) will be locally stable if

- 1) $\sigma_1(\hat{m}, \hat{k}) \geq 1$, or
- 2) $k_1(\hat{m}, \hat{k}) \geq k_2(\hat{m}, \hat{k})$ and $\sigma_1(\hat{m}, \hat{k}) \geq \sigma_2(\hat{m}, \hat{k})$.

As the following example shows, the stability of an I.M. steady state does not necessarily yield its stability in the N.I.M. (5) - (10).

$$\text{Let } f_1(k_1) = \left[\frac{3}{1 + 2^{24}} \right]^{1/23} \left[\frac{2^{24}}{1 + 2^{24}} k_1^{-23} + \frac{1}{1 + 2^{24}} \right]^{-1/23}$$

$$f_2(k_2) = 3/2 k_2^{1/2}$$

$$s = \frac{1}{2} , \quad n = 9/14 .$$

A steady state for this economy is given by:

$$\hat{k} = 7/5 , \quad \hat{m} = 2/5$$

corresponding to:

$$\hat{k}_1=2, \hat{k}_2=1, \hat{f}_1=1, \hat{f}_2=3/2, \hat{f}_2'=3/4, \hat{f}_1'=1/3, \hat{p}=4/9, \hat{\omega}=1, \hat{\sigma}_1=1/24, \hat{\sigma}_2=1 .$$

The capital intensity condition is fulfilled, $\hat{k}_1 > \hat{k}_2$, however if we substitute those values in (17) we find:

$$\left[\frac{\partial k_2}{\partial k_1} \right]_x = \frac{2}{6/5 + 24/26 \cdot 1/5} > \frac{2}{7/5} = \frac{\omega + k_2}{k}$$

$$\text{By (16)} \quad \left[\frac{\partial \dot{k}}{\partial k} \right]_{(2/5, 7/5)} > 0 .$$

Therefore the behavior of the system at $(2/5, 7/5)$ is cyclic. There is always a value of c for which the trace of the Jacobian at $(2/5, 7/5)$ is positive, i.e. the cycles will be divergent.

We remark that the $\dot{m} = 0$ and $\dot{k} = 0$ locus do not have a simple behavior outside of the steady state. It seems that to obtain results analogous to Proposition 1 with respect to global stability would require more specific forms of the H function.

B) Classical Saving Assumption

Given m and k , the momentary equilibrium of the economy is determined by:

$$m k_1 + (1-m) k_2 = k \quad (19)$$

$$(1-m) f_2(k_2) = f_2'(k_2) k \quad (20)$$

$$f_1'(k_1) = p f_2'(k_2) \quad (21)$$

Under the assumptions in (1) the model (19) - (20) has a unique solution in k_1 and k_2 .

The paths of $m(t), k(t)$ are described by (9) - (10) above.

Under the condition $\dot{m} = 0$, $\omega_1 = \omega_2 = \omega$. (19) and (20) can be expressed as (see Uzawa [7], Inada [3]).

$$k = \varphi(\omega) = \frac{\omega + k_2(\omega)}{\omega + k_1(\omega)} k_1(\omega) \quad (22)$$

$$\text{and} \quad m = \frac{\omega}{\omega + k_1(\omega)} \quad (23)$$

Suppose that at $\bar{x}$, $\dot{m} = 0$. Then:

$$a) \quad \text{If } [\varphi'(\omega)]_{\bar{x}} > 0, \text{ then } \left[\frac{\partial \dot{m}}{\partial m} \right]_{\bar{x}} < 0.$$

Proof: We assume that $\sigma_1(\bar{x}) \neq 1$. The proof when $\sigma_1(\bar{x}) = 1$ is given in A-5 of the Appendix.

Differentiating (22), it is easily checked that:

$$\text{sign} \left[\frac{dk}{d\omega} \right]_{\dot{m}=0}^{\bar{x}} = \text{sign} (1 - \sigma_1(\bar{x})).$$

If $\sigma_1(\bar{x}) \neq 1$ we can write:

$$\text{sign} \left[\frac{d\omega}{dm} \right]_{\dot{m}=0}^{\bar{x}} = \text{sign} [1 - \sigma_1(\bar{x})].$$

Therefore if $[\varphi'(\omega)]_{\bar{x}} > 0$ and $\sigma_1(\bar{x}) \neq 1$ k is, locally, a function of m and,

$$\left[\frac{\partial \dot{m}}{\partial m} \right]_{\bar{x}} = - \left[\frac{dk}{dm} \right]_{\dot{m}=0}^{\bar{x}} \left[\frac{\partial \dot{m}}{\partial k} \right]_{\bar{x}} = - [\varphi'(\omega)]_{\bar{x}} \left[\frac{d\omega}{dm} \right]_{\dot{m}=0}^{\bar{x}} \left[\frac{\partial \dot{m}}{\partial k} \right]_{\bar{x}}$$

However it is proved in A-6 of the Appendix that

$$\text{sign} \left[\frac{\partial \dot{m}}{\partial k} \right]_{\bar{x}} = \text{sign} [1 - \sigma_1(\bar{x})] = \text{sign} \left[\frac{d\omega}{dm} \right]_{\dot{m}=0, \bar{x}},$$

yielding $\left[\frac{\partial \dot{m}}{\partial k} \right]_{\bar{x}} < 0$ q.e.d.

