**A NONLOCAL DIFFUSION EQUATION
ARISING IN TERMINALLY ATTACHED
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IMA Preprint Series # 647

June 1990

A NONLOCAL DIFFUSION EQUATION ARISING IN TERMINALLY ATTACHED POLYMER CHAINS

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Abstract. We consider a polymer melt in a domain Ω whereby each polymer chain is attached at one endpoint to a fixed surface S contained in $\partial\Omega$. Denote by $G(x, t; y)$ the normalized number density of all subchains from x to y of length t . Then, according to the selfconsistent mean field theory, G satisfies, for each $y : G_t - \nabla^2 G + \sigma\varphi G = 0$ where σ is a real parameter, and φ is a functional of $G(\cdot, \cdot; \cdot)$ both nonlocal and nonlinear. We establish existence of G and C^∞ regularity of φ , as a function of x .

Introduction. This paper is concerned with a heat equation involving a nonlocal, nonlinear term. The equation models the distribution of polymer chains whereby each chain's endpoint is attached to a given surface S ; the chains are contained in a domain Ω , and S is a part of the boundary $\partial\Omega$ of Ω .

It is customary to consider polymer chains as consisting of polygonal arcs; each side has a fixed length l called Kuhn's segment and is made of several monomers. Adjacent segments are considered to have directions independent of one another. Consequently the polymer chain may be viewed as a random walk with, say, N steps. We consider here the situation where the initial point of each polymer is attached to S , and the chain is a random walk which is constrained to be in Ω .

Denote by $G(x, j; x')$ the probability of a subchain which starts at x' and ends at x after j segments. G is affected by all the chains in Ω . The self-consistent mean field theory (*SCF*) due to Dolan and Edwards (1974, 1975) derives, after some approximations, the following equation for G :

$$(0.1) \quad G_t - \Delta G + \sigma\varphi G = 0$$

where

$$(0.2) \quad G(x, 0; x') = \delta_{x'}(x) \quad (\text{the Dirac measure at } x'),$$

and $\varphi(x)$ is a potential representing the probability of all the chains passing through x ; it is given by

$$(0.3) \quad \varphi(x) = \frac{\int_0^T ds \int_S G(x, s; \eta) dS_\eta \int_\Omega G(\xi, T - s; x) d\xi}{\int_S dS_\eta \int_\Omega G(\xi, T; \eta) d\xi}.$$

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Here jl was replaced by the time variable t , and $T = Nl$. More precisely, $\Delta G \equiv \Sigma \partial^2 G / \partial x_i^2$ should be replaced by $l^2 \Delta G$ and σ is a constant proportional to the excluded volume coefficient as well as to the surface density of the anchored polymers on S ; see Muthukumar and Ho (1989). However, after non-dimensionalizing x and t we arrive at (0.1)–(0.3) with $\sigma = O(1)$. The parameter σ , as well as the excluded volume parameter, depend on the temperature θ ; $\sigma > 0$ (good solvent) for $\theta > \theta_c$ and $\sigma < 0$ (poor solvent) for $\theta < \theta_c$. We refer the reader to recent articles by Patel (1989) and Ploehn and Russel (1989) which explain the SCF model and include extensive bibliographies.

If the boundary $\partial\Omega$ is a hard wall then one imposes the absorbing boundary condition

$$(0.4) \quad G(x, t; x') = 0 \quad \text{for } x \in \partial\Omega .$$

On the other hand if $\partial\Omega$ is partially reflecting and partially absorbing then the appropriate boundary condition is (see de Gennes (1969))

$$(0.5) \quad \left(\frac{\partial G}{\partial \nu} + aG \right) (x, t; x') = 0 \quad \text{for } x \in \partial\Omega , \quad a \geq 0$$

where ν is the outward unit normal to $\partial\Omega$.

Numerical work in the one-dimensional case for the absorbing boundary condition is described in Muthukumar and Ho (1989) and Patel (1989); the main interest of the computation is to discover the shape of the potential function $\varphi(x)$.

In the present paper we shall consider the general case where Ω is 3- or n -dimensional. For definiteness we shall establish all our results for the boundary condition (0.5); the case (0.4) requires rather minor modifications (the definition of φ needs to be slightly modified in this case, since $G(\xi, T; \eta) = 0$ if $\eta \in \partial\Omega$).

In §§1,2 we consider the case where $\sigma > 0$, and prove the existence of a solution G with φ in C^∞ . The case $\sigma < 0$ is considered in §3.

§1. Auxiliary estimates. Let Ω be a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) with C^2 boundary and let $a(x)$ be a non-negative Hölder continuous function defined on the boundary $\partial\Omega$ of Ω .

Let φ be any function defined in $\overline{\Omega}$ and satisfying:

$$(1.1) \quad \begin{aligned} &\varphi \geq 0, \\ &\varphi \in L^p(\Omega) \quad \text{for some } 1 < p < \infty, \\ &\int_{\Omega} \varphi = \sigma T, \quad \sigma > 0, \quad T > 0. \end{aligned}$$

We denote by Ω_s the set $\Omega \times \{0 < t < s\}$.

Consider the parabolic problem:

$$(1.2) \quad u_t - \Delta u + \varphi u = 0 \quad \text{in } \Omega_T ,$$

$$(1.3) \quad u_\nu + a(x)u = 0 \quad \text{on } \partial\Omega \times (0, T) ,$$

$$(1.4) \quad u(x, 0) = u_0(x) , \quad u_0(x) \geq 0 .$$

The existence and uniqueness of a weak solution is valid for any measure u_0 .

