LAGRANGIAN BLOCK DIAGONALIZATION

By

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1 INTRODUCTION

Consider a Lagrangian system with symmetry group G . Given some configuration $q_e \in Q$ and some element ξ of the Lie algebra $\mathcal{G}$ of G , we wish to determine conditions on q_e and ξ under which the configuration q_e will undergo a steady group motion with group velocity ξ , i.e., that the evolution of q_e according to the dynamics of the Lagrangian system will be given by

$$q_t = \exp(t\xi) \cdot q_e, \quad (1.1)$$

with velocity

$$\dot{q}_t = \xi_Q(q_t) := \frac{\partial}{\partial t} \exp(t\xi) \cdot q_e. \quad (1.2)$$

For example, if $G = SO(3)$, the group of rigid rotations in $\mathbf{R}^3$, then an element $\xi \in \mathcal{G}$ can be identified with a vector in $\mathbf{R}^3$ and the motion q_t consists of a steady rotation about the axis ξ . Once we have identified a pair (q_e, ξ) satisfying (1.1), we wish to consider the stability of this motion. Generally, such motions cannot be strictly stable, since the motion is insensitive to perturbations in particular directions along the group orbit; however, by suitably relaxing the criteria for stability, one can obtain a physically meaningful and verifiable standard of stability for such motions. The two key ingredients of this analysis are the augmented Lagrangian $L_\xi : Q \rightarrow \mathbf{R}$ and the generalized inertia mapping $\Pi : Q \times \mathcal{G} \rightarrow \mathcal{G}^*$. The augmented Lagrangian L_ξ is constructed by identifying each configuration $q \in Q$ with the velocity vector $\xi_Q(q) \in TQ$ obtained by letting q move with group velocity ξ . The value of L_ξ at q is simply the value of the Lagrangian L at $\xi_Q(q)$. We shall show that the augmented Lagrangian can be used to determine equilibria of the form $v_e = \xi_Q(q_e)$. The generalized inertia mapping expresses the momentum associated to group motions; $\Pi(q, \xi)$ is the group momentum of the vector $\xi_Q(q)$. This construction makes possible a simple characterization of the level sets of the momenta which is central to the stability analysis.

The variational structure of Hamiltonian and Lagrangian dynamics is the key to energy stability analyses. Given a relative equilibrium, there exists a conserved quantity, the energy-momentum functional, for which the equilibrium is a critical point. If the energy-momentum functional can be modified so as to produce a conserved quantity for which the relative equilibrium is a local minimum modulo symmetries, then this new conserved quantity can be used, much like a Liapunov function, to demonstrate nonlinear stability of the equilibrium modulo symmetries. The Lagrangian energy-momentum method presented here is a direct extension of the energy-momentum and reduced energy-momentum (REM) methods described in Simo et al. [1989a] and Simo et al. [1989b]. The energy-momentum methods are applicable to simple mechanical systems, i.e. Hamiltonian systems on a cotangent bundle T^*Q with Hamiltonian

$$H(z) = \frac{1}{2} |z|_{g^{-1}}^2 + V(q), \quad (1.3)$$

where g is a group invariant metric on Q and the group invariant potential V depends only on the basepoint q of z . These methods specify a quotient manifold $P_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$ on which the induced energy-momentum functional *may* have a local extremum at $[v_e]$. Conservation

of energy and momentum implies that if $[v_e]$ is a local extremum of the energy-momentum functional, then it remains a local extremum under the induced dynamics; hence the energy-momentum functional can be used as a Liapunov function. If certain regularity conditions on the equilibrium momentum are satisfied, then equivariance of the dynamics imply that the dynamics on TQ are accurately modeled (modulo G_μ) by the induced dynamics on P_μ .

The REM method is an improved version of the energy-momentum method; the two methods provide identical stability conditions, but the REM method is simpler and more efficient to apply. The REM method provides sufficient conditions for (formal) orbital stability of a relative equilibrium with configuration q_e and group momentum μ as follows: By identifying the configuration manifold Q with a subset of the level set of the group momenta, it is possible to construct the *amended potential* V_μ , which is due to Smale [1970a,b]; if certain non-degeneracy assumptions are satisfied, definiteness of the second variation of V_μ modulo momentum conserving group motions, i.e. modulo the orbit of the isotropy subgroup G_μ , implies orbital stability of the relative equilibrium. The energy-momentum method and REM method have been applied to coupled rigid bodies and rods (Simo et al. [1989a]), pseudo-rigid bodies (Lewis and Simo [1989a]), and multiple coupled rigid bodies (Patrick [1990]), among other examples; a trio of simple applications, intended to illustrate the REM method, can be found in Lewis and Simo [1989b]. A related energy method is the energy-Casimir method, in which the quotient $\mathbf{J}^{-1}(\mu)/G_\mu$ is replaced by the quotient T^*Q/G . See Holm et al. [1985] for a detailed discussion of the energy-Casimir method and numerous applications.

We shall show that the block diagonalization results obtained for simple mechanical systems are valid for a much larger class of systems, including all natural Lagrangian systems, as characterized by Gantmacher [1975], §11. It is not necessary for stability that the quotient P_μ or the level set $\mathbf{J}^{-1}(\mu)$ be manifolds. In the current treatment, we test the restriction of the second variation, $D^2E_\xi(v_e)$, of the Lagrangian energy-momentum function to $\ker D\mathbf{J}(v_e)$ for definiteness modulo variations tangent to the orbit of the isotropy subgroup G_μ . (If P_μ is a manifold, this is equivalent to testing the second variation of the induced functional for definiteness on $T_{[v_e]}P_\mu$.) We then use a straightforward extension of a result due to Patrick [1990] to show that, for finite dimensional systems satisfying appropriate conditions on the isotropy subgroups G_μ and G_{v_e} , definiteness of the restricted second variation implies nonlinear stability modulo G_μ . In the infinite dimensional case, additional analysis is required to establish nonlinear stability.

The most notable change in the derivation of the block diagonalization constructions is the use of the Lagrangian L in place of the Hamiltonian $H = \text{'kinetic'} + \text{'potential'}$. The key to the extension of the reduced energy-momentum method is the observation that the appropriate generalizations of a G invariant metric are the Legendre transformation and linearized Legendre transformation. The existence of an energy dependent pairing between variations (in particular, between group variations) is central to the construction of the augmented Lagrangian and stability form; we show that a metric is not necessary for these constructions – it is sufficient that the linearized Legendre transformation satisfy a non-degeneracy condition on the space of group variations. An additional, relatively

minor generalization of the method is the observation that an arbitrary complement to the tangent to the orbit of the isotropy subgroup may be chosen, rather than the orthogonal complement specified in the original treatment of the reduced energy momentum method. The stability conditions obtained are independent of the choice of complement; hence the most convenient choice for the application at hand may be used.

The motivation for the present treatment is two-fold: first, the desire to encompass as large a class of practical applications as possible; second, the theoretical consideration of the assumptions necessary for block diagonalization. In the interest of the latter, we have attempted to minimize the assumptions made in the general treatment of block diagonalization. (For example, we do not assume that relative equilibria are orbit regular, as defined in §4.a, even though we do not presently know of a physically reasonable example which fails to be orbit regular.) We note, however, that many of the central constructions and proofs are actually simplified by a more general treatment. We shall attempt where possible to indicate the simplest and most common cases; the case of a simple mechanical system with a free left $SO(3)$ action (i.e., material frame indifference) may be kept in mind as a prototype.

2 RELATIVE EQUILIBRIA

2.a Definitions

We now briefly describe some of the geometric constructions to be used in the derivation of the equilibrium and stability conditions. For a more detailed description of Hamiltonian and Lagrangian mechanics in the presence of symmetry, see, e.g., Abraham and Marsden [1978]. Our configurations are elements of a manifold Q ; our velocities, or configuration-velocity pairs, are elements of the tangent bundle TQ , while momenta, or configuration-momentum pairs, are elements of the cotangent bundle T^*Q . We assume that a Lie group G acts on Q on the left; this action is denoted by

$$g \cdot q = \Psi_g(q) \quad (2.1)$$

for $g \in G$ and $q \in Q$. The induced action on TQ is given by

$$g \cdot v := T_q \Psi_g \cdot v \quad (2.2)$$

for $v \in T_q Q$. Let $\pi : TQ \rightarrow Q$ denote the canonical fiber bundle projection; in coordinates, $\pi(q, \mathbf{v}) = q$. We require a notation for variation within a *fixed* tangent space fiber, i.e. a variation of the velocity v which leaves the configuration q fixed. Given $v \in T_q Q$, let $\text{vert}_v : T_q Q \rightarrow T_v TQ$ denote the *vertical lift*

$$\text{vert}_v(w) := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} v + \epsilon w \quad \forall w \in T_q Q; \quad (2.3)$$

in coordinates: $\text{vert}_v(w) = (\mathbf{0}, w) \in T_{(q,v)} TQ$.

Let $L : TQ \rightarrow \mathbb{R}$ be a Lagrangian which is invariant under the the action of G on Q , i.e., which satisfies $L(g \cdot v) = L(v)$ for all $g \in G$ and $v \in TQ$. Let $\mathbf{FL} : TQ \rightarrow T^*Q$ denote the Legendre transformation (fiber derivative) of L :

$$\mathbf{FL}(v) \cdot w := DL(v) \cdot \text{vert}_v(w) \quad \forall w \in T_{\pi(v)}Q. \quad (2.4)$$

Let $E : TQ \rightarrow \mathbb{R}$ denote the energy

$$E(v) := \mathbf{FL}(v) \cdot v - L(v) \quad (2.5)$$

associated to the Lagrangian L . The following equivariance results are well-known; we include the proofs both for the sake of completeness and to illustrate in a simple setting the techniques that will be used throughout the paper.

Proposition 2.1 *The Legendre transformation $\mathbf{FL}$ is G equivariant and the energy E is G invariant, i.e.*

$$\mathbf{FL}(T_q \Psi_g \cdot v) = T_q^* \Psi_{g^{-1}} \cdot \mathbf{FL}(v) \quad \text{and} \quad E(g \cdot v) = E(v) \quad (2.6)$$

for all $g \in G$ and $v \in TQ$.

Proof: Given v and $w \in T_q Q$, let v_ϵ be a curve in $T_q Q$ such that $w = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} v_\epsilon$. G invariance of L implies that

$$\left(T_q \Psi_g^* \cdot DL(T_q \Psi_g \cdot v) \right) \cdot w = DL(T_q \Psi_g \cdot v) \cdot (T_q \Psi_g \cdot w) \quad (2.7)$$

$$\begin{aligned} &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L(T_q \Psi_g \cdot v_\epsilon) \\ &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L(v_\epsilon) \\ &= DL(v) \cdot w; \end{aligned} \quad (2.8)$$

hence

$$\mathbf{FL}(T_q \Psi_g \cdot v) = T_q^* \Psi_{g^{-1}} \cdot \mathbf{FL}(v). \quad (2.9)$$

It follows from the equivariance of $\mathbf{FL}$ that

$$\begin{aligned} \mathbf{FL}(g \cdot v) \cdot (g \cdot v) &= \left(T_q^* \Psi_{g^{-1}} \cdot \mathbf{FL}(v) \right) \cdot (T_q \Psi_g \cdot (v)) \\ &= \mathbf{FL}(v) \cdot v; \end{aligned} \quad (2.10)$$

hence, since L is G invariant, E is G invariant. $\checkmark$

We now give a brief summary of the relationship between the dynamics and the Lagrangian L . (For a more detailed description, see Abraham and Marsden [1978].) The Lagrange two-form ω_L is given by

$$\omega_L := \mathbf{FL}^* \omega, \quad (2.11)$$

where ω denotes the canonical symplectic form on the cotangent bundle T^*Q . A Lagrangian vector field is a vector field X_E on TQ satisfying

$$\iota_{X_E} \omega_L = DE, \quad (2.12)$$

i.e.

$$\omega_L(X_E(v), \delta v) = DE(v) \cdot \delta v \quad (2.13)$$

for all $v \in TQ$ and $\delta v \in T_v TQ$. If L is regular, then X_E exists and is second order, i.e. X_E satisfies $T\pi \circ X_E = \text{Id}$. We shall assume throughout that a second order Lagrangian vector field X_E exists and that the dynamics of the system are determined by the relation

$$\dot{v} = X_E(v). \quad (2.14)$$

If Q is finite dimensional and X_E is second order, then (2.14) is equivalent to the Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}. \quad (2.15)$$

Given a configuration q_e undergoing a steady group motion with total velocity ξ , i.e. satisfying (1.1), we say that the associated configuration-velocity pair $v_e \in TQ$ is a *relative equilibrium*. In this case, the evolution of v_e under the Lagrangian dynamics $\dot{v} = X_E(v)$ is

$$v_t = \exp(t\xi) \cdot v_e. \quad (2.16)$$

Hence v_e , as well as q_e , evolves by a steady group motion. We define the *infinitesimal generator* $\xi_Q : Q \rightarrow TQ$ associated to an element ξ of the Lie algebra $\mathcal{G}$ by

$$\xi_Q(q) := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \exp(\epsilon\xi) \cdot q. \quad (2.17)$$

The infinitesimal generator $\xi_{TQ} : TQ \rightarrow T(TQ)$ is defined analogously. Thus, if q_t satisfies (1.1), then $v_e := \xi_Q(q_e)$ and

$$\dot{v}_t = X_E(v_t) = \xi_{TQ}(v_t). \quad (2.18)$$

2.b Relative equilibria as critical points

It is clear from (2.12) that equilibria of the system determined by the Lagrangian vector field X_E must be critical points of the energy E . (If L is regular, then critical points and equilibria are in exact correspondence.) Relative equilibria are also critical points of an appropriate functional. This functional is obtained from the energy by adding a component of the total momentum to the energy E .

Define the *momentum map* $\mathbf{J} : TQ \rightarrow \mathcal{G}^*$ by

$$\mathbf{J}(v) \cdot \eta := \mathbf{FL}(v) \cdot \eta_Q(\pi(v)). \quad (2.19)$$

Given $\xi \in \mathcal{G}$, define the momentum map $J_\xi : TQ \rightarrow \mathbb{R}$ in the direction of ξ by

$$J_\xi(v) := \mathbf{J}(v) \cdot \xi. \quad (2.20)$$

The momentum map satisfies

$$\iota_{\xi_Q} \omega_L = DJ_\xi \quad (2.21)$$

for all $\xi \in \mathcal{G}$. Noether's theorem states that $\mathbf{J}$ is an integral of the Lagrangian dynamics. (See, e.g., Abraham and Marsden [1978].)

The *relative equilibrium theorem* for Hamiltonian systems states that $z_e \in T^*Q$ is a relative equilibrium if and only if there exists $\xi \in \mathcal{G}$ such that z_e is a critical point of the *Hamiltonian energy-momentum function* $H_\xi : T^*Q \rightarrow \mathbf{R}$ given by

$$H_\xi(z) := H(z) - (\mathbf{J}(z) - \mu_e) \cdot \xi, \quad (2.22)$$

where $\mathbf{J}$ denotes, for the moment, the momentum map on the cotangent bundle T^*Q and $\mu_e := \mathbf{J}(z_e)$. An analogous result holds for Lagrangian systems as well. Define the *Lagrangian energy-momentum function* $E_\xi : TQ \rightarrow \mathbf{R}$ by

$$E_\xi(v) := E(v) - J_\xi(v). \quad (2.23)$$

Proposition 2.2 *If $v_e \in TQ$ is a relative equilibrium, i.e. if v_e satisfies (2.18) for some element $\xi \in \mathcal{G}$, then v_e is a critical point of the energy-momentum functional E_ξ . If the Lagrangian L is regular, then v_e is a relative equilibrium if and only if v_e is a critical point of E_ξ for some ξ .*

Proof: The first claim of the theorem follows from (2.21) and (2.12), since

$$\iota_{X_E} \omega_L = DE \quad \text{and} \quad \iota_{\xi_{TQ}} \omega_L = DJ_\xi \quad (2.24)$$

imply

$$\begin{aligned} \mathbf{0} &= \iota_{(X_E(v_e) - \xi_{TQ}(v_e))} \omega_L(v_e) \\ &= \iota_{X_E} \omega_L(v_e) - \iota_{\xi_{TQ}} \omega_L(v_e) \\ &= DE(v_e) - DJ_\xi(v_e) \\ &= DE_\xi(v_e). \end{aligned} \quad (2.25)$$

If L is regular, then (2.18) holds if and only if $DE_\xi(v_e) = \mathbf{0}$, since nondegeneracy of ω_L implies that

$$\iota_{(X_E(v_e) - \xi_{TQ}(v_e))} \omega_L(v_e) = \mathbf{0} \quad \Longleftrightarrow \quad X_E(v_e) = \xi_{TQ}(v_e). \quad (2.26)$$

✓

The relative equilibrium theorem can be phrased in the following fashion: Given a velocity v_e , does there exist an algebra element ξ such that v_e is a critical point of E_ξ ? In many contexts, a more natural question is the following: Given a specific algebra element $\xi \in \mathcal{G}$, for what $q_e \in Q$ is $v_e := \xi_Q(q_e)$ a relative equilibrium? This question can be partially answered by determining the set of critical points of E_ξ of the form $v_e = \xi_Q(q_e)$; the set of relative equilibria with total velocity ξ is contained within this set. (If L is regular, then these two sets coincide.) In fact, relative equilibria with total velocity ξ can be identified with critical points of a simpler function, which depends only on the configuration q . Given $\xi \in \mathcal{G}$, define the *augmented Lagrangian* $L_\xi : Q \rightarrow \mathbf{R}$ by

$$L_\xi(q) := L(\xi_Q(q)). \quad (2.27)$$

We shall show below that $v_e := \xi_Q(q_e)$ is a critical point of E_ξ if and only if q_e is a critical point of L_ξ . (If q_e has non-trivial symmetry, the algebra element ξ need not be uniquely determined.)