We note that for the steady state $\hat{k}$ corresponding to the I.M. to be (even locally) stable we need $\varphi'(\omega) > 0$ for all ω . Otherwise the momentary equilibrium may not be uniquely defined (see Inada [3]). In the model (19) - (21) the momentary equilibrium is always uniquely determined. Therefore $\varphi'(\hat{\omega}) > 0$ is sufficient for local stability.

We can apply Lemma 1 and Lemma 2 as in the case examined before. In order to apply Lemma 2 we need $\varphi'(\omega) > 0$ for all ω .

Suppose $(\hat{m}, \hat{k})$ -- for short $\hat{x}$ -- is a steady state of the system. (It is in fact unique.) Then:

$$\text{b) } \left[\frac{\partial \dot{k}}{\partial k} \right]_{\hat{x}} < 0$$

Proof: By (20),

$$\dot{k} = f_2'(k_2) k - nk,$$

at $\hat{x}$, $\text{sign} \frac{\partial \dot{k}}{\partial k} = \text{sign} \frac{\partial (k/k)}{\partial k} = \text{sign} f_2''(k_2) \frac{\partial k_2}{\partial k} < 0$, since in A-5 of the Appendix it is proved that $\frac{\partial k_2}{\partial k} > 0$. In fact, b) is a necessary and sufficient condition. Since, in turn, $\varphi'(\omega) > 0$ is a necessary and sufficient condition for the stability of the instantaneous adjustment model (see Drandakis [1]) we conclude, in view of a), b) and Lemma 1:

Proposition 2:

If a steady state $\hat{k}$ corresponding to the model (1) with classical savings assumption and instantaneous adjustment of labor is locally stable, the steady state $(m(\hat{k}), \hat{k})$ corresponding to the model (9) - (10) and

(19) - (21) is locally stable.

The steady state will be stable if (see Drandakis [1])

- 1) $\sigma_1(\hat{x}) + \sigma_2(\hat{x}) \geq 1$ or
- 2) $k_1(\hat{x}) \geq k_2(\hat{x})$

PART II

In this part we will assume that capital does not move instantaneously or that it is not shiftable at all. Competitive conditions prevail in each sector. The short-run rate of return on capital, r_1 , may differ between sectors. We will assume that investment functions are such that the relative amount of capital in each sector varies in direction according to the current rate-of-return differential. Under those conditions the steady states of the model coincide with the steady states of a model with perfect mobility of capital.⁷

In order to make the treatment quite simple we will use a notation which will bring out the formal analogy with the models treated in Part I.

$$\text{Call } u = \frac{K_1}{K}, \quad t_1 = \frac{L_1}{K_1}, \quad f_1(t_1) = F_1(1, t_1),$$

$$r_1 = f_1 - t_1 f_1', \quad w_1 = f_1', \quad w_2 = p f_2', \quad \bar{\omega}_1 = \frac{r_1}{w_1} = 1/\omega_1$$

Observe that:

$$\sigma_1 = \frac{dk_1}{d\omega} \frac{\omega}{k_1} = \frac{dt_1}{d\omega} \frac{\bar{\omega}}{t_1} \quad (25)$$

u and $t_1 (= 1/k)$ are state variables.

7. With classical saving assumption and elasticity of substitution greater than one in the first sector, Inada [4] has established the global stability of a model with non-shiftable capital. Except for differentiability his specific investment functions yield the restrictions we impose. Therefore the following can be viewed as a generalization of some of the stability results of Inada with respect to saving behavior and the admissible class of investment functions.

