Quantum strings in $AdS_5 \times S^5$ and gauge-string duality

Arkady Tseytlin

- Review
- Quantum string corrections to dimension of “short” operators
  [R. Roiban, AT, to appear]

Summary: planar $\mathcal{N}$=4 SYM $\lambda = g^2_{\rm YM} N_c$

cusp anomalous dimension

$$f(\lambda \ll 1) = \frac{\lambda}{2\pi^2}\Big[1 - \frac{\lambda}{48} + \frac{11\lambda^2}{2^8 \cdot 45} - (\frac{73}{630} + \frac{4(\zeta(3))^2}{\pi^6})\frac{\lambda^3}{2^7} + ...\Big]$$

$$f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi}\Big[1 - \frac{3\ln 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} - ...\Big]$$

BES integral equation: any number of terms in expansions known

anomalous dimension of Konishi operator

$$\gamma(\lambda \ll 1) = \frac{12\lambda}{(4\pi)^2}\Big[1 - \frac{4\lambda}{(4\pi)^2} + \frac{28\lambda^2}{(4\pi)^4} + [-208 + 48\zeta(3) - 120\zeta(5)]\frac{\lambda^3}{(4\pi)^6} + ...\Big]$$

$$\gamma(\lambda \gg 1) = 2\sqrt{\sqrt{\lambda}}\ \Big[1 + \frac{b}{\sqrt{\lambda}} + O(\frac{1}{(\sqrt{\lambda})^2})\Big]$$

b - leading correction to mass of "lightest" $AdS_5 \times S^5$ string state

higher order terms? integral equation for γ?

AdS/CFT:

progress largely using limited tools of

supergravity + classical probe actions

To go beyond: understand quantum string theory in $AdS_5 \times S^5$

Problems for string theory:

- find spectrum of states:

  energies/dimensions as functions of $\lambda = g^2_{\rm YM} N_c$
- construct vertex operators: closed and open (?) strings
- compute their correlation functions – scattering amplitudes
- compute expectation values of Wilson loops
- generalizations to simplest less supersymmetric cases

  ....

“tree-level” $AdS_5 \times S^5$ superstring = planar $\mathcal{N} = 4$ SYM

Recent remarkable progress in quantitative understanding

interpolation from weak to strong ‘t Hooft coupling

based on/checked by perturbative gauge theory (4-loop in λ)

and perturbative string theory (2-loop in $\frac{1}{\sqrt{\lambda}}$) “data”

and (strong evidence of) exact integrability

string energies = dimensions of local $\mathrm{Tr}(...)$ operators

$$E(\sqrt{\lambda}, C, m, ...) = \Delta(\lambda, C, m, ...)$$

C - “charges” of $SO(2,4) \times SO(6)$: $S_1, S_2;\ J_1, J_2, J_3$

m - windings, folds, cusps, oscillation numbers, ...

Operators: $\mathrm{Tr}(\Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3} D_+^{S_1} D_\perp^{S_2} ... F_{mn} ... \Psi ...)$

Solve supersymmetric 4-d CFT

= Solve string in curved R-R background (2-d CFT):

compute $E = \Delta$ for any λ (and any C,m)

Problem: perturbative expansions are opposite

$\lambda \gg 1$ in perturbative string theory

$\lambda \ll 1$ in perturbative gauge theory

weak-coupling expansion convergent – defines $\Delta(\lambda)$

need to go beyond perturbation theory: integrability

Last 7 years – remarkable progress for subclass of states:

"semiclassical" string states with large quantum numbers

dual to "long" SYM operators (canonical dim. $\Delta_0 \gg 1$)

[BMN 02, GKP 02, FT 03,...]

$E = \Delta$ – same (in some cases !) dependence on $C, m, ...$

with coefficients = "interpolating" functions of λ

Current status:

1. "Long" operators = strings with large quantum numbers:

Asymptotic Bethe Ansatz (ABA) [Beisert, Eden, Staudacher 06]

firmly established (including non-trivial phase factor)

2. "Short" operators = general quantum string states:

Partial progress based on improving ABA by
"Luscher corrections" [Janik et al 08]
Attempts to generalize ABA to TBA [Arutyunov, Frolov 08]
Very recent (complete ?) proposal for underlying "Y-system"
[Gromov, Kazakov, Vieira 09]

To justify from first principles
need better understanding of quantum
$AdS_5 \times S^5$ superstring theory:
1. Solve string theory on a plane $R^{1,1}$ $\rightarrow$
relativistic 2d S-matrix $\rightarrow$ asymptotic BA for the spectrum
2. Generalize to finite-energy closed strings – the theory on $R \times S^1$
$\rightarrow$ TBA (cf. integrable sigma models)

Reformulation in terms of currents with Virasoro conditions solved
("Pohlmeyer reduction") is promising approach
[Grigoriev, AT 07; Roiban, AT 09]

Superstring theory in $AdS_5 \times S^5$

bosonic coset $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

generalized to supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ [Metsaev, AT 98]

$$S = T \int d^2\sigma \Big[G_{mn}(x)\partial x^m \partial x^n + \bar\theta(D + F_5)\theta\partial x + \bar\theta\theta\bar\theta\theta\partial x\partial x + ... \Big]$$

tension $T = \frac{\mathrm{R}^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical integrability of coset σ-model (Luscher-Pohlmeyer 76)

true for $AdS_5 \times S^5$ superstring [Bena, Polchinski, Roiban 02]

Progress in understanding of implications of (semi)classical integrability [Kazakov, Marshakov, Minahan, Zarembo 04,...]