We now discuss some of the decompositions of variations which are central to the block diagonalization results. A crucial construction is the decomposition of a general variation in $T(TQ)$ into the image of a variation in TQ under a vector field and the remaining ‘vertical’ component.

Proposition 2.3

- (i) *Given a vector field $\mathbf{v} : Q \rightarrow TQ$ and a variation $\delta v \in T_{\mathbf{v}(q)}TQ$, there exists $\Delta v \in T_q Q$ such that*

$$DE_\xi(\mathbf{v}(q)) \cdot \delta v = D(E_\xi \circ \mathbf{v})(q) \cdot (T_{\mathbf{v}(q)}\pi \cdot \delta v) + D_v^2 L(\mathbf{v}(q))(\Delta v, \mathbf{v}(q) - \xi_Q(q)), \quad (2.28)$$

where D_v denotes differentiation within the fiber $T_q Q$. In particular,

$$DE_\xi(\mathbf{v}(q)) \cdot \text{vert}_{\mathbf{v}(q)}(v) = D_v^2 L(\mathbf{v}(q))(v, \mathbf{v}(q) - \xi_Q(q)) \quad (2.29)$$

for all $v \in T_q Q$.

- (ii) *If $v_e \in T_{q_e} Q$ is a relative equilibrium, then there exists $\xi \in \mathcal{G}$ such that $v_e = \xi_Q(q_e)$ and q_e is a critical point of L_ξ .*
- (iii) *If L is regular at v_e , then v_e is a relative equilibrium if and only if there exists $\xi \in \mathcal{G}$ such that $v_e = \xi_Q(q_e)$ and q_e is a critical point of L_ξ .*

Proof: Let $v := \mathbf{v}(q)$. Consider a variation $\delta v \in T_v TQ$ and define $\delta q := T_v \pi \cdot \delta v$;

$$T_v \pi \cdot (\delta v - T_q \mathbf{v} \cdot \delta q) = \mathbf{0} \quad (2.30)$$

implies there exists $\Delta v \in T_q Q$ such that

$$\delta v = T_q \mathbf{v} \cdot \delta q + \text{vert}_v(\Delta v). \quad (2.31)$$

Thus

$$\begin{aligned} DE_\xi(v) \cdot \delta v &= DE_\xi(v) \cdot (T_q \mathbf{v} \cdot \delta q) + DE_\xi(v) \cdot \text{vert}_v(\Delta v) \\ &= D(E_\xi \circ \mathbf{v})(q) \cdot \delta q + (D_v \text{FL}(v) \cdot \Delta v) \cdot [\mathbf{v}(q) - \xi_Q(q)]. \end{aligned} \quad (2.32)$$

This proves the first claim.

Proposition 2.2 states that if v_e is a relative equilibrium, then there exists $\xi \in \mathcal{G}$ such that v_e is a critical point of E_ξ (if L is regular at v_e , then v_e is a relative equilibrium if and only if such a ξ exists). It follows from (2.32) that $DE_\xi(v_e) = \mathbf{0}$ implies $v_e = \xi_Q(q_e)$, where $q_e := \pi(v_e)$. Hence, to establish the second and third claims, we need to show that v_e is a critical point of E_ξ if and only if q_e is a critical point of L_ξ . We note that

$$\begin{aligned} E_\xi(v) &= \text{FL}(v) \cdot v - L(v) - J_\xi(v) \\ &= \text{FL}(v) \cdot (v - \xi_Q(q)) - L(v) \end{aligned} \quad (2.33)$$

implies

$$E_\xi(\xi_Q(q)) = -L_\xi(q). \quad (2.34)$$

Thus, applying the previous result,

$$\begin{aligned} DE_\xi(\xi_Q(q)) \cdot \delta v &= D(E_\xi \circ \xi_Q)(q) \cdot \delta q \\ &= -DL_\xi(q) \cdot \delta q, \end{aligned} \quad (2.35)$$

so $DE_\xi(v_e) = \mathbf{0}$ if and only if $DL_\xi(q_e) = \mathbf{0}$. $\checkmark$

3 FORMAL STABILITY OF RELATIVE EQUILIBRIA

3.a The energy-momentum method and the configuration-velocity split

Throughout this section, we shall assume that v_e is a relative equilibrium, i.e. a critical point of E_ξ . The argument presented in the preceding section shows that the augmented Lagrangian can be used to identify relative equilibria with a specified velocity; however, the augmented Lagrangian does *not* provide reliable stability information. In the construction of the augmented Lagrangian, a configuration q is identified with the velocity $\xi_Q(q)$, which need not lie on the momentum level set $\mathbf{J}^{-1}(\mu)$. This failure to satisfy the momentum constraint can lead to artificially restrictive stability conditions. In the case of a simple mechanical system, the augmented Lagrangian equals the augmented potential V_ξ ; the second variation of V_ξ is known to yield overly conservative stability conditions. (See Simo et al. [1989b].) As in the simple mechanical case, we must appropriately incorporate the momentum constraints to obtain realistic stability conditions. In the asymmetric simple mechanical case, it was shown that associated to each configuration q and total momentum $\mu \in \mathcal{G}^*$ there is a *unique* algebra element $\eta \in \mathcal{G}$ such that the configuration q moving with total velocity η has total momentum μ . This fact was used to identify the configuration manifold Q with a subset of $\mathbf{J}^{-1}(\mu)$ consisting solely of rigid (group) motions. In the current work, we wish to relax the preceding assumptions, while preserving the essential features of the construction.

The central construction to be used in identifying Q (or a subset of Q) with a subset of $\mathbf{J}^{-1}(\mu)$ is the generalized inertia mapping. This mapping assigns to a pair consisting of a configuration q and an element of the Lie algebra $\mathcal{G}$ the total momentum of the configuration q moving with total velocity ξ . We shall show that for a very general class of Lagrangian systems, the inertia mapping is invertible, or nearly invertible, with respect to the algebra element. This allows us to associate to each configuration a ‘rigid motion’ with total momentum (nearly) equal to μ .

We now introduce some of the terminology associated to the action of the symmetry group G on the manifold Q . The *group orbit* $G \cdot q_e = \{g \cdot q_e : g \in G\}$ has tangent space

$$\mathcal{G} \cdot q_e := \{\eta_Q(q_e) : \eta \in \mathcal{G}\}. \quad (3.1)$$

Let G_{q_e} denote the isotropy subgroup (symmetry group) of q_e :

$$G_{q_e} := \{g \in G : g \cdot q_e = q_e\}. \quad (3.2)$$

The algebra $\mathcal{G}_{q_e}$ of G_{q_e} satisfies

$$\mathcal{G}_{q_e} = \{\eta \in \mathcal{G} : \eta_Q(q_e) = \mathbf{0}\}; \quad (3.3)$$

hence $\mathcal{G}_{q_e}$ is the kernel of the mapping $\eta \mapsto \eta_Q(q_e)$ taking algebra elements to infinitesimal group motions of q_e .

Define the generalized inertia mapping $\mathbb{I} : Q \times \mathcal{G} \rightarrow \mathcal{G}^*$ by

$$\mathbb{I}(q, \eta) := \mathbf{J}(\eta_Q(q)), \quad (3.4)$$

i.e

$$\mathbb{I}(q, \eta) \cdot \zeta = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L((\eta + \epsilon\zeta)_Q(q)). \quad (3.5)$$

The Lagrangian block diagonalization procedure can be carried out if there exists a neighborhood U of q_e and a map $\Xi : U \rightarrow \mathcal{G}$ such that $\Xi(q_e) = \xi$ and the map $\mu : Q \rightarrow \mathcal{G}^*$ defined by

$$\mu(q) := \mathbb{I}(q, \Xi(q)) \quad (3.6)$$

satisfies

$$(D\mu(q_e) \cdot \delta q) \cdot \eta \neq \mathbf{0} \implies \eta \in \mathcal{G}_{q_e} \quad (3.7)$$

for all $\delta q \in T_{q_e}Q$. It is often possible to construct a map Ξ which *exactly* satisfies the condition

$$\mathbb{I}(q, \Xi(q)) = \mu \quad (3.8)$$

for some specified value of μ . (We shall always consider the equilibrium momentum $\mu := \mathbf{J}(v_e)$.) In this case, the map Ξ can be used to identify Q with a subset of $\mathbf{J}^{-1}(\mu)$, as discussed below. This identification is, conceptually, the goal of the following constructions. Condition (3.8) can be satisfied, for example, for simple mechanical systems on which the symmetry group G acts freely. However, (3.8) is substantially more restrictive than (3.7) and is, in fact, more restrictive than is necessary. We shall show below that condition (3.7) is sufficient to enable us to identify a subspace of $T_{q_e}Q$ with a subspace of $T_{v_e}\mathbf{J}^{-1}(\mu)$. This more general identification is sufficient for application of the energy-momentum method. We describe in Section 4.a a fairly general condition under which a map Ξ satisfying (3.7) exists. Given such maps Ξ and μ , the analysis proceeds as follows.

The space $\ker D\mathbf{J}(v_e)$ can be decomposed into two subspaces, one modeled on configuration variations, the other modeled on velocity variations. We first construct the model spaces $\mathcal{Q}$ and $(\mathcal{G} \cdot q_e)^\perp$; both are subspaces of $T_{q_e}Q$, but elements of $\mathcal{Q}$ are to be interpreted as configuration variations, while elements of $(\mathcal{G} \cdot q_e)^\perp$ are viewed as velocities. Define the *locked momentum map* $\mathbb{I}_\xi : Q \rightarrow \mathcal{G}^*$ by

$$\mathbb{I}_\xi(q) := \mathbb{I}(q, \xi) \quad (3.9)$$

and define

$$\mathcal{Q} := \left\{ \delta q \in T_{q_e}Q : D\mathbb{I}_\xi(q_e) \cdot \delta q \in \mathcal{G}_{q_e}^A \right\}, \quad (3.10)$$

where $\mathcal{G}_{q_e}^A \subset \mathcal{G}^*$ denotes the annihilator of the isotropy algebra of q_e . Let $\langle\langle \cdot, \cdot \rangle\rangle_e : T_{q_e}Q \times T_{q_e}Q \rightarrow \mathbb{R}$ denote the bilinear form induced by the linearized fiber derivative:

$$\langle\langle v, w \rangle\rangle_e := \left. \frac{d}{d\tau} \right|_{\tau=0} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L(\xi_Q(q_e) + \tau v + \epsilon w) \quad (3.11)$$

for all $v, w \in T_{v_e}TQ$ and let $(\mathcal{G} \cdot q_e)^\perp$ denote the $\langle\langle \cdot, \cdot \rangle\rangle_e$ orthogonal complement to $\mathcal{G} \cdot q_e$:

$$(\mathcal{G} \cdot q_e)^\perp := \{v \in T_{q_e}Q : \langle\langle v, \eta_Q(q_e) \rangle\rangle_e = 0 \quad \forall \eta \in \mathcal{G}\}. \quad (3.12)$$

The variations associated to $\mathcal{Q}$ capture the configuration-velocity coupling induced by the momentum constraint; the elements of $(\mathcal{G} \cdot q_e)^\perp$ are associated to the remaining ‘uncoupled’ velocity variations. The subspace $\mathcal{Q}$ is embedded in $\ker DJ(v_e)$ by means of the the *locked total momentum* vector field $\mathbf{v}_\mu : U \rightarrow TQ$, given by

$$\mathbf{v}_\mu(q) := [\Xi(q)]_Q(q). \quad (3.13)$$

The designation ‘locked momentum’ is motivated by the fact that

$$\mathbf{J}(\mathbf{v}_\mu(q)) = \mu(q) \quad (3.14)$$

for all $q \in U$, which follows from (3.4) and (3.6). The space $(\mathcal{G} \cdot q_e)^\perp$ is simply vertically lifted to $\ker DJ(v_e)$.

Proposition 3.1 *Define*

$$\mathcal{K}_q := \{T_{q_e}\mathbf{v}_\mu \cdot \delta q : \delta q \in \mathcal{Q}\} \quad (3.15)$$

and

$$\mathcal{K}_v := \left\{ \text{vert}_{v_e}(v) : v \in (\mathcal{G} \cdot q_e)^\perp \right\}. \quad (3.16)$$

Then

$$\ker DJ(v_e) = \mathcal{K}_q \oplus \mathcal{K}_v. \quad (3.17)$$

Proof: Given a element $\delta v \in \ker DJ(v_e)$, define $\delta q := T_{v_e}\pi \cdot \delta v$,

$$\delta v_q := T_{q_e}\mathbf{v}_\mu \cdot \delta q \quad \text{and} \quad \delta v_v := \delta v - \delta v_q. \quad (3.18)$$

By construction, $\delta v_v = \text{vert}_{v_e}(v)$ for some $v \in T_{q_e}Q$. In addition, $\delta v \in \ker DJ(v_e)$ and (3.14) imply

$$\begin{aligned} \langle\langle v, \eta_Q(q_e) \rangle\rangle_e &= (DJ(v_e) \cdot \delta v_v) \cdot \eta \\ &= (DJ(v_e) \cdot \delta v - DJ(v_e) \cdot \delta v_q) \cdot \eta \\ &= -(D\mu(q_e) \cdot \delta q) \cdot \eta. \end{aligned} \quad (3.19)$$

By assumption, $\eta \notin \mathcal{G}_{q_e}$ implies $(D\mu(q_e) \cdot \delta q) \cdot \eta = \mathbf{0}$ and hence $\langle\langle v, \eta_Q(q_e) \rangle\rangle_e = \mathbf{0}$. On the other hand, $\eta \in \mathcal{G}_{q_e}$ if and only if $\eta_Q(q_e) = \mathbf{0}$; hence $\eta \in \mathcal{G}_{q_e}$ implies

$$0 = \langle\langle v, \eta_Q(q_e) \rangle\rangle_e = -(D\mu(q_e) \cdot \delta q) \cdot \eta. \quad (3.20)$$

Thus $\delta v \in \ker DJ(v_e)$ implies $\delta v_v \in \mathcal{K}_v$ and $D\mu(q_e) \cdot \delta q = \mathbf{0}$.

