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OF A SCALAR PARABOLIC P.D.E.**

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IMA Preprint Series # 644

May 1990

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* This research was done in part at the Instituto Superior Técnico and the Institute for Mathematics and its Applications and was partially supported by the Fundação Luso-Americana para o Desenvolvimento and J.N.I.C.T. (Portugal) under grant no. 87437 / MATM.

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1. INTRODUCTION

The scalar reaction diffusion equation

$$\begin{aligned} u_t &= u_{xx} + f(x, u, u_x) \quad , \quad 0 \leq x \leq 1 \\ u_x(t, 0) &= u_x(t, 1) = 0 \end{aligned} \tag{1}$$

with a nonlinear function f satisfying suitable qualitative conditions provides one of the best understood examples in the geometric theory of infinite dimensional dynamical systems [Ha1]. If $f \in C^k([0, 1] \times \mathbb{R}^2, \mathbb{R})$, $k \geq 3$, satisfies a growth condition of the form $|f(x, u, v)| \leq c(1 + |v|^\gamma)$ with $c > 0$ and $0 \leq \gamma < 2$ uniformly for x and u in compact sets, and a dissipative condition of the form $f(x, u, 0) \cdot u < 0$ for $|u|$ large, then (1) generates a global semigroup $T_f(t): X \rightarrow X$, $t \geq 0$, in the Sobolev space $X = H^1(0, 1)$ [Am] which is point dissipative and compact for $t > 0$ (see [Ha1] and [He1]). In this case $T_f(t)$ has a compact, connected, invariant set $A_f \subset X$ which is globally uniformly asymptotically stable. This set is called the global attractor [Ha1] and it contains all the essential information on the asymptotic behavior of $T_f(t)$.

Let $\mathfrak{X}$ denote the set of functions f described above endowed with the C^k strong Whitney topology [Hi] and let $\mathfrak{X}_0$ denote the subset corresponding to functions f depending only on u . One important problem in the qualitative or geometric theory of infinite dimensional dynamical systems is the characterization of the flow of $T_f(t)$ on the attractor A_f and its dependence on $f \in \mathfrak{X}$ (or $f \in \mathfrak{X}_0$) considered as a parameter.

The problem of characterizing the attractor A_f for the problem (1) with either Neumann or Dirichlet boundary conditions has been addressed by many authors (see for example: [CI],

[CS], [He2], [BF1] and [BF2] for $f \in \mathfrak{X}_0$; [Ma1], [FH], [HR], [R] and [FR] for $f \in \mathfrak{X}$; and also [AF], [FM-P] and [Ma2] for other boundary conditions). This attractor is entirely composed of equilibria and orbits connecting them [Ha1]. In addition, it is well known that generically in $\mathfrak{X}$ (or in $\mathfrak{X}_0$) $T_f(t)$ has the Morse-Smale property, that is: (i) there exist only a finite number of equilibria, henceforth denoted by $e_j, j=1, \dots, n$; (ii) all the equilibria are hyperbolic with a finite dimensional unstable manifold; and (iii) all the stable and unstable manifolds of the equilibria intersect transversally [He2], [An1]. In this case, from a gradient-like structure of the flow [Z] and the point dissipative property of $T_f(t)$, it follows that A_f is the union of the unstable manifolds of all the equilibria [Ha2]. Moreover, a theorem of Oliva [HMO] asserts that infinite dimensional Morse-Smale systems are structurally stable in the sense that the flows restricted to the corresponding attractors A_f are stable under small perturbations of the semigroup $T_f(t)$. We denote by Σ the subset of $f \in \mathfrak{X}$ corresponding to the structurally stable systems. This set is open and dense in $\mathfrak{X}$ and its complement $\mathfrak{X} \setminus \Sigma$ is the bifurcation set. Similarly, we define $\Sigma_0 \subset \mathfrak{X}_0$.

The first objective of this paper is to show that for all $f \in \Sigma$ the attractor A_f is the graph of a function Φ over a compact subset of a finite dimensional subspace of X . Furthermore, the dimension N of this subspace is equal to the maximum dimension of the unstable manifolds of the equilibria, $N = \max \{ \dim W^u(e_j), j=1, \dots, n \}$. In the particular case of $f \in \Sigma_0$ this result was shown by Jolly [J] (for f of the Chafee-Infante type [He1]) and by Brunovsky [B]. In addition, Brunovsky has also proved that for $f \in \Sigma_0$ the function Φ is C^1 smooth.

