Holography and the speed of sound

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Holography and hydrodynamics

- Experimental motivation: Relativistic Heavy Ion Collisions (v_2)
- Traditionally: kinetic description $\Rightarrow$ hydrodynamics
- Discovery of sQGP: hydrodynamics but no kinetic description
  - i.e QFT $\Rightarrow$ hydrodynamics.
- Strong coupling regime of some susy gauge theories can be studied using AdS/CFT (holographic) correspondence.
  - i.e., instead of QFT $\Rightarrow$ kinetic description (Boltzmann) $\Rightarrow$ hydrodynamics,
    QFT $\Rightarrow$ holographic description $\Rightarrow$ hydrodynamics
- Kinetic coefficients computed in $\mathcal{N} = 4$ sYM: η, also τ, κ, λ_1, λ_2, λ_3 – 2nd order.
- Bulk visc. ζ in some non-conformal susy theories.

Speed of sound

$$c_s^2 = \frac{dp}{d\epsilon}$$

Plays an important role in hydrodynamics.

- Example: Bjorken boost-inv. expansion: $T \sim \tau^{-c_s^2}$.
- Mach cone angle
- $(c_s^2 - 1/3)$ – a measure of the non-conformality of a relativistic system.

- As $T \to \infty$, $p \sim T^4$ by dimension, i.e., $\epsilon = T dp/dT - p \to 3p$ and $c_s^2 \to 1/3$.
- Question: Does c_s^2 always approach $1/3$ from below?
  - Examples: QCD ($N_f = 0$)

$$c_s^2 \simeq \frac{1}{3} + \frac{5N_c}{36\pi}\,\beta(\alpha) < \frac{1}{3}$$

follows from asymptotic freedom: $\beta < 0$.

- Strongly interacting systems?

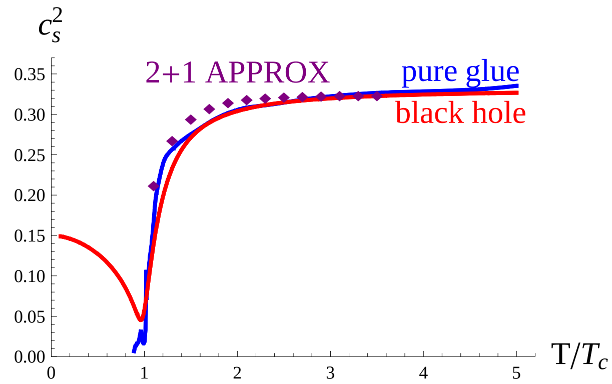
More $c_s^2 < 1/3$ examples

- $\mathcal{N} = 2^*$ susy Yang-Mills plasma (Benincasa, Buchel, Starinets)

$$c_s = \frac{1}{\sqrt{3}}\left(1 - \frac{\left[\Gamma\left(\frac{3}{4}\right)\right]^4}{3\pi^4}\ \left(\frac{m_f}{T}\right)^2 - \frac{1}{18\pi^4}\ \left(\frac{m_b}{T}\right)^4 + \cdots\right).$$

- Lattice QCD (Karsch et al) and holographic mimicking (Gubser-Nellore):

lattice data vs. black holes



- Not a universal bound!
  Counterexample QCD + μ_{isospin}, low T:

$$c_s^2 = \frac{1-(m_\pi/\mu)^4}{1+3(m_\pi/\mu)^4} \to 1.$$

$(m_\pi = 0,\, n \sim \mu^1)$

- How universal is the result $c_s^2 < 1/3$? What are the prerequisites?

Non-conformality

- Introduce $\theta = \epsilon - 3p$ (as measure of non-conformality)

and $w = \epsilon + p$ (heat function, proxy to T, $dw/dT = c_v + s > 0$).

Express

$$\epsilon = (3w + \theta)/4; \qquad p = (w - \theta)/4.$$

$$c_s^2 = \frac{dp}{d\epsilon} = \frac{1 - d\theta/dw}{3 + d\theta/dw}\,.$$

Then

$$c_s^2 < 1/3 \qquad \Leftrightarrow \qquad d\theta/dw > 0$$

- Compare/contrast
  - Bjorken argues: $p/T^4 \sim \#$d.o.f., thus

$$\frac{d}{dT}\left(\frac{p}{T^4}\right) = \frac{\theta}{T^5} > 0$$

  - Appelquist-Cohen-Schmaltz-Shrock:

$$\left[\frac{p}{T^4}\right]_{\mathrm{IR}}^{UV} = \int_{\mathrm{IR}}^{\mathrm{UV}} \frac{dT}{T^5}\,\theta > 0$$

Bjorken on $\theta > 0$

Other general features of the hydrodynamic expansion follow from the positivity condition on the trace of the energy-momentum tensor

$$T^{\mu}_{\mu} \geq 0 \tag{33}$$

which is true under quite general circumstances.[9]

[9]This is far from guaranteed, but is always true when the fluid can be considered a collection of noninteracting quanta.