Define the *linearized inertia tensor* $\mathbb{I}_e : \mathcal{G} \rightarrow \mathcal{G}^*$ by

$$\mathbb{I}_e \eta := D_\xi \mathbb{I}(q_e, \xi) \cdot \eta. \quad (3.21)$$

We can use (3.5) to derive the following expressions for $\mathbb{I}_e$: Given η and $\zeta \in \mathcal{G}$,

$$\begin{aligned} \zeta \cdot \mathbb{I}_e \eta &= \left. \frac{d}{d\tau} \right|_{\tau=0} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L[\xi + \tau\eta + \epsilon\zeta]_Q(q_e) \\ &= \langle\langle \eta_Q(q_e), \zeta_Q(q_e) \rangle\rangle_e. \end{aligned} \quad (3.22)$$

Next, note that application of the chain rule to $\mu(q) = \mathbb{I}(q, \Xi(q))$ yields

$$\begin{aligned} D\mu(q_e) \cdot \delta q &= D\mathbb{I}(q_e, \xi) \cdot \delta q + D_\xi \mathbb{I}(q_e, \xi) \cdot (D\Xi(q_e) \cdot \delta q) \\ &= D\mathbb{I}_\xi(q_e) \cdot \delta q + \mathbb{I}_e(D\Xi(q_e) \cdot \delta q). \end{aligned} \quad (3.23)$$

Hence $D\mu(q_e) \cdot \delta q = \mathbf{0}$ implies

$$D\mathbb{I}_\xi(q_e) \cdot \delta q = -\mathbb{I}_e(D\Xi(q_e) \cdot \delta q). \quad (3.24)$$

Since (3.22) implies $\mathbb{I}_e|_{\mathcal{G}_{q_e}} = \mathbf{0}$,

$$(D\mu(q_e) \cdot \delta q)|_{\mathcal{G}_{q_e}} = (D\mathbb{I}_\xi(q_e) \cdot \delta q)|_{\mathcal{G}_{q_e}}. \quad (3.25)$$

Thus $\delta v \in \ker DJ(v_e)$ implies $\delta q = T_{v_e} \pi \cdot \delta v \in \mathcal{Q}$ and, hence, $\delta v_q \in \mathcal{K}_q$. It follows that $\ker DJ(v_e) = \mathcal{K}_q \oplus \mathcal{K}_v$. $\checkmark$

Lemma 3.1

$$\begin{aligned} (D\mathbb{I}_\xi(q_e) \cdot \delta q) \cdot \zeta &= D^2 L(v_e) (\text{vert}_{v_e}(\zeta_Q(q_e)), T_{q_e} \xi_Q \cdot \delta q) \\ &\quad + DL(v_e) \cdot (T_{q_e} \zeta_Q \cdot \delta q). \end{aligned} \quad (3.26)$$

Proof: Let $q_\epsilon \in U$ be a curve tangent to δq at q_e . Then

$$\begin{aligned} (D\mathbb{I}_\xi(q_e) \cdot \delta q) \cdot \zeta &= \left. \frac{d}{d\tau} \right|_{\tau=0} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L((\xi + \tau\zeta)_Q(q_\epsilon)) \\ &= \left. \frac{d}{d\tau} \right|_{\tau=0} DL((\xi + \tau\zeta)_Q(q_e)) \cdot (T_{q_e}(\xi + \tau\zeta)_Q \cdot \delta q) \\ &= D^2 L(v_e) \left(\text{vert}_{\xi_Q(q_e)}(\zeta_Q(q_e)), T_{q_e} \xi_Q \cdot \delta q \right) \\ &\quad + DL(v_e) \cdot (T_{q_e} \zeta_Q \cdot \delta q). \end{aligned} \quad (3.27)$$

$\checkmark$

Lemma 3.1 implies that

$$\mathcal{Q} = \{\delta q \in T_{q_e} Q : DL(v_e) \cdot (T_{q_e} \zeta_Q \cdot \delta q) = 0 \quad \forall \zeta \in \mathcal{G}_{q_e}\}. \quad (3.28)$$

In particular, $\mathcal{Q}$ contains all variations which preserve the symmetry of the equilibrium configuration.

3.b The amended energy

We can identify the space $\mathcal{K}_q$ of configuration-velocity variations with its model space $\mathcal{Q}$ of configuration variations by pulling back the function E_ξ from TQ to Q . Consideration of variations of the function E_ξ with respect to vectors of the form $\delta v = T_{q_e} \mathbf{v}_\mu \cdot \delta q$ naturally suggests the function $E_\xi \circ \mathbf{v}_\mu$. We define the *amended energy* $E_\mu : Q \rightarrow \mathbb{R}$ by

$$\begin{aligned} E_\mu(q) &:= E_\xi(\mathbf{v}_\mu(q)) \\ &= \mathbf{FL}(\mathbf{v}_\mu(q)) \cdot \mathbf{v}_\mu(q) - L(\mathbf{v}_\mu(q)) - \mathbf{J}(\mathbf{v}_\mu(q)) \cdot \xi \\ &= \mu(q) \cdot (\Xi(q) - \xi) - L(\mathbf{v}_\mu(q)). \end{aligned}$$

Proposition 3.2

(i)

$$DE_\mu(q) \cdot \delta q = [D\mu(q) \cdot \delta q] \cdot (\hat{\xi} - \xi) - D(L \circ \hat{\xi}_Q)(q) \cdot \delta q, \quad (3.29)$$

where $\hat{\xi} := \Xi(q)$, for all $q \in U$ and $\delta q \in T_q Q$. In particular,

$$DE_\mu(q_e) = \mathbf{0}. \quad (3.30)$$

(ii)

$$D^2 E_\mu(q_e)(\delta q, \Delta q) = -D^2 L_\xi(q_e)(\delta q, \Delta q) - (D\mathbb{I}_\xi(q_e) \cdot \Delta q)(D\Xi(q_e) \cdot \delta q). \quad (3.31)$$

Proof:

$$\begin{aligned} DE_\mu(q) \cdot \delta q &= (D\mu(q) \cdot \delta q) \cdot (\hat{\xi} - \xi) + \mu(q) \cdot (D\Xi(q) \cdot \delta q) \\ &\quad - DL(\mathbf{v}_\mu(q)) \cdot \left[\text{vert}_{\hat{\xi}_Q(q)} \left([D\Xi(q) \cdot \delta q]_Q(q) \right) \right] \\ &= (D\mu(q) \cdot \delta q) \cdot (\hat{\xi} - \xi) + \mu(q) \cdot (D\Xi(q) \cdot \delta q) \\ &\quad - \mathbf{FL}(\mathbf{v}_\mu(q)) \cdot [D\Xi(q) \cdot \delta q]_Q(q) - DL(\hat{\xi}_Q(q)) \cdot (T_q \hat{\xi}_Q \cdot \delta q) \\ &= (D\mu(q) \cdot \delta q) \cdot (\hat{\xi} - \xi) - D(L \circ \hat{\xi}_Q)(q) \cdot \delta q, \end{aligned} \quad (3.32)$$

since $\mathbf{v}_\mu(q) = \hat{\xi}_Q(q)$ and

$$\begin{aligned} \mathbf{FL}(\mathbf{v}_\mu(q)) \cdot \eta_Q(q) &= \mathbf{J} \left([\Xi(q)]_Q(q) \right) \cdot \eta \\ &= \mathbb{I}(q, \Xi(q)) \cdot \eta \\ &= \mu(q) \cdot \eta \end{aligned} \quad (3.33)$$

for all $\eta \in \mathcal{G}$.

To prove the second claim, let $\tilde{\delta}q : U \rightarrow TQ$ denote an arbitrary vector field extension of the variation δq , let q_ϵ be a curve tangent to Δq , and define $\xi^\epsilon := \Xi(q_\epsilon)$; using (3.24), (3.27), and (3.29), we derive the relationship

$$\begin{aligned} D^2 E_\mu(q_e)(\delta q, \Delta q) &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} [DE_\mu(q_\epsilon) \cdot \tilde{\delta}q(q_\epsilon)] \end{aligned}$$

$$\begin{aligned}
&= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left[\left(D\mu(q_\epsilon) \cdot \tilde{\delta}q(q_\epsilon) \right) \cdot (\xi^\epsilon - \xi) - DL(\xi_Q^\epsilon(q_\epsilon)) \cdot \left(T_{q_\epsilon} \xi_Q^\epsilon \cdot \tilde{\delta}q(q_\epsilon) \right) \right] \\
&= -D^2 L(\xi_Q(q_e)) (T_{q_e} \xi_Q \cdot \delta q, T_{q_e} \xi_Q \cdot \Delta q) \\
&\quad - D^2 L(\xi_Q(q_e)) \left(\text{vert}_{\xi_Q(q_e)} \left([D\Xi(q_e) \cdot \Delta q]_Q(q_e) \right), T_{q_e} \xi_Q \cdot \delta q \right) \\
&\quad - DL(\xi_Q(q_e)) \left(T_{q_e} [D\Xi(q_e) \cdot \Delta q]_Q \cdot \delta q \right) \\
&= -D^2 L_\xi(q_e)(\delta q, \Delta q) - (D\mathbb{I}_\xi(q_e) \cdot \Delta q) (D\Xi(q_e) \cdot \delta q).
\end{aligned} \tag{3.34}$$

3.c Block diagonalization and formal stability conditions

Proposition 3.3

(i) The second variation $D^2 E_\xi(v_e)|_{\ker DJ(v_e)}$ block diagonalizes with respect to $\mathcal{K}_q$ and $\mathcal{K}_v$, i.e.

$$D^2 E_\xi(v_e)(\delta v_q, \delta v_v) = 0 \quad \forall (\delta v_q, \delta v_v) \in \mathcal{K}_q \times \mathcal{K}_v. \tag{3.35}$$

(ii) The second variation of E_ξ restricted to $\mathcal{K}_q$ takes the form

$$D^2 E_\xi(v_e)(T_{q_e} \mathbf{v}_\mu \cdot \delta q, T_{q_e} \mathbf{v}_\mu \cdot \Delta q) = \mathcal{B}_e(\delta q, \Delta q), \tag{3.36}$$

where the stability two form $\mathcal{B}_e : \mathcal{Q} \times \mathcal{Q}$ is given by

$$\mathcal{B}_e(\delta q, \Delta q) := (D\Xi(q_e) \cdot \Delta q) \cdot \mathbb{I}_e(D\Xi(q_e) \cdot \delta q) - D^2 L_\xi(q_e)(\delta q, \Delta q). \tag{3.37}$$

(iii) The second variation of E_ξ restricted to $\mathcal{K}_v$ takes the form

$$D^2 E_\xi(v_e)(\text{vert}_{v_e}(v_1), \text{vert}_{v_e}(v_2)) = \langle\langle v_1, v_2 \rangle\rangle_e. \tag{3.38}$$

Proof: Given $\delta v_q \in \mathcal{K}_q$ and $\delta v_v \in \mathcal{K}_v$, let $\delta q \in \mathcal{Q}$ and $v \in (\mathcal{G} \cdot q_e)^\perp$ denote the associated configuration variation and velocity. Let $\mathbf{v} : U \rightarrow T^*Q$ denote a vector field such that $\mathbf{v}(q_e) = v$ and let q_ϵ be a curve tangent to δq . Then, using (2.29) and the fact that v_e is a critical point of E_ξ , we see that

$$\begin{aligned}
D^2 E_\xi(v_e)(\delta v_v, \delta v_q) &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left[DE_\xi(\mathbf{v}_\mu(q_\epsilon)) \cdot \text{vert}_{\mathbf{v}_\mu(q_\epsilon)}(\mathbf{v}(q_\epsilon)) \right] \\
&= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} [D_v \mathbf{FL}(\mathbf{v}_\mu(q_\epsilon)) \cdot \mathbf{v}(q_\epsilon)] \cdot (\mathbf{v}_\mu(q_\epsilon) - \xi_Q(q_\epsilon)) \\
&= \left\langle\left\langle v, [D\Xi(q_e) \cdot \delta q]_Q(q_e) \right\rangle\right\rangle_e \\
&= 0.
\end{aligned} \tag{3.39}$$

Proposition 3.2 and (3.24) imply

$$\begin{aligned}
&D^2 E_\xi(v_e)(T_{q_e} \mathbf{v}_\mu \cdot \delta q, T_{q_e} \mathbf{v}_\mu \cdot \Delta q) \\
&= D^2 E_\mu(q_e)(\delta q, \Delta q) \\
&= -D^2 L_\xi(q_e)(\delta q, \Delta q) - (D\mathbb{I}_\xi(q_e) \cdot \Delta q) (D\Xi(q_e) \cdot \delta q) \\
&= -D^2 L_\xi(q_e)(\delta q, \Delta q) + (D\Xi(q_e) \cdot \delta q) \cdot \mathbb{I}_e(D\Xi(q_e) \cdot \Delta q) \\
&= \mathcal{B}_e(\delta q, \Delta q)
\end{aligned} \tag{3.40}$$

for $\delta q, \Delta q \in \mathcal{Q}$, which proves the second claim.

The proof of the third claim is straightforward:

$$\begin{aligned} D^2 E_\xi(v_e)(\text{vert}_{v_e}(v), \text{vert}_{v_e}(\hat{v})) &= D_v^2 L(v_e)(v, \hat{v}) \\ &= \langle\langle v, \hat{v} \rangle\rangle_e. \end{aligned} \quad (3.41)$$

✓

The G_μ invariance of E_ξ implies that

$$D^2 E_\xi(v_e)(\zeta_Q(q_e), \delta v) = 0 \quad (3.42)$$

for all $\zeta \in \mathcal{G}_\mu$ and $\delta v \in T_{v_e}(TQ)$. (See Proposition 3.5 for a direct proof of this fact.) Hence, $D^2 E_\xi(v_e)$ can never be definite; instead, we consider definiteness modulo the tangent to the G_μ orbit. This can be accomplished either by constructing an explicit complement $\mathcal{S}$ to the tangent space $\mathcal{G}_\mu \cdot v_e \subset \ker DJ(v_e)$ or by replacing the usual condition that the second variation be definite with the condition that the second variation be semi-definite, with kernel $\mathcal{G}_\mu \cdot v_e$. We briefly describe both approaches.

Let $\tilde{\mathcal{V}}$ be a complement to the linearized isotropy orbit $\mathcal{G}_\mu \cdot q_e$ in $T_{q_e}Q$ and define

$$\mathcal{V} := \tilde{\mathcal{V}} \cap \mathcal{Q} \quad (3.43)$$

and

$$\mathcal{S} := \{\delta v \in \ker DJ(v_e) : T_{q_e}\pi \cdot \delta v \in \mathcal{V}\}. \quad (3.44)$$

Clearly, $\mathcal{K}_v \subset \mathcal{S}$, since $T_{q_e}\pi \cdot \text{vert}_{v_e}(v) = \mathbf{0}$ for all $v \in T_{q_e}Q$. The complement condition does, however, restrict the configuration variations:

$$T_{q_e}\mathbf{v}_\mu \cdot \delta q \in \mathcal{S} \iff \delta q \in \mathcal{V}. \quad (3.45)$$

Thus, the stability conditions may be summarized as follows: The relative equilibrium v_e is formally stable, in the sense that the second variation $D^2 E_\xi(v_e)|_{\mathcal{S}}$ is positive (negative) definite, if

(i) $\langle\langle \cdot, \cdot \rangle\rangle_e|_{(\mathcal{G} \cdot q_e)^\perp}$ is positive (negative) definite

(ii) $\mathcal{B}_e|_{\mathcal{V}}$ is positive (negative) definite.

The stability conditions are independent of the choice of complement. Hence the second condition may be replaced by the following:

(ii)' $\mathcal{B}_e$ is positive (negative) semi-definite, with kernel $\mathcal{G}_\mu \cdot q_e$.

3.d Decomposition of configuration variations

There is a further decomposition available which can be used to simplify the determination of the definiteness of the the stability form $\mathcal{B}_e$. The configuration variations in the space $\mathcal{V}$ can be decomposed into ‘rigid variations’ and ‘internal variations’. This further decomposition not only reduces the size of the blocks to be checked for definiteness; it also exploits the group invariance of the Lagrangian to obtain a relatively simple expression for the form $\mathcal{B}_e$ on the subspace of group variations.

3.d.i Group variations

While the original Lagrangian L is G invariant, and hence variations of L with respect to group motions are trivial, the modified functions L_ξ , E_ξ , and E_μ are *not* G invariant, since the introduction of the total velocity ξ and total momentum μ results in a ‘preferred direction’ which breaks the symmetry of the original system. However, the variations of the augmented Lagrangian L_ξ and the inertia tensor $\mathbb{I}$ with respect to variations along the group orbit $G \cdot q_e$ take a particularly simple form which depends only on the fiber derivatives of the Lagrangian.