A) Uniform Saving Ratio

For given $0 < u < 1$, $t > 0$ the momentary equilibrium of the economy is determined by:

$$ut_1 + (1-u)t_2 = t \quad (26)$$

$$f_1'(t_1) = pf_2'(t_2) \quad (27)$$

$$(1-s)(uf_1 + (1-u)pf_2) = uf_1 \quad (28)$$

(27) and (28) reduce to:

$$su(\bar{\omega}_1 + t_1) = (1-s)(1-u)(\bar{\omega}_2 + t_2) \quad (29)$$

The path of $u(t)$, $t(t)$ is described by the following dynamic equations:

$$\dot{u} = cH(u, t); \text{ sign } \dot{u} = \text{sign } \{r_1 - r_2\}, c > 0 \quad (30)$$

$$\dot{t} = nt - (1-u)f_2(t_2)t \quad (31)$$

We assume that H is continuously differentiable and (30) - (31) has a unique and continuous solution for initial (u_0, t_0) , $0 < u_0 < 1$, $t_0 > 0$. Suppose $(\hat{u}, \hat{t}) \equiv \hat{x}$ is a steady-state equilibrium of the system (26) - (31). Suppose that at $\bar{x} = (\bar{u}, \bar{t})$, $\dot{u} = 0$. Then:

$$a) \quad \left[\frac{\partial \dot{u}}{\partial u} \right]_{\bar{x}} < 0$$

Proof: Equations (26), (29) and (30) are formally identical with equations (5), (8) and (9). Taking into account (25) and that

$$\text{sign } \frac{dt}{d\omega} = \text{sign } \frac{dk}{d\omega},$$

a simple relabelling of symbols in A-1 and A-2 of the Appendix yields the result.

$$b) \quad \left[\frac{\partial \dot{t}}{\partial t} \right]_{\hat{x}} < 0$$

Proof: Since at $\hat{x}$, $i = 0$,

$$\text{sign} \left[\frac{\partial i}{\partial t} \right]_{\hat{x}} = \text{sign} \left[\frac{\partial (i/i)}{\partial t} \right]_{\hat{x}} = \text{sign} \{ -(1-u)f'_2(i_2) \frac{\partial i_2}{\partial t} \}$$

by differentiation of (31). By A-1 of the Appendix, after relabelling,

$$\frac{\partial i_2}{\partial t} > 0. \text{ Therefore } \left[\frac{\partial i}{\partial t} \right]_{\hat{x}} < 0 \quad \text{q.e.d.}$$

Combining a), b) and Lemma 1 we may state (recall that $i = 1/k$) the following.

Proposition 3:

If the steady state $\hat{k}$ corresponding to the model (1) with a uniform rate of savings and instantaneous shiftability of capital is locally stable, then the steady state $(u(\hat{k}), \hat{k})$ corresponding to the model (26) - (31) is locally stable.

Therefore, using the results of Uzawa [8] and Drandakis [1], this will be the case if

- 1) $\sigma_1(\hat{x}) \geq 1$ or
- 2) $k_1(\hat{x}) \geq k_2(\hat{x})$

B) Classical Savings Assumption

For given (u, i) the momentary equilibrium of the economy is determined by

$$ui_1 + (1-u)i_2 = i \quad (32)$$

$$uf_1(i_1) = f'_1(i_1)i \quad (33)$$

$$f'_1(i_1) = pf'_2(i_2) \quad (34)$$

As in Part I, (32) - (34) have a unique solution in i_1 and i_2 . The dynamic path is described by (30) - (31).

When $\dot{u} = 0$, $\bar{\omega}_1 = \bar{\omega}_2 = \bar{\omega}$ and we can define, manipulating (32) - (33), $i = \bar{\varphi}(\bar{\omega})$. If at $\bar{x}$, $\dot{u} = 0$, then:

$$a) \quad \left[\frac{\partial \dot{u}}{\partial u} \right]_{\bar{x}} < 0 \quad \text{if} \quad [\bar{\varphi}'(\bar{\omega})]_{\bar{x}} > 0.$$

The proof follows step by step⁸ the proof given in Part I and A-5 and A-6 of the Appendix. We note that in this case it is σ_2 (not σ_1) which plays the crucial role in determining the sign of $\left[\frac{du}{d\omega} \right]_{\dot{u}=0}$. If $(\hat{u}, \hat{i}) \equiv \hat{x}$ is a steady state of the economy, relying again on the formal analogy of the system (30) - (34) with the system (9) - (10) and (19) - (21), we conclude

$$b) \quad \left[\frac{\partial i}{\partial i} \right]_{\hat{x}} < 0$$

Combining a), b) and Lemma 1 and taking into account that $\text{sign } \bar{\varphi}'(\bar{\omega}) = \text{sign } \varphi'(\omega)$, we can state the following proposition.

Proposition 4:

If the steady state $\hat{k}$ corresponding to the model (1) with classical savings assumption and complete and instantaneous shiftability of capital is locally stable, the steady state $(u(\hat{k}), \hat{k})$ corresponding to the model (30) - (34) is locally stable.