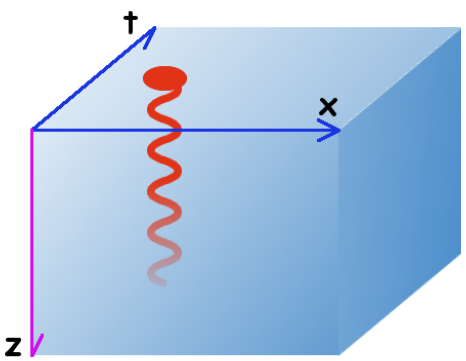
Holographic model

- Bottom-up approach: Take a "minimal" set of operators: $T^{\mu\nu}$ and a scalar $\mathcal{O}$ on the r.h.s. of scale (trace) anomaly eqn. (in QCD $\theta = \frac{\beta}{8\pi\alpha^2}\langle \mathrm{tr}\, F^2 \rangle$) to describe conformality violation.

- Holographic duality requires 5d fields: $g_{\mu\nu}$ (dual to $T^{\mu\nu}$) and ϕ (dual to $\mathcal{O}$).

$$S_5 = \frac{1}{2\kappa^2}\left[\int_M d^5x\,\sqrt{-g}\left(R - V(\phi) - \frac{1}{2}(\partial\phi)^2\right) - 2\int_{\partial M} d^4x\,\sqrt{-\gamma}\,K\right],$$

with *generic* scalar potential $V(\phi)$ (tuned by Gubser-Nellore).



- $1/\kappa^2 \sim$ # d.o.f.

- Recipe for calculating a correlator of, e.g., $T^{\mu\nu}$:

Vary $z = 0$ boundary value, $g^{\mu\nu}(x, 0)$, then

$$\langle T^{\mu\nu}(x)\rangle \sim \frac{\delta S}{\delta g_{\mu\nu}(x,0)}.$$

Background solution

- Most general (up to coordinate transforms):

$$ds^2 = \frac{1}{z^2}\left(-f(z)dt^2 + d\vec{x}^2\right) + e^{2B(z)}\frac{dz^2}{z^2 f(z)}.$$

where f, B, and ϕ obey

$$\dot{B} = -\tfrac{1}{6}\,\dot{\phi}^2,$$

$$\ddot{f} = \left(4 + \dot{B}\right)\dot{f},$$

$$-6\dot{f} + f\left(24 - \dot{\phi}^2\right) + 2e^{2B}\,V(\phi) = 0,$$

$$\ddot{\phi}f + \dot{\phi}\left(\dot{f} - f(4 + \dot{B})\right) - e^{2B}\,V'(\phi) = 0,$$

where a dot denotes a $\log z$ derivative, e.g., $\dot{\phi} = z\,d\phi/dz$, while $V' = dV/d\phi$.

- The only boundary conditions at $z = \varepsilon \to 0$:

$$f(\varepsilon) = 1 \quad \text{and} \quad \phi(\varepsilon) = c\,\varepsilon^{\Delta_-}\,.$$

where c is the source for $\mathcal{O}$, i.e., $[c] = \Delta_-;\quad [\mathcal{O}] = \Delta_+;\quad \Delta_+ + \Delta_- = 4.$

Background solution

- Equation $\ddot{f} = \left(4 + \dot{B}\right) \dot{f}$ can be integrated once:

$$\dot{f} = -w z^4 e^B .$$

integration constant $w > 0$ to have a horizon $f(z_H) = 0$.

- Eq. $\ddot{\phi}\, f + \ldots$ linearizes near $z = 0$ and is solved by (with $\phi(\varepsilon) = c\, \varepsilon^{\Delta_-}$)

$$\phi(z) \to (c - d\, \varepsilon^{\Delta_+ - \Delta_-})\, z^{\Delta_-} + d\, z^{\Delta_+} + \ldots ,$$

where $\Delta_\pm = 2 \pm \sqrt{4 + m^2}$, with $m^2 \equiv V''(0)$.

d is fixed by condition $|\phi(z_H)| < \infty$.