Proposition 3.4

(i)

$$D(L \circ \zeta_Q)(q) \cdot \eta_Q(q) = \text{ad}_\zeta^* \mathbb{I}(q, \zeta) \cdot \eta \quad (3.46)$$

for all η and $\zeta \in \mathcal{G}$ and $q \in Q$.

(ii) $v_e = \xi_Q(q_e)$ implies $\xi \in \mathcal{G}_\mu$.

(iii)

$$DE_\mu(q) \cdot \eta_Q(q) = -\text{ad}_{\Xi(q)}^* \mu(q) \cdot \eta + [D\mu(q) \cdot \eta_Q(q)] \cdot (\Xi(q) - \xi) \quad (3.47)$$

for all $\eta \in \mathcal{G}$ and $q \in U$.

(iv)

$$D\mathbb{I}_\xi(q_e) \cdot \eta_Q(q)_e = -(\mathbb{I}_e(\text{ad}_\eta \xi) + \text{ad}_\eta^* \mu). \quad (3.48)$$

Proof: The following equivariance property is the key to the proposition: G invariance of L implies

$$\begin{aligned} L(\zeta_Q(g \cdot q)) &= L\left(g \cdot \left([\text{Ad}_{g^{-1}} \zeta]_Q(q)\right)\right) \\ &= L\left([\text{Ad}_{g^{-1}} \zeta]_Q(q)\right) \end{aligned} \quad (3.49)$$

for all $\zeta \in \mathcal{G}$, $g \in G$, and $q \in Q$.

(3.49) implies

$$\begin{aligned} D(L \circ \zeta_Q)(q) \cdot \eta_Q(q) &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L(\zeta_Q(\exp(\epsilon \eta) \cdot q)) \\ &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L([\text{Ad}_{\exp(-\epsilon \eta)} \zeta]_Q(q)) \\ &= -\mathbf{FL}(\zeta_Q(q)) \cdot [\text{ad}_\eta \zeta]_Q(q) \\ &= -\mathbb{I}(q, \zeta) \cdot \text{ad}_\eta \zeta \\ &= \text{ad}_\zeta^* \mathbb{I}(q, \zeta) \cdot \eta. \end{aligned} \quad (3.50)$$

In particular,

$$0 = DL_\xi(q_e) \cdot \eta_Q(q_e) = \text{ad}_\xi^* \mu \cdot \eta \quad (3.51)$$

for all $\eta \in \mathcal{G}$ implies $\xi \in \mathcal{G}_\mu$. Combining the first result with Proposition 3.2 and $\mu(q) = \mathbb{I}(q, \Xi(q))$ yields

$$DE_\mu(q) \cdot \eta_Q(q) = [D\mu(q) \cdot \eta_Q(q)] \cdot (\Xi(q) - \xi) - \text{ad}_{\Xi(q)}^* \mu(q) \cdot \eta, \quad (3.52)$$

which proves the third claim.

Given $\eta \in \mathcal{G}$, we derive the expression for $D\mathbb{I}_\xi(q_e) \cdot \eta_Q(q_e)$ as follows: (3.5) and (3.49) imply

$$\begin{aligned}
\mathbb{I}(g \cdot q, \zeta) \cdot \omega &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L((\zeta + \epsilon\omega)_Q(g \cdot q)) \\
&= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L\left(\left[\text{Ad}_{g^{-1}}(\zeta + \epsilon\omega)\right]_Q(q)\right) \\
&= \text{FL}\left(\left[\text{Ad}_{g^{-1}}\zeta\right]_Q(q)\right) \cdot \left[\text{Ad}_{g^{-1}}\omega\right]_Q(q) \\
&= \mathbb{I}(q, \text{Ad}_{g^{-1}}\zeta) \cdot \text{Ad}_{g^{-1}}\omega.
\end{aligned} \tag{3.53}$$

for all $q \in \mathcal{Q}$, $g \in G$, and ζ and $\omega \in \mathcal{G}$. Application of this result to the curve $g_\epsilon := \exp(\epsilon\eta)$ in G yields

$$\begin{aligned}
(D\mathbb{I}(q, \zeta) \cdot \eta_Q(q)) \cdot \omega &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathbb{I}((\exp(\epsilon\eta)) \cdot q, \zeta) \cdot \omega \\
&= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left[\mathbb{I}(q, \text{Ad}_{\exp(-\epsilon\eta)}\zeta) \cdot \text{Ad}_{\exp(-\epsilon\eta)}\omega\right] \\
&= (D_\zeta \mathbb{I}(q, \zeta) \cdot (-\text{ad}_\eta \zeta)) \cdot \omega + \mathbb{I}(q, \zeta) \cdot (-\text{ad}_\eta \omega) \\
&= -\left[D_\zeta \mathbb{I}(q, \zeta) \cdot (\text{ad}_\eta \zeta) + \text{ad}_\eta^* \mathbb{I}(q, \zeta)\right] \cdot \omega.
\end{aligned} \tag{3.54}$$

Hence (3.48) holds. $\checkmark$

We now characterize the ‘rigid variations’, which are a subspace of the group orbit $\mathcal{G} \cdot q_e$. Define the space

$$\mathcal{Q}_{\text{rig}} := \mathcal{G} \cdot q_e \cap \mathcal{Q} \tag{3.55}$$

Our immediate goal is the specification of a subspace $\mathcal{G}_{\mathcal{Q}} \subseteq \mathcal{G}$ such that $\mathcal{Q}_{\text{rig}} = \mathcal{G}_{\mathcal{Q}} \cdot q_e$. Since $\mathcal{G}_{q_e} \subseteq \ker \mathbb{I}_e$, it follows from Proposition 3.4.i that $D\mathbb{I}_\xi(q_e) \cdot \eta_Q(q) \in \mathcal{G}_{q_e}^A$, and hence $\eta_Q(q_e) \in \mathcal{Q}$, if and only if $\text{ad}_\eta^* \mu \in \mathcal{G}_{q_e}^A$. Hence, if we define the subspace $\mathcal{G}_{\mathcal{Q}} \subseteq \mathcal{G}$ by

$$\mathcal{G}_{\mathcal{Q}} := \left\{ \eta \in \mathcal{G} : \text{ad}_\eta^* \mu \in \mathcal{G}_{q_e}^A \right\}, \tag{3.56}$$

then

$$\mathcal{Q}_{\text{rig}} = \mathcal{G}_{\mathcal{Q}} \cdot q_e. \tag{3.57}$$

Note that $\mathcal{G}_{\mathcal{Q}} \supseteq \mathcal{G}_\mu \cup \mathcal{G}_{q_e}$, since $\mathcal{G}_{q_e}$ is a subalgebra and $\mu \in \mathcal{G}_{q_e}^A$.

We now show that the stability two form $\mathcal{B}_e$ takes a particularly simple form on group variations. Define the generalized Arnold form $\mathcal{A} : \mathcal{G}_{\mathcal{Q}} \times \mathcal{G}_{\mathcal{Q}} \rightarrow \mathbb{R}$ by

$$\mathcal{A}(\zeta, \eta) := \text{ad}_\zeta^* \mu \cdot (D\Xi(q_e) \cdot \eta_Q(q_e)). \tag{3.58}$$

Proposition 3.5

$$D^2 E_\mu(q_e)(\eta_Q(q_e), \delta q) = \text{ad}_\eta^* \mu \cdot (D\Xi(q_e) \cdot \delta q) \tag{3.59}$$

for all $\eta \in \mathcal{G}$ and $\delta q \in \mathcal{Q}$. In particular,

$$D^2 E_\mu(q_e)(\eta_Q(q_e), \zeta_Q(q_e)) = \mathcal{A}(\eta, \zeta) \quad \forall \eta, \zeta \in \mathcal{G}_{\mathcal{Q}}. \tag{3.60}$$

Proof: Consider $\eta \in \mathcal{G}$ and $\delta q \in \mathcal{Q}$; let q_ϵ be a curve tangent to δq . Proposition 3.4.iii implies

$$\begin{aligned}
D^2 E_\mu(q_e)(\eta_Q(q_e), \delta q) &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} D E_\mu(q_\epsilon) \cdot \eta_Q(q_\epsilon) \\
&= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left[-\text{ad}_{\Xi(q_e)}^* \mu(q_\epsilon) \cdot \eta + [D\mu(q_\epsilon) \cdot \eta_Q(q_\epsilon)] \cdot (\Xi(q_\epsilon) - \xi) \right] \\
&= -\text{ad}_{(D\Xi(q_e) \cdot \delta q)}^* \mu \cdot \eta \\
&= \text{ad}_\eta^* \mu \cdot (D\Xi(q_e) \cdot \delta q),
\end{aligned} \tag{3.61}$$

since $D\mu(q_e) \cdot \delta q = \mathbf{0}$ for all $\delta q \in \mathcal{Q}$. $\checkmark$

3.d.ii Internal variations

The next step in the decomposition is the definition of a space $\mathcal{Q}_{\text{int}}$ of ‘internal variations’ such that the stability form $\mathcal{B}_e$ block diagonalizes with respect to $\mathcal{Q}_{\text{rig}}$ and $\mathcal{Q}_{\text{int}}$. The space $\mathcal{Q}_{\text{int}}$ is simply the orthogonal complement to $\mathcal{Q}_{\text{rig}}$ with respect to the bilinear form $\mathcal{B}_e$; this apparent tautology is justified by the fact that the simple form (3.59) taken by $\mathcal{B}_e|_{\mathcal{Q}_{\text{rig}} \times \mathcal{Q}}$ makes the determination of $\mathcal{Q}_{\text{int}}$ relatively simple. For a large class of systems, there is a simple physical interpretation of the space $\mathcal{Q}_{\text{int}}$.

Proposition 3.4.iv and (3.57) imply that

$$D^2 E_\mu(q_e)(\Delta q, \delta q) = 0 \quad \forall \Delta q \in \mathcal{Q}_{\text{rig}} \tag{3.62}$$

if and only if

$$\text{ad}_{(D\Xi(q_e) \cdot \delta q)}^* \mu \cdot \eta = 0 \quad \forall \eta \in \mathcal{G}_{\mathcal{Q}}. \tag{3.63}$$

Hence, we define the subspace $\mathcal{Q}_{\text{int}} \subset \mathcal{Q}$ by

$$\mathcal{Q}_{\text{int}} := \left\{ \delta q \in \mathcal{Q} : \text{ad}_{(D\Xi(q_e) \cdot \delta q)}^* \mu \in \mathcal{G}_{\mathcal{Q}}^A \right\}. \tag{3.64}$$

By construction, $D^2 E_\mu(q_e)|_{\mathcal{Q}_{\text{rig}} \times \mathcal{Q}_{\text{int}}} = \mathbf{0}$.

Remark: For a large class of relative equilibria, including asymmetric equilibria, the definitions of the spaces $\mathcal{Q}_{\text{rig}}$ and $\mathcal{Q}_{\text{int}}$ take a somewhat simpler form. Specifically, if $G_\mu \supseteq G_{q_e}$, then $\mathcal{G}_{\mathcal{Q}} = \mathcal{G}$; hence

$$\mathcal{Q}_{\text{rig}} = \mathcal{G} \cdot q_e \quad \text{and} \quad \mathcal{Q}_{\text{int}} = \{ \delta q \in \mathcal{Q} : D\Xi(q_e) \cdot \delta q \in \mathcal{G}_\mu \}. \tag{3.65}$$

Define the subspaces $\mathcal{V}_{\text{rig}}$ and $\mathcal{V}_{\text{int}}$ of $\mathcal{V}$ by

$$\mathcal{V}_{\text{rig}} := \mathcal{Q}_{\text{rig}} \cap \tilde{\mathcal{V}} \quad \text{and} \quad \mathcal{V}_{\text{int}} := \mathcal{Q}_{\text{int}} \cap \tilde{\mathcal{V}}. \tag{3.66}$$

Proposition 3.6 *If G is finite dimensional and $\mathcal{A}(\zeta, \cdot) = \mathbf{0}$ implies $\zeta \in \mathcal{G}_\mu \cup \mathcal{G}_{q_e}$, e.g. if $D^2 E_\mu(q_e)|_{\mathcal{V}_{\text{rig}} \times \mathcal{V}_{\text{rig}}}$ is definite, then*

$$\mathcal{Q} = \mathcal{Q}_{\text{rig}} + \mathcal{Q}_{\text{int}}, \quad \mathcal{Q}_{\text{rig}} \cap \mathcal{Q}_{\text{int}} = \mathcal{G}_\mu \cdot q_e, \tag{3.67}$$

and hence

$$\mathcal{V} = \mathcal{V}_{\text{rig}} \oplus \mathcal{V}_{\text{int}}. \quad (3.68)$$

If G is infinite dimensional, we must additionally assume that

$$\text{Range } \mathcal{A} + \mathcal{G}_Q^A = (\mathcal{G}_\mu \cup \mathcal{G}_{q_e})^A, \quad (3.69)$$

where $\mathcal{A}$ is viewed as a linear map from $\mathcal{G}_Q$ to $\mathcal{G}^*$, to guarantee that the decompositions are valid.

Proof: To simplify the notation, let $\mathcal{G}_\iota := \mathcal{G}_\mu \cup \mathcal{G}_{q_e} \subseteq \mathcal{G}_Q$. Equations (3.59) and (3.60) imply that $\ker \mathcal{A} \supseteq \mathcal{G}_\iota$ and, since $\mathcal{A}$ is symmetric,

$$\text{Range } \mathcal{A} + \mathcal{G}_Q^A \subseteq \mathcal{G}_\iota^A. \quad (3.70)$$

It follows from (3.60) and (3.64) that $\zeta_Q(q_e) \in \mathcal{Q}_{\text{int}}$ implies $\mathcal{A}(\zeta, \cdot) = \mathbf{0}$ and, since we assume that $\mathcal{A}$ is nondegenerate on the complement in $\mathcal{G}_Q$ to $\mathcal{G}_\iota$,

$$\mathcal{Q}_{\text{rig}} \cap \mathcal{Q}_{\text{int}} = \mathcal{G}_\iota \cdot q_e = \mathcal{G}_\mu \cdot q_e. \quad (3.71)$$

Nondegeneracy of $\mathcal{A}$ modulo $\mathcal{G}_\iota$ implies that $\text{Range } \mathcal{A} \cap \mathcal{G}_Q^A = \mathbf{0}$. Hence, if G is finite dimensional, $\text{Range } \mathcal{A} + \mathcal{G}_Q^A = \mathcal{G}_\iota^A$, since

$$\begin{aligned} \text{codim}_{\mathcal{G}_\iota^A} \mathcal{G}_Q^A &\leq \dim \mathcal{G}_Q - \dim \mathcal{G}_\iota \\ &= \dim \text{Range } \mathcal{A} \end{aligned} \quad (3.72)$$

implies

$$\begin{aligned} \text{codim}_{\mathcal{G}_\iota^A} (\text{Range } \mathcal{A} + \mathcal{G}_Q^A) &= \dim \text{Range } \mathcal{A} - \text{codim}_{\mathcal{G}_\iota^A} \mathcal{G}_Q^A - \dim (\text{Range } \mathcal{A} \cap \mathcal{G}_Q^A) \\ &= 0. \end{aligned} \quad (3.73)$$

Let $\delta q \in \mathcal{Q}$. Proposition (3.5) implies

$$\text{ad}_{(D\Xi(q_e) \cdot \delta q)}^* \mu \in \mathcal{G}_\iota^A; \quad (3.74)$$

hence there exists $\eta \in \mathcal{G}_Q$ such that $\tilde{\delta q} := \delta q - \eta_Q(q_e)$ satisfies

$$\begin{aligned} \text{ad}_{(D\Xi(q_e) \cdot \tilde{\delta q})}^* \mu &= \text{ad}_{(D\Xi(q_e) \cdot \delta q)}^* \mu - \text{ad}_{(D\Xi(q_e) \cdot \eta_Q(q_e))}^* \mu \\ &= \text{ad}_{(D\Xi(q_e) \cdot \delta q)}^* \mu - \mathcal{A}(\eta, \cdot) \\ &\in \mathcal{G}_Q^A. \end{aligned} \quad (3.75)$$

Thus $\tilde{\delta q} \in \mathcal{Q}_{\text{int}}$ and, since $\eta_Q(q_e) \in \mathcal{Q}_{\text{rig}}$, $\mathcal{Q} = \mathcal{Q}_{\text{rig}} + \mathcal{Q}_{\text{int}}$. Finally, $\mathcal{V} \cap (\mathcal{G}_\mu \cdot q_e) = \mathbf{0}$ implies $\mathcal{V} = \mathcal{V}_{\text{rig}} \oplus \mathcal{V}_{\text{int}}$. $\checkmark$

Let $\tilde{\mathcal{G}}_Q \subset \mathcal{G}_Q$ be a subspace satisfying the conditions

$$\mathcal{V}_{\text{rig}} = \tilde{\mathcal{G}}_Q \cdot q_e \quad \text{and} \quad \tilde{\mathcal{G}}_Q \cap \mathcal{G}_{q_e} = \mathbf{0}. \quad (3.76)$$

The relative equilibrium v_e is formally stable, in the sense that the second variation $D^2 E_\xi(v_e)|_S$ is positive (negative) definite, if

- (i) $\langle\langle \cdot, \cdot \rangle\rangle_e|_{(\mathcal{G} \cdot q_e)^\perp}$ is positive (negative) definite
- (ii) $\mathcal{A}|_{\tilde{\mathcal{G}}_Q}$ is positive (negative) definite
- (iii) $\mathcal{B}_e|_{\mathcal{V}_{\text{int}}}$ is positive (negative) definite.
- (iv) (3.69) holds; e.g., G is finite dimensional.