1-loop quantum superstring corrections
[Frolov, AT; Park, Tirziu, AT, 02-04, ...]
used as an input data to fix 1-loop term
in strong-coupling expansion of the phase $\theta(\lambda)$ in ABA
[Beisert, AT 05; Hernandez, Lopez 06]

2-loop quantum superstring corrections
[Roiban, Tirziu, AT; Roiban, AT 07]
– check of finiteness of the GS superstring
– implicit check of integrability of quantum string theory
– non-trivial confirmation of BES phase in ABA
[Basso, Korchemsky, Kotansky 07]

Gauge states vs string states: principles of comparison

1. compare states with same global $SO(2,4) \times SO(6)$ charges

e.g., (S, J) – "sl(2) sector" – $\mathrm{Tr}(D_+^S \Phi^J)$

J=twist= spin-chain length

2. assume no "level crosing" while changing λ

min/max energy (S, J) states should be in correspondence

Gauge theory:

$\Delta \equiv E = S + J + \gamma(S, J, m, \lambda)$,

$\gamma = \sum_{k=1}^{\infty} \lambda^k \gamma_k(S, J, m)$

fix $S, J, ...$ and expand in λ;

then may expand in large/small $S, J, ...$

Semiclassical string theory:

$E = S + J + \gamma(\mathcal{S}, \mathcal{J}, m, \sqrt{\lambda})$,

$\gamma = \sum_{k=-1}^{\infty} \frac{1}{(\sqrt{\lambda})^k} \widetilde{\gamma}_k(\mathcal{S}, \mathcal{J}, m)$

fix semiclassical parameters $\mathcal{S} = \frac{S}{\sqrt{\lambda}}, \ \mathcal{J} = \frac{J}{\sqrt{\lambda}}, \ m$

and expand in $\frac{1}{\sqrt{\lambda}}$

To match in general will need to resum – beyond ABA

Various special limits studied:

(i) "Fast strings" – "locally-BPS" long operators

GT: $J \gg 1, \quad \frac{S}{J}$=fixed

ST: $\mathcal{J} \gg 1, \quad \frac{\mathcal{S}}{\mathcal{J}}$=fixed

[BMN; Frolov, AT03; Beisert, Minahan, Staudacher, Zarembo03]

$$E = S + J + \frac{\lambda}{J}\Big[h_1(\tfrac{S}{J}, m) + \tfrac{1}{J} h_2(\tfrac{S}{J}, m) + ...\Big] + ...$$

(ii) "Slow long strings" – long non-BPS operators [GKP02]

GT: $\ln S \gg J \gg 1$

ST: $\ln \mathcal{S} \gg \mathcal{J}, \ \mathcal{J} = 0$ or $\mathcal{J}$=fixed

$$E = S + f(\lambda) \ln S + ...$$

$$f(\lambda \gg 1) = a_1\sqrt{\lambda} + ... , \qquad f(\lambda \ll 1) = c_1 \lambda + ...$$

(iii) "Fast long strings"

GT: $S \gg J \gg 1, \ j \equiv \frac{J}{\ln S}$=fixed

ST: $\mathcal{S} \gg \mathcal{J} \gg 1, \ \ell \equiv \frac{\mathcal{J}}{\ln \mathcal{S}}$=fixed = $\frac{j}{\sqrt{\lambda}}$

[Belitsky, Gorsky, Korchemsky 06; Frolov, Tirziu, AT 06;...]

Key example of "long" operators – $\mathrm{Tr}(\Phi D_+^S \Phi)$

dual to spinning string

Folded spinning string in flat space:

$X_1 = \epsilon \ \sin\sigma \ \cos\tau, \quad X_2 = \epsilon \ \sin\sigma \ \sin\tau$

$$ds^2 = -dt^2 + dX_i dX_i = -dt^2 + d\rho^2 + \rho^2 d\phi^2$$

$$t = \epsilon\tau \ , \quad \rho = \epsilon \sin\sigma \ , \quad \phi = \tau$$

tension $T = \frac{1}{2\pi\alpha'} \equiv \frac{\sqrt{\lambda}}{2\pi}$

energy $E = \epsilon\sqrt{\lambda}$ and spin $S = \frac{\epsilon^2}{2}\sqrt{\lambda}$ – Regge relation:

$$E = \sqrt{2\sqrt{\lambda}S}$$

Folded spinning string in AdS_5:

[Gubser, Klebanov, Polyakov 02]

$$ds^2 = -\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\phi^2$$

$$t = \kappa\tau, \quad \phi = w\tau, \quad \rho = \rho(\sigma)$$

$$\sinh\rho = \epsilon\, \mathrm{sn}(\kappa\epsilon^{-1}\sigma,\ -\epsilon^2)\ , \quad 0 < \rho < \rho_{\max}$$

$$\coth\rho_{\max} = \frac{w}{\kappa} \equiv \sqrt{1 + \frac{1}{\epsilon^2}}$$

ϵ measures length of the string

$$\kappa = \epsilon\ {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2)$$

classical energy $E_0 = \sqrt{\lambda}\mathcal{E}_0$ and spin $S = \sqrt{\lambda}\mathcal{S}$

$$\mathcal{E}_0 = \epsilon\ {}_2F_1(-\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2), \qquad \mathcal{S} = \frac{\epsilon^2\sqrt{1+\epsilon^2}}{2}\ {}_2F_1(\frac{1}{2}, \frac{3}{2}; 2; -\epsilon^2)$$

solve for ϵ – analog of Regge relation

$$\mathcal{E}_0 = \mathcal{E}_0(\mathcal{S})\ , \qquad E_0 = \sqrt{\lambda}\, \mathcal{E}_0(\frac{S}{\sqrt{\lambda}})$$

short/long string – flat space/AdS interpolation:

$\mathcal{E}_0(\mathcal{S} \ll 1) = \sqrt{2\mathcal{S}} + ...$

$\mathcal{E}_0(\mathcal{S} \gg 1) = \mathcal{S} + \frac{1}{\pi} \ln \mathcal{S} + ...$

$\mathcal{S} \to \infty$: folds reach the boundary ($\rho = \infty$)

solution drastically simplifies: length $\kappa \sim \ln \mathcal{S} \to \infty$

$$t = \kappa\tau, \quad \phi \approx \kappa\tau, \quad \rho \approx \kappa\sigma \ , \quad \kappa \sim \epsilon \sim \ln \mathcal{S} \to \infty$$

$E = S$ from massless end points at AdS boundary (null geodesic)

$E - S \approx \frac{\sqrt{\lambda}}{\pi} \ln S$ from tension/stretching of the string

quantum superstring corrections to E respect $S + \ln S$ form:

Semiclassical string theory limit

$$1. \ \lambda \gg 1 \ , \qquad \mathcal{S} = \frac{S}{\sqrt{\lambda}} = \text{fixed}; \qquad 2. \ \mathcal{S} \gg 1$$

$$E = S + f(\lambda) \ln S + ... \ ,$$

$$f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi} \Big[1 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{(\sqrt{\lambda})^2} + ... \Big]$$

a_n–Feynmann graphs of 2d CFT – $AdS_5 \times S^5$ superstring

$a_1 = -3 \ln 2$: Frolov, AT 02

$a_2 = -K$: Roiban, AT 07

$K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$ = 0.915 (2-loop σ-model integrals)