On what concerns the dependence of $T_f(t)$ on the parameter, we mentioned that the notion of structural stability used corresponds to the equivalence of the flows on the attractors of neighboring semigroups. The flows of $T_f(t)$ and $T_g(t)$ are equivalent if there is a homeomorphism h between the corresponding attractors, $h: A_f \rightarrow A_g$, mapping orbits to orbits and preserving the sense of direction in time. Hence, to characterize completely the dependence of the attractor A_f on the parameter one should try to classify all possible structurally stable attractors up to topological equivalence. In addition, one should compare Σ and Σ_0 and try to

understand better the attractors occurring for $f \in \Sigma$ that are not observable for $f \in \Sigma_0$, that is, attractors with special features essentially due to the dependence of f on x and u_x . For example, it is well known that if $f \in \mathfrak{X}_0$ the problem (1) has no nonconstant stable solutions [CH], while there exist $f \in \mathfrak{X}$ such that (1) has nonconstant stable equilibria [Y], [FH], [HR]. Another major difference between both of the forementioned parameter spaces is that the generic type of bifurcation observed in $\mathfrak{X}$ is the saddle-node bifurcation of one equilibrium, while it can be shown that in $\mathfrak{X}_0$, due to the Neumann boundary conditions, the generic bifurcation associated to the appearance of nonconstant solutions is either a pitchfork bifurcation from an unstable constant equilibrium or a pair of simultaneous saddle-node bifurcations.

In [FR], Fusco and Rocha introduced a permutation σ that seems to be important for the classification problem of all the possible structurally stable dynamics of equation (1). This permutation is defined by ordering the equilibria of (1) first by their values at $x=0$ and secondly by their values at $x=1$. That is, if we let $a_j = e_j(0)$, $b_j = e_j(1)$, $j=1, \dots, n$ and the equilibria are ordered in such way that $j > i \Leftrightarrow a_j < a_i$ then $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is defined by $\sigma(j) > \sigma(i) \Leftrightarrow b_{\sigma(j)} < b_{\sigma(i)}$. Clearly, σ also depends on the parameter f and not every possible permutation can be obtained from (1) with some $f \in \mathfrak{X}$. A permutation σ is said to be admissible in $\mathfrak{X}$ if there exists an $f \in \mathfrak{X}$ such that σ corresponds to the permutation obtained from (1) with this f (and similarly for $\mathfrak{X}_0$). One of the results in [FR] is the characterization of a large class Π of permutations admissible in $\mathfrak{X}$.

In the following, $f \in \Sigma$ is said to be of the pitchfork type in $\mathfrak{X}$ with respect to $f_0 \in \Sigma$ if there exists a family $f_\mu \in \mathfrak{X}$, $\mu \in [0, 1]$, such that $f_1 = f$ and $f_\mu \in \Sigma$ except at a finite number of points $\mu_1, \dots, \mu_n$, $n \geq 1$, at which a supercritical pitchfork bifurcation takes place. We also have a similar definition in $\mathfrak{X}_0$ for $f \in \Sigma_0$. The interesting fact about this definition is that if f is of the pitchfork type with respect to some f_0 , then all the heteroclinic connections in A_f are easily determined if the connections in A_{f_0} are known (see [FR]).

The class of admissible permutations Π introduced in [FR] covers exactly all the cases in which the dynamics on A_f can be obtained from the situation where the attractor is reduced to

a point, say $A_{f_0} = \{0\}$, through a finite sequence of supercritical pitchfork bifurcations.

Every permutation admissible in $\mathfrak{X}_0$ is in the class Π . Furthermore, it follows from [FR] that, at least in the subset of $f \in \mathfrak{X}_0$ having only three zeros, the admissible permutations allow only the possibility of pitchfork bifurcations from the equilibrium corresponding to the middle zero, excluding the possibility of secondary pitchfork bifurcations. This fact can be used to explain the nonexistence of stable nonconstant solutions in the case of $f \in \mathfrak{X}_0$ (at least for f with only three zeros).

The second objective of this paper is to show the existence of permutations that are admissible in $\mathfrak{X}$ but do not belong to the class Π . It follows that there exist nonlinear functions f in Σ such that: (i) the corresponding attractor A_f is not a point; and (ii) f is not of the pitchfork type with respect to some $f_0 \in \Sigma$ with a simpler attractor A_{f_0} . This implies that, in general, to solve the problem of determining all the connections in the attractor A_f , it is *not sufficient* to understand the connections appearing through a pitchfork bifurcation, but one needs to solve the more global problem of computing the new connections appearing through saddle-node bifurcations.

In the last section of this paper we construct a family of nonlinear functions $f_\delta = f_\delta(x, u)$, depending only on x and u , from which we obtain a permutation $\sigma \notin \Pi$ admissible in $\mathfrak{X}$. Furthermore, the attractor A_{f_δ} corresponding to this permutation is a two dimensional graph and, using the maximum principle and the transversality of stable and unstable manifolds of equilibria, we are able to determine all the connections in A_{f_δ} , thus providing an example of an attractor that cannot be reduced to a simpler one through a pitchfork bifurcation.