Therefore, the steady state will be locally stable if

- 1) $\sigma_1(\hat{x}) + \sigma_2(\hat{x}) \geq 1$ or
- 2) $k_1(\hat{x}) \geq k_2(\hat{x})$

As in Part I for c high enough the steady states of both models are locally stable if they are so in the I.M. Applying Lemma 2 one may conclude the corresponding global results.

8. After substituting σ_2 for σ_1 .

PART III

In this part we will combine the analysis of the two preceding parts. We will assume that both capital and labor do not move instantaneously between sectors. Therefore the capital-labor ratio k and the proportions of labor and capital in the first sector, m and u respectively, are treated as state variables.

The results will formalize some aspects of the common contention⁹ that labor adjusts quickly relative to capital which, in turn, has a high speed of adjustment relative to the pace of capital accumulation. In order not to be repetitive we consider only the case with uniform saving ratio.

Let $x \equiv (t, u, m)$ be state variables, $(t = \frac{1}{k})$. Given x the momentary equilibrium is determined by:

$$t_1 = t \frac{m}{u} \quad ; \quad t_2 = t \frac{1-m}{1-u} \quad (35)$$

$$\text{suf}_1(t_1) = p(1-s)(1-u)f_2(t_2) \quad (36)$$

As before we assume $0 < m_0 < 1$, $0 < u_0 < 1$. The dynamics of the model is given by:

$$\dot{t} = n t - (1 - u) f_2 t \quad (37)$$

$$\dot{u} = c_1 H_1(t, u, m), \quad c_1 > 0, \quad \text{sign } \dot{u} = \text{sign } \{r_1 - r_2\} \quad (38)$$

$$\dot{m} = c_2 H_2(t, u, m), \quad c_2 > 0, \quad \text{sign } \dot{m} = \text{sign } \{w_1 - w_2\} \quad (39)$$

We assume the H_i 's are continuously differentiable and that the system (37) - (39) has a unique and continuous solution $x(t)$ for given

9. See, for example, Inada [4].

x_0 . Inada conditions will guarantee that $x(t)$ stays in the feasible region. The steady states of this model, $\hat{x}$, coincide with the ones of the I.M.

Observe that $\frac{\partial p}{\partial m} > 0$ (from (36)).

Assume that the Inada conditions hold. Then from (35) - (36) and (39):

a) If $\bar{i}$, $\bar{u}$ are given ($0 < \bar{u} < 1$) there is a unique $\bar{m}$ such that $\dot{m}(\bar{x}) = 0$. Moreover $\left[\frac{\partial \dot{m}}{\partial m} \right]_{\bar{x}} < 0$.

Given a steady state $\hat{x}$ let $\Delta_{\hat{x}} \subset R^3$ represent a compact rectangle containing $\hat{x}$ and such that if $x \in \Delta_{\hat{x}}$ then $x_1 > 0$, $0 < x_2 < 1$, $0 < x_3 < 1$.

We can state the following.

Proposition V:

Suppose $\hat{x} = (\hat{k}, u(\hat{k}), m(\hat{k}))$ is a globally stable steady state of the I.M. For every $\Delta_{\hat{x}}$ there is $M > 0$ such that if $c_1 > M$ and $c_2 > c_1 M$, $\hat{x}$ is a globally stable steady state of (37) - (39) in the region $\Delta_{\hat{x}}$ (i.e., if $x_0 \in \Delta_{\hat{x}}$, $\lim_{t \rightarrow \infty} x(t) = \hat{x}$, where $x(t)$ is the solution corresponding to x_0).

Proof: Condition a) above and Aa) in Part II guarantee that all the assumptions of Lemma 2 are fulfilled.

Referring to local stability a stronger result may be given.

Proposition VI:

If $\hat{k}$ is a locally stable steady state of the I.M., there is $\bar{c}_2 \geq 0$ such that if $c_2 > \bar{c}_2$ the steady state $(\hat{k}, u(\hat{k}), m(\hat{k}))$ is locally stable in the system (37) - (39).

Proof: The system (37) - (39) plays the same role as system [B] in the Introduction. By Part II local stability in the I.M. yields local stability of the model with non-instantaneous adjustment of capital, which corresponds to system [B¹] in the Introduction. By a) above Lemma 3 can be applied and the conclusion follows.

PART IV

We will assume that the wage rental ratio is rigid in the short run and that its rate of adjustment is inversely related to the quantity of labor unemployed and directly related to the quantity of unused capital. The state variables are k and ω . This model has been briefly considered by Uzawa [7] whose general approach we use.