Gauge theory: dual operators

$\mathrm{Tr}(\Phi D_+^S \Phi)$, $\Delta - S - 2 = O(\lambda)$

same $\ln S$ asymptotics of anomalous dimensions

from symmetry argument [Alday, Maldacena 07]

Perturbative gauge theory limit:

$$1.\ \lambda \ll 1\ , \quad S = \text{fixed}; \qquad 2.\ S \gg 1$$

$$\Delta - S - 2 = f(\lambda) \ln S + \ldots$$

$$f(\lambda \ll 1) = c_1 \lambda + c_2 \lambda^2 + c_3 \lambda^3 + c_4 \lambda^4 + \ldots$$

$$= \frac{1}{2\pi^2} \Big[\lambda - \frac{\lambda^2}{48} + \frac{11\lambda^3}{2^8 \times 45} - \Big(\frac{73}{630} + \frac{4(\zeta(3))^2}{\pi^6}\Big) \frac{\lambda^4}{2^7} + \ldots \Big]$$

c_n are given by Feynmann graphs of 4d CFT – N=4 SYM
c_3: Kotikov, Lipatov, et al 03;
c_4: Bern, Czakon, Dixon, Kosower, Smirnov 06;

String and gauge limits are formally different
but leading $\ln S$ term is universal $\rightarrow$
single $f(\lambda)$ provides smooth interpolation
Remarkably, both expansions are reproduced from
single BES integral equation for $f(\lambda)$
[strong coupling expansion:
numerical – Benna, Benvenuti, Klebanov, Scardicchio 07;
analytic – Basso, Korchemsky, Kotansky 07]
both expansions thus known in principle to any order
exact expression for $f(\lambda)$ from BES equation?
asymptotic nature of strong-coupling expansion:
non-perturbative $e^{-\frac{1}{2}\sqrt{\lambda}}$ terms [BKK]
fixed by $AdS_5 \rightarrow S^5$, $\sqrt{\lambda} \rightarrow -\sqrt{\lambda}$ symmetry ?

Subleading terms in large S expansion

string has large but finite length: does not reach boundary

$$E_0(\mathcal{S} \gg 1) = S + a_0 \ln \mathcal{S} + a_1 + \frac{1}{\mathcal{S}}(a_2 \ln \mathcal{S} + a_3)$$
$$+ \frac{1}{\mathcal{S}^2}(a_4 \ln^2 \mathcal{S} + a_5 \ln \mathcal{S} + a_6) + O(\frac{\ln^3 \mathcal{S}}{\mathcal{S}^3})$$

$a_0 = \frac{\sqrt{\lambda}}{\pi}, \quad a_1 = \frac{\sqrt{\lambda}}{\pi} \ln(8\pi) - 1, \ ....$

coeffs of $\frac{\ln^k \mathcal{S}}{\mathcal{S}^k}$ happen to be related to coeff of $\ln \mathcal{S}$:

$a_2 = \frac{1}{2} a_0^2, \ \ a_4 = -\frac{1}{8} a_0^3, ...$

according to "functional relation" [Basso, Korchemsky 06]

$$E - S = f(E + S) = a_0 \ln(S + \frac{1}{2} a_0 \ln S + ...) + ...$$

Why? In near-boundary limit for large $\mathcal{S}$

string end moves along nearly null line at the boundary:

pp-wave limit – cusp anomaly as "pp-wave anomaly"

pp-wave limit effectively establishes contact with
collinear conformal group in the boundary theory
[Kruczenski, AT 08; Ishizeki, Kruczenski, Titziu, AT 08]

Some of coefficients in large S expansion are related
due to reciprocity property in gauge theory
true also at strong coupling
[Basso, Korchemski 06; Beccaria, Forini, Tirziu, AT 08]

Comparison of semiclassical string theory expansion
to large S gauge theory expansion:
$E = S + f(\lambda) \ln S + h(\lambda) + O(\frac{1}{S})$
f and h are controlled by infinite length limit – ABA
but subleading coefficients require knowledge of
wrapping / finite size corrections
comparison in general will require resummation of the series

Dimensions of short operators
= energies of quantum string states:

progress in understanding spectrum of conformal dimensions
of planar $N = 4$ SYM or spectrum of strings in $AdS_5 \times S^5$
based on quantum integrability
Spectrum of states with large quantum numbers –
solution of ABA equations
key example: cusp anomaly function
Recent proposal of how to extend this to "short" states
with any quantum numbers – TBA or "Y-system" approach
so far not checked/compared to direct quantum string results

Aim: compute leading $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ correction to dimension of
"lightest" massive string state dual to
Konishi operator in SYM theory
– data for checking future (numerical) prediction of "Y-system"

Konishi operator:

operators (long multiplet) related to singlet $[0,0,0]^2_{(0,0)}$ by susy

$\Delta = \Delta_0 + \gamma(\lambda), \qquad \Delta_0 = 2, \frac{5}{2}, 3, ..., 10$

– same anomalous dimension γ

singlet eigen-state of anom. dim. matrix with lowest eigenvalue

examples:

$\mathrm{Tr}(\bar{\Phi}_i \Phi_i), \quad i = 1, 2, 3, \quad \Delta_0 = 2$

$\mathrm{Tr}([\Phi_1, \Phi_2]^2)$ in $su(2)$ sector $\Delta_0 = 4$

$\mathrm{Tr}(\Phi_1 D_+^2 \Phi_1)$ in $sl(2)$ sector $\Delta_0 = 4$

Weak-coupling expansion of $\gamma(\lambda)$: $\qquad \lambda = g^2_{\text{YM}} N_c$

$$\gamma(\lambda) = 12\Big[\frac{\lambda}{(4\pi)^2} - 4\frac{\lambda^2}{(4\pi)^4} + 28\frac{\lambda^3}{(4\pi)^6} + [-208 + 48\zeta(3) - 120\zeta(5)]\frac{\lambda^4}{(4\pi)^8} + ...\Big]$$

[Fiamberti,Santambrogio,Sieg,Zanon; Bajnok,Janik; Velizhanin 08]

Finite radius of convergence ($N_c = \infty$) – if we could sum up and then re-expand at large λ – what to expect? (cf. $f(\lambda)$)

AdS/CFT duality: Konishi operator dual to
"lightest" among massive $AdS_5 \times S^5$ string states
large $\sqrt{\lambda} = \frac{\mathrm{R}^2}{\alpha'}$:
– "small" string at center of AdS_5 – in nearly flat space

$$\lambda \gg 1: \qquad \Delta(\Delta - 4) = 4\sqrt{\lambda} + a + O(\tfrac{1}{\sqrt{\lambda}})$$

$$\Delta - 2 = 2\sqrt{\sqrt{\lambda}}\left[1 + \frac{b}{\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\right], \qquad b = \frac{1}{8}(a+4)$$