2. THE ATTRACTOR IS A GRAPH

Due to the dissipative condition on f , problem (1) always has at least one stationary solution. Since, by a change of variables, any stationary solution can be transformed into the trivial solution $u \equiv 0$, from now on we assume without loss of generality that f satisfies $f(x, 0, 0) = 0$.

We denote by $z(u)$ the zero number, that is, the number of strict sign changes, of the continuous function u defined on $[0, 1]$ (see for example [B], [BF3], [Ma3], [An2]). When the zeros of u are nondegenerate in the sense that $u_x(x) \neq 0$ if $u(x) = 0$, we simply have $z(u) = \text{card}\{x \in (0, 1) : u(x) = 0\}$.

The key result of this section is the following:

THEOREM 1: If N denotes the maximum dimension of the unstable manifolds of the equilibria of (1), $N = \max \{\dim W^u(e_j), j=1, \dots, n\}$, then for every pair of equilibria e_j, e_k with $j \neq k$, we have

$$z(e_j - e_k) < N.$$

We postpone the proof of this theorem and present as a corollary the main result of this section. Let L denote a second order differential operator $L: D(L) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ of the form $L(u) = u_{xx} + p(x)u_x + q(x)u$ with p and q continuous, satisfying Neumann boundary conditions, and let $\psi_1, \psi_2, \psi_3, \dots$, denote the corresponding eigenfunctions. Let also $P: L^2(0, 1) \rightarrow L^2(0, 1)$ denote the spectral projection onto the space spanned by the first $N-1$ eigenfunctions and $Q = I - P$ denote the projection onto the complement $Y = \text{span}\{\psi_N, \psi_{N+1}, \dots\}$. Then, we have:

THEOREM 2: The attractor A_f of problem (1) is the graph of a function $\Phi: PA_f \rightarrow X \cap Y$,

$$A_f = \text{graph } \Phi.$$

The proof of this theorem follows from Theorem 1 as in [J] or [B], and here we only

sketch the argument. Multiplying by a suitable exponential factor, the eigenvalue problem for L is easily transformed into a Sturm-Liouville problem and from classical arguments we obtain $z(Qw) \geq N$ for any continuous w such that $Qw \neq 0$. Let u and v be different elements of A_f and define $w(t) = T_f(t)u - T_f(t)v$. Using Matano's principle, it can be shown that $z(w(t))$ is a nonincreasing function of t (see for example [He2] and [An2]). We remark that w satisfies an appropriate linear second-order parabolic equation with coefficients depending on t and x . The orbits through u and v are globally defined and the corresponding α -limit sets are equilibria of (1), say e_j and e_k respectively. If j is equal to k , using the linearization of (1) around e_j we have $z(u-v) \leq i(e_j) < N$. If $j \neq k$, from Theorem 1 we also have $z(u-v) \leq z(e_j - e_k) < N$. Then, if we assume that $Pu = Pv$, we obtain $z(u-v) = z(Q(u-v)) < N$ which is a contradiction. This completes the argument. Since the operator L is arbitrary, we can choose any linearization of (1) around an equilibrium or, for example, take $p \equiv q \equiv 0$ and have $\psi_k(x) = \cos(k-1)\pi x$, $k=1,2,\dots$, and P, Q the corresponding orthogonal projections.

To proceed we need to introduce some notation used in [FR]. Consider the initial value problem corresponding to the stationary problem for (1):

$$\begin{aligned} u_{xx} + f(x, u, u_x) &= 0 \\ u(0, a) &= a, \quad u_x(0, a) = 0. \end{aligned} \tag{2}$$

Since we are only interested in studying the properties of the attractor A_f , which is compact, here we assume that the function f satisfies appropriate growth conditions ensuring that the solutions of (2) are defined for all $x \in [0, 1]$. Let M_f denote the two dimensional manifold of all the solutions of (2) with $a \in \mathbb{R}$:

$$M_f = \{(x, u, v): u = u(x, a), v = v(x, a), \text{ where } u(., a) \text{ solves (2) and } (x, a) \in [0, 1] \times \mathbb{R}\},$$

and let γ_f denote the section curve of M_f at $x=1$, $\gamma_f = M_f|_{x=1}$. Then, the equilibria e_j , $j=1, \dots, n$ of (1) are in one-to-one correspondence with the intersection points between γ_f and the plane $v=0$,

and the equilibrium e_j is hyperbolic if and only if the corresponding intersection is transversal. Moreover, the permutation σ referred to in the introduction can be read out from γ_f [FR] (see Figure 1 for an example). Our objective in the rest of this section is to show that γ_f also determines completely $z(e_j - e_k)$ for all $1 \leq j, k \leq n$, and establish Theorem 1.