If $u_0(x) = \delta_{x'}(x)$ then we set

$$G(x, t; x') = u \quad \text{if } x' \in \Omega ;$$

when $x' \in \partial\Omega$ we take $G(x, t; x')$ to be the limit of $G(x, t; y)$, $y \in \Omega, y \rightarrow x'$. Let S be a nonempty open portion of $\partial\Omega$, and set

$$(1.5) \quad B(x, t) = \int_S G(x, t; \eta) dS_\eta .$$

If $u_0(x) = 1$ in (1.4) then we denote the solution by $A(x, t)$.

We are interested in studying the functions A, B .

Denote by $\Gamma(x, t; x')$ the Green function of

$$\begin{aligned} v_t - \Delta v &= 0 \quad \text{in } \Omega_T , \\ v_\nu + a(x)v &= 0 \quad \text{on } \partial\Omega \times (0, T) . \end{aligned}$$

By symmetry,

$$(1.6) \quad \begin{aligned} \Gamma(x, t; x') &= \Gamma(x', t; x) , \\ G(x, t; x') &= G(x', t; x) . \end{aligned}$$

If $\varphi \in C^\alpha(\overline{\Omega})$, both (1.2) and (1.3) are satisfied in the usual sense. Then, by the maximum principle,

$$(1.7) \quad u \geq 0 ,$$

and

$$(1.8) \quad G(x, t; x') \leq \Gamma(x, t; x') .$$

By approximation we can establish (1.7), (1.8) also in case φ is only in L^p . Since Γ is a bounded function for t uniformly positive, $\Gamma\varphi$ belongs to L^p and, by (1.8), also $G\varphi$ belongs to L^p . It follows that (1.2) is satisfied in the a.e. sense and that G_t and $D_x^2 G$ belong to $L_{\text{loc}}^p(\Omega_T)$.

In the sequel we shall denote by $C(T)$ various positive constants which depend on T but are otherwise independent of φ .

LEMMA 1.1. The function $A(x, t) = \int_{\Omega} G(\xi, t; x) d\xi$ satisfies:

$$(1.9) \quad A \leq 1 ,$$

$$(1.10) \quad \iint_{\Omega_T} A_t^2 + \int_{\Omega} |\nabla A(x, t)|^2 dx \leq C(T) .$$

Proof. The estimate (1.9) follows by comparison, since $u \equiv 1$ is a supersolution.

Next, multiplying

$$(1.11) \quad A_t - \Delta A + \varphi A = 0$$

by A_t and integrating over Ω_t we get, after integration by parts,

$$\iint_{\Omega_t} A_t^2 + \int_0^t \int_{\partial\Omega} (-A_\nu) A_t + \iint_{\Omega_t} \frac{1}{2} \frac{d}{dt} |\nabla A|^2 + \iint_{\Omega_t} \frac{1}{2} \varphi \frac{d}{dt} A^2 = 0 .$$

Since $A_\nu + aA = 0$, we obtain

$$(1.12) \quad \iint_{\Omega_t} A_t^2 + \frac{1}{2} \int_{\Omega} |\nabla A(x, t)|^2 dx + \frac{1}{2} \int_{\partial\Omega} a(x) A^2(x, t) dS_x + \frac{1}{2} \int_{\Omega} \varphi(x) A^2(x, t) dx \\ = \frac{1}{2} \int_{\partial\Omega} a + \frac{1}{2} \int_{\Omega} \varphi = C(T) ,$$

and (1.10) follows.

LEMMA 1.2. The function

$$w = \int_0^t B(x, s) ds$$

satisfies:

$$(1.13) \quad \iint_{\Omega_T} w_t^2 + \int_{\Omega} |\nabla w(x, t)|^2 dx \leq C(T) .$$

Proof. Multiplying

$$B_t - \Delta B + \varphi B = 0$$

by

$$\int_t^\tau B(x, s) ds$$

and integrating over Ω_τ we get

(1.14)

$$\begin{aligned} & \int_{\Omega} dx \int_0^\tau B_t(x, t) \left(\int_t^\tau B(x, s) ds \right) dt \\ & - \int_0^\tau dt \int_{\Omega} \Delta B(x, t) \left(\int_t^\tau B(x, s) ds \right) dx + \iint_{\Omega_\tau} \varphi(x) B(x, t) \int_t^\tau B(x, s) ds = 0 . \end{aligned}$$

The first integral is equal to

$$\begin{aligned} & \int_{\Omega} dx \left[B(x, t) \int_t^\tau B(x, s) ds \right]_0^\tau + \int_{\Omega} dx \int_0^\tau B^2(x, t) dt \\ & = - \int_S \int_0^\tau B(\eta, s) ds dS_\eta + \int_{\Omega_t} w_t^2 , \end{aligned}$$

since $B = w_t$. The second term on the left-hand side of (1.14) is equal to

$$\begin{aligned} & \int_0^\tau dt \left(\int_{\Omega} \nabla B(x, t) \int_t^\tau \nabla B(x, s) ds \right) dx - \int_0^\tau dt \int_{\partial\Omega} B_\nu(x, t) \left(\int_t^\tau B(x, s) ds \right) dS_x \\ & = \frac{1}{2} \int_{\Omega} dx \left| \int_0^\tau \nabla B(x, s) ds \right|^2 + \frac{1}{2} \int_{\partial\Omega} a(x) \left(\int_0^\tau B(x, s) ds \right)^2 dS_x , \end{aligned}$$

since $B_\nu + aB = 0$. Finally, the last term on the left-hand side of (1.14) is equal to

$$- \iint_{\Omega_\tau} \frac{1}{2} \varphi \frac{d}{dt} \left(\int_t^\tau B(x, s) ds \right)^2 = \frac{1}{2} \int_{\Omega} \varphi(x) \left(\int_0^\tau B(x, s) ds \right)^2 .$$