$a =$ first correction to mass of dual string state

Evidence below: $\quad a = -4, \quad b = 0$

Flat space case:

$m^2 = \frac{4(n-1)}{\alpha'}, \quad n = \frac{1}{2}(N + \bar{N}) = 1, 2, ..., \ \ N = \bar{N}$

$n = 1$: massless IIB supergravity (BPS) level

l.c. vacuum $|0>$: $(8 + 8)^2 = 256$ states

$n = 2$: first massive level (many states, highly degenerate)

$[(a^i_{-1} + S^a_{-1})|0>]^2 = [(8 + 8) \times (8 + 8)]^2$

in $SO(9)$ reps:

$([2, 0, 0, 0] + [0, 0, 1, 0] + [1, 0, 0, 1])^2 = (44 + 84 + 128)^2$

e.g. $44 \times 44 = 1 + 36 + 44 + 450 + 495 + 910$

$84 \times 84 = 1 + 36 + 44 + 84 + 126 + 495 + 594 + 924 + 1980 + 2772$

switching on $AdS_5 \times S^5$ background fields lifts degeneracy

states with “lightest mass” at 1-st excited level

should correspond to Konishi multiplet

string spectrum in $AdS_5 \times S^5$:

long multiplets $\mathcal{A}^{\Delta}_{[k,p,q](j,j')}$ of $PSU(2,2|4)$

highest weight states: $[k,p,q](j,j')$ labels of $SO(6) \times SO(4)$

Remarkably, flat-space string spectrum can be re-organized
in multiplets of $SO(2,4) \times SO(6) \subset PSU(2,2|4)$
[Bianchi, Morales, Samtleben 03]
$SO(4) \times SO(5) \subset SO(9)$ rep.
lifted to $SO(4) \times SO(6)$ rep. of $SO(2,4) \times SO(6)$

Konishi long multiplet
$\widehat{T}_1 = (1 + Q + Q \wedge Q + ...)[0,0,0]_{(0,0)}$
determines the KK “floor” of 1-st excited string level
$H_1 = \sum_{J=0}^{\infty} [0,J,0]_{(0,0)} \times \widehat{T}_1$

One expects for scalar massive state in AdS_5

$(-\nabla^2 + m^2)\Phi + ... = 0$

$\Delta(\Delta - 4) = (m\mathrm{R})^2 + O(\alpha') = 4(n-1)\frac{\mathrm{R}^2}{\alpha'} + O(\alpha')$

$\Delta = 2 + \sqrt{(m\mathrm{R})^2 + 4 + O(\alpha')}$

$$\Delta(\lambda \gg 1) = \sqrt{4(n-1)\sqrt{\lambda}} + ...$$

[Gubser, Klebanov, Polyakov 98]

e.g., for first massive level:

$n = 2: \qquad \Delta = 2\sqrt{\sqrt{\lambda}} + ...$

Subleading corrections?

Comparison between gauge and string theory states non-trivial:

GT ($\lambda \ll 1$): operators built out of free fields,
canonical dimension Δ_0 determines states that can mix
ST ($\lambda \gg 1$): near-flat-space string states built out of
free oscillators, level n determines states that can mix

meaning of Δ_0 at strong coupling?
meaning of n at weak coupling?

1. relate states with same global charges;
2. assume "non-intersection principle" [Polyakov 01]:
no level crossing for states with same quantum numbers
as λ changes from strong to weak coupling

Approaches to computation of corrections to string masses:

(i) semiclassical approach:
identify short string state as small-spin limit of
semiclassical string state
– reproduce the structure of strong-coupling corrections
to short operators
[Frolov, AT 03; Tirziu, AT 08]

(ii) vertex operator approach:
use $AdS_5 \times S^5$ string sigma model perturbation theory to find
leading terms in anomalous dimension of corresponding
vertex operator
[Polyakov 01; AT 03]

(iii) space-time effective action approach:
use near-flat-space expansion and NSR vertex operators
to reconstruct $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections to corresponding
massive string state equation of motion
[Burrington, Liu 05]

(iv) "light-cone" quantization approach:
start with light-cone gauge $AdS_5 \times S^5$ string action
and compute corrections to energy of
corresponding flat-space oscillator string state
[Metsaev, Thorn, AT 00]

Semiclassical expansion: spinning strings

$$E = E(\frac{J}{\sqrt{\lambda}}, \sqrt{\lambda}) = \sqrt{\lambda}\mathcal{E}_0(\mathcal{J}) + \mathcal{E}_1(\mathcal{J}) + \frac{1}{\sqrt{\lambda}}\mathcal{E}_2(\mathcal{J}) + ...$$

in "short" string limit $\mathcal{J} \ll 1$

$$\mathcal{E}_n = \sqrt{\mathcal{J}}\ (a_{0n} + a_{1n}\mathcal{J} + a_{2n}\mathcal{J}^2 + ...)$$

expansion valid for $\sqrt{\lambda} \gg 1$ and $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ fixed: $J \sim \sqrt{\lambda} \gg 1$
but if knew all terms in this expansion – could express $\mathcal{J}$
in terms of J, fix J to finite value and re-expand in $\sqrt{\lambda}$

$$E = \sqrt{\sqrt{\lambda}J}\Big[a_{00} + \frac{a_{10}J + a_{01}}{\sqrt{\lambda}} + \frac{a_{20}J^2 + a_{11}J + a_{02}}{(\sqrt{\lambda})^2} + ...\Big]$$

to trust the coeff of $\frac{1}{(\sqrt{\lambda})^n}$ need coeff of up to n-loop terms
e.g. classical a_{10} and 1-loop a_{01} sufficient to fix $\frac{1}{\sqrt{\lambda}}$ term
[cf. "fast string" expansion $\mathcal{J} \gg 1$ [Frolov, AT 03]: for fixed J
positive powers of $\sqrt{\lambda}$ – need to resum]