Let $\zeta = \zeta(a)$ denote the point of γ_f corresponding to the initial value problem (2), $\zeta(a) = (1, u(1, a), u_x(1, a)) \in \gamma_f$. As a goes from ∞ to $-\infty$ the point $\zeta(a)$ describes the curve γ_f . We also define $\theta = \theta(a)$ as the angle swept clockwise by the unit vector tangent to the section curve $M_f|_{x=\bar{x}}$ at the point $(\bar{x}, u(\bar{x}, a), u_x(\bar{x}, a))$ as $\bar{x}$ describes the interval $[0, 1]$ from 0 to 1. We have $\theta(a) = s(1, a)$ where $s = s(x, a)$ satisfies the differential equation

$$s_x = \sin^2 s - p(x, a) \sin s \cos s + q(x, a) \cos^2 s, \quad s(0, a) = 0$$

and $p(x, a) = f_v(x, u(x, a), u_x(x, a))$, $q(x, a) = f_u(x, u(x, a), u_x(x, a))$. Moreover, if the equilibrium e_j of (1) with initial value $a_j = e_j(0)$ is hyperbolic, then its Morse index $i(e_j)$ defined as the dimension of the unstable manifold of e_j is given by $i(e_j) = 1 + \text{Integer part of } [\theta(a_j)/\pi]$ (see for example [R]). Since $s = s(x, a)$ is continuous, we remark that $\theta = \theta(a)$ can also be computed from the angle swept clockwise by the unit vector tangent to γ_f at $\zeta = \zeta(a)$ as $\zeta(a)$ describes γ_f (that is, the tangent indicatrix of γ_f [Ca]).

Figure 1: Example corresponding to $\sigma : (1, 2, 3, 4, 5, 6, 7) \rightarrow (1, 6, 5, 2, 3, 4, 7)$

As a goes from a_j to a_k , $\zeta(a)$ describes γ_f from $(1, b_j, 0)$ to $(1, b_k, 0)$ with $b_i = e_i(1)$, $i = 1, \dots, n$. Let θ_j denote the unit tangent vector to γ_f at $\zeta(a_j)$ and l_j denote the vertical line at $\zeta(a_j)$, $l_j = \{(1, b_j, c) : c \in \mathbb{R}\}$. From the results of [FR] it follows that we can have γ_f everywhere transversal to l_j by taking a small perturbation of f if necessary. This perturbation is constructed by deforming the solution manifold M_f locally near l_j and it is clear that it does not affect the value of $z(e_j - e_k)$ for $1 \leq j < k \leq n$ if it is sufficiently small. In this case, since e_j is hyperbolic, we

also have that θ_j belongs to one of the four open quadrants, avoiding the vertical and the horizontal lines (see Figure 2.a for an example).

PROPOSITION 3: If r_{jk} , $1 \leq j < k \leq n$, denotes the number of times that $\zeta(a)$ crosses l_j as a goes from a_j to a_k , clockwise around $\zeta(a_j)$ and taking into account the signs, we have that

$$z(e_j - e_k) = i(e_j) + r_{jk}$$

if θ_j belongs to the odd quadrants, and

$$z(e_j - e_k) = i(e_j) - 1 + r_{jk}$$

if θ_j belongs to the even quadrants.

(a): Example corresponding to $r_{jk}=3$ (b): Changing the number of intersections

Figure 2

Proof: Consider the integer valued function V_j defined on $\mathbb{R} \setminus \{a_j\}$ by $V_j(a) = z(e_j - u(., a))$ where $u = u(x, a)$ is the solution of (2). We are interested in the values of V_j for $a_j < a \leq a_k$. In fact, we want to determine $V_j(a_k) = z(e_j - e_k)$. Let $w = w(x, a)$ denote the difference $w = u - e_j$. Using (2) we can show that w satisfies the differential equation

$$w_{xx} + \bar{p}(x, a) w_x + \bar{q}(x, a) w = 0$$

$$w(0, a) = a - a_j, \quad w_x(0, a) = 0$$

where $\bar{p}(x, a) = \int_0^1 f_v(x, \rho(x, a, \mu), \rho_x(x, a, \mu)) d\mu$, $\bar{q}(x, a) = \int_0^1 f_u(x, \rho(x, a, \mu), \rho_x(x, a, \mu)) d\mu$ and $\rho(., a, \mu) = \mu u(., a) + (1 - \mu)e_j$. Performing the change of variables $w = \bar{r} \cos \bar{s}$, $w_x = -\bar{r} \sin \bar{s}$ we have that $\bar{s}$ satisfies the differential equation

$$\bar{s}_x = \sin^2 \bar{s} - \bar{p}(x, a) \sin \bar{s} \cos \bar{s} + \bar{q}(x, a) \cos^2 \bar{s}, \quad \bar{s}(0, a) = 0$$