The stability criteria can be expressed without making a specific choice of the complement to $\mathcal{G}_\mu \cdot q_e$. If we define $\mathcal{B}_{\text{int}} := \mathcal{B}_e|_{\mathcal{Q}_{\text{int}}}$, then conditions (ii) and (iii) may be replaced by:

- (ii)' $\mathcal{A}$ is positive (negative) semi-definite, with kernel $\mathcal{G}_\mu \cup \mathcal{G}_{q_e}$
- (iii)' $\mathcal{B}_{\text{int}}$ is positive (negative) semi-definite, with kernel $\mathcal{G}_\mu \cdot q_e$.

4 APPLICATION OF BLOCK DIAGONALIZATION

In the preceding section, our goal was the derivation of the block diagonalization algorithm in as general a context as possible. In the present section we shall address the following practical considerations. First, are there simple, readily verifiable conditions which guarantee that the block diagonalization procedure can be carried out? Second, under what conditions does formal stability imply nonlinear orbital stability? We first describe a sufficiency condition, referred to as orbit regularity, for applicability of block diagonalization. It is straightforward to test for this condition if the group G is finite dimensional; if G is infinite, verification may be more delicate. We then discuss the conditions under which formal stability implies orbital stability with respect to an appropriate symmetry group. If the configuration manifold Q is finite dimensional, we can apply an extension of a result due to Patrick [1990]; if appropriate conditions on the isotropy subgroups G_μ and G_{v_e} are satisfied, then formal stability implies orbital stability with respect to G_μ . While the spirit of these results carries over to the infinite dimensional case, analytic difficulties can arise if the topology of TQ is poorly matched to the Lagrangian. Hence, we make no general claims regarding the relationship between formal and orbital stability in the infinite dimensional case.

We note that if the relative equilibrium v_e has nontrivial isotropy, then the equilibrium group velocity is not uniquely defined. In this case, it is necessary to consider all possible choices of group velocity to obtain optimal stability conditions.

4.a Orbit regular equilibria

In this section, we describe a class of relative equilibria for which the block diagonalization procedure can be carried out, i.e., for which a map $\Xi : Q \rightarrow \mathcal{G}$ satisfying (3.7) can be found. We shall say that a triplet (L, ξ, q_e) consisting of a C^1 G invariant Lagrangian $L : Q \rightarrow \mathbb{R}$, an element ξ of the Lie algebra $\mathcal{G}$, and a point q_e is *orbit regular* if there exists a complement $\tilde{\mathcal{G}}$ to $\mathcal{G}_{q_e}$ in $\mathcal{G}$ such that $\xi \in \tilde{\mathcal{G}}$ and $\Pi_e|_{\tilde{\mathcal{G}}}$ is an isomorphism between $\tilde{\mathcal{G}}$ and $\mathcal{G}_{q_e}^A$, where $\mathcal{G}_{q_e}^A$ denotes the annihilator of $\mathcal{G}_{q_e}$. Since

$$\eta \cdot \Pi_e \zeta = \langle\langle \eta_Q(q_e), \zeta_Q(q_e) \rangle\rangle_e = 0 \quad (4.1)$$

for all $\zeta \in \mathcal{G}_{q_e}$ and $\eta \in \mathcal{G}$, $\ker \mathbb{I}_e \supseteq \mathcal{G}_{q_e}$; hence orbit regularity can be interpreted as the condition that $\mathbb{I}_e$ have no degeneracies other than those due to the symmetry of the equilibrium configuration.

If G is finite dimensional, then (L, ξ, q_e) is orbit regular if

$$\ker \mathbb{I}_e = \mathcal{G}_{q_e}, \quad (4.2)$$

since

$$\dim \mathcal{G}_{q_e}^A = \dim \mathcal{G} - \dim \mathcal{G}_{q_e} = \dim \tilde{\mathcal{G}} \quad (4.3)$$

implies $\tilde{\mathbb{I}}_e$ is an isomorphism if it is one to one. If G is finite dimensional and $\langle\langle \cdot, \cdot \rangle\rangle_e|_{\mathcal{G} \cdot q_e}$ is definite, then (L, ξ, q_e) is orbit regular, since $\langle\langle \cdot, \cdot \rangle\rangle_e|_{\mathcal{G} \cdot q_e}$ definite implies $\zeta \in \ker \mathbb{I}_e$ if and only if $\zeta_Q(q_e) = 0$, i.e. if $\zeta \in \mathcal{G}_{q_e}$. In particular, simple mechanical systems with finite dimensional symmetry groups are orbit regular at all points. (If the group G is infinite dimensional, orbit regularity may be influenced by the choice of the dual space $\mathcal{G}^*$.)

Proposition 4.1 *If (L, ξ, q_e) is orbit regular, then there exists a neighborhood $U \subset Q$ of q_e and a C^1 map $\Xi : Q \rightarrow \mathcal{G}$ such that $\Xi(q_e) = \xi$ and the mapping $\mu(q) := \mathbb{I}(q, \Xi(q))$ satisfies*

$$(D\mu(q_e) \cdot \delta q) \cdot \zeta \neq 0 \quad \implies \quad \zeta \in \mathcal{G}_{q_e} \quad (4.4)$$

for all $\delta q \in T_{q_e}Q$. If q_e has nontrivial isotropy subalgebra $\mathcal{G}_{q_e}$, then for each choice of a complement $\tilde{\mathcal{G}}$ to $\mathcal{G}_{q_e}$ isomorphic to $\mathcal{G}_{q_e}^A$ and containing ξ there is an associated mapping Ξ . If q_e is a point without isotropy, then there is a unique map Ξ such that $\mu(q) \equiv \mu$.

Proof: If q_e has trivial isotropy, then the first claim of the proposition is a straightforward application of the implicit function theorem to $\mathbb{I}$. If q_e has nontrivial isotropy, then $\mathbb{I}_e : \mathcal{G} \rightarrow \mathcal{G}_{q_e}^A \neq \mathcal{G}^*$; hence $\mathbb{I}$ must be modified before the implicit function theorem can be applied. Let a complement $\tilde{\mathcal{G}}$ to $\mathcal{G}_{q_e}$ be given and let $\tilde{\mathbb{P}}$ denote the projection $\tilde{\mathbb{P}} : \mathcal{G} \rightarrow \tilde{\mathcal{G}}$. Define the restricted inertia mapping $\tilde{\mathbb{I}} : Q \times \tilde{\mathcal{G}} \rightarrow \mathcal{G}_{q_e}^A$

$$\tilde{\mathbb{I}}(q, \eta) := \mathbb{I}(q, \eta) \circ \tilde{\mathbb{P}}. \quad (4.5)$$

$\tilde{\mathbb{I}}(q_e, \eta) = \mathbb{I}(q_e, \eta)$ for all $\eta \in \mathcal{G}$; in particular, $\mu \in \mathcal{G}_{q_e}^A$. Hence we can apply the implicit function theorem to $\tilde{\mathbb{I}}$ to find a curve $\Xi : U \rightarrow \tilde{\mathcal{G}}$ such that

$$\tilde{\mathbb{I}}(q, \Xi(q)) \equiv \mu. \quad (4.6)$$

It follows that $\mu(q) := \mathbb{I}(q, \Xi(q))$ satisfies $(D\mu(q_e) \cdot \delta q) \cdot \eta = 0$ for all $\eta \in \tilde{\mathcal{G}}$ and $\delta q \in T_{q_e}Q$.
✓

If the triplet (L, ξ, q_e) is orbit regular, then not only do we know that a curve Ξ exists; we also have a very simple expression for its first variation at q_e in terms of the linearized inertia tensor and the first variation of the locked momentum mapping. This provides expressions for the stability two form and the Arnold form which are straightforward generalizations of the forms derived in the simple mechanical case.

Proposition 4.2 *If (L, ξ, q_e) is orbit regular, then*

(i) Let $\tilde{\mathbb{I}}_e : \tilde{\mathcal{G}} \rightarrow (\mathcal{G}_{q_e})^A$ denote the restriction of $\mathbb{I}_e$ to $\tilde{\mathcal{G}}$. $\tilde{\mathbb{I}}_e$ is invertible and

$$D\Xi(q_e) \cdot \delta q = \tilde{\mathbb{I}}_e^{-1} (D\mathbb{I}_\xi(q_e) \cdot \delta q). \quad (4.7)$$

(ii) The stability two form $\mathcal{B}_e : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathbb{R}$ takes the form

$$\mathcal{B}_e(\delta q, \Delta q) = (D\mathbb{I}_\xi(q_e) \cdot \delta q) \cdot \tilde{\mathbb{I}}_e^{-1} (D\mathbb{I}_\xi(q_e) \cdot \Delta q) - D^2 L_\xi(q_e)(\delta q, \Delta q). \quad (4.8)$$

(iii) The generalized Arnold form $\mathcal{A} : \mathcal{G}_Q \times \mathcal{G}_Q \rightarrow \mathbb{R}$ takes the form

$$\mathcal{A}(\zeta, \eta) = \text{ad}_\zeta^* \mu \cdot \left(\text{ad}_\eta \xi + \tilde{\mathbb{I}}_e^{-1} \text{ad}_\eta^* \mu \right). \quad (4.9)$$

The forms $\mathcal{A}$ and $\mathcal{B}_e$ do not depend on the choice of the subspace $\tilde{\mathcal{G}}$.

Proof: The first claim follows directly from (3.24). By substituting (4.7) into (3.37), we obtain (4.8). To establish the third claim, we note that (3.48) and (4.7) imply

$$\begin{aligned} D\Xi(q_e) \cdot \eta_Q(q_e) &= \tilde{\mathbb{I}}_e^{-1} \cdot \left(\mathbb{I}_e \text{ad}_\eta \xi + \text{ad}_\eta^* \mu \right) \\ &= \tilde{\mathbb{P}} \text{ad}_\eta \xi + \tilde{\mathbb{I}}_e^{-1} \text{ad}_\eta^* \mu. \end{aligned} \quad (4.10)$$

Inserting (4.10) into (3.58) yields

$$\mathcal{A}(\zeta, \eta) = \text{ad}_\zeta^* \mu \cdot \left(\text{ad}_\eta \xi + \tilde{\mathbb{I}}_e^{-1} \text{ad}_\eta^* \mu \right). \quad (4.11)$$

The independence of the definitions from the choice of the complement follows from the condition that $D\mathbb{I}_\xi(q_e) \cdot \delta q \in \mathcal{G}_{q_e}^A$ for all $\delta q \in \mathcal{Q}$.

4.b Orbital stability

Once formal stability of a relative equilibrium has been established, we can attempt to demonstrate actual orbital stability of the relative equilibrium. Orbital stability will typically be taken with respect to the isotropy subgroup G_μ , although in cases in which the relative equilibrium possesses nontrivial isotropy, orbital stability may be determined with respect to a subgroup of G_μ . We discuss here methods of proving orbital stability by means of a Liapunov function. Hence, we need to determine conditions under which formal stability of a relative equilibrium v_e implies that v_e is, in some appropriate sense, a local minimum of a conserved functional. For infinite dimensional systems, definiteness of the second variation of a function at a critical point need not imply that the critical point is a local extremum. Such results are highly dependent on the underlying function spaces; a point which is a local minimum with respect to a specific topology may fail to be a minimum with respect to other topologies. These issues are discussed, for example, in Ball [1978]. To simplify matters, we shall, for the remainder of this section, consider only finite dimensional systems.

Given a local extremum of a constrained variational problem, one can typically construct an *augmented* functional for which the extremum becomes a local minimum of an

unconstrained variational problem. The augmented functional is obtained by adding an appropriate positive function of the constraints to the original functional; see Hestenes [1975]. The systems we consider cannot be directly treated by such methods. Due to the G_μ invariance of the energy-momentum functional, we do not have true extrema even for the constrained system; the energy-momentum functional attains its (locally) optimal value along the entire orbit $G_\mu \cdot v_e$. To obtain an isolated extremum, we must, in some fashion, ‘factor out’ the G_μ symmetry of the system.

If the quotient T^*Q/G_μ is a manifold in a neighborhood of the equivalence class $[v_e]$, then we can apply the usual augmentation techniques to the induced energy-momentum function on the quotient manifold to show that $[v_e]$ is a local extremum of some conserved functional. If the quotient is *not* a manifold, then an alternative approach is to construct a slice with respect to G_μ and demonstrate the existence of a local minimum on the slice. Recent works which apply slice techniques to the analysis of the stability and bifurcation of relative equilibria include Field [1980], Krupa [1990], and Patrick [1990]. Patrick addresses the orbital stability of relative equilibria of Hamiltonian systems, combining augmentation and slice techniques. We quote without proof (adapting notation) the following result from Patrick [1990], where it appears as Theorem 1:

Let $(P, \omega, H, G, \mathbf{J})$ be a Hamiltonian system with symmetry. Suppose z_e is a regular relative equilibrium with evolution $t \mapsto \exp(\xi t) \cdot z_e$, $\mathbf{J}(z_e) = \mu$, the action of G_μ on P is proper, and $\mathcal{G}$ admits an inner product invariant under the Ad action of G_μ . Then $D(H - J_\xi)(z_e) = \mathbf{0}$, and z_e is stable if it is formally stable; that is, if $D^2(H - J_\xi)(z_e)|_{T_{z_e}\mathbf{J}^{-1}(\mu)}$ is positive or negative definite on some (and hence any) complement to $G_\mu \cdot z_e$ in $T_{z_e}\mathbf{J}^{-1}(\mu)$.

Here stability is taken modulo G_μ .