Example: circular rotating string in S^5 with $J_1 = J_2 = J$:

cf. Konishi descendant with $J_1 = J_2 = 2$: $\mathrm{Tr}([\Phi_1, \Phi_2]^2)$

try represent it by "short" classical string with same charges

flat space $R_t \times R^4$: circular string solution

$$x_1 + ix_2 = a\, e^{i(\tau+\sigma)}\ , \qquad x_3 + ix_4 = a\, e^{i(\tau-\sigma)}$$

$$E = \sqrt{\frac{4}{\alpha'} J}, \qquad J = \frac{a^2}{\alpha'}$$

this solution can be directly embedded into

$R_t \times S^5$ in $AdS_5 \times S^5$ [Frolov, AT 03] :

string on *small* sphere inside S^5: $X_1^2 + ... + X_6^2 = 1$

$$X_1 + iX_2 = a\, e^{i(\tau+\sigma)}, \quad X_3 + iX_4 = a\, e^{i(\tau-\sigma)}\ ,$$
$$X_5 + iX_6 = \sqrt{1 - 2a^2}, \qquad t = \kappa\tau$$
$$\mathcal{J} = \mathcal{J}_1 = \mathcal{J}_2 = a^2, \qquad \mathcal{E}^2 = \kappa^2 = 4\mathcal{J}$$

Remarkably, exact E_0 is just as in flat space

$$E_0 = \sqrt{\lambda}\mathcal{E} = \sqrt{4\sqrt{\lambda} J}\ , \qquad J = \sqrt{\lambda}\mathcal{J}$$

[cf. another (unstable) branch of $J_1 = J_2$ solution with $\mathcal{J} > \frac{1}{2}$:

$E_0 = \sqrt{J^2+\lambda} = \sqrt{\lambda}(1 + \frac{J^2}{2\sqrt{\lambda}} + ...)$]

1-loop quantum string correction to the energy:

sum of bosonic and fermionic fluctuation frequencies ($n = 0, 1, 2, ...$)

Bosons (2 massless + massive):

$$AdS_5: \quad 4\times \qquad \omega_n^2 = n^2 + 4\mathcal{J}$$

$$S^5: \quad 2\times \qquad \omega_{n\pm}^2 = n^2 + 4(1-\mathcal{J}) \pm 2\sqrt{4(1-\mathcal{J})n^2 + 4\mathcal{J}^2}$$

Fermions:

$$4\times \qquad \omega_{n\pm}^{2\,f} = n^2 + 1 + \mathcal{J} \pm \sqrt{4(1-\mathcal{J})n^2 + 4\mathcal{J}}$$

$$E_1 = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}\Big[4\omega_n + 2(\omega_{n+} + \omega_{n-}) - 4(\omega^f_{n+} + \omega^f_{n-})\Big]$$

expand in small $\mathcal{J}$ and do sums (UV divergences cancel)

$$E_1 = \frac{1}{\sqrt{\mathcal{J}}}\Big[-\mathcal{J} - [3+\zeta(3)]\mathcal{J}^2 - \tfrac{1}{4}\big[5 + 6\zeta(3) + 30\zeta(5)\big]\mathcal{J}^3 + \ldots\Big]$$

$$E = E_0 + E_1 = 2\sqrt{\sqrt{\lambda}J}\Big[1 - \frac{1}{2\sqrt{\lambda}} - \frac{3J}{4(\sqrt{\lambda})^2}(1 + 2\zeta(3)) + ...\Big]$$

if we could interpolate to fininte $J = J_1 = J_2 = 2$
that would suggest for Konishi state

$$E = 2\sqrt{\sqrt{\lambda}}\,\Big[1 - \frac{1}{2\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\Big]$$

But: above still valid for large E and J;
need to account for quantization of c.o.m. modes – reinterpret as

$$E(E-4) = 4\sqrt{\lambda}(J-1) - 4 + O(\tfrac{1}{\sqrt{\lambda}})$$

$$J = 2: \qquad E - 2 = 2\sqrt{\sqrt{\lambda}}\,\Big[1 + 0 + O(\tfrac{1}{(\sqrt{\lambda})^2})\Big]$$

same result will be found by different methods below

Spectrum of quantum string states from target space anomalous dimension operator

Flat space: $k^2 = m^2 = \frac{4(n-1)}{\alpha'}$

e.g. leading Regge trajectory $(\partial x \bar{\partial} x)^{S/2} e^{ikx}, \quad n = S/2$

spectrum in (weakly) curved background:

solve marginality (1,1) conditions on vertex operators

e.g. scalar anomalous dimension operator $\widehat{\gamma}(G)$

on $T(x) = \sum c_{n...m} x^n ... x^m$ or on coefficients $c_{n...m}$

differential operator in target space

found from β-function for the corresponding perturbation

$$I = \frac{1}{4\pi\alpha'} \int d^2 z [G_{mn}(x) \partial x^m \bar{\partial} x^n + T(x)]$$

$$\beta_T = -2T - \frac{\alpha'}{2} \widehat{\gamma}\, T + O(T^2)$$

$$\widehat{\gamma} = \Omega^{mn} D_m D_n + ... + \Omega^{m...k} D_m ... D_k + ...$$

$$\Omega^{mn} = G^{mn} + p_1 \alpha' R^{mn} + O(\alpha'^3)$$

$p_1 = 0, ...$ in DR; $\Omega^{\cdots\cdot} \sim \alpha'^n R^p_{....} H^q_{...}$
Solve $-\widehat{\gamma}\, T + m^2 T = 0$: diagonalize $\widehat{\gamma}$

similarly for massless (graviton, ...) and massive states
e.g. $\beta^G_{mn} = \alpha' R_{mn} + O(\alpha'^3)$
gives Lichnerowitz operator as anomalous dimension operator

$$(\widehat{\gamma} h)_{mn} = -D^2 h_{mn} + 2R_{mknl} h^{kl} - 2R_{k(m} h^k_{n)} + O(\alpha'^3)$$

Massive string states in curved background:

$$\int d^D x \sqrt{g}\, \Big[\Phi_{...} (-D^2 + m^2 + X) \Phi_{...} + ... \Big]$$
$$m^2 = \tfrac{4}{\alpha'} (n-1)\ , \qquad X = R_{....} + O(\alpha')$$

case of $AdS_5 \times S^5$ background

$$R_{mn} - \tfrac{1}{96} (F_5 F_5)_{mn} = 0, \quad R = 0\ , \quad F_5^2 = 0$$

leading-order term in X should vanish for scalar state leading α' correction to scalar string state mass =0 (?!)

$$[-D^2 + m^2 + O(\tfrac{1}{\sqrt{\lambda}})]\Phi \ = 0$$

$$\Delta = 2 + \sqrt{4(n-1) + 4 + O(\tfrac{1}{\sqrt{\lambda}})}$$

$$\Delta_{(n=2)} = 2 + 2\sqrt{\sqrt{\lambda}}\left[1 + \tfrac{1}{2\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\right]$$

prediction (?) for leading term in strong-coupling expansion
of singlet Konishi state dimension
... but possible subtleties... 10d scalar vs singlet state...