and since $\bar{p}$, $\bar{q}$ are continuous and $\bar{p}(x, a_j) = p(x, a_j)$, $\bar{q}(x, a_j) = q(x, a_j)$ we conclude that $\bar{z}(1, a_j) = \theta(a_j)$. This implies that, for $\varepsilon > 0$ sufficiently small, taking $a_j < a < a_j + \varepsilon$ we have that $z(w(., a)) = \text{Integer part of } [\theta(a_j)/\pi]$ if θ_j belongs to the even quadrants, and $z(w(., a)) = \text{Integer part of } [\theta(a_j)/\pi] + 1$ if θ_j belongs to the odd quadrants. Thus, for $a \in (a_j, a_j + \varepsilon)$, we conclude that either $V_j(a) = i(e_j) - 1$ or $V_j(a) = i(e_j)$ according to whether θ_j belongs to the even or odd quadrants.

Next, we consider the points in $(a_j, a_k]$ at which V_j changes its value. If $a_0 \in (a_j, a_k]$ denotes one such point, then there is either an $x_0 \in (0, 1)$ at which $u(., a_0)$ and e_j have a nontransversal intersection (see Figure 2.b) or $u(x_0, a_0) = e_j(x_0)$ with $x_0 = 0$ or 1 . Clearly, $x_0 \neq 0$ since $a_0 \neq a_j$. Moreover, since both $u(., a_0)$ and e_j satisfy the same second order differential equation (2) we must have $x_0 \notin (0, 1)$, otherwise the nontransversal intersection given by $u(x_0, a_0) = e_j(x_0)$ and $u_x(x_0, a_0) = de_j(x_0)/dx$ would imply $u(., a_0) \equiv e_j$. Hence, we conclude that $x_0 = 1$. This implies that $\zeta(a_0) \subset l_j \cap \gamma_f$, which by hypothesis is a point of transversal intersection, and that $a_0 \neq a_k$. Finally, we analyse the change of value of V_j at a_0 as $\zeta(a)$ crosses l_j at $\zeta(a_0)$. By transversality we have that $u_a(1, a_0) \neq 0$. Since $u_x(1, a_0) \neq 0$, we have that V_j increases by one if $u_x(1, a_0) \cdot u_a(1, a_0) > 0$, that is, if $\zeta(a)$ crosses l_j at $\zeta(a_0)$ clockwise with respect to $\zeta(a_j) = (1, b_j, 0)$, and that V_j decreases by one if $u_x(1, a_0) \cdot u_a(1, a_0) < 0$, that is, if $\zeta(a)$ crosses l_j at $\zeta(a_0)$ in the opposite sense. This completes the proof of the proposition.

PROPOSITION 4: For $1 \leq j < k \leq n$ we have that $z(e_j - e_k) < N_{jk}$ where

$$N_{jk} = \max \{i(e_j), i(e_{j+1}), \dots, i(e_k)\}.$$

Proof: We consider only the case of θ_j belonging to an even quadrant since the other case is similar. If $r_{jk} \leq 0$ the result follows immediately from Proposition 3, hence we assume that $r_{jk} > 0$. In this case, there is a sequence of integers $j = i_0 < i_1 < \dots < i_m \leq k$ with $m = r_{jk}$ such that

$$r_{ji_{l+1}} = r_{ji_l} + 1 \quad (3)$$

and

$$r_{ji_l} > r_{j,s} \quad (4)$$

for $s=j+1, \dots, i_l-1$ and $l=1, \dots, m-1$. We will show that

$$i(e_{i_{l+1}}) = i(e_{i_l}) + 1 \quad (5)$$

for $l=0, 1, \dots, m-1$, thus concluding that $i(e_{i_m}) = i(e_j) + r_{jk}$ and completing the proof in view of Proposition 3.

We recall that $i(e_{i_l})$ is given by $1 + \text{Integer part of } [\theta(a_{i_l})/\pi]$ where $\theta(a)$ is computed from the tangent map τ of the plane curve γ_f given by $\tau(a) = \zeta'(a)/\|\zeta'(a)\|$. From (4), we have that $r_{ji_l} = r_{j,i_l-1} + 1$ and we conclude that the point $\zeta(a)$ describing the curve γ_f crosses the (u,x) -plane for $a=a_{i_1}, \dots, a_{i_m}$ clockwise with respect to the point $(1, b_j, 0)$. Hence, to prove (5) for $0 < l \leq m$ we only need to show that

$$0 < \theta(a_{i_{l+1}}) - \theta(a_{i_l}) < 2\pi. \quad (6)$$

Moreover, the same argument applies to the case $l=0$.