Substituting this solution back into S_5 one finds (Klebanov-Witten):

$$\langle \mathcal{O} \rangle = \partial S_5 / \partial c = -d\, (\Delta_+ - \Delta_-).$$

- Two-parameter (w, c) family of solutions.

Scaling $z \to z/\lambda$, $c \to c\, \lambda^{\Delta_-}$ and $w \to w\, \lambda^4$ relates different solutions.

One-point functions

- Consider variable boundary condition on the metric

$$g_{\mu\nu}(\varepsilon) = h_{\mu\nu}\,\varepsilon^{-2}\,.$$

where $h_{\mu\nu}$ is the metric in 4D theory (in general, curved).

- Holographic correspondence then gives us

$$\langle T^{\mu\nu}\rangle = -2\,\frac{\delta S_5}{\delta h_{\mu\nu}} \qquad \Rightarrow \qquad \langle T^{tt}\rangle = -\frac{6}{\varepsilon^4}e^{-B(\varepsilon)}; \qquad \langle T^{xx}\rangle = w - \langle T^{tt}\rangle.$$

and (subtracting T-indep. constant) $\quad (\, B(\varepsilon) \leftarrow -6\dot{f} + f\left(24 - \dot{\phi}^2\right) + 2e^{2B}\,V(\phi) = 0\,)$

$$\epsilon = \frac{3}{4}\,w - \frac{1}{4}\,c\,d\,\Delta_-\,(\Delta_+ - \Delta_-) \quad \text{and} \quad p = \frac{1}{4}\,w + \frac{1}{4}\,c\,d\,\Delta_-\,(\Delta_+ - \Delta_-)\,.$$

or

$$\theta = \epsilon - 3p = -c\,d\,\Delta_-\,(\Delta_+ - \Delta_-) = c\,\langle\mathcal{O}\rangle\,\Delta_-\,,$$

- Recognizable and derivable directly on the field-theory side.

Speed of sound

- Recall

$$c_s^2 = \frac{dp}{d\epsilon} = \frac{1 - d\theta/dw}{3 + d\theta/dw}\,.$$

with $\theta = \Delta_- \, c \, \langle \mathcal{O} \rangle$ and $\langle \mathcal{O} \rangle = -d \, (\Delta_+ - \Delta_-)$, and no approximation,

$$c_s^2 = \frac{dp}{d\epsilon} = \frac{1 + c \, \Delta_- \, (\Delta_+ - \Delta_-) \, (\partial d/\partial w)}{3 - \; c \, \Delta_- \, (\Delta_+ - \Delta_-) \, (\partial d/\partial w)}\,.$$

High temperatures

At large w (i.e., high T), z_h is small and $B \approx 0$:

$$f(z) = 1 - w\,z^4/4.$$

$$\left(1 - \frac{1}{4}\,wz^4\right)\phi'' - \left(\frac{3}{z} + \frac{wz^4}{4}\right)\phi' - \frac{m^2}{z^2}\,\phi = 0.$$

$$\begin{aligned}\phi(z) = c\,z^{\Delta_-}\ {}_2F_1\left(\Delta_-/4,\ \Delta_-/4,\ \Delta_-/2,\ wz^4/\,4\right)\\ + d\,z^{\Delta_+}\ {}_2F_1\left(\Delta_+/4,\ \Delta_+/4,\ \Delta_+/2,\ wz^4/4\right),\end{aligned}$$

and $|\phi(z_h)| < \infty$ gives

$$d = -c\,w^{(\Delta_+ - \Delta_-)/4}\,D(\Delta_-)\,,$$

where the function $D(\Delta_-) = 1/D(\Delta_+)$ is

$$D(\Delta_-) = \frac{\pi\,2^{\Delta_-}}{2-\Delta_-}\cot\left(\pi\Delta_-/4\right)\frac{\Gamma(\Delta_-/2)^2}{\Gamma(\Delta_-/4)^4} > 0\,.$$

High temperatures

- Substituting we find our main result

$$c_s^2 = \frac{1}{3} - \frac{1}{9}\, c^2\, w^{-\Delta_-/2}\, \Delta_-(\Delta_+ - \Delta_-)^2 D(\Delta_-) + \ldots < \frac{1}{3}.$$

universally for all $0 < \Delta < 4$.

  - Same result, different approach — Cherman, Cohen, Nellore.