What about non-singlet Konishi descendant states ?

How to find $\widehat{\gamma}$: Effective action approach

derive equation of motion for a massive string field
in curved background from quadratic effective action S
reconstructed from flat-space NSR S-matrix
Example: totally symmetric NS-NS 10-d tensor
– state on leading Regge trajectory in flat space

symmetric tensor $\Phi_{\mu_1 \ldots \mu_{2n}}$ $\quad (m^2 = \frac{4(n-1)}{\alpha'})$
in metric+RR background

$$L = R - \frac{1}{2\cdot 5!} F_5^2 + O(\alpha'^3)$$
$$-\frac{1}{2}(D_\mu \Phi D^\mu \Phi + m^2 \Phi^2) + \sum_{k\geq 1} (\alpha')^{k-1} \Phi X_k(R, F_5, D)\Phi + \ldots$$

assumption: $\alpha' n R \ll 1, \quad i.e. \quad n \ll \sqrt{\lambda}$:
small massive string in the middle of AdS_5:
near-flat-space expansion should be applicable

then eq. for Φ to leading α' order [Burrington, Liu 05]

$$
\begin{aligned}
&[-D^2 + m^2 + X_1 + O(\alpha')]\Phi_{\mu_1\cdots\mu_{2n}} = 0 \\
&\Phi X_1 \Phi = c_1 \Phi_{\mu_1\mu_2\cdots\mu_{2n}} R^{\mu_1\nu_1\mu_2\nu_2} \Phi_{\nu_1\nu_2}{}^{\mu_3\cdots\mu_{2n}} \\
&\qquad + c_2 \Phi_{\mu_1\cdots\mu_{2n}} F^{\mu_1\nu_1\alpha_3\cdots\alpha_5} F^{\mu_2\nu_2}{}_{\alpha_3\cdots\alpha_5} \Phi_{\nu_1\nu_2}{}^{\mu_3\cdots\mu_{2n}} \\
&\qquad + c_3 \Phi_{\mu_1\mu_2\cdots\mu_{2n}} F^{\mu_1\alpha_2\cdots\alpha_5} F^{\nu_1}{}_{\alpha_2\cdots\alpha_5} \Phi_{\nu_1}{}^{\mu_2\cdots\mu_{2n}} \\
&c_1 = n^2 \,, \quad c_2 = -\tfrac{1}{4!} \,, \qquad c_3 = -\tfrac{1}{4\times 4!}
\end{aligned}
$$

check: reproduces eq for graviton perturbation around

$R_{\mu\nu} - \frac{1}{4\times 4!}(F_5 F_5)_{\mu\nu} = 0$

$AdS_5 \times S^5$ background: $R_{ab} = -\frac{4}{\mathrm{R}^2} g_{ab}, \; R_{mn} = \frac{4}{\mathrm{R}^2} g_{mn}$

$\mu, \nu, \ldots = 0, 1, ...9; \quad a, b, \ldots$ in AdS_5 and $m, n, \ldots$ in S^5

$$
\begin{aligned}
L &= \tfrac{1}{2} \Phi_{\mu_1\cdots\mu_{2n}} (-D^2 + m^2) \Phi^{\mu_1\cdots\mu_{2n}} \\
&\quad + \frac{n^2}{\mathrm{R}^2} \left(\Phi_{a_1 a_2 \mu_3\cdots\mu_{2n}} \Phi^{a_1 a_2 \mu_3\cdots\mu_{2n}} - \Phi_{m_1 m_2 \mu_3\cdots\mu_{2n}} \Phi^{m_1 m_2 \mu_3\cdots\mu_{2n}} \right) + \ldots
\end{aligned}
$$

background is direct product – can consider particular tensor with S indices in AdS_5 and K indices in S^5:

end up with anomalous dimension operator

$$[-D^2 + (m^2 + \frac{K^2 - S^2}{2\mathrm{R}^2})]\Phi = 0 \ , \qquad D^2 = D^2_{AdS_5} + D^2_{S_5}$$
$$m^2 = \frac{4}{\alpha'}(n-1) = \frac{2}{\alpha'}(S + K - 2), \quad 2n = S + K$$

symmetric transverse traceless tensor – highest-weight state –
Young table labels $(\Delta, S, 0; J, K, 0), \quad J \geqslant K$
extract AdS_5 radius R and set $\sqrt{\lambda} = \frac{\mathrm{R}^2}{\alpha'}$

$$(-D^2_{AdS_5} + M^2)\Phi = 0$$
$$M^2 = 2\sqrt{\lambda}(S + K - 2) + \frac{1}{2}(K^2 - S^2) + J(J+4) - K$$

For symmetric traceless rank S tensor in AdS_5:

$$\Delta - 2 = \sqrt{M^2 + S + 4}$$
$$= \sqrt{2\sqrt{\lambda}(S + K - 2) + \frac{1}{2}(S + K - 2)(K - S) + J(J+4) + 4 + O(\frac{1}{\sqrt{\lambda}})}$$

To summarize:

condition of marginality of (1,1) vertex operator

for $(\Delta, S_1, S_2; J_1, J_2, J_3) = (\Delta, S, 0; J, K, 0)$ state

$$0 = -\sqrt{\lambda}(S + K - 2) + \frac{1}{2}[\Delta(\Delta - 4) + \frac{1}{2}S(S-2) - \frac{1}{2}K(K-2) - J(J+4)] + O(\tfrac{1}{\sqrt{\lambda}})$$

BPS level $n = \frac{1}{2}(S + K) = 1$: $\quad J = K + J', \quad J' = 0, 1, 2, ...$

$S = 2, K = 0$: $\Delta = 4 + J'$; $\quad K = 2, S = 0$: $\Delta = 6 + J'$; etc

First massive level: $n = \frac{1}{2}(S + K) = 1$

case of minimal dimension shift

$S = 4,\ K = J = 0$: $\quad$ dual to $\Delta_0 = 6$ Konishi state $[0,0,0]_{(2,2)}$

$$\Delta - \Delta_0 = 2\sqrt{\sqrt{\lambda} + O(\tfrac{1}{\sqrt{\lambda}})} = 2\sqrt{\sqrt{\lambda}}\left[1 + O(\tfrac{1}{(\sqrt{\lambda})^2})\right]$$

what about other states in Konishi multiplet?