As before, we let $b_s, j \leq s \leq k$, denote the value of the abscissa on the u -axis of the point of intersection of γ_f with the (u,x) -plane corresponding to the equilibrium e_s and, for definiteness, in the following we assume that $b_{i_l} < b_{i_{l+1}}$ (see Figure 3).

Figure 3: Example of a nonsimple curve Γ_l

Let γ_l denote the arc of γ_f described by ζ for $a_{i_l} \leq a \leq a_{i_{l+1}}$, let $\tilde{\gamma}_l$ denote the open line segment $\tilde{\gamma}_l = \{(1, b, 0) : b_{i_l} < b < b_{i_{l+1}}\}$, and define the plane curve Γ_l as the union $\Gamma_l = \gamma_l \cup \tilde{\gamma}_l$. If γ_l does not intersect $\tilde{\gamma}_l$, then Γ_l is a piecewise differentiable Jordan curve and (6) follows immediately from an application to Γ_l of the turning tangent theorem [Ca], [S]. If $\gamma_l \cap \tilde{\gamma}_l \neq \emptyset$ the curve Γ_l is not simple. Considering the orientation of γ_f , let α denote the abscissa of the first

intersection point of γ_l with $\tilde{\gamma}_l$. From (4) we have that α must satisfy $b_{i_l} < \alpha < b_j$ and that $\zeta(a)$ crosses the (u,x) -plane anticlockwise with respect to the point $(1, b_j, 0)$ at the corresponding intersection. Furthermore, by a successive application of the Jordan curve theorem, (4) also implies that γ_l must intersect again $\tilde{\gamma}_l$ at a point with abscissa β satisfying $\alpha < \beta < b_j$ and $\zeta(a)$ crosses the (u,x) -plane clockwise with respect to the point $(1, b_j, 0)$ at the corresponding intersection. Then, we can decompose Γ_l as the union of an odd number of Jordan curves, say $2K+1$, such that $K+1$ are positively oriented and K have the negative orientation, and (6) will follow again from an application of the turning tangent theorem.

3. A SPECIAL TWO DIMENSIONAL ATTRACTOR

In this section we prove the existence of nonlinear functions f in $\mathfrak{X}$ for which the corresponding permutations σ do not belong to the class Π . Moreover, for the particular examples presented here we determine all the connections in the corresponding attractors A_f . The method applied is a phase-energy method exploring the knowledge of the curves γ_f when f is a cubic polynomial and we refer to [R] and [HR] for details.

Let $g(u)=u-u^3$, let $\phi_\varepsilon:\mathbb{R}\rightarrow\mathbb{R}$ denote a C^∞ -smooth monotonically increasing function satisfying $\phi_\varepsilon = 0$ on $(-\infty, -\varepsilon]$ and $\phi_\varepsilon = 1$ on $[\varepsilon, \infty)$ and define the function $f:[0, L]\times\mathbb{R}\rightarrow\mathbb{R}$ given by $f(x, u)=\delta_1 g(u)$ for $0\leq x<\delta_4$ and $f(x, u)=\phi_\varepsilon(u)\delta_2 g(2u-1)+[1-\phi_\varepsilon(u)]\delta_3 g(2u+1)$ for $0\leq x-\delta_4\leq\delta_5$ where $\delta=(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)\in\mathbb{R}_+^5$ and $L=\delta_4+\delta_5$. We consider the problem

$$u_t = u_{xx} + f(x, u) \quad , \quad 0 \leq x \leq L \quad (7)$$

satisfying Neumann boundary conditions. Clearly f is not continuous in x , but the initial value problem is still well posed in $H^1(0, L)$ [LSU]. Furthermore, considering the ordinary differential equation satisfied by the stationary solutions of (7) and its linear variational equation, the corresponding results for functions f in $\mathfrak{X}$ follow from the continuous dependence of the solutions on initial conditions and parameters (see for example [Fi]).

PROPOSITION 5: There exist an $\varepsilon_0>0$ and an open set $G\subset\mathbb{R}_+^5$ such that for $0<\varepsilon<\varepsilon_0$ and $\delta\in G$ the problem (7) has a two dimensional attractor A_f containing 11 equilibria, where 4 are stable, 5 have Morse index 1 and 2 have Morse index 2. Moreover, the corresponding permutation σ_f is given by $\sigma_f: (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11) \rightarrow (1, 8, 7, 2, 3, 6, 9, 10, 5, 4, 11)$.