A) Uniform Saving Ratio

Let $m_1 = \frac{L_1}{L}$; given $0 < \omega < \infty$, $k > 0$ the momentary equilibrium is determined by:

$$s (m_1 f_1 + m_2 f_2) = p m_2 f_2 \quad (40)$$

$$f_i = (\omega + k_i) f'_i \quad i = 1, 2 \quad (41)$$

$$f'_1 = p f'_2 \quad (42)$$

$$k_1 m_1 + k_2 m_2 \leq k \quad (43)$$

$$m_1 + m_2 \leq 1 \quad (44)$$

The model is closed when either (43) or (44) holds with equality.

(40) - (42) reduces to:

$$\frac{1-s}{s} \frac{\omega + k_2(\omega)}{\omega + k_1(\omega)} = \frac{m_1}{m_2} \quad (45)$$

Let $k(\omega)$ designate the -- uniquely determined -- capital labor ratio that makes expressions (43) and (44) to hold with equality, $\frac{dk}{d\omega} > 0$ (Uzawa [8]). If $k < k(\omega)$ there is labor unemployment ($m_1 + m_2 < 1$). If $k > k(\omega)$ there is capital underutilization ($m_1 k_1 + m_2 k_2 < k$).

The dynamic path of the model is described by:

$$\dot{\omega} = c H(\omega, k) ; \quad \text{sign } \dot{\omega} = \text{sign}(k - k(\omega)) , \quad c > 0 \quad (46)$$

$$\dot{k} = m_2 f_2(k_2) - nk \quad (47)$$

We assume (46) - (47) have a unique solution for every (ω_0, k_0) , which is continuous with respect to initial conditions.

Define $\psi(\omega) = \frac{1}{n} s f_2'(k_2(\omega)) (k(\omega) + \omega)$.

When $k \geq k(\omega)$ $n \psi(\omega) = m_2 f_2$ and (47) may be expressed as (see Uzawa [8]):

$$\dot{k} = n \psi(\omega) - nk \quad \text{if} \quad k \geq k(\omega) \quad (48)$$

$$\dot{k} = n \psi(\omega) \frac{k}{k(\omega)} - nk \quad \text{if} \quad k < k(\omega) \quad (49)$$

The steady states of this model are the same that the steady states of a model with instantaneous adjustment of ω . Let $\hat{x}$ be a steady state of (46) - (47), then:

$$a) \quad \left[\frac{\partial \dot{\omega}}{\partial \omega} \right]_{\hat{x}} < 0 . \quad \text{This follows from the fact that } \frac{dk(\omega)}{d\omega} > 0 .$$

Since the system (46) - (47) is not differentiable at $\hat{x}$, in order to study local stability at $\hat{x}$ we will rely on phase-diagram analysis. It can be drawn as in Figure 1. In a steady state $\psi(\hat{\omega}) = k(\hat{\omega})$. We observe at once that if $\psi(\hat{\omega})$ intersects $k(\hat{\omega})$ from above the corresponding steady state will be locally stable for c high enough, and for any $c > 0$ if $\left[\frac{d \psi(\omega)}{d\omega} \right]_{\hat{\omega}} > 0$.

If $\hat{x}$ is locally stable in the I.M., (48) (for $k = k(\omega)$) yields that $\psi(\omega)$ intersects $k(\omega)$ from above.

Figure 1

We can therefore state the following Proposition.

Proposition VII:

A steady state $(\hat{\omega}, \hat{k})$ of the model (40) - (47) is locally stable if:

$$1) \quad \sigma_1(\hat{\omega}, \hat{k}) \geq 1 \quad \text{and} \quad \sigma_2(\hat{\omega}, \hat{k}) \geq 1 \quad \text{or}$$

$$ii) \quad k_1(\hat{\omega}, \hat{k}) \geq k_2(\hat{\omega}, \hat{k}) \quad \text{and} \quad \sigma_1(\hat{\omega}, \hat{k}) \geq 1.$$

Proof: Both i) and ii) are sufficient conditions for stability of the

I.M. We need only prove $\left[\frac{d\psi(\omega)}{d\omega} \right]_{\hat{\omega}} > 0$.

$$i) \quad \text{If} \quad \sigma_1 \geq 1 \quad \frac{dk_1(\omega)}{d\omega} \geq \frac{k_1(\omega)}{\omega} \quad (50)$$

(See Drandakis [1])

The logarithmic differentiation of $\psi(\omega)$ gives:

$$\frac{1}{\psi} \frac{d\psi}{d\omega} = - \frac{1}{k_1(\omega) + \omega} + \frac{\frac{dk_1(\omega)}{d\omega} + 1}{k_1(\omega) + \omega} > 0$$

by (50), completing the proof of i).