The result as stated is not directly applicable to relative equilibria of Lagrangian systems which fail to be regular either in the sense that the Lagrangian itself fails to be regular, and hence cannot be transformed to an equivalent Hamiltonian system, or in the sense that the relative equilibrium is a singular point of the momentum map, i.e. that the relative equilibrium has nontrivial isotropy. (See Arms et al. [1981] for a discussion of the relationship between isotropy and singularities of the level sets of the momentum map.) The following two observations indicate that the theorem can be easily extended to a very general class of Lagrangian relative equilibria, including some equilibria with nontrivial isotropy. First, we note that the result relies on the conservation of energy and momentum, not on the existence of a symplectic structure. Hence the theorem is applicable to Lagrangian systems, with the energy replacing the Hamiltonian. Second, we observe that regularity of the relative equilibrium, in the sense that the level set $\mathbf{J}^{-1}(\mu)$ is a manifold, is used only to guarantee the existence of a smooth map $\tilde{\Psi} : G_\mu \cdot z_e \rightarrow \mathcal{G}$ which yields the velocities of the relative equilibria along the orbit $G_\mu \cdot z_e$. This assumption can be relaxed as follows: Let S and $\pi : G_\mu \cdot S \rightarrow G_\mu \cdot z_e$ (or, in the Lagrangian case, v_e), denote, respectively, the slice and the orbit translation map defined in Patrick [1990]. Obviously, G_μ acts transitively on $G_\mu \cdot z_e$; hence the isotropy subgroup $G_{\mu, z_e} := G_\mu \cap G_{z_e}$ is a closed Lie subgroup of

G_μ , $G_\mu/G_{\mu,z_e}$ is a smooth manifold diffeomorphic to $G_\mu \cdot z_e$, and every element of G_μ lies in the image of a local section of $G_\mu/G_{\mu,z_e}$ into G_μ . (See, for example, Boothby [1975], §IV.9 for proofs of these claims.) It follows that $G_\mu \rightarrow G_\mu/G_{\mu,z_e}$ is a principal bundle with fiber group G_{μ,z_e} . If there exists a smooth global section $\sigma : G_\mu/G_{\mu,z_e} \rightarrow G_\mu$, i.e., if the bundle $G_\mu \rightarrow G_\mu/G_{\mu,z_e}$ is trivial, then the conclusions of Patrick's theorem hold even in the absence of regularity. This can be seen as follows: Let $\phi : G_\mu \cdot v_e \rightarrow G_\mu/G_{\mu,z_e}$ denote the diffeomorphism between the orbit and the quotient manifold and define $\gamma := \sigma \circ \phi \circ \pi$. The smooth map $\hat{\Psi} : G_\mu \cdot S \rightarrow \mathcal{G}$ given by

$$\hat{\Psi}(z) := \text{Ad}_{\gamma(z)} \xi \quad (4.12)$$

replaces Patrick's map Ψ in the proof. This minor modification allows us to take advantage of the power of slice techniques, by allowing us to consider relative equilibrium with nontrivial isotropy. (Since we assume that G_μ acts properly, it is only in the neighborhood of a symmetric point that the quotient T^*Q/G_μ can fail to be a manifold.)

Thus, we can state the stability result as follows:

Corollary 4.1 *Let $v_e \in TQ$ be a formally stable relative equilibrium with group momentum μ of a G equivariant finite dimensional Lagrangian system. If*

- (i) *the Lie algebra $\mathcal{G}$ possesses an $\text{Ad } G_\mu$ invariant inner product*
- (ii) *G_μ acts properly on TQ*
- (iii) *the principal bundle $G_\mu \rightarrow G_\mu/(G_\mu \cap G_{v_e})$ is trivial,*

then v_e is orbitally stable with respect to G_μ .

Clearly, this statement also holds for a relative equilibrium $z_e \in T^*Q$ of an equivariant Hamiltonian system.

As a simple example, consider a trio of Lie groups

$$G_1 \supseteq G_2 \supseteq H \quad (4.13)$$

and a formally stable relative equilibrium v_e for which $G_\mu = G_1 \times G_2$ and $G_\mu \cap G_{v_e}$ is the diagonal of $H \times H$. This situation arises for mechanical systems possessing both spatial and body symmetries; e.g., if we consider isotropic homogeneous elasticity with respect to a reference configuration possessing an axis of symmetry, then $G_1 = G_2 = H = S^1$. The cosets of $G_\mu/(G_\mu \cap G_{v_e})$ are determined by the relation

$$(\tilde{g}_1, \tilde{g}_2) \approx (g_1, g_2) \iff (\tilde{g}_1, \tilde{g}_2) = (hg_1, hg_2) \quad \text{for some } h \in H. \quad (4.14)$$

Hence a global section is given by

$$\sigma([(g_1, g_2)]) := (g_2^{-1}g_1, \text{Id}), \quad (4.15)$$

where Id denotes the identity element of G_1 . If $\mathcal{G}$ possesses an invariant inner product, then we can apply Corollary 4.1 to show that v_e is orbitally stable.

4.c One dimensional momentum isotropy subgroups

If the isotropy subgroup G_μ of the equilibrium momentum μ is one dimensional, several convenient simplifications take place. This situation arises in the case of an $SO(3)$ or S^1 action, e.g. in the case of a mechanical system possessing no symmetries other than material frame indifference.

We first note that one of the assumptions on G_μ given in Corollary 4.1 is automatically satisfied in this case. The subgroup $G_\mu \cap G_{v_e}$ is either zero dimensional or equal to G_μ ; hence the bundle $G_\mu \rightarrow G_\mu/(G_\mu \cap G_{v_e})$ is trivial. Thus, if Q is finite dimensional and G_μ acts properly on TQ , and $\mathcal{G}$ possesses a G_μ invariant inner product, formal stability implies nonlinear orbital stability.

If G_μ is one dimensional and $G_\mu \supseteq G_{q_e}$, then the determination of the formal stability conditions associated to internal variations is substantially simplified. By combining the equilibrium conditions and the defining conditions of the space $\mathcal{Q}_{\text{int}}$ of internal variations, we shall show that $\mathcal{B}_{\text{int}}$ can be expressed in terms of variations of scalar functions. Specifically, define $f_\xi : Q \rightarrow \mathbb{R}$ by

$$f_\xi(q) := \Pi_\xi(q) \cdot \xi \quad (4.16)$$

and set

$$\kappa_\xi := \xi \cdot \Pi_e \xi. \quad (4.17)$$

Note that, for simple mechanical systems, $f_\xi = -2C_\xi$ and $\kappa_\xi = -2C_\xi(q_e)$, where C_ξ denotes the centrifugal potential $C_\xi(q) := -\frac{1}{2} |\xi_Q(q)|_g^2$. If $\kappa_\xi = 0$, then $\mathcal{B}_{\text{int}} = D^2 L_\xi|_{\mathcal{Q}_{\text{int}}}$; otherwise

$$\mathcal{B}_{\text{int}}(\delta q, \Delta q) = \frac{1}{\kappa_\xi} (Df_\xi(q_e) \cdot \delta q)(Df_\xi(q_e) \cdot \Delta q) - D^2 L_\xi(q_e)(\delta q, \Delta q). \quad (4.18)$$

This can be seen as follows: The equilibrium conditions imply that $\xi \in \mathcal{G}_\mu$; hence, if $\mathcal{G}_\mu$ is one dimensional, then ξ spans $\mathcal{G}_\mu$. Thus, using (3.65), we see that

$$\delta q \in \mathcal{Q}_{\text{int}} \iff D\Xi(q_e) \cdot \delta q = \lambda_{\delta q} \xi \quad \text{for some } \lambda_{\delta q} \in \mathbb{R} \quad (4.19)$$

and

$$(D\Xi(q_e) \cdot \delta q) \cdot \Pi_e (D\Xi(q_e) \cdot \Delta q) = \lambda_{\delta q} \lambda_{\Delta q} \kappa_\xi. \quad (4.20)$$

The identity (3.24) implies

$$\begin{aligned} Df_\xi(q_e) \cdot \delta q &= \xi \cdot (D\Pi_\xi(q_e) \cdot \delta q) \\ &= -\xi \cdot \Pi_e (D\Xi(q_e) \cdot \delta q) \\ &= -\lambda_{\delta q} \kappa_\xi \end{aligned} \quad (4.21)$$

Hence, if $\kappa_\xi \neq 0$, then

$$\lambda_{\delta q} = -\frac{Df_\xi(q_e) \cdot \delta q}{\kappa_\xi} \quad (4.22)$$

and, using (3.37),

$$\begin{aligned} \mathcal{B}_{\text{int}}(\delta q, \Delta q) &= (D\Xi(q_e) \cdot \delta q) \cdot \Pi_e (D\Xi(q_e) \cdot \Delta q) - D^2 L_\xi(q_e)(\delta q, \Delta q) \\ &= \frac{1}{\kappa_\xi} (Df_\xi(q_e) \cdot \delta q)(Df_\xi(q_e) \cdot \Delta q) - D^2 L_\xi(q_e)(\delta q, \Delta q). \end{aligned} \quad (4.23)$$

5 EXAMPLES

5.a Natural mechanical systems

Gantmacher [1975], §11, defines a natural mechanical system as a Lagrangian system with Lagrangian of the form

$$L(v) := \frac{1}{2} |v|_g^2 + \langle v, Y(q) \rangle_g - V(q), \quad (5.1)$$

where $q := \pi(v)$ is the basepoint of the tangent vector v . We first treat the general case of a natural mechanical system, showing, in particular, that all relative equilibria of a natural system with a finite dimensional symmetry group are orbit regular. We then apply the results obtained for the general case to the analysis of a particle moving in a electromagnetic field.

5.a.i Definitions

Let Q be a manifold acted upon by a Lie group G . Let the Lagrangian L be of the form (5.1), where g is a G invariant metric on Q , $Y : Q \rightarrow TQ$ is a G equivariant vector field, and $V : Q \rightarrow \mathbb{R}$ is a G invariant function. The Legendre transformation $\mathbf{FL} : TQ \rightarrow T^*Q$ associated to L is determined by

$$\mathbf{FL}(v) \cdot w = \langle v + Y(q), w \rangle_g \quad (5.2)$$

for all $v \in TQ$ and $w \in T_{\pi(v)}Q$. The energy associated to L is

$$\begin{aligned} E(v) &= \mathbf{FL}(v) \cdot v - L(v) \\ &= \langle v + Y(q), v \rangle_g - \frac{1}{2} |v|_g^2 - \langle v, Y(q) \rangle_g + V(q) \\ &= \frac{1}{2} |v|_g^2 + V(q). \end{aligned} \quad (5.3)$$

Note that E is independent of Y .

Let $\mathbf{J}_g : TQ \rightarrow \mathcal{G}^*$ denote the map determined by

$$\mathbf{J}_g(v) \cdot \eta := \langle v, \eta_Q(q) \rangle_g. \quad (5.4)$$

The augmented Lagrangian $L_\xi : Q \rightarrow \mathbb{R}$ is given by

$$\begin{aligned} L_\xi(q) &:= L(\xi_Q(q)) \\ &= \frac{1}{2} \xi \cdot \mathbb{I}_g(q) \xi + \beta(q) \cdot \xi - V(q), \end{aligned} \quad (5.5)$$

where $\beta := \mathbf{J}_g \circ Y$ maps $Q \rightarrow \mathcal{G}^*$ and $\mathbb{I}_g$ denotes the inertia tensor associated to g , i.e.

$$\mathbb{I}_g(q) \xi := \mathbf{J}_g(\xi_Q(q)). \quad (5.6)$$

The inertia tensor $\mathbb{I} : Q \times \mathcal{G} \rightarrow \mathcal{G}^*$ is affine with respect to $\mathcal{G}$:

$$\begin{aligned} \mathbb{I}(q, \xi) &:= \mathbf{J}_g(\xi_Q(q) + Y(q)) \\ &= \mathbf{J}_g(\xi_Q(q)) + \mathbf{J}_g(Y(q)) \\ &= \mathbb{I}_g(q) \xi + \beta(q). \end{aligned} \quad (5.7)$$

It follows that the linearized inertia tensor $\mathbb{I}_e$ is simply the metric inertia tensor at equilibrium, i.e.

$$\mathbb{I}_e = \mathbb{I}_g(q_e). \quad (5.8)$$

If G is finite dimensional, then the triplet (L, ξ, q_e) is orbit regular for all $q_e \in Q$ and all $\xi \in \mathcal{G}$, since $\zeta \in \ker \mathbb{I}_e$ implies

$$|\zeta_Q(q_e)|_g^2 = \zeta \cdot \mathbb{I}_g(q_e) \zeta = 0 \quad (5.9)$$

and hence $\zeta \in \mathcal{G}_{q_e}$.

5.a.ii Formal stability

Let $v_e = \xi_Q(q_e)$ be a relative equilibrium. The analysis of the ‘vertical’ block associated to $(\mathcal{G} \cdot q_e)^\perp$ is straightforward, since

$$\langle\langle v, v \rangle\rangle_e = |v|_g^2 > 0 \quad (5.10)$$

for all $v \neq \mathbf{0}$. Assume (L, ξ, q_e) is orbit regular.

The stability two form $\mathcal{B}_e$ is given by

$$\begin{aligned} \mathcal{B}_e(\delta q, \Delta q) &= (D(\mathbb{I}(q_e) \cdot \delta q) \xi) \cdot \tilde{\mathbb{I}}_e^{-1} (D(\mathbb{I}(q_e) \cdot \Delta q) \xi) \\ &\quad + D^2 V(q_e)(\delta q, \Delta q) - \frac{1}{2} \xi \cdot (D^2 \mathbb{I}(q_e)(\delta q, \Delta q) \xi) \\ &= [D(\mathbb{I}_g(q_e) \cdot \delta q) \xi + D\beta(q_e) \cdot \delta q] \cdot \tilde{\mathbb{I}}_e^{-1} [D(\mathbb{I}_g(q_e) \cdot \Delta q) \xi + D\beta(q_e) \cdot \Delta q] \\ &\quad + D^2 V(q_e)(\delta q, \Delta q) - \frac{1}{2} \xi \cdot [D^2 \mathbb{I}_g(q_e)(\delta q, \Delta q) \xi - D^2 \beta(q_e)(\delta q, \Delta q)]. \end{aligned} \quad (5.11)$$

If we define the set Q_α of asymmetric points by

$$Q_\alpha := \{q \in Q : \eta_Q(q) \neq \mathbf{0} \quad \forall \eta \in \mathcal{G}\}, \quad (5.12)$$

then the restriction of $\mathbb{I}$ to $Q_\alpha \times \mathcal{G}$ is invertible with respect to $\mathcal{G}$, with inverse $\mathbb{I}^{-1} : Q_\alpha \times \mathcal{G}^* \rightarrow \mathcal{G}$ given by

$$\mathbb{I}^{-1}(q, \mu) = \mathbb{I}_g^{-1}(q)(\mu - \beta(q)). \quad (5.13)$$

The constant momentum vector field $\mathbf{v}_\mu : Q_\alpha \rightarrow TQ$ is given by

$$\begin{aligned} \mathbf{v}_\mu(q) &:= \left(\mathbb{I}^{-1}(q, \mu) \right)_Q(q) \\ &= \left[\mathbb{I}_g^{-1}(q)(\mu - \beta(q)) \right]_Q(q). \end{aligned} \quad (5.14)$$

Hence, the restriction of the amended energy E_μ to Q_α has the form

$$\begin{aligned} E_\mu(q) &:= E_\xi(\mathbf{v}_\mu(q)) \\ &= \frac{1}{2} |\mathbf{v}_\mu(q)|_g^2 + V(q) - \mu \cdot \xi \\ &= \frac{1}{2} \left\| \left[\mathbb{I}_g^{-1}(q)(\mu - \beta(q)) \right]_Q(q) \right\|_g^2 + V(q) - \mu \cdot \xi \\ &= \frac{1}{2} (\mu - \beta(q)) \mathbb{I}_g^{-1}(q)(\mu - \beta(q)) + V(q) - \mu \cdot \xi. \end{aligned} \quad (5.15)$$

Remark: Note that if β is constant, then the equilibrium configurations and their stabilities are identical to those of the *simple mechanical system* with metric g and potential V . For example, if the vector field Y is *horizontal*, i.e. if $Y(q)$ lies within the horizontal subspace associated to the simple mechanical connection α , then $\beta \equiv \mathbf{0}$. If β is constant to first order at q_e , then the equilibrium conditions are unaffected by Y and the stability conditions differ from those of the associated simple mechanical system by the term $\frac{1}{2}D^2\beta(q_e)(\delta q, \Delta q) \cdot \xi$.

5.a.iii A particle in an electromagnetic field

As an example of a natural mechanical system, we now consider a point mass moving in a electromagnetic field. In this case, the configuration space is $Q = \mathbb{R}^3$ and the tangent bundle TQ is identified with $\mathbb{R}^3 \times \mathbb{R}^3$. We assume that the fields in question are cylindrically symmetric, i.e. there exists an axis (e.g., the $\hat{\mathbf{z}}$ axis) such that the fields are invariant under rotations about the axis and translations along the axis. The symmetry group is then $G = S^1 \times \mathbb{R}$, with group action

$$(\theta, \zeta) \cdot q := R_\theta q + \zeta \hat{\mathbf{z}}, \quad (5.16)$$

where $R_\theta \in SO(3)$ is the matrix associated to rotation through the angle θ about the axis $\hat{\mathbf{z}}$.

The Lagrangian is given by

$$L(v) = \frac{m}{2} |v|^2 + \varepsilon m \langle v, A(q) \rangle - e\varphi(q), \quad (5.17)$$

where

- (i) m is the mass of the particle
- (ii) e is the electric charge of the particle
- (iii) $\varepsilon := \frac{e}{c}$, where c is the speed of light
- (iv) A is the magnetic vector potential
- (v) φ is the electric scalar potential
- (vi) $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^3$.