Proof: One immediately recognizes that the constant functions $u\equiv-1, 0, 1$ are equilibria of (7). In addition, every stationary solution must satisfy $-1\leq u(x)\leq 1$ and be a solution of the ordinary differential equation

$$u_{xx} + f(x, u) = 0 \quad , \quad 0 \leq x \leq L . \quad (8)$$

For this equation we consider the initial value problem on $0 \leq x \leq \delta_4$ determining the transition curve τ_+ given by

$$\tau_+ = \{(u(\delta_4), u_x(\delta_4)): u \text{ satisfies (8) with } u_x(0)=0 \text{ and } -1 \leq u(0) \leq 1\},$$

and, similarly, we consider the end value problem on $\delta_4 \leq x \leq L$ determining the transition curve τ_- given by

$$\tau_- = \{(u(\delta_4), u_x(\delta_4)): u \text{ satisfies (8) with } u_x(L)=0 \text{ and } -1 \leq u(L) \leq 1\}.$$

Then, as in [R] the set of equilibrium solutions of (7) is in one-to-one correspondence with the set $\tau_+ \cap \tau_-$ and the permutation σ_f can be determined by ordering the intersection points firstly along τ_+ and secondly along τ_- . Furthermore, the Morse indices of the equilibria are determined from the angles formed by the curves at the corresponding intersections and can be computed from the permutation σ_f .

Since for $0 \leq x \leq \delta_4$ the function f is given by the cubic $f(x, u) = u - u^3$, the transition curve τ_+ can be written explicitly in terms of Jacobian elliptic functions as in [HR]:

$$\tau_+ = \{(u_0 \operatorname{cn} \chi, -u_0 \beta \delta_1^{1/2} \operatorname{sn} \chi \operatorname{dn} \chi): u_0 \in [-1, 1]\},$$

where the arguments of the elliptic functions are given by $\chi = (\beta \delta_1^{1/2} \delta_4 | m)$, that is, we have $\operatorname{cn} \chi = \operatorname{cn}(\beta \delta_1^{1/2} \delta_4 | m)$, ..., with $\beta = (1 - u_0^2)^{1/2}$ and $m = -\beta^{-2} u_0^2 / 2$. This curve is contained in the closed set bounded by the curve $\Gamma = \operatorname{cl}(\Gamma_+ \cup \Gamma_-)$ where $\Gamma_{\pm} = \{(u_0, \pm(\delta_1/2)^{1/2}(1 - u_0^2)): u_0 \in (-1, 1)\}$ are heteroclinic curves in the phase portrait of (8). We are interested on the dependence of τ_+ on the parameters. If we let $\lambda_0 = \delta_1^{1/2}$, $\lambda_1 = \delta_1^{1/2} \delta_4$ and write $\tau_+ = \tau_+(\lambda_0, \lambda_1)$, one immediately recognizes that λ_0 has the effect of a change of scale in the u_x -axis while λ_1 affects only the argument of the elliptic functions. Writing τ_+ in polar coordinates we can easily verify that the polar phase depends monotonically on λ_1 . In fact, if we write $u = \rho \cos \theta$, $u_x = -\lambda_0 \rho \sin \theta$, with $\theta = \theta(u_0, \lambda_1)$ continuously defined on $[-1, 1] \times \mathbb{R}_+$ with values in $\mathbb{R}$ and satisfying $\theta(1, \lambda_1) = 0$, we can verify that $\partial \theta(u_0, \lambda_1) / \partial \lambda_1 > 0$ for $u_0 \in (-1, 1)$. Moreover, $\theta(0, \lambda_1) = \lambda_1$ and

$u_0 \partial \theta(u_0, \lambda_1) / \partial u_0 > 0$ on $u_0 \in [-1, 1] \setminus \{0\}$ [HR]. Another fact concerning the dependence of τ_+ on λ_1 is that, as λ_1 takes values on $(0, \infty)$, the point of τ_+ corresponding to $u_0 \in (-1, 1)$ describes periodically the curve given by $\{(u, v): \lambda_0^{-2} v^2 + u^2 - u^4/2 = u_0^2 - u_0^4/2\}$.

Completely similar observations can be made about the curve τ_- after being restricted to the left or the right half plane, since for $\delta_4 \leq x \leq L$ the function f is given on each interval $[-1, -\varepsilon]$ and $[\varepsilon, 1]$ by the cubic $g(2u \pm 1)$. It turns out that, apart from an ε -neighborhood of Γ , the curve τ_+ under the changes of variables $(2u \pm 1, 2v) \rightarrow (u, -v)$ will correspond to the appropriate restriction of the curve τ_- for a suitable choice of parameters.

Figure 4: The curves τ_+ and τ_- for $\delta = (16, 32, 32, 2/3, 1/3)$.