ii) $k_1 \geq k_2$ and $\sigma_1 > 1$. $\psi(\omega)$ intersects $k(\omega)$ from above so that in a small enough neighborhood around $\hat{\omega}$, if $\omega' < \hat{\omega}$ then $\psi(\omega') > k(\omega')$ and $\sigma_1(\omega') > 1$. Therefore

$$n \psi(\omega') = m_2 f_2(k_2) \quad \text{and} \quad m_1 + m_2 = 1 \quad (51)$$

Differentiating (48) taking (51) into account we find:

$$\left[\frac{\partial m_2}{\partial \omega} \right]_{\omega'} = \frac{\left[s(1 - m_2) \left(1 + \frac{dk_1}{d\omega} \right) - (1 - s) m_2 \left(1 + \frac{dk_2}{d\omega} \right) \right]}{(1 - s)(k_2 + \omega) + s(k_1 + \omega)} \Big|_{\omega'}$$

The numerator reduces to (after using (48)):

$$\frac{1 + \frac{\sigma_1 k_1}{\omega'}}{k_1 + \omega'} - \frac{1 + \frac{\sigma_2 k_2}{\omega'}}{k_2 + \omega'}$$

which is positive if $\sigma_1 > 1$ and $\sigma_2 \leq 1$.

In this case,¹⁰ since:

$$\left[\frac{n \cdot d\psi}{d\omega} \right]_{\hat{\omega}} = \left[\frac{d m_2 f_2}{d \omega} \right]_{\hat{\omega}} = \lim_{\omega' \rightarrow \hat{\omega}} \left[f_2 \frac{\partial m_2}{\partial \omega} + f_2' \frac{d k_2}{d \omega} \right]_{\omega'}$$

10. By the assumption in (1) $\psi(\omega)$ is continuously differentiable.

where the minus sign in the middle term means left hand derivative. Therefore

$\left[\frac{d}{d\omega} \right] \hat{\omega} > 0$ follows. The case $\sigma_1 \geq 1$ and $\sigma_2 \geq 1$ has been covered in i). The $\sigma_1 = 1$ case follows by a judicious use of ε 's and δ 's in the argument above.

B) Classical Savings Assumption

The uniquely determined momentary equilibrium is given by

$$f_1' = pf_2' ; \quad \frac{f_1}{f_1'} = \omega + k_1 \quad i = 1, 2 \quad (52)$$

$$pm_2 f_2(k_2) = f_1(k_1) \bar{k} , \text{ where } \bar{k} = k_1 m_1 + k_2 m_2 \quad (53)$$

$$m_1 k_1 + m_2 k_2 \leq k \quad (54)$$

$$m_1 + m_2 \leq 1 \quad (55)$$

Equations (52) - (53) reduce to

$$m_1/m_2 = \omega/k_1 (\omega) \quad (56)$$

Let $k(\omega)$ be the uniquely determined capital-labor ratio corresponding to the full employment of labor and capital at the wage-rental ratio ω .

The dynamic path is given by (46) - (47).

If $k \geq k(\omega)$ labor is fully employed. From (56) and $m_1 + m_2 = 1$, we get $m_2 = \frac{k_1(\omega)}{k_1(\omega) + \omega}$. Substituting in (47) we obtain

$$\dot{k} = \frac{k_1(\omega) f_2(k_2(\omega))}{k_1(\omega) + \omega} - nk \equiv n \Phi(\omega) - nk , k \geq k(\omega) \quad (57)$$

Similarly,

$$\dot{k} = \frac{k f_2(k_2(\omega))}{\omega + k_2(\omega)} - nk , \quad k < k(\omega) \quad (58)$$

The steady state $\hat{x}$ is unique. From (22) and (57) follows that $\Phi'(\hat{\omega}) < k'(\hat{\omega})$.

When $\left[\frac{dk(\omega)}{d\omega} \right]_{\hat{x}} > 0$ the phase diagram -- in the vicinity of $\hat{x}$ -- has the same form as that corresponding to the steady states of Figure 1.

Figure 2

We can state the following proposition.

Proposition VIII:

The steady state $(\hat{\omega}, \hat{k})$ corresponding to the model (46) - (47), (52) - (56), is locally stable if $\sigma_1(\hat{\omega}, \hat{k}) \geq 1$.

Proof: If $\sigma_1(\hat{\omega}, \hat{k}) \geq 1$, $\left[\frac{dk(\omega)}{d\omega} \right]_{\hat{x}} > 0$ (see Drandakis [1]).

Following the same steps as in the proof of Proposition V, we only need to show that the slope of $\Phi(\omega)$ (defined by (57)) is positive at $\hat{\omega}$. f_2 and k_1 are bounded in a vicinity of $\hat{\omega}$.