- Dimensionful parameters: c (b.c. on ϕ) and w (integration const).

At high T, dimensionless expansion parameter is c/T^{Δ_-}. High T is small c.

- Origin of the universality:

$z_H \sim w^{-1/4} \to 0$. Thus ϕ remains small $\Rightarrow$ only $V''(0) = \Delta(\Delta - 4)$ matters.

- Why $c_s^2 < 1/3$?

More general argument

- Recall

$$c_s^2 < 1/3 \qquad \Leftrightarrow \qquad d\theta/dw > 0$$

- At high T (small c):

$$\theta = \Delta_-\, c\,\langle\mathcal{O}\rangle = \Delta_-\, c^2\,\chi_{\mathcal{O}} + \dots\,.$$

The susceptibility $\chi_{\mathcal{O}} \equiv \partial\langle\mathcal{O}\rangle/\partial c$ can be related to the effective potential for $\langle\mathcal{O}\rangle$, (in holography $W = -S_5$)

$$\Gamma(\langle\mathcal{O}\rangle) \equiv W(c) + c\langle\mathcal{O}\rangle,$$

as

$$\chi_{\mathcal{O}} = 1/\Gamma''(0).$$

- Stability implies $\Gamma''(0) > 0$, and thus $\chi_{\mathcal{O}} > 0$. This means $\theta > 0$. (cf. Bjorken)

- For $d\theta/dw > 0$ curvature $\Gamma''(0)$ must *decrease* with T. Strange? Opposite to ϕ^4.

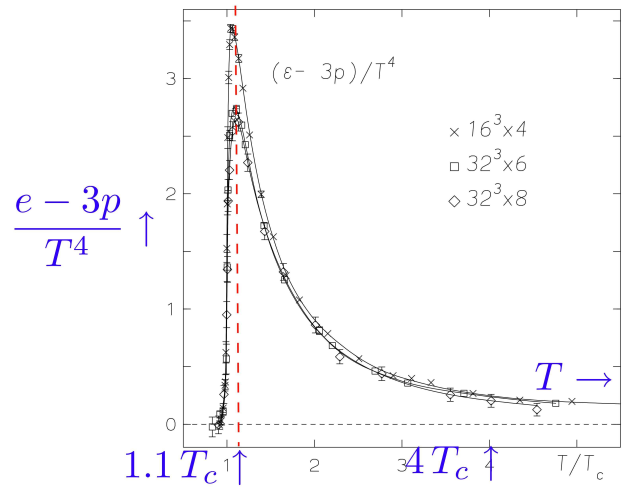
Dimension counting: $[\chi_{\mathcal{O}}] = \Delta_+ - \Delta_- > 0$, which means $\chi_{\mathcal{O}} \sim T^{\Delta_+ - \Delta_-}$ and increases with T.

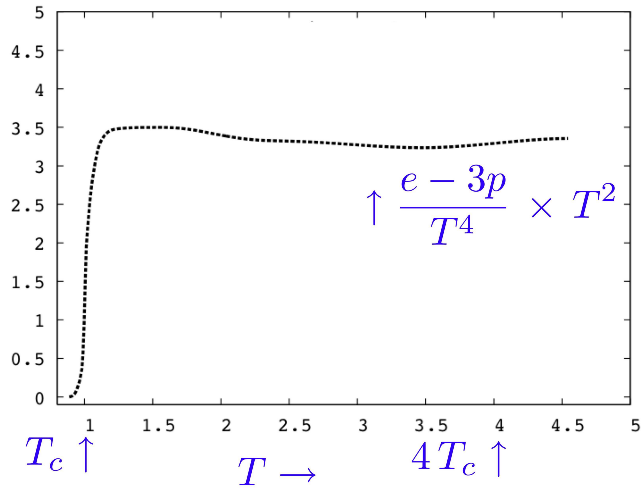
Conclusions and comments

- In a wide class of strongly interacting theories $c_s^2 \to 1/3$ from below.
- In holography, the correction is a universal (negative) function of Δ, times c^2, times $w^{-\Delta_-/2}$ (i.e., $T^{-2\Delta_-}$).
- Can one extend the argument to all T?
- $\mu \neq 0$?

Comments

- Looking at lattice data on YM Pisarski observes: $\theta \sim w^{1/2}$.





This would take $\Delta_- \approx 1$. Is there a $[\mathcal{O}] \approx 3$ operator?

Happy Birthday, Misha!