Combining these expressions, we get from (1.14)

$$\begin{aligned} & \iint_{\Omega_\tau} w_t^2 + \frac{1}{2} \int_{\Omega} |\nabla w(x, \tau)|^2 + \frac{1}{2} \int_{\Omega} \varphi w^2(x, \tau) + \frac{1}{2} \int_{\partial\Omega} a w^2(x, \tau) dS_x \\ (1.15) \quad & = \int_S w(\eta, \tau) dS_\eta . \end{aligned}$$

The right-hand side is bounded by

$$\varepsilon \int_{\Omega} [w^2(x, \tau) + |\nabla w(x, \tau)|^2] + C_\varepsilon$$

whereas, since $w(x, 0) = 0$,

$$w(x, \tau) = \int_0^\tau w_t(x, t) dt$$

so that

$$\int_\Omega w^2(x, \tau) \leq C(T) \iint_{\Omega_\tau} w_t^2(x, t) .$$

Choosing ε small enough we then deduce from (1.15) that (1.13) holds with $C(T)$ depending on T but not on φ .

LEMMA 1.3. *The function B satisfies:*

$$(1.16) \quad B(x, t) \leq \frac{C(T)}{\sqrt{t}} .$$

Proof. From (1.5), (1.8),

$$(1.17) \quad B(x, t) \leq \int_S \Gamma(x, t; y) dS_y \leq C \int_S \frac{e^{-\frac{|x-y|^2}{4t}}}{t^{n/2}} dS_y$$

where C is a positive constant; the last inequality is a well known estimate for the Green function Γ , and follows from the construction of Γ in the form $\Gamma = \Gamma_0 + h$ where Γ_0 is the fundamental solution of the heat operator and h is a “regular” term which makes Γ satisfy the boundary condition (1.3).

Suppose first that S is a domain in $y_n = 0$. Setting $y' = (y_1, \dots, y_{n-1})$, $x' = (x_1, \dots, x_{n-1})$, we can estimate the right-hand side of (1.17) by

$$C \int_{|y'| < C} \frac{e^{-\frac{c(x_n - y_n)^2}{t}}}{\sqrt{t}} \frac{e^{-\frac{c|x' - y'|^2}{t}}}{t^{(n-1)/2}} dy' \leq \frac{C}{\sqrt{t}} .$$

By partitioning S into a finite number of domains, each locally diffeomorphic to a plane region, we find that the above estimate is valid in general.

Later on we shall need to work with the function

$$(1.18) \quad E(x) = \int_0^T B(x, s) A(x, T - s) ds = \int_0^T w_t(x, s) A(x, T - s) ds .$$

LEMMA 1.4. *The function E satisfies:*

$$(1.19) \quad 0 \leq E(x) \leq C(T) ,$$

$$(1.20) \quad \int_\Omega |\nabla E(x)| \leq C(T) .$$

Proof. The estimate (1.19) follows from (1.9) and Lemma 1.3. To prove (1.20) we write

$$\begin{aligned}
D_x E &= \int_0^T D_x w_t(x, s) \cdot A(x, T-s) \, ds + \int_0^T w_t(x, s) D_x A(x, T-s) \, ds \\
&= [D_x w(x, s) \cdot A(x, T-s)]_0^T + \int_0^T D_x w(x, s) A_t(x, T-s) \, ds \\
&\quad + \int_0^T w_t(x, s) D_x A(x, T-s) \, ds .
\end{aligned}$$

By Lemmas 1.1, 1.2, each of the last two integrals is in $L^1(\Omega)$ with L^1 -norm bounded by $C(T)$. Since also

$$\int_{\Omega} \left| [D_x w(x, s) \cdot A(x, T-s)]_0^T \right| = \int_{\Omega} |D_x w(x, T)| \leq C(T) ,$$

the assertion (1.20) follows.

Let ψ be the principal eigenfunction of

$$\begin{aligned}
(1.21) \quad & -\Delta \psi + \varphi \psi = \lambda \psi \quad \text{in } \Omega , \\
& \psi_\nu + a \psi = 0 \quad \text{on } \partial\Omega , \\
& \psi > 0 \quad \text{in } \Omega, \quad \int_{\Omega} \psi^2 = 1 .
\end{aligned}$$

Then ψ is the minimizer (up to multiplication by ± 1) and λ is the minimum of the problem

$$(1.22) \quad \lambda = \min_{\int_{\Omega} v^2 = 1} \left\{ \int_{\Omega} \left(\frac{1}{2} |\nabla v|^2 + \frac{1}{2} \varphi v^2 \right) + \int_{\partial\Omega} \frac{1}{2} a v^2 \right\} .$$