From (57), if $\omega^* < \hat{\omega}$,

$$\text{sign } \left[\frac{dk}{d\omega} \right]_{\omega^*} = \text{sign } \left[f_2' \frac{dk_2}{d\omega} \left(1 + \frac{\omega}{k_1} \right) - f_2 \frac{d(\omega/k_1)}{d\omega} \right]_{\omega^*}$$

and for arbitrarily small $\varepsilon > 0$ we can find ω^* such that

$$\left[\frac{d(\omega/k_1)}{d\omega} \right]_{\omega^*} = \frac{1 - \sigma_1(\omega^*)}{k_1} < \varepsilon \quad \text{q.e.d.}$$

APPENDIX

A-1

Differentiating totally the system (5) - (7) and using (8) we find:

$$\begin{aligned}\frac{\partial k_1}{\partial k} &= \frac{1}{\Delta} \{-pf_2'' (1-m)(1-s) f_2 + (1-m)(1-s) p(f_2')^2\} = \\ &= \frac{1}{A} \frac{1 + \frac{\omega_2}{\sigma_2 + k_2}}{\omega_2 + k_2}\end{aligned}$$

$$\begin{aligned}\frac{\partial k_2}{\partial k} &= \frac{1}{\Delta} \{-f_1'' (1-m)(1-s) f_2 + smf_1' f_2'\} = \\ &= \frac{1}{A} \frac{1 + \omega_1/\sigma_1 k_1}{\omega_1 + k_1}\end{aligned}$$

$$\text{Where } \Delta = \begin{vmatrix} m & (1-m) & 0 \\ f_1'' & -pf_2'' & -f_2' \\ -smf_1' & (1-m)(1-s)pf_2' & (1-m)(1-s)f_2 \end{vmatrix} > 0$$

$$\text{and } A = \frac{\Delta}{f_2' f_1' sm} = \frac{m}{\sigma_2 k_2} \left\{ \frac{\omega_2 + \sigma_2 k_2}{\omega_2 + k_2} \right\} + \frac{1-m}{\sigma_1 k_1} \left\{ \frac{\omega_1 + \sigma_1 k_1}{\omega_1 + k_1} \right\}$$

$$\text{taking into account that } -\frac{f_1''(k_1)}{f_1'(k_1)} = \frac{\omega_1}{\sigma_1 k_1} \frac{1}{\omega_1 + k_1}$$

$$\text{Clearly } \frac{\partial k_1}{\partial k} > 0, \quad \frac{\partial k_2}{\partial k} > 0.$$

A-2

Substituting the expressions from A-1, we get:

$$\left[\frac{d\omega_1}{dk_1} \frac{\partial k_1}{\partial k} - \frac{d\omega_2}{dk_2} \frac{\partial k_2}{\partial k} \right]_{\omega_1 = \omega_2} = \frac{1}{A} \left\{ \frac{\omega}{\sigma_1 k_1} \left(-\frac{f_2''}{f_2'} + \frac{f_2'}{f_2} \right) - \frac{\omega}{\sigma_2 k_2} \left(-\frac{f_1''}{f_1'} + \frac{f_1'}{f_1} \right) \right\}$$

Manipulating this expression reduces to:

$$\frac{1}{A} \left\{ \frac{\omega}{\sigma_1 k_1 \sigma_2 k_2} \frac{\omega + \sigma_2 k_2}{\omega + k_2} - \frac{\omega}{\sigma_1 k_1 \sigma_2 k_2} \frac{\omega + \sigma_1 k_1}{\omega + k_1} \right\} ,$$

which establishes the claim.

A-3

Taking the expression for $\frac{\partial k_2}{\partial k}$ in A-1 and substituting A into it we have:

$$\frac{\partial k_2}{\partial k} = \frac{1}{\frac{mk_1}{k_2} \frac{\omega_2/\sigma_2 + k_2}{\omega_1/\sigma_1 + k_1} \frac{\omega_1 + k_1}{\omega_2 + k_2} + (1 - m)}$$

In $\hat{x}$, $\omega_1 = \omega_2 = \omega$, yielding:

$$\frac{\partial k_2}{\partial k} = \frac{\omega + k_2}{\sigma(m, k) mk_1 + (1 - m) k_2 + (1 - m) \omega}$$

A-4

This is immediate. Differentiating totally (19) and (20),

$$\frac{\partial k_2}{\partial k} = \frac{1}{\Delta} f'_2(k_2) ,$$

where

$$\Delta = (1-m)f'_2 - f''_2 k > 0 .$$

$$\frac{\partial k_1}{\partial k} = \frac{1}{\Delta} (-f''_2 k)$$

A-5

Case $\sigma_1(\hat{x}) = 1$.