From the preceding observations it follows that we can find $\delta \in \mathbb{R}_+^5$ such that the curves τ_+ and τ_- , in addition to the three points $(-1, 0)$, $(0, 0)$ and $(1, 0)$, intersect exactly at four more pairs of points (see Figure 5). Furthermore, by changing the parameters δ_i , each of these intersection pairs can be created or destroyed independently, corresponding to four different saddle-node bifurcations of equilibria. Ordering the intersection points along τ_+ and τ_- we obtain the permutation $\sigma_f: (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11) \rightarrow (1, 8, 7, 2, 3, 6, 9, 10, 5, 4, 11)$ from which all the other assertions of the Proposition follow. In particular, we have that the equilibria numbered 1, 3, 9 and 11 are stable, the equilibria 2, 4, 6, 8 and 10 have one dimensional unstable manifolds and the equilibria 5 and 7 have two dimensional unstable manifolds. From Theorem 2 we conclude that A_f is a graph over a two dimensional space.

As referred to in the first section, the class of admissible permutations Π correspond to the situations that can arise through a sequence of supercritical pitchfork bifurcations. This implies that in every permutation belonging to this class we can find at least a triple of consecutive numbers. Since this criteria rules out the permutation σ_f , we conclude that $\sigma_f \notin \Pi$. This fact impairs our ability to determine the heteroclinic connections in the attractor A_f of (7)

using the method of [FR] and makes the knowledge of this attractor more interesting. In this case, however, all the connections in the attractor can be determined using transversality arguments as in [R], and the results of Brunovsky and Fiedler [BF1], [BF2].

Figure 5: (a) Representation of the attractor A_f ; (b) Sketch of the equilibria.

PROPOSITION 6: For $\varepsilon_0 > 0$ and G given in Proposition 5, the flow on the attractor A_f of problem (7) with $0 < \varepsilon < \varepsilon_0$ and $\delta \in G$ is shown in Figure 5.a where the connections between the equilibria are sketched.

Proof: By transversality, for $0 < \varepsilon < \varepsilon_0$ and $\delta \in G$ all the flows of (7) are equivalent. Since we can choose f satisfying $f(x, u) = -f(x, -u)$, using this symmetry we only need to establish the connections starting at the equilibria e_j with $1 \leq j \leq 6$. Moreover, since e_1 and e_3 are stable, we only consider e_2, e_4, e_6 which have one dimensional unstable manifolds, and e_5 which has a two dimensional unstable manifold. In the following we denote by $e_i \rightarrow e_j$ a connection orbit going from e_i at $t = -\infty$ to e_j at $t = +\infty$.

Considering the pairs of equilibria involved in the saddle-node bifurcations referred to before, we conclude immediately that there are connections $e_2 \rightarrow e_3$ and $e_5 \rightarrow e_4$. In order to establish other connections we need some qualitative information on the equilibria. It is sufficient to know the exact number of intersections between each pair of equilibria which we can compute using Proposition 3. In Figure 5.b we sketch the equilibria showing the corresponding number of intersections. From this information and a simple application of the maximum principle we conclude that there is also a connection $e_2 \rightarrow e_1$, which together with e_2 and $e_2 \rightarrow e_3$ completes the unstable manifold of the equilibrium e_2 . Similarly, we can establish the connections $e_4 \rightarrow e_3$, $e_4 \rightarrow e_{11}$, $e_6 \rightarrow e_3$ and $e_6 \rightarrow e_9$, obtaining the the unstable manifolds of e_4 and e_6 , and also the connections $e_5 \rightarrow e_3$ and $e_5 \rightarrow e_{11}$. This completes the study of the connection

orbits starting at the equilibria e_2 , e_4 and e_6 . To consider the missing connections starting at e_5 we recall that for some $\delta \notin G$ the equilibria e_9 and e_{10} undergo a saddle-node bifurcation. Let δ be such that exactly these two equilibria are missing. In this case, by Corollary 3.2 and §6 of [BF1] we have that e_5 connects to e_6 . It turns out, again by transversality, that this connection is preserved when the saddle-node bifurcation of e_9 and e_{10} occurs. This establishes the existence of $e_5 \rightarrow e_6$ and, since we already have the connection $e_6 \rightarrow e_9$, we also obtain the connections $e_5 \rightarrow e_9$ and $e_5 \rightarrow e_{10}$ from the transversality and a saddle-node argument in the last case. Finally, we need to show that there are no other equilibria that e_5 can connect to. In fact, the other possibilities for connections starting at e_5 would be $e_5 \rightarrow e_8$, which by Lemma 4.1 and §6 of [BF1] is blocked by e_6 , and $e_5 \rightarrow e_1$, $e_5 \rightarrow e_2$, which similarly are blocked by e_3 . This completes the proof of the Proposition.

ACKNOWLEDGEMENTS

I would like to thank Sigurd Angenent, Pavol Brunovsky, Bernold Fiedler and Peter Polačik for helpful discussions during my visits to the Institute for Mathematics and its Applications and the Institut für Angewandte Mathematik in Heidelberg. I would also like to thank Miguel Abreu who helped with numerical computer work.

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Figure 1

(a)

(b)

Figura 2

Figure 3

Figure 4

(a)

(b)

Figure 5