Since $\int_{\Omega} \varphi = \sigma T$, by taking a bounded function v in (1.22) we deduce that

$$(1.23) \quad \lambda \leq C(T) .$$

It then follows that

$$(1.24) \quad \int_{\Omega} |\nabla \psi|^2 \leq C(T) .$$

LEMMA 1.5. *There holds:*

$$(1.25) \quad \psi(x) \leq C(T) .$$

Proof. By (1.24) and Sobolev's inequality,

$$(1.26) \quad \int_{\Omega} |\psi|^q \leq C(T), \quad q = \frac{2n}{n-2} ,$$

if $n \geq 3$; if $n = 2$ then q is any positive number and if $n = 1$ then (1.25) holds. Without loss of generality we may assume that $n \geq 2$. We shall compare ψ with the solution χ to

$$\begin{aligned} -\Delta \chi &= \lambda \psi \quad \text{in } \Omega , \\ \chi_{\nu} + a\chi &= 0 \quad \text{on } \partial\Omega . \end{aligned}$$

Since $-\Delta \psi \leq \lambda \psi$, we deduce that

$$0 \leq \psi \leq \chi .$$

Applying L^p -elliptic estimates to χ and recalling (1.26), we get

$$|\chi|_{W^{2,q}} \leq C(T) .$$

It follows by Sobolev's imbedding that

$$|\chi|_{L^{q_1}(\Omega)} \leq C(T) ,$$

where

$$q_1 = \begin{cases} \frac{nq}{n-2q} & \text{if } q < \frac{n}{2} \\ \infty & \text{if } q > \frac{n}{2} \end{cases}$$

and q_1 is any positive number if $q = \frac{n}{2}$. Consequently

$$|\psi|_{L^{q_1}(\Omega)} \leq C(T) .$$

If $q_1 < \infty$ then we can repeat the above process and deduce that

$$|\psi|_{L^{q_2}} \leq C(T)$$

with

$$q_2 = \begin{cases} \frac{nq_1}{n-2q_1} & \text{if } q_1 < \frac{n}{2} \\ \infty & \text{if } q_1 > \frac{n}{2} \end{cases}$$

and q_2 any positive number if $q_1 = \frac{n}{2}$. Since the sequence $q_1, q_2, \dots, q_m$ cannot remain finite if m is large enough, the assertion (1.25) follows.

LEMMA 1.6. *The function B satisfies:*

$$(1.27) \quad \int_{\Omega} B(x, T) \, dx \geq c > 0$$

where c is a constant depending on T , but otherwise independent of φ .

The proof will exploit Lemma 1.5 and the following three lemmas.

LEMMA 1.7. *If $u > 0$ in Ω and*

$$\int_{\Omega} u^2 = 1, \quad \int_{\Omega} u^p \leq C_1 \quad \text{for some } p > 2,$$

then there exist positive constants α, b depending only on C_1 and Ω such that

$$(1.28) \quad \text{meas}\{u(x) > \alpha, x \in \Omega\} > b.$$

Proof. Writing

$$\begin{aligned} 1 &= \int_{\{u < \alpha\}} u^2 + \int_{\{u > \alpha\}} u^2 < \alpha^2 |\Omega| + \left\{ \int_{\{u > \alpha\}} u^p \right\}^{\frac{2}{p}} \{\text{meas}\{u > \alpha\}\}^{\frac{1}{q}} \\ &< \alpha^2 |\Omega| + C_1^{\frac{2}{p}} \{\text{meas}\{u > \alpha\}\}^{\frac{1}{q}} \end{aligned}$$

where $1/q + 2/p = 1$, (1.28) follows by choosing $\alpha^2 |\Omega| = 1/2$.

LEMMA 1.8. *The function ψ satisfies:*

$$(1.29) \quad \int_{\Omega} |\nabla(\log \psi)|^2 \leq C(T).$$

Proof. Multiplying the equation for ψ by $1/(\psi + \varepsilon)$ ($\varepsilon > 0$) and integrating over Ω , we get

$$\begin{aligned} \int_{\Omega} (\lambda - \varphi) \frac{\psi}{\psi + \varepsilon} &= - \int_{\Omega} \frac{\Delta \psi}{\psi + \varepsilon} = - \int_{\partial \Omega} \psi_{\nu} \frac{1}{\psi + \varepsilon} + \int_{\Omega} \nabla \psi \cdot \nabla \frac{1}{\psi + \varepsilon} \\ &= \int_{\partial \Omega} \frac{a \psi}{\psi + \varepsilon} - \int_{\Omega} \frac{|\nabla \psi|^2}{(\psi + \varepsilon)^2} = \int_{\partial \Omega} \frac{a \psi}{\psi + \varepsilon} - \int_{\Omega} |\nabla \log(\psi + \varepsilon)|^2. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we find that

$$\int_{\Omega} |\nabla(\log \psi)|^2 = \int_{\partial \Omega} a + \int_{\Omega} (\varphi - \lambda) \leq C(T).$$

LEMMA 1.9. *The function ψ satisfies:*

$$(1.30) \quad \int_S \psi(x) dS_x \geq e^{-C(T)} .$$

Proof. By Lemma 1.7 applied to ψ , if Ω_0 is a compact subdomain of Ω such that $\text{meas}(\Omega \setminus \Omega_0) < b/2$, then

$$\text{meas } \Omega_* > \frac{b}{2} \quad \text{where} \quad \Omega_* = \{x \in \Omega_0, \psi(x) > \alpha\}.$$

In the sequel we choose Ω_0 to have smooth boundary.