Differentiating totally (19) and (20):

$$\frac{\partial k_2}{\partial m} = \frac{1}{\Delta} m f_2(k_2)$$

$$\begin{aligned}
 \frac{\partial k_1}{\partial m} &= \frac{1}{\Delta} \{ (k_2 - k_1) ((1-m)f_2' - f_2''k) - (1-m)f_2 \} = \quad , \text{ using (20)} \\
 &= \frac{f_2'}{\Delta} \{ (k_2 - k_1) [(1-m) + \frac{f_2'}{f_2} \frac{k\omega_2}{k_2\sigma_2}] - k \} = \quad , \text{ using (20)} \\
 &= \frac{f_2'}{\Delta} \{ (k_2 - k_1) (1-m) [1 + \frac{\omega_2}{k_2\sigma_2}] - k \} = \quad , \text{ using (19)} \\
 &= \frac{f_2'}{\Delta} \{ \frac{k - k_1}{k_2} \frac{\omega_2}{\sigma_2} - k_1 \} = \quad , \text{ using (19) and (23)} \\
 &= \frac{k_1 f_2'}{\Delta} \{ \frac{k_2 - k_1}{k_2} \frac{m}{\sigma_2} - 1 \} .
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \text{sign} \left[\frac{\partial \dot{m}}{\partial m} \right]_{\hat{x}} &= \text{sign} \left[\frac{d\omega_1}{dk_1} \frac{\partial k_1}{\partial m} - \frac{d\omega_2}{dk_2} \frac{\partial k_2}{\partial m} \right]_{\hat{x}} = \\
 &= \text{sign} \left[\frac{\omega k_1}{k_1} \left\{ \frac{k_2 - k_1}{k_2} \frac{m}{\sigma_2} - 1 \right\} - \frac{\omega}{\sigma_2 k_2} m \frac{f_2'}{f_2} \right]_{\hat{x}} = \\
 &= \text{sign} \left[\frac{m(k - k_1)}{(1-m)\sigma_2 k_2} - \frac{mk}{(1-m)\sigma_2 k_2} - 1 \right]_{\hat{x}} < 0 ,
 \end{aligned}$$

where use has been made of (19), (20) and (23).

A-6

$$\text{sign} \left[\frac{\partial \dot{m}}{\partial m} \right]_{\hat{x}} = \text{sign} \left[\frac{d\omega_1}{dk_1} \frac{\partial k_1}{\partial k} - \frac{d\omega_2}{dk_2} \frac{\partial k_2}{\partial k} \right]_{\hat{x}} =$$

Substituting from A-4,

$$\begin{aligned}
 &= \text{sign} \left[\frac{d\omega_1}{dk_1} k f_2'' - \frac{d\omega_2}{dk_2} m f_2' \right]_{\hat{x}} = \\
 &= \text{sign} \left[\frac{\omega}{\sigma_1 k_1} \frac{\omega}{\sigma_2 k_2} (1-m) - \frac{\omega}{\sigma_2 k_2} m \right]_{\hat{x}} = \text{sign} \left[\frac{\omega}{\sigma_1 k_1} (1-m) - m \right]_{\hat{x}}
 \end{aligned}$$

But by (23),

$$\frac{\omega}{k_1} = \frac{m}{1-m} .$$

Therefore

$$\text{sign} \left[\frac{\partial \dot{m}}{\partial k} \right]_{\hat{x}} = \text{sign} [(1/\sigma_1)m - m]_{\hat{x}} = \text{sign} (1 - \sigma_1(\hat{x})) .$$

REFERENCES

- [1] Drandakis, E., "Factor Substitution in the Two-Sector Growth Model", REStud, 30, (October 1963), 217-228.
- [2] Gantmacher, The Theory of Matrices, (Second Volume), Chelsea Publishing House, New York, 1964.
- [3] Inada, K., "On a Two-Sector Model of Economic Growth: Comments and a Generalization", REStud, 30, (June 1963), 119-127.
- [4] Inada, K., "Investment in Fixed Capital and the Stability of Growth Equilibrium", REStud, 33, (January 1966), 19-30.
- [5] McFadden, D., "On Hicksian Stability" in Value, Capital and Growth, P. Wolfe ed. Oxford University Press, 1966.
- [6] Shell, K., M. Sidrauski, and J. E. Stiglitz, "Capital Gains, Income and Savings", REStud, 36, (January 1969), 15-26.
- [7] Uzawa, H., "On a Two Sector Model of Economic Growth: I", REStud, 29, (October 1961), 40-47.
- [8] Uzawa, H., "On a Two Sector Model of Economic Growth: II", REStud, 30, (June 1963), 105-118.