Thus $\langle \cdot, \cdot \rangle_g = m \langle \cdot, \cdot \rangle$, $Y(q) := \varepsilon A(q)$ and $V(q) := e\varphi(q)$. G equivariance implies the vector potential A is of the form

$$A(q) = \nu(\rho) \mathbb{P}q + \sigma(\rho)(\hat{\mathbf{z}} \times q) + \tau(\rho)\hat{\mathbf{z}}, \quad (5.18)$$

where $\mathbb{P}q := (\hat{\mathbf{z}} \times q) \times \hat{\mathbf{z}}$ denotes the projection of q onto the x - y plane and $\rho := |\hat{\mathbf{z}} \times q|^2$, and the scalar potential φ is a function of ρ . The magnetic field is given by

$$\nabla \times A(q) = (2\rho\sigma'(\rho) + \sigma(\rho))\hat{\mathbf{z}} - 2\tau'(\rho)(\hat{\mathbf{z}} \times q). \quad (5.19)$$

The Lie group G has Lie algebra $\mathcal{G} \approx \mathbb{R}^2$, with infinitesimal generator

$$(\xi, \zeta)_Q(q) = (q, \xi \hat{\mathbf{z}} \times q + \zeta \hat{\mathbf{z}}). \quad (5.20)$$

The momentum map associated to the action of G on TQ and the metric g is

$$\mathbf{J}_g(q, v) = m(v \cdot (\hat{\mathbf{z}} \times q), v \cdot \hat{\mathbf{z}}); \quad (5.21)$$

hence the metric inertia tensor $\mathbb{I}_g : \mathcal{G} \rightarrow \mathcal{G}^*$ is given by

$$\begin{aligned} \mathbb{I}_g(q)(\xi, \zeta) &:= \mathbf{J}_g((\xi, \zeta)_Q(q)) \\ &= \mathbf{J}_g(q, \xi \hat{\mathbf{z}} \times q + \zeta \hat{\mathbf{z}}) \\ &= m(\rho \xi, \zeta). \end{aligned} \quad (5.22)$$

Using (5.21), we see that the map $\beta := \mathbf{J}_g \circ Y$ is given by

$$\beta(q) = \varepsilon m(\rho \sigma(\rho), \tau(\rho)). \quad (5.23)$$

Hence (5.7) implies

$$\mathbb{I}_{\xi, \zeta}(q) = m(\rho(\xi + \varepsilon \sigma(\rho)), \zeta + \varepsilon \tau(\rho)). \quad (5.24)$$

The augmented Lagrangian is given by

$$L_{\xi, \zeta}(q) = \frac{m}{2} (\xi^2 \rho + \zeta^2) + \varepsilon m(\xi \rho \sigma(\rho) + \zeta \tau(\rho)) - e\varphi(\rho). \quad (5.25)$$

The group G is abelian, hence $G_\mu = G$. We consider a point q_e such that $\rho_e \neq 0$; q_e has trivial isotropy. The tangent to the orbit, $\mathcal{G} \cdot q_e$, is spanned by the vectors $\hat{\mathbf{z}} \times q_e$ and $\hat{\mathbf{z}}$, thus the complement $\mathcal{V}$ to the orbit can be chosen as the span of q_e .

The augmented Lagrangian has first variation

$$DL_{\xi, \zeta}(q_e) = 2 \left[m\xi \left(\frac{\xi}{2} + \varepsilon(\sigma(\rho_e) + \rho_e \sigma'(\rho_e)) \right) - \varepsilon m \zeta \tau'(\rho_e) - e\varphi'(\rho_e) \right] q_e, \quad (5.26)$$

where $'$ denotes variation with respect to ρ .

Hence q_e is a critical point of $L_{\xi, \zeta}$ if

$$\varphi'(\rho_e) = \varepsilon m \left[\xi \left(\frac{\xi}{2\varepsilon} + \sigma(\rho_e) + \rho_e \sigma'(\rho_e) \right) - \zeta \tau'(\rho_e) \right]. \quad (5.27)$$

The augmented Lagrangian has second variation

$$D^2 L_{\xi, \zeta}(q_e) = 4(\varepsilon m[\xi(2\sigma'(\rho_e) + \rho_e \sigma''(\rho_e)) + \zeta \tau''(\rho_e)] - e\varphi''(\rho_e)) q_e \otimes q_e. \quad (5.28)$$

The first variation of $\mathbb{I}_{\xi, \zeta}(q)$ at q_e is given by

$$D\mathbb{I}_{\xi, \zeta}(q_e) = 2\varepsilon m(\varepsilon^{-1}\xi + \sigma(\rho_e) + \rho_e \sigma'(\rho_e), \tau'(\rho_e)) q_e. \quad (5.29)$$

The linearized inertia tensor is given by $\mathbb{I}_e = \text{diag}[m\rho_e, m]$; hence

$$\begin{aligned} \mathcal{B}_e(q_e^2) &= 4\rho_e^2 \left(\varepsilon m^2 \left[\frac{1}{\rho_e} (\varepsilon^{-1}\xi + \sigma(\rho_e) + \rho_e \sigma'(\rho_e))^2 + (\tau'(\rho_e))^2 \right] \right. \\ &\quad \left. - \varepsilon m[\xi(2\sigma'(\rho_e) + \rho_e \sigma''(\rho_e)) + \zeta \tau''(\rho_e)] + e\varphi''(\rho_e) \right). \end{aligned} \quad (5.30)$$

Thus $v_e = (q_e, \xi \hat{\mathbf{z}} \times q_e + \zeta \hat{\mathbf{z}})$ is a formally stable relative equilibrium if (5.27) holds and (5.30) is positive.

Since G is abelian, the coadjoint action is trivial and the Euclidean inner product on $\mathcal{G} \approx \mathbb{R}^2$ is trivially G_μ invariant. Since v_e has trivial isotropy, Corollary 4.1 implies that a formally stable relative equilibrium is nonlinearly orbitally stable.

5.b The symmetric rigid body

As a simple example of a symmetric configuration, we consider a rigid body with one axis of symmetry. The symmetry group is $SO(3) \times S^1$, with action

$$(\Lambda, \theta) \cdot Q = \Lambda Q R_\theta^T, \quad (5.31)$$

where $R_\theta \in SO(3)$ is the matrix associated to rotation through the angle θ about the axis of symmetry. If the body is symmetric about its axis of rotation, then it is sufficient to apply the energy-momentum method using the subgroup $SO(3)$ with the usual left action. If the body is symmetric about an axis σ in the plane of rotation, then the stability two form associated to the pure left action is indefinite. In this case, it is necessary to consider the full group G to obtain definite stability conditions.

The configuration space of the rigid body is the rotation group $SO(3)$. The Lagrangian is given by

$$L(\dot{\Lambda}) := \frac{1}{2} \langle \dot{\Lambda}, \dot{\Lambda} \rangle_{\mathcal{B}}, \quad (5.32)$$

where

$$\langle U, V \rangle_{\mathcal{B}} := \int_{\mathcal{B}} \rho_{\text{ref}} \langle U \mathbf{x}, V \mathbf{x} \rangle d^3 x. \quad (5.33)$$

G has Lie algebra $\mathcal{G} \approx \mathbb{R}^3 \times \mathbb{R}$ with infinitesimal generator

$$(\xi, \alpha)_Q(\Lambda) = \hat{\xi} \Lambda + \alpha(\text{Id} - \Lambda)\sigma. \quad (5.34)$$

Let $\mathbb{I}_L(\Lambda)$ denote the usual inertia diadic associated to the left action of $SO(3)$. We assume that the coordinate axes are chosen such that $\mathbb{I}_L$ is diagonal; the symmetry assumption implies that $\mathbb{I}_L = \text{diag}[I_1, I_2, I_1]$ for some constants $I_1 \neq I_2 \in \mathbb{R}$. Additionally, we assume that the body is in steady rotation with angular velocity ξ such that ξ is *not* parallel to σ . Given $(\xi, 0) \in \mathcal{G}$,

$$\begin{aligned} L_{\xi,0}(\Lambda) &= \frac{1}{2} \langle (\xi, 0)_Q(\Lambda), (\xi, 0)_Q(\Lambda) \rangle_{\mathcal{B}} \\ &= \frac{1}{2} (\xi, 0) \cdot \mathbb{I}(\Lambda) (\xi, 0) \end{aligned} \quad (5.35)$$

$$= \frac{1}{2} \xi \cdot \mathbb{I}_L(\Lambda) \xi. \quad (5.36)$$

If we identify $\delta\Lambda \in T_{\Lambda_e} SO(3)$ with $\delta\theta \in \mathbb{R}^3$ as usual, then $L_{\xi,0}$ has first variation

$$DL_{\xi,0}(\Lambda_e) \cdot \delta\theta = \frac{1}{2} \xi \cdot [D\mathbb{I}_L(\Lambda_e) \cdot \delta\theta] \xi. \quad (5.37)$$

Hence the usual rigid body equilibrium conditions imply that ξ is parallel to a principal axis of $\mathcal{B}$.

The ‘locked momentum map’ is given by

$$\mathbb{I}_\xi(\Lambda) \cdot (\eta, \alpha) = \xi \cdot \mathbb{I}_L(\Lambda)(\eta + \alpha(\text{Id} - \Lambda)\sigma). \quad (5.38)$$

The equilibrium momentum is given by

$$\mathbb{I}_\xi(\Lambda_e)(\xi, 0) = (\mu, 0), \quad (5.39)$$

where

$$\mu := \mathbb{I}_L(\Lambda_e) = \mathbb{I}_1 \xi. \quad (5.40)$$

The locked momentum map has first variation

$$D\mathbb{I}_\xi(\Lambda_e) \cdot \delta\theta = (\delta\theta \times \mu - \mathbb{I}_L(\Lambda_e)(\delta\theta \times \xi), \mu \cdot (\sigma \times \delta\theta)) \quad (5.41)$$

at Λ_e .

The relevant isotropy subalgebras are

$$(i) \ \mathcal{G}_{\Lambda_e} = \text{span} \{(\mathbf{0}, 1)\}$$

$$(ii) \ \mathcal{G}_{v_e} = (\mathbf{0}, 0)$$

$$(iii) \ \mathcal{G}_\mu = \text{span} \{(\xi, 0), (\mathbf{0}, 1)\}.$$

Hence

$$\mathcal{Q} = \left\{ \delta\theta \in \mathbb{R}^3 : \delta\theta \cdot (\xi \times \sigma) = 0 \right\} = \text{span} \{ \sigma, \xi \}, \quad (5.42)$$

$$\begin{aligned} \mathcal{G}_\mathcal{Q} &= \left\{ (\eta, \alpha) \in \mathbb{R}^3 \times \mathbb{R} : (\xi \times \eta) \cdot \sigma = 0 \right\} \\ &= \text{span} \{ (\xi, 0), (\mathbf{0}, 1), (\sigma, 0) \} \\ &= \mathcal{G}_\mu \cup \mathcal{G}_{\Lambda_e} + \text{span} \{ (\sigma, 0) \}, \end{aligned} \quad (5.43)$$

and

$$\mathcal{Q} = \mathcal{G}_\mathcal{Q} \cdot \Lambda_e. \quad (5.44)$$

$\mathcal{G} \cdot \Lambda_e = T_{\Lambda_e} SO(3)$ implies $(\mathcal{G} \cdot \Lambda_e)^\perp = \mathbf{0}$. Thus $\mathcal{A}$ can be either positive or negative semi-definite and

$$\mathcal{A}((\sigma, 0), (\sigma, 0)) = \frac{\mathbb{I}_1 - \mathbb{I}_2}{\mathbb{I}_1 \mathbb{I}_2} |\mu \times \sigma|^2 \neq 0 \quad (5.45)$$

implies the equilibrium is formally stable. The Euclidean inner product on $\mathcal{G} \approx \mathbb{R}^3 \times \mathbb{R}$ is G_μ invariant. If the equilibrium v_e is not in rotation about the axis of symmetry, then v_e has trivial isotropy, so we can apply Corollary 4.1. If v_e is in rotation about the axis of symmetry, then, as stated earlier, v_e is formally stable if we replace G with $SO(3)$, which acts freely. Hence we can again apply the corollary and we have shown that any relative equilibrium is orbitally stable.

5.c $SO(3)$ invariant Lagrangians on $T\mathbb{R}^3$

As our final example, we consider a general $SO(3)$ equivariant Lagrangian system on $T\mathbb{R}^3$. $SO(3)$ invariance implies that the Lagrangian L takes the form

$$L(q, \dot{q}) = f(\rho, \chi, \kappa), \quad (5.46)$$

where

$$\rho := |q|^2 \quad \chi := \langle q, \dot{q} \rangle \quad \kappa := \frac{1}{2} |\dot{q}|^2 \quad (5.47)$$

for some function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$. L has partial derivatives

$$\begin{aligned} L_{\dot{q}} &= f_{\chi} q + f_{\kappa} \dot{q} \\ L_{\dot{q}\dot{q}} &= f_{\chi\chi} q \otimes q + f_{\chi\kappa} (q \otimes \dot{q} + \dot{q} \otimes q) + f_{\kappa} \text{Id} + f_{\kappa\kappa} \dot{q} \otimes \dot{q} \\ L_q &= 2f_{\rho} q + f_{\chi} \dot{q} \\ L_{qq} &= 2f_{\rho} \text{Id} + 4f_{\rho\rho} q \otimes q + f_{\chi\chi} \dot{q} \times \dot{q} \\ L_{\dot{q}q} &= 2f_{\rho\kappa} \dot{q} \otimes q + f_{\chi} \text{Id} + f_{\chi\chi} (q \otimes \dot{q} + \dot{q} \otimes q), \end{aligned} \quad (5.48)$$

where all derivatives of L are evaluated at the point $(q, \dot{q})$ and all derivatives of f are evaluated at the corresponding triplet $(\rho, \langle q, \dot{q} \rangle, \frac{1}{2} |\dot{q}|^2)$.