Take a plane Π tangent to S at some interior point x_0 of S , and move it in the direction of the inner normal at x_0 . We denote by Π_τ the position of Π at distant τ from x_0 . Since

$$\int \text{meas}(\Pi_\tau \cap \Omega_*) d\tau > \frac{b}{2}$$

there exists a τ_0 such that

$$\text{meas}(\Pi_{\tau_0} \cap \Omega_*) > \frac{b}{2R} \quad (R = \text{diameter of } \Omega).$$

We now connect an S -neighborhood S_0 of x_0 to $\Pi_{\tau_0} \cap \Omega$ by smooth curves l_ζ ($\zeta \in S_0$) such that at each point (ζ, s) on l_ζ (s is the length parameter on l_ζ)

$$(1.31) \quad \frac{1}{C} dS_\zeta ds \leq dx \leq C dS_\zeta ds .$$

Denote by S_1 the subset of points of S_0 for which l_ζ meets $\Pi_{\tau_0} \cap \Omega_*$.

Integrating $\log \psi$ along the curves l_ζ with $\zeta \in S_1$ and using Jensen's inequality, we get

$$(1.32) \quad \begin{aligned} \log \int_{S_1} \psi(x) \frac{dS_x}{|S_1|} &\geq \int_{S_1} \log \psi(x) \frac{dS_x}{|S_1|} \\ &= \frac{1}{|S_1|} \left(\int_{\Pi_{\tau_0} \cap \Omega_*} \log \psi + \iint \frac{\partial}{\partial l_\zeta} \log \psi \right) . \end{aligned}$$

The double integral in (1.32) is taken over the set generated by $\{l_\zeta, \zeta \in S_1\}$. By Lemma 1.8 and (1.31),

$$\left| \iint \frac{\partial}{\partial l_\zeta} \log \psi \right| \leq C \int_\Omega |\nabla \log \psi| \leq C(T) .$$

Since also $\log \psi > \log \alpha$ on $\Pi_{\tau_0} \cap \Omega_*$, (1.32) yields

$$\log \int_{S_1} \psi \geq -C(T) .$$

Therefore also $\log \int_S \psi \geq -C(T)$, and the lemma follows.

Proof of Lemma 1.6. Consider the function

$$g(t) = \int_{\Omega} B(x, t) \psi(x) \, dx .$$

Clearly

$$\begin{aligned} g'(t) &= \int_{\Omega} B_t \psi = \int_{\Omega} (\Delta B - \varphi B) \psi \\ &= \int_{\Omega} (\Delta \psi - \varphi \psi) B = - \int_{\Omega} \lambda \psi B = -\lambda g , \end{aligned}$$

so that

$$(1.33) \quad g(T) \geq e^{-\lambda T} g(0) .$$

But

$$\begin{aligned} g(0) &= \int_{\Omega} \int_S G(x, 0; \eta) \, dS_{\eta} \, \psi(x) \, dx \\ &= \int_{\Omega} \int_S \delta_{\eta}(x) \psi(x) \, dx = \int_S \psi(\eta) \geq e^{-C(T)} \end{aligned}$$

by Lemma 1.9, whereas

$$g(T) = \int_{\Omega} B(x, T) \psi(x) \, dx \leq C \int_{\Omega} B(x, T) \, dx$$

by Lemma 1.5. Hence (1.33) yields the assertion (1.27).

We conclude this section by studying the function

$$(1.34) \quad \Psi(x) = \frac{\int_0^T B(x, s) A(x, T-s) ds}{\int_{\Omega} B(y, T) \, dy} .$$

LEMMA 1.10. *The function Ψ satisfies:*

$$(1.35) \quad \int_{\Omega} \Psi(x) \, dx = T.$$

Proof. By the semigroup property (or by uniqueness of solutions of (1.2)–(1.4)) and by the symmetry of G ,

$$\int_{\Omega} G(x, s; \eta) G(\xi, T - s; x) \, dx = G(\xi, T; \eta) .$$

Integrating with respect to $\eta \in S$ and $\xi \in \Omega$, we find that

$$\int_{\Omega} B(x, s) A(x, T - s) \, dx = \int_{\Omega} B(\xi, T) \, d\xi ,$$

from which (1.35) follows.

LEMMA 1.11. *The function Ψ satisfies:*

$$(1.36) \quad 0 \leq \Psi \leq C(T) ,$$

$$(1.37) \quad \int_{\Omega} |\nabla \Psi(x)| \, dx \leq C(T) .$$

Indeed, this follows immediately from Lemmas 1.4, 1.6.

§2. Existence in case $\sigma > 0$. In this section we establish the existence of a solution to (0.1)–(0.3), (0.5) in case $\sigma > 0$. For convenience we replace $\sigma\varphi$ by φ . Then the system becomes

$$(2.1) \quad \left(\frac{\partial}{\partial t} - \Delta + \varphi(x) \right) G(x, t; x') = 0 , \quad (x, t) \in \Omega_T ,$$

$$(2.2) \quad (\partial_{\nu} + a(x)) G(x, t; x') = 0 , \quad x \in \partial\Omega, \, 0 < t < T ,$$

$$(2.3) \quad G(x, 0; x') = \delta_{x'}(x) , \quad x \in \Omega$$

when $\varphi(x)$ is defined by

$$(2.4) \quad \varphi = \sigma \Psi ,$$

$$\Psi(x) = \frac{\int_0^T ds \int_S G(x, s; \eta) \, dS_{\eta} \int_{\Omega} G(\xi, T - s; x) \, d\xi}{\int_S dS_{\eta} \int_{\Omega} G(\xi, T; \eta) \, d\xi}$$

THEOREM 2.1. *Let $\sigma > 0$. Then there exists a solution of (2.1)–(2.4) satisfying:*

$$(2.5) \quad \varphi \geq 0, \quad \int_{\Omega} \varphi = T, \quad \varphi \in C(\overline{\Omega}), \quad \nabla \varphi \in L^1(\Omega).$$

Proof. Fix any $1 < p < \infty$ and introduce the set, in $L^p(\Omega)$:

$$\mathcal{A} = \left\{ \varphi \in L^p(\Omega), \quad |\varphi|_{L^p(\Omega)} \leq C^*, \quad \varphi \geq 0, \quad \int_{\Omega} \varphi = T \right\},$$

where C^* is a positive constant to be determined.