Vertex operator approach [Polyakov 01; AT 03]

$I = \frac{1}{4\pi\alpha'} \int d^2\xi \; G_{mn}(x) \partial x^m \bar\partial x^n + ...$ perturbed by
$V(f) = f_{m_1 ... m_s}(x) \partial^{k_1} x^{m_1} .... \bar\partial^{k_h} x^{m_s}$
compute the renormalization of $f_{m_1 ... m_j}$ and set $\beta_f = \widehat\gamma f + ...$=0
$\widehat\gamma f = [2 - J + \frac{1}{2}\alpha' D^2 + \sum c_k \alpha'^k (R....)^n ... D^p] f = 0$
diagonalize "anomalous dimension" operator
but $\widehat\gamma$ for generic f and G not known even to α' order
calculate anomalous dimensions from "first principles"
superstring theory in $AdS_5 \times S^5$:

$$I = \frac{\sqrt\lambda}{4\pi} \int d^2\sigma \Big[\partial N_p \bar\partial N^p + \partial n_k \bar\partial n_k + \text{fermions} \Big]$$

$$N_+ N_- - N_u N_u^* - N_v N_v^* = 1 \, , \qquad n_x n_x^* + n_y n_y^* + n_z n_z^* = 1$$

$$N_\pm = N_0 \pm i N_5, \quad N_u = N_1 + i N_2, \; ..., \; n_x = n_1 + i n_2, ...$$

construct marginal (1,1) operators in terms of N_p and n_k

e.g. vertex operator for dilaton sugra mode

$$\mathrm{V}_J(\xi) = \ (N_+)^{-\Delta}\ (n_x)^J \left[-\partial N_p \bar\partial N^p + \partial n_k \bar\partial n_k + \text{fermions} \right]$$

$N_+ \equiv N_0 + iN_5 = \frac{1}{z}(z^2 + x_m x_m) \sim e^{it}$
$n_x \equiv n_1 + in_2 \sim e^{i\varphi}$

$$0 = 2 - 2 + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta - 4) - J(J+4)] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

i.e. $\Delta = 4 + J$ (BPS)

candidate operators for states on leading Regge trajectory:

$$\mathrm{V}_J(\xi) = (N_+)^{-\Delta} \big(\partial n_x \bar\partial n_x\big)^{J/2}, \qquad n_x \equiv n_1 + in_2$$

$$\mathrm{V}_S(\xi) = (N_+)^{-\Delta} \big(\partial N_u \bar\partial N_u\big)^{S/2}, \qquad N_u \equiv N_1 + iN_2$$

+ fermionic terms

+ $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms from diagonalization of anom. dim. op.

how they mix with ops with same charges and dimension?

in general $\left(\partial n_x \bar{\partial} n_x\right)^{J/2}$ mixes with singlets

$$(n_x)^{2p+2q}(\partial n_x)^{J/2-2p}(\bar{\partial} n_x)^{J/2-2q}(\partial n_m \partial n_m)^p(\bar{\partial} n_k \partial n_k)^q$$

$S^5 = SO(6)/SO(5)$ sigma model

$$S = \frac{\sqrt{\lambda}}{4\pi}\int d^2\xi\, \partial n_m \bar{\partial} n_m\ , \qquad n_m n_m = 1$$

$$\dot{\rm g} = -\epsilon {\rm g} + 4{\rm g}^2 + 4{\rm g}^3 + ...\ , \qquad {\rm g} \equiv \frac{1}{\sqrt{\lambda}} = \frac{\alpha'}{{\rm R}^2}\ , \qquad \epsilon = d-2$$

running is cancelled if embedded into $AdS_5 \times S^5$ string σ-model

ops. for states on leading Regge trajectory

$$O_{\ell,s} = f_{k_1...k_\ell m_1...m_{2s}} n_{k_1}...n_{k_\ell} \partial n_{m_1} \bar{\partial} n_{m_2} ... \partial n_{m_{2s-1}} \bar{\partial} n_{m_{2s}}$$

their renormalization studied before [Wegner 90] renormalization of composite operators to leading order in $\frac{1}{\sqrt{\lambda}}$

use "pairing rules" (and ignore "on-shell" operators):

$$< AB >=< A > B + A < B > + < A, B >$$

$$< A, B >= \int d^2\xi d^2\xi' < n_k(\xi), n_m(\xi') > \frac{\delta A}{\delta n_k(\xi)} \frac{\delta B}{\delta n_m(\xi')}$$

$$< A(n) >= \frac{1}{2} \int d^2\xi d^2\xi' < n_k(\xi), n_m(\xi') > \frac{\delta^2 A}{\delta n_k(\xi) \delta n_m(\xi')}, \text{ etc.}$$

$$< n_k >= -\frac{5}{2} I n_k \ , \ \ < n_k, n_l >= -I(n_k n_l - \delta_{kl}) \ , \qquad I = -\frac{1}{2\pi\epsilon} \to \infty$$

$$< n_k, \partial n_l >= -I \partial n_k n_l \ , \ \ < n_k, \bar{\partial} n_l >= -I \bar{\partial} n_k n_l \ ,$$

$$< \partial n_k, \partial n_l >= I n_k n_l \partial n_m \partial n_m \ , \qquad < \bar{\partial} n_k, \bar{\partial} n_l >= I n_k n_l \bar{\partial} n_m \bar{\partial} n_m \ ,$$

$$< \partial n_k, \bar{\partial} n_l >= -I(\bar{\partial} n_k \partial n_l - \delta_{kl} \partial n_m \bar{\partial} n_m)$$

$$< (\partial n_k \bar{\partial} n_k) >= 0, \ \ < (\partial n_k \partial n_k) >= -4I \partial n_k \partial n_k \ , \qquad < (\bar{\partial} n_k \bar{\partial} n_k) >= -4I \bar{\partial} n_k \bar{\partial} n_k$$

simplest case: $f_{k_1...k_\ell} n_{k_1}...n_{k_\ell}$ with traceless $f_{k_1...k_\ell}$
same anom. dim. $\widehat{\gamma}$ as its highest-weight rep $V_J = (n_x)^J$

$$\widehat{\gamma} = 2 - \frac{1}{2\sqrt{\lambda}}[5J + J(J-1)] + ... = 2 - \frac{1}{2\sqrt{\lambda}} J(J+4) + ...$$