We consider now a candidate relative equilibrium $v_e = (q_e, \xi \times q_e)$ for some $\xi \in \mathcal{G} \approx \mathbb{R}^3$ such that $\xi \times q_e \neq 0$. Henceforth, all derivatives of f are to be evaluated at the point $(\rho, 0, \frac{1}{2} \xi^2 \rho)$. The augmented Lagrangian has first variation

$$\begin{aligned} DL_{\xi}(q_e) \cdot \delta q &= \langle L_q, \delta q \rangle + \langle L_{\dot{q}}, \xi \times \delta q \rangle \\ &= \langle 2f_{\rho} q - f_{\chi} \xi \times (\xi \times q_e), \delta q \rangle \\ &= \left\langle \left(2f_{\rho} + |\xi|^2 f_{\kappa} \right) q_e - f_{\kappa} \langle \xi, q_e \rangle \xi, \delta q \right\rangle. \end{aligned} \quad (5.49)$$

Hence $v_e = (q_e, \xi \times q_e)$ is a relative equilibrium if

$$f_{\kappa} = f_{\rho} = 0 \quad \text{or} \quad \rho_e f_{\rho} + \kappa_e f_{\kappa} = \langle \xi, q_e \rangle = 0. \quad (5.50)$$

We now compute the linearized Legendre transformation and inertia tensor. $\langle\langle \cdot, \cdot \rangle\rangle_e = L_{\dot{q}\dot{q}}(q_e, \xi \times q_e)$; hence, if we take $\{q_e/\sqrt{\rho_e}, (\xi \times q_e)/\sqrt{2\kappa_e}, \xi/|\xi|\}$ as a basis of $T_{q_e}Q$, then

$$\langle\langle \cdot, \cdot \rangle\rangle_e = \begin{pmatrix} f_{\kappa} + \rho_e f_{\chi\chi} & \sqrt{2\rho_e \kappa_e} f_{\chi\kappa} & 0 \\ \sqrt{2\rho_e \kappa_e} f_{\chi\kappa} & f_{\kappa} + 2\kappa_e f_{\kappa\kappa} & 0 \\ 0 & 0 & f_{\chi} \end{pmatrix}. \quad (5.51)$$

Define $\Pi_g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$\Pi_g(q) := |q|^2 \text{Id} - q \otimes q. \quad (5.52)$$

The linearized inertia tensor is determined by

$$\eta \cdot \Pi_e \zeta = \langle\langle \eta \times q_e, \zeta \times q_e \rangle\rangle_e; \quad (5.53)$$

hence, if we take $\{q_e/\sqrt{\rho_e}, (\xi \times q_e)/\sqrt{2\kappa_e}, \xi/|\xi|\}$ as a basis of $\mathcal{G}$, then (5.51) implies

$$\mathbb{I}_e = \text{diag} \left[0, \rho_e^2 f_\kappa, \rho_e (f_\kappa + 2\kappa_e f_{\kappa\kappa}) \right]. \quad (5.54)$$

(L, ξ, q_e) is orbit regular if $\ker \mathbb{I}_e = \mathcal{G}_{q_e} = \text{span} \{q_e\}$; hence (L, ξ, q_e) is orbit regular if

$$-2\kappa_e f_{\kappa\kappa} \neq f_\kappa \neq 0. \quad (5.55)$$

We now consider the locked momentum map $\mathbb{I}_\xi$, which takes the form

$$\mathbb{I}_\xi(q) = f_\kappa(\rho, 0, \frac{1}{2}\xi^2 \rho) \mathbb{I}_g(q) \xi. \quad (5.56)$$

The equilibrium momentum μ is

$$\mu = \mathbb{I}_\xi(q_e) = \rho_e f_\kappa \xi, \quad (5.57)$$

with isotropy subalgebra $\mathcal{G}_\mu = \text{span} \{\xi\}$. At q_e , $\mathbb{I}_\xi(q)$ has first variation

$$\begin{aligned} D\mathbb{I}_\xi(q_e) \cdot \delta q &= f_\kappa(D\mathbb{I}_g(q_e) \cdot \delta q) \xi + 2f_{\rho\kappa} \mathbb{I}_g(q_e) \xi \langle q_e, \delta q \rangle \\ &\quad + f_{\kappa\kappa} \left(\frac{1}{2} \xi \cdot (D\mathbb{I}_g(q_e) \cdot \delta q) \xi \right) \mathbb{I}_g(q_e) \xi \\ &= 2(f_\kappa + \rho_e f_{\rho\kappa} + \kappa_e f_{\kappa\kappa}) \langle q_e, \delta q \rangle \xi - f_\kappa \langle \xi, \delta q \rangle q_e. \end{aligned} \quad (5.58)$$

The space $\mathcal{Q}$ of admissible variations is computed as follows: $\mathcal{G}_{q_e} = \text{span} \{q_e\}$ implies

$$\begin{aligned} \mathcal{Q} &:= \left\{ \delta q \in T_{q_e} Q : D\mathbb{I}_\xi(q_e) \cdot \delta q \in \mathcal{G}_{q_e}^\perp \right\} \\ &= \left\{ \delta q \in T_{q_e} Q : q_e \cdot (D\mathbb{I}_\xi(q_e) \cdot \delta q) = 0 \right\} \\ &= \left\{ \delta q \in T_{q_e} Q : \rho_e f_\kappa \langle \xi, \delta q \rangle = 0 \right\} \\ &= \text{span} \{q_e, \xi \times q_e\}. \end{aligned} \quad (5.59)$$

If we choose $\tilde{\mathcal{V}} = \text{span} \{q_e, \xi\}$ as a complement to $\mathcal{G}_\mu \cdot q_e = \text{span} \{\xi \times q_e\}$, then $\mathcal{V} = \text{span} \{q_e\}$. We now proceed to the computation of the stability tensor $\mathcal{B}_e$.

$$D\mathbb{I}_\xi(q_e) \cdot q_e = 2\rho_e (f_\kappa + \rho_e f_{\rho\kappa} + \kappa_e f_{\kappa\kappa}) \xi. \quad (5.60)$$

For convenience, we choose $\tilde{\mathcal{G}} = \text{span} \{\xi, \xi \times q_e\}$; (5.54) implies

$$\begin{aligned} \xi \cdot \tilde{\mathbb{I}}_e^{-1} \xi &= \frac{|\xi|^2}{\rho_e (f_\kappa + 2\kappa_e f_{\kappa\kappa})} \\ &= \frac{2\kappa_e}{\rho_e^2 (f_\kappa + 2\kappa_e f_{\kappa\kappa})} \\ &= \rho_e^{-2} \left(\frac{f_\kappa}{2\kappa_e} + f_{\kappa\kappa} \right)^{-1} \end{aligned} \quad (5.61)$$

and hence

$$(D\mathbb{I}_\xi(q_e) \cdot q_e) \cdot \tilde{\mathbb{I}}_e^{-1} (D\mathbb{I}_\xi(q_e) \cdot q_e) = 4 \left(\frac{f_\kappa}{2\kappa_e} + f_{\kappa\kappa} \right)^{-1} (f_\kappa + \rho_e f_{\rho\kappa} + \kappa_e f_{\kappa\kappa})^2. \quad (5.62)$$

The second variation of the augmented Lagrangian is

$$\begin{aligned} D^2 L_\xi(q_e)(q_e^2) &= \left(f_{\rho\rho} + \frac{1}{2}|\xi|^2 f_{\rho\kappa} + \frac{1}{4}f_{\kappa\kappa}\right) (2|q_e|^2)^2 \\ &= 4 \left(\rho_e^2 f_{\rho\rho} + \rho_e \kappa_e f_{\rho\kappa} + \kappa_e^2 f_{\kappa\kappa}\right). \end{aligned} \quad (5.63)$$

Hence

$$B_e(q_e^2) = 4 \left[\left(\frac{f_\kappa}{2\kappa_e} + f_{\kappa\kappa} \right)^{-1} (f_\kappa + \rho_e f_{\rho\kappa} + \kappa_e f_{\kappa\kappa})^2 - \left(\rho_e^2 f_{\rho\rho} + \rho_e \kappa_e f_{\rho\kappa} + \kappa_e^2 f_{\kappa\kappa} \right) \right]. \quad (5.64)$$

To obtain the final stability conditions, it is necessary to determine the definiteness of $\langle\langle \cdot, \cdot \rangle\rangle_e$ on $(\mathcal{G} \cdot q_e)^\perp = \text{span} \{q_e\}$.

$$\langle\langle q_e, q_e \rangle\rangle_e = \rho_e (f_\kappa + \rho_e f_{\chi\chi}); \quad (5.65)$$

hence $\langle\langle \cdot, \cdot \rangle\rangle_e|_{(\mathcal{G} \cdot q_e)^\perp}$ is positive (negative) definite if $f_\kappa + \rho_e f_{\chi\chi} > (<) 0$.

To show that formal stability implies orbital stability, we note that $\mathcal{G} \approx \mathbb{R}^3$ possesses an invariant inner product. $\xi \times q_e \neq \mathbf{0}$ implies q_e and $\xi \times q_e$ are not parallel; hence the relative equilibrium has trivial isotropy and we can apply Corollary 4.1.

To summarize, if q_e and ξ satisfy $\xi \times q_e \neq \mathbf{0}$ and

(i) $\langle \xi, q_e \rangle = \rho_e f_\rho + \kappa_e f_\kappa = 0$
(equilibrium)

(ii) $-2\kappa_e f_{\kappa\kappa} \neq f_\kappa \neq 0$
(orbit regularity)

(iii) $f_\kappa + \rho_e f_{\chi\chi} > (<) 0$ and

$$\left(\frac{f_\kappa}{2\kappa_e} + f_{\kappa\kappa} \right)^{-1} (f_\kappa + \rho_e f_{\rho\kappa} + \kappa_e f_{\kappa\kappa})^2 > (<) \rho_e^2 f_{\rho\rho} + \rho_e \kappa_e f_{\rho\kappa} + \kappa_e^2 f_{\kappa\kappa} \quad (5.66)$$

(definiteness),

then q_e and ξ determine an orbitally stable relative equilibrium.

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APPENDIX: SUMMARY OF THE METHOD

For the convenience of the reader, we present here a brief summary of the method in the case that Q and G are both finite dimensional and the equilibrium is orbit regular. For the appropriate stability conditions when Q or G is infinite dimensional or the equilibrium is not known to be orbit regular, refer to the body of the text.

Let $L : TQ \rightarrow \mathbb{R}$ be a Lagrangian which is invariant under the tangent lift of the action of a Lie group G on Q . Let there be a second order Lagrangian vector field X_E associated to L ; e.g., let L be regular. Given $\xi \in \mathcal{G}$, define the infinitesimal generator $\xi_Q : Q \rightarrow TQ$ by

$$\xi_Q(q) := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \exp(\epsilon \xi) \cdot q \quad (\text{A.1})$$

and the augmented Lagrangian $L_\xi : Q \rightarrow \mathbb{R}$ by

$$L_\xi(q) := L(\xi_Q(q)). \quad (\text{A.2})$$

$v_e \in T_{q_e}Q$ is a relative equilibrium if and only if there exists $\xi \in \mathcal{G}$ such that $v_e := \xi_Q(q_e)$ and q_e is a critical point of L_ξ .

The stability conditions can be specified as follows: Let v_e be a relative equilibrium with associated total velocity ξ . Define

(i) the Legendre transformation $\mathbf{FL} : TQ \rightarrow T^*Q$

$$\mathbf{FL}(v) \cdot w := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L(v + \epsilon w) \quad (\text{A.3})$$

(ii) the locked momentum mapping $\mathbb{I}_\xi : Q \rightarrow \mathcal{G}^*$

$$\mathbb{I}_\xi(q) \cdot \eta := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} L((\xi + \epsilon \eta)_Q(q)) \quad \forall \eta \in \mathcal{g} \quad (\text{A.4})$$

(iii) the equilibrium momentum $\mu := \mathbb{I}_\xi(q_e)$

(iv) the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle_e$ on $T_{q_e}Q$ induced by the linearized Legendre transformation

$$\langle\langle v, w \rangle\rangle_e := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left. \frac{d}{d\tau} \right|_{\tau=0} L(\xi_Q(q_e) + \epsilon v + \tau w) \quad (\text{A.5})$$

(v) the linearized inertia tensor $\mathbb{I}_e : \mathcal{G} \rightarrow \mathcal{G}^*$

$$\zeta \cdot \mathbb{I}_e \eta := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \left. \frac{d}{d\tau} \right|_{\tau=0} L((\xi + \epsilon \eta + \tau \zeta)_Q(q_e)) \quad (\text{A.6})$$

(vi) the $\langle\langle \cdot, \cdot \rangle\rangle_e$ -orthogonal complement $(\mathcal{G} \cdot q_e)^\perp$ to the tangent space $\mathcal{G} \cdot q_e$ of the group orbit

$$(\mathcal{G} \cdot q_e)^\perp := \{v \in T_{q_e}Q : \langle\langle v, \eta_Q(q_e) \rangle\rangle_e = 0 \quad \forall \eta \in \mathcal{G}\} \quad (\text{A.7})$$

(vii) the isotropy subgroup G_μ of μ with respect to the coadjoint action and its algebra $\mathcal{G}_\mu$

$$G_\mu := \{g \in G : \text{Ad}_g^* \mu = \mu\} \quad \text{and} \quad \mathcal{G}_\mu := \{\eta \in \mathcal{G} : \text{ad}_\eta^* \mu = 0\} \quad (\text{A.8})$$

(viii) the orbit $\mathcal{G}_\mu \cdot q_e$ of the subalgebra $\mathcal{G}_\mu$

$$\mathcal{G}_\mu \cdot q_e := \{\eta_Q(q_e) : \eta \in \mathcal{G}_\mu\} \quad (\text{A.9})$$

(ix) the isotropy subgroup G_{q_e} of the equilibrium configuration q_e and its subalgebra $\mathcal{G}_{q_e}$

$$G_{q_e} := \{g \in G : g \cdot q_e = q_e\} \quad \text{and} \quad \mathcal{G}_{q_e} := \{\eta \in \mathcal{G} : \eta_Q(q_e) = 0\} \quad (\text{A.10})$$

(x) the isotropy subgroup G_{v_e} of the equilibrium velocity v_e

$$G_{v_e} := \{g \in G : g \cdot v_e = v_e\} \quad (\text{A.11})$$

(xi) the subspace $\mathcal{G}_Q \subset \mathcal{G}$ of admissible group motions

$$\mathcal{G}_Q := \{\eta \in \mathcal{G} : \text{ad}_\eta^* \mu \in \mathcal{G}_{q_e}^A\}. \quad (\text{A.12})$$

If the group G is finite dimensional, then the assumption that the triplet (L, ξ, q_e) is orbit regular is equivalent to the assumption that $\ker \mathbb{I}_e = \mathcal{G}_{q_e}$. Let $\tilde{\mathcal{G}}$ be a complement of $\mathcal{G}_{q_e}$ and let $\tilde{\mathbb{I}}_e$ denote the restriction of $\mathbb{I}_e$ to $\tilde{\mathcal{G}}$. (If q_e has trivial isotropy, then $\tilde{\mathbb{I}}_e = \mathbb{I}_e$.) Define

(i) the generalized Arnold form $\mathcal{A} : \mathcal{G}_Q \times \mathcal{G}_Q \rightarrow \mathbb{R}$

$$\mathcal{A}(\zeta, \eta) := \text{ad}_\zeta^* \mu \cdot \left(\text{ad}_\eta \xi + \tilde{\mathbb{I}}_e^{-1} \text{ad}_\eta^* \mu \right). \quad (\text{A.13})$$

(ii) the space $\mathcal{Q}_{\text{int}}$ of ‘internal’ variations

$$\mathcal{Q}_{\text{int}} = \left\{ \delta q \in T_{q_e}Q : D\mathbb{I}_\xi(q_e) \cdot \delta q \in \mathcal{G}_{\text{int}}^A \right\}, \quad (\text{A.14})$$

where

$$\mathcal{G}_{\text{int}} := \mathcal{G}_{q_e} \cup \left\{ \tilde{\mathbb{I}}_e^{-1} (\text{ad}_\eta^* \mu) : \eta \in \mathcal{G}_Q \right\} \quad (\text{A.15})$$

(iii) the ‘internal’ stability two form $\mathcal{B}_{\text{int}} : \mathcal{Q}_{\text{int}} \times \mathcal{Q}_{\text{int}} \rightarrow \mathbb{R}$

$$\mathcal{B}_{\text{int}}(\delta q, \Delta q) := (D\mathbb{I}_\xi(q_e) \cdot \delta q) \cdot \tilde{\mathbb{I}}_e^{-1} (D\mathbb{I}_\xi(q_e) \cdot \Delta q) - D^2 L_\xi(q_e)(\delta q, \Delta q). \quad (\text{A.16})$$

A relative equilibrium v_e is formally stable if

- (i) $\langle\langle \cdot, \cdot \rangle\rangle_e$ is positive (negative) definite on $(\mathcal{G} \cdot q_e)^\perp$
- (ii) $\mathcal{A}$ is positive (negative) semi-definite, with kernel $\mathcal{G}_\mu \cup \mathcal{G}_{q_e}$
- (iii) $\mathcal{B}_{\text{int}}$ is positive (negative) semi-definite, with kernel $\mathcal{G}_\mu \cdot q_e$.

v_e is nonlinearly stable modulo G_μ if

- (i) v_e is formally stable
- (ii) there is a G_μ invariant inner product on $\mathcal{G}$
- (iii) there is a smooth global section $\sigma : G_\mu / (G_\mu \cap G_{v_e}) \rightarrow G_\mu$,
i.e. the principal bundle $G_\mu \rightarrow G_\mu / (G_\mu \cap G_{v_e})$ is trivial.

Remarks: $\mathcal{A}$, $\mathcal{Q}_{\text{int}}$, and $\mathcal{B}_{\text{int}}$ do not depend on the choice of $\tilde{\mathcal{G}}$. Note that if $\mathcal{G}_{q_e} \subset \mathcal{G}_\mu$, then $\mathcal{G}_Q = \mathcal{G}$ and

$$\mathcal{Q}_{\text{int}} = \left\{ \delta q \in T_{q_e} Q : \tilde{\mathbb{I}}_e^{-1}(D\mathbb{I}_\xi(q_e) \cdot \delta q) \in \mathcal{G}_\mu \right\}. \quad (\text{A.17})$$

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