For any $\varphi \in \mathcal{A}$ we can solve (2.1)–(2.3) as in §1 and define a mapping $\tilde{\varphi} = W\varphi$ where $\tilde{\varphi} = \Psi$, Ψ as in (1.34). We need to prove that σW has a fixed point φ and that $\varphi \in C(\overline{\Omega})$, $\nabla \varphi \in L^1(\Omega)$.

We first notice that, by Lemma 1.10, 1.11,

$$(2.6) \quad \sigma W \text{ maps } \mathcal{A} \text{ into itself}$$

provided C^* is chosen large enough (depending only on T).

We next show that

$$(2.7) \quad \sigma W(\mathcal{A}) \text{ is a compact subset of } \mathcal{A},$$

where $\mathcal{A}$ is endowed with the $L^p(\Omega)$ -norm. Indeed, take a sequence φ_j in $\mathcal{A}$ and set $\tilde{\varphi}_j = W\varphi_j$. Then

$$|\tilde{\varphi}_j|_{W^{1,1}} \leq C(T), \quad \text{by Lemma 1.11.}$$

Since $W^{1,1}$ is compactly imbedded in L^q , $1 < q < \frac{n}{n-1}$, there exists a subsequence φ_j such that

$$|\tilde{\varphi}_j - \hat{\psi}|_{L^q} \rightarrow 0, \quad \hat{\psi} \in L^q(\Omega).$$

By Lemma 1.11 we have $0 \leq \psi_j \leq C(T)$, and the same must also hold for $\hat{\psi}$. If we choose $1 < q < p$ then, by interpolation,

$$\begin{aligned} |\tilde{\varphi}_j - \hat{\psi}|_{L^p} &\leq C |\tilde{\varphi}_j - \hat{\psi}|_{L^q}^{\mu} |\tilde{\varphi}_j - \hat{\psi}|_{L^{\infty}}^{1-\mu} \\ &\leq C(T) |\tilde{\varphi}_j - \hat{\psi}|_{L^q}^{\mu} \rightarrow 0 \quad \text{if } j \rightarrow \infty \end{aligned}$$

(for some $0 < \mu < 1$), which implies the asserted compactness.

We finally claim that

$$(2.8) \quad \sigma W \text{ is a continuous mapping,}$$

that is,

$$\text{if } |\varphi_j - \varphi|_{L^p} \rightarrow 0 \quad \text{then} \quad |\tilde{\varphi}_j - \tilde{\varphi}|_{L^p} \rightarrow 0$$

where $\tilde{\varphi}_j = W\varphi_j$, $\tilde{\varphi} = W\varphi$.

Denote by A_j, B_j, w_j the functions A, B, w corresponding to φ_j . Then by Lemmas 1.1, 1.2,

$$\begin{aligned} \iint_{\Omega_T} A_{j,t}^2 + \int_{\Omega} |\nabla A_j(x, t)|^2 &\leq C, \\ \iint_{\Omega_T} w_{j,t}^2 + \int_{\Omega} |\nabla w_j(x, t)|^2 &\leq C. \end{aligned}$$

It is also clear by uniqueness of solutions to (1.2)–(1.4) that

$$A_j \rightarrow A \quad \text{and} \quad B_j \rightarrow B$$

in the appropriate weak sense. It follows that

$$\int_0^T B_j(x, s) A_j(x, T-s) ds \rightarrow \int_0^T B(x, s) A(x, T-s) ds$$

in $L^1(\Omega)$ and

$$\int_{\Omega} B_j(x, T) dx \rightarrow \int_{\Omega} B(x, T) dx.$$

Hence $\tilde{\varphi}_j \rightarrow \tilde{\varphi}$ in L^1 . Since further

$$|\tilde{\varphi}_j|_{L^\infty} \leq C(T),$$

we can use interpolation as before to deduce that

$$|\tilde{\varphi}_j - \tilde{\varphi}|_{L^p} \rightarrow 0.$$

Having proved that σW satisfies (2.6), (2.7) and (2.8), we appeal to the Schauder fixed point theorem and deduce that σW has a fixed point φ . It remains to show that $\varphi \in C(\overline{\Omega})$.

Since $\varphi \in L^\infty(\Omega)$, A_t and $D_x A$ belong to $L^p(\Omega_T)$ for any $p < \infty$. Similarly B_t and $D_x B$ belong to $L^p(\Omega_T \setminus \Omega_\varepsilon)$ for any $p < \infty$, $\varepsilon > 0$. Also, by Lemma 1.3

$$\int_0^\tau B(x, s) ds \leq C\sqrt{\tau}.$$

It follows that the integral in (1.18) is convergent (at $s = 0$) uniformly in x , and the integrand is continuous (or even Hölder continuous) in x for $0 < s \leq T$. Hence $E(x)$ is continuous in $\overline{\Omega}$, and the same is true of $\varphi(x)$.