scalar spherical harmonic that solves Laplace eq. on S^5
similarly for AdS_5 or $SO(2,4)$ model:
replacing n_x^J and $\partial n_m \bar{\partial} n_m$ with $N_+^{-\Delta}$ and $\partial N^p \bar{\partial} N_p$, with
$J = -\Delta$ and g $= \frac{1}{\sqrt{\lambda}} \to -\frac{1}{\sqrt{\lambda}}$
e.g. dimension of $n_x^J \partial n_m \bar{\partial} n_m$: $\widehat{\gamma} = -\frac{1}{2\sqrt{\lambda}} J(J+4) + O(\frac{1}{(\sqrt{\lambda})^2})$
dimension of $N_+^{-\Delta} \partial N^p \bar{\partial} N_p$: $\widehat{\gamma} = \frac{1}{2\sqrt{\lambda}} \Delta(\Delta - 4) \ + O(\frac{1}{(\sqrt{\lambda})^2})$

example of scalar higher-level operator:

$$N_+^{-\Delta}[(\partial n_k \bar{\partial} n_k)^r + ...] , \qquad r = 1, 2, ...$$

[Kravtsov, Lerner, Yudson 89; Castilla, Chakravarty 96]

$$0 = -2(r-1) + \frac{1}{2\sqrt{\lambda}}[\Delta(\Delta-4) + 2r(r-1)]$$
$$+ \frac{1}{(\sqrt{\lambda})^2}[\tfrac{2}{3}r(r-1)(r-\tfrac{7}{2}) + 4r] + ...$$

$r = 1$: ground level

fermionic contributions should make $r = 1$ exact zero of $\widehat{\gamma}$

$r = 2$: first excited level

candidate for singlet Konishi state $\Delta_0 = 2$

$$\Delta(\Delta - 4) = 4\sqrt{\lambda} - 4 + O(\tfrac{1}{\sqrt{\lambda}}) ,$$
$$\Delta - \Delta_0 = 2\sqrt{\sqrt{\lambda}} \left[1 + O(\tfrac{1}{(\sqrt{\lambda})^2})\right]$$

same as for $(S = 4, K = 0)$ Konishi state with $\Delta_0 = 6$

Operators with two spins J, K in S^5:

$$V_{K,J} = N_+^{-\Delta} \sum_{u,v=0}^{K/2} c_{uv} M_{uv}$$
$$M_{uv} \equiv n_y^{J-u-v} n_x^{u+v} (\partial n_y)^u (\partial n_x)^{K/2-u} (\bar\partial n_y)^v (\bar\partial n_x)^{K/2-v}$$

highest and lowest eigen-values of 1-loop anom. dim. matrix

$$\widehat\gamma_{\rm min} = 2 - K + \frac{1}{2\sqrt\lambda}[\Delta(\Delta-4) - \frac{1}{2}K(K+10) - J(J+4) - 2JK] + O(\tfrac{1}{(\sqrt\lambda)^2})$$
$$\widehat\gamma_{\rm max} = 2 - K + \frac{1}{2\sqrt\lambda}[\Delta(\Delta-4) - \frac{1}{2}K(K+6) - J(J+4)] + O(\tfrac{1}{(\sqrt\lambda)^2})$$

fermions may alter terms linear in K

$K = 4$: first massive level – Konishi state
identify operators with right representations
– more evidence for $b = 0$
[R.Roiban, AT, in progress]

Conclusions

Beginning of understanding
quantum string spectrum in $AdS_5 \times S^5$
= spectrum of "short" SYM operators

more progress expected soon
aiding/checking integrability approach

Happy Birthday Misha!

Thank you for your inspiration, insight and help!

Long Konishi multiplet

$\Delta_{0\ min} = 2, \quad [m, n, k](s, s') = [0, 0, 0](0, 0)$

$SO(6)$ and $SO(4)$ labels

[Andreanopoli,Ferrara 98; Bianchi,Morales,Samtleben 03]

Δ_0	
2	$[0,0,0]_{(0,0)}$
$\frac{5}{2}$	$[0,0,1]_{(0,\frac{1}{2})} + [1,0,0]_{(\frac{1}{2},0)}$
3	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(0,0)}$
$\frac{7}{2}$	$[0,0,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)} + [0,1,1]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [1,0,0]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})} + [1,0,2]_{(\frac{1}{2},0)}$ $+[1,1,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [2,0,1]_{(0,\frac{1}{2})}$
4	$[0,0,0]_{(0,0)+(0,2)+(1,1)+(2,0)} + [0,0,2]_{(\frac{1}{2},\frac{1}{2})+(\frac{3}{2},\frac{1}{2})} + [0,1,0]_{2(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})} + [2,0,2]_{(0,0)}$ $+[0,1,2]_{(1,0)} + [0,2,0]_{2(0,0)+(1,1)} + [1,0,1]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [1,1,1]_{2(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(\frac{1}{2}}$
6	$[0,0,0]_{3(0,0)+3(1,1)+(2,2)} + [0,0,2]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})} + [0,1,0]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})+}$ $+[0,1,2]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [0,2,0]_{3(0,0)+(0,1)+(0,2)+(1,0)+3(1,1)+(2,0)} + [0,2,2]_{(\frac{1}{2},\frac{1}{2})}$ $+[0,3,0]_{2(\frac{1}{2},\frac{1}{2})} + [0,4,0]_{(0,0)} + [1,0,1]_{(0,0)+3(0,1)+3(1,0)+4(1,1)+(1,2)+(2,1)} + [1,0,3]_{(\frac{1}{2},\frac{1}{2})} + [0,$ $+[1,1,1]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})} + [1,2,1]_{(0,0)+(0,1)+(1,0)} + [2,0,0]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}$ $+[2,0,2]_{(0,0)+(1,1)} + [2,1,0]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [2,2,0]_{(\frac{1}{2},\frac{1}{2})} + [3,0,1]_{(\frac{1}{2},\frac{1}{2})} + [4,0,0]_{(0,0}$
$\frac{17}{2}$	$[0,0,1]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})} + [0,1,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [1,0,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)} + [1,0,2]_{(0,\frac{1}{2})}$ $+[1,1,0]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [2,0,1]_{(\frac{1}{2},0)}$
9	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(0,0)}$
$\frac{19}{2}$	$[0,0,1]_{(\frac{1}{2},0)} + [1,0,0]_{(0,\frac{1}{2})}$
10	$[0,0,0]_{(0,0)}$