REMARK 2.1. Using the form $G = \Gamma + h$ of G where h has weaker singularity than Γ , one can actually prove that $\varphi \in C^\alpha(\overline{\Omega})$ for any $0 < \alpha < \frac{1}{2}$.

We shall now establish further the regularity of $\varphi(x)$:

THEOREM 2.2. *For any solution of (2.1)–(2.4) with $\varphi \in L^\infty(\Omega)$,*

$$(2.9) \quad \varphi \in C^\infty(\Omega) .$$

Proof. Since $\varphi \in L^\infty(\Omega)$, we can apply L^p estimates to solutions of

$$(2.10) \quad u_t - \Delta u + \varphi(x)u = 0 \quad \text{in} \quad \Omega_T .$$

Recalling that $A(x, t), B(x, t)$ are solutions satisfying

$$A(x, 0) = 1, \quad B(x, 0) = 0 \quad \text{if} \quad x \in \Omega ,$$

it follows that

$$(2.11) \quad D_t A, \nabla_x A, D_t B \quad \text{and} \quad \nabla_x B$$

belong to $L^p(Q_0)$ for any $1 < p < \infty$ and for any compact subdomain Q_0 of

$$\Omega_T \cup \{\Omega \times (t = 0)\} \cup \{\Omega \times (t = T)\} .$$

Hence A and B belong to $C^\alpha(Q_0)$ and, by (2.4), $\varphi \in C^\alpha(\Omega)$.

We can now apply the Schauder estimates to (2.10) and conclude that the functions in (2.11) belong to $C^{2+\alpha}(Q_0)$, Q_0 as before. It follows that $\varphi \in C^{2+\alpha}(\Omega)$. Repeating the last argument step-by-step it follows that $\varphi \in C^{2m+\alpha}$ for all positive integers m .

REMARK 2.2. If σ is small then the mapping σW is a contraction and, consequently, the solution is unique.

REMARK 2.3. Theorem 2.1 extends to the boundary condition

$$(2.12) \quad u = 0 \quad \text{on} \quad \partial\Omega \times (0, T) ,$$

but one must replace

$$(2.13) \quad \int_S G(x, s; \eta) dS_\eta$$

by

$$(2.14) \quad \int_{S^\delta} G(x, s; \eta) dS_\eta$$

where S^δ is obtained from S by moving along the inner normals a distance δ ; δ is positive and small. To explain this notice that if φ is any “reasonable” function and G is the Green function for (2.1) with the boundary condition (2.12), then $G(x, s; \eta) = 0$ for any $\eta \in \partial\Omega$ so that the integral in (2.13) vanishes. Thus there is clearly a mathematical need to modify (2.13) in the definition of φ as given in (0.3). The reason why (2.14) should be the correct replacement for (2.13) is the following: Each polymer chain is terminally attached to S , that is, the first Kuhn segment is fixed. Thus the probabilistic considerations of the SCF theory should disregard the first segment. This means that (2.13) should be replaced by (2.14) where δ is the Kuhn length.

One can check that Theorem 2.1 extends to the present boundary condition (2.12); the main difference occurs in the proof of Lemma 1.6 or, more specifically, in the proof of the analog of Lemma 1.9,

$$(2.15) \quad \int_{S^\delta} \psi(x) dS_x \geq e^{-C(T)} .$$

To prove (2.15) we can proceed essentially as before provided we can establish the following version of Lemma 1.8:

$$(2.16) \quad \int_{\Omega} |\nabla(\log \psi)|^2 \zeta^2 \leq C(T)$$

where ζ is any function in $C_0^\infty(\Omega)$.

Proof of (2.16). Multiplying the equation for ψ by ζ^2/ψ and integrating over Ω , we get

$$\begin{aligned} \int_{\Omega} (\varphi - \lambda) \zeta^2 &= \int_{\Omega} \frac{\Delta \psi}{\psi} \zeta^2 = - \int_{\Omega} \nabla \psi \cdot \nabla \left(\frac{\zeta^2}{\psi} \right) \\ &= \int_{\Omega} \zeta^2 \nabla \psi \cdot \frac{\nabla \psi}{\psi^2} - 2 \int_{\Omega} \frac{\nabla \psi}{\psi} \zeta \cdot \nabla \zeta \\ &\geq \frac{1}{2} \int_{\Omega} \zeta^2 \frac{|\nabla \psi|^2}{\psi^2} - C \int_{\Omega} |\nabla \zeta|^2 , \end{aligned}$$

and (2.16) follows.

Theorem 2.2 also extends to the boundary condition (2.12).

§3. The case $\sigma < 0$. In this case G satisfies

$$(3.1) \quad \left(\frac{\partial}{\partial t} - \Delta - \varphi(x) \right) G(x, t; x') = 0, \quad (x, t) \in \Omega_T$$

and, otherwise, (2.2)–(2.4) remain the same with $\varphi = -\sigma\Psi = |\sigma|\Psi$ in (2.4). Similarly, A and B satisfy the equation

$$(3.2) \quad u_t - \Delta u - \varphi(x)u = 0$$

and otherwise are defined as before; for simplicity we shall designate from now on $|\sigma|$ by σ , so that our system becomes (3.2), (2.2)–(2.4) with $\sigma > 0$.

THEOREM 3.1. *If $n = 1$, then there exists a solution of (3.1), (2.2)–(2.4) (with σ positive in (2.4)) satisfying:*

$$(3.3) \quad \varphi \geq 0, \quad \int_{\Omega} \varphi = T, \quad \varphi \in C(\overline{\Omega}), \quad \nabla \varphi \in L^1(\Omega),$$

and $\varphi \in C^\infty(\Omega)$.

Proof. Since $n = 1$, for any $\varepsilon > 0$

$$(3.4) \quad |v^2|_{L^\infty(\Omega)} < \varepsilon \int_{\Omega} |\nabla v|^2 + C_\varepsilon \int_{\Omega} v^2.$$

Since $\int \varphi = T$, this implies that

$$(3.5) \quad \frac{1}{4} \int_{\Omega} |\nabla v|^2 - \int_{\Omega} \varphi v^2 \geq -C(T) \int_{\Omega} v^2.$$

Analogously to (1.12) we have

$$\iint_{\Omega_t} A_t^2 + \frac{1}{2} \int_{\Omega} |\nabla A(x, t)|^2 \leq \iint_{\Omega_t} \varphi A^2 + C(T).$$

Using (3.5) to estimate the right-hand side, and using also the relation

$$A(x, t) = 1 + \int_0^t A_t(x, s) ds,$$

one can easily derive, by Gronwall's inequality, the bound

$$(3.6) \quad \iint_{\Omega_T} A_t^2 + \int_{\Omega} |\nabla A(x, t)|^2 dx + \int_{\Omega} A^2(x, t) dx \leq C(T) .$$

From Sobolev's imbedding we then also get

$$(3.7) \quad 0 \leq A(x, t) \leq C(T) .$$

We next consider $B(x, t)$, or $w(x, t)$. Using (3.5) and proceeding analogously to (1.15), we get

$$(3.8) \quad \iint_{\Omega_T} w_t^2 + \int_{\Omega} |\nabla w(x, t)|^2 dx \leq C(T) + C(T) \int_{\Omega} w^2(x, t) dx .$$

Using the relations

$$\begin{aligned} \int_{\Omega} w^2(x, t) dx &= \iint_{\Omega_t} 2ww_t(x, t) dx \\ &\leq 2 \iint_{\Omega_T} w^2 + \frac{1}{2} \iint_{\Omega_T} w_t^2 \end{aligned}$$

to estimate the right-hand side of (3.8), we then obtain, by Gronwall's inequality,

$$(3.9) \quad \iint_{\Omega_t} w_t^2 + \int_{\Omega} |\nabla w(x, t)|^2 dx \leq C(T) .$$

The estimates (3.6), (3.7) and (3.9) replace Lemmas 1.1 and 1.2. From these estimate we conclude, as in §1, that

$$(3.10) \quad \int_{\Omega} |\nabla E(x)| \leq C(T) .$$

Since also $\int_{\Omega} E(x) \leq C(T)$ and $n = 1$, it follows that

$$(3.11) \quad 0 \leq E(x) \leq C(T) ;$$

thus Lemma 1.4 is also valid.

We next need an analog of Lemma 1.6. But this is very easy: since $G \geq \Gamma$ (by comparison),

$$(3.12) \quad \int_{\Omega} B(x, T) dx \geq \int_S \int_{\Omega} \Gamma(x, t; x') dx dS_{x'} \geq C(T) > 0;$$

of course, since $n = 1$, S' consists of just one or the two endpoints of Ω .

With the above estimates on E and $\int B(x, T) dx$, we can repeat the proof of Theorem 2.1, and then also of Theorem 2.2, and thus establish Theorem 3.1.

REMARK 3.1. If σ is sufficiently small then one can establish existence and uniqueness for (3.1), (2.2)–(2.4) for any $n \geq 1$. Indeed, we define

$$\mathcal{A} = \{\varphi \in L^\infty(\Omega), 0 \leq \varphi \leq M, \int_{\Omega} \varphi = T\}, \quad M > T/|\Omega|.$$

Then the proofs of Lemma 1.1, 1.2 remain valid for our present equation (3.2) with constants $C(M)$ which depend on M . But then also (3.6)–(3.8) hold with $C = C(M)$, and the same for (3.10), (3.11). Since (3.12) is also valid, W is a bounded operator from $\mathcal{A}$ into $\mathcal{A}$ (with the L^∞ -norm), and σW is then a contraction.

REMARK 3.2. It is not clear whether uniqueness is true in case $\sigma > 0$. For σ negative and large in absolute value, numerical calculations show non-uniqueness when the Laplacian is discretized and the number of interior partition points is 2 or 3.

Acknowledgement. We would like to thank Mathew Tirrell for introducing us to the topic studied in this paper.

REFERENCES

- [1] P.-G. DE GENNES, *Some conformation problems for long macromolecular*, Rep. Prog. Phys., 32 (1969), 187–205.
- [2] A. DOLAN AND S.F. EDWARDS, *Theory of the stabilization of colloids by adsorbed polymer*, Proc. Roy. Soc. London, A. 337 (1974), 509–516.
- [3] A. DOLAN AND S.F. EDWARDS, *The effect of excluded volume on polymer dispersant action*, Proc. Royal Soc. London, A. 343 (1975), 427–442.
- [4] M. MUTHUKUMAR AND J.S. HO, *Self-consistent field theory of surfaces with terminally attached chains*, Macromolecules, 22 (1989), 965–973.
- [5] S.S. PATEL, *Self-consistent field theory of terminally attached polymers: Dolan–Edwards revisited*, submitted to Macromolecules.
- [6] H.J. PLOEHN AND W.B. RUSSEL, *Interaction between colloidal particles and soluble polymer*, submitted to Advances in Chemical Engineering.

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