**ROTATION SETS AND MONOTONE PERIODIC
ORBITS FOR ANNULUS HOMEOMORPHISMS**

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Abstract. If f is a homeomorphism of the annulus and p/q is a rational in lowest terms that is contained in the rotation set of f then f has a (p, q) -topologically monotone periodic orbit. In addition, if f has a p/q -period orbit that is not topologically monotone then the Farey interval of p/q is contained in the rotation set of f .

Section 1. There are many theorems which give information about periodic orbits for maps of the annulus. The Poincaré-Birkhoff theorem implies that an area preserving homeomorphism always has periodic orbits with rotation numbers equal to every rational number between the rotation numbers of the boundary circles. With the addition of the monotone twist hypothesis one gets more information. In this case, the Aubry-Mather theorem yields the existence of periodic orbits whose radial order is preserved by the map. Such orbits are called Birkhoff ([K]) or monotone.

Using topological techniques, Hall showed that whenever a monotone twist map has a p/q -periodic orbit, it has a monotone p/q -periodic orbit ([H1]). In [H2] he pointed out the appropriate generalization of this notion to a homeomorphism of the annulus and asked whether the appropriate version of the Aubry-Mather theorem for periodic orbits was true in this more general context. Recently, Le Calvez answered this to the affirmative under the area preserving hypothesis ([LC]). This paper contains a proof of the general case. In addition, it is shown that if the homeomorphism has a p/q -periodic orbit that is not “topologically monotone” then a certain interval depending on simple arithmetic properties of p/q must be in the rotation set. This second result was proved for monotone twist maps without the area preserving hypothesis in [Bd1] using Lemma 4 of [B+H].

The main tool used in the proofs is the Thurston-Nielson theory of surface automorphisms. The proofs are clarified and simplified by encoding the theorem in a statement about a partial order on the set of periodic orbits for homeomorphisms of the disk. This partial order is the analog of the partial order in Sharkovski’s Theorem. For background information on this partial order on braid types see [Bd2].

If p and q are relatively prime integers. Define m/n to be the maximum of $\{r/s : r/s < p/q, s > q \text{ and } (r, s) = 1\}$ and k/l to be the minimum of $\{r/s : r/s > p/q, s > q \text{ and } (r, s) = 1\}$. The *Farey interval* of p/q is $I(p/q) = [m/n, k/l]$. The endpoints of the Farey interval satisfy $mq - pn = 1$ and $kq - pl = -1$ and are the last two convergents of the continued fraction of p/q if the continual fraction ends in a one (see [HW] chapter 3). The notation $\rho(f)$ means the rotation set of f .

MAIN THEOREM. *If f is an orientation and boundary preserving homeomorphism of the annulus and $p/q \in \rho(f)$ with p and q positive and relatively prime then f has a (p, q) -topologically monotone periodic orbit. If f has a (p, q) -orbit that is not topologically monotone then $I(p/q) \subseteq \rho(f)$.*

Recalling the interpretation of rotation number as a frequency of oscillation, one could make an analogy between this result and the fact that an oscillating physical system or field that supports a complicated vibration at a given frequency must also support the simplest, least excited state with that frequency (cf. [Hg]).

It is natural to ask whether there is a similar theorem about irrational numbers in the rotation set. One version of this might be: If α is an irrational number with $\alpha \in \rho(f)$ does f have a minimal set that is semiconjugate to rigid rotation by α on a circle. Under the monotone twist hypothesis the answer is yes. One way to prove this is by taking Hausdorff limits of monotone periodic orbits ([K]).

For a general homeomorphism the answer is no. In [Hn1], Handel constructed an area preserving, C^∞ -diffeomorphism, h , with an irrational $\alpha \in \rho(h)$ such that every point with rotation number α is contained in a single minimal set, X . The set X is the Hausdorff limit of topologically monotone periodic orbits but h restricted to X is not semiconjugate to rigid rotation by α on a circle. The set X is topologically a pseudocircle. This leads to: If α is an irrational in the rotation set of f , does f have a minimal set that is semiconjugate to a homeomorphism on the circle or pseudo circle?

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Section 2. We begin with some basic notation and definitions. A homeomorphism f of the annulus $A = S^1 \times [0, 1]$ will always be isotopic to the identity. A lift to the universal cover $\tilde{A} = \mathbb{R} \times [0, 1]$ is denoted $\tilde{f}$, and $T : \tilde{A} \rightarrow \tilde{A}$ given by $T(x, y) = (x + 1, y)$ is the deck transformation. The projection onto the first factor is $\pi_x : \tilde{A} \rightarrow \mathbb{R}$.

Given $\tilde{x} \in \tilde{A}$, we define its rotation number under $\tilde{f}$ as

$$\rho(\tilde{x}, \tilde{f}) = \lim_{n \rightarrow \infty} \frac{\pi_x(\tilde{f}^n(\tilde{x})) - \pi_x(\tilde{x})}{n}$$

if the limit exists. Note that we include the dependence on the lift in our definition of rotation number. The set of rotation numbers of $\tilde{f}$ is

$$\rho(\tilde{f}) = \{\rho(\tilde{x}, \tilde{f}) : \tilde{x} \in \tilde{A}\}.$$

A theorem of Handel ([Hn4]) states that $\rho(\tilde{f})$ is a closed set. A periodic orbit, $o(x)$, is called a (p, q) periodic orbit if its period is q and there are lifts $\tilde{x}$ and $\tilde{f}$ with $T^{-p}\tilde{f}^q(\tilde{x}) = \tilde{x}$.

Note that if x is a (p, q) -periodic orbit there always exists a p' with $0 \leq p'/q < 1$ such that x is also a (p'/q) -periodic orbit.

Remark. One may also define the rotation interval for any point $\tilde{x}$ as

$$\bar{\rho}(\tilde{x}, \tilde{f}) = \left[\liminf_{n \rightarrow \infty} \frac{\pi_x(\tilde{f}^n(\tilde{x})) - \pi_x(\tilde{x})}{n}, \limsup_{n \rightarrow \infty} \frac{\pi_x(\tilde{f}^n(\tilde{x})) - \pi_x(\tilde{x})}{n} \right]$$

and then define $\bar{\rho}(\tilde{f}) = \cup \bar{\rho}(\tilde{x}, \tilde{f})$. A theorem of Franks ([F1]) and Handel ([Hn3]) shows that $p/q \in \bar{\rho}(\tilde{f})$ with p and q relatively prime implies that f has a (p, q) -periodic orbit. Thus $\bar{\rho}(\tilde{f}) \cap \mathbb{Q} = \rho(\tilde{f}) \cap \mathbb{Q}$ and since $\rho(\tilde{f})$ is closed, a simple argument yields $\bar{\rho}(\tilde{f}) = \rho(\tilde{f})$.

We next define braid types and the partial order. Fix a copy of the annulus minus n interior points and call it A_n . The group of isotopy classes of orientation preserving homeomorphisms of A_n is denoted G_n . The isotopies fix the boundary setwise but not necessarily pointwise. If $f : A \rightarrow A$ is a homeomorphism with a period n -periodic orbit $o(x, f) \subseteq \text{Int}(A)$, let $A_x = A - o(x, f)$ and $f_x = f|_{A_x}$. Pick a homeomorphism $h : A_x \rightarrow A_n$ and let $[hf_x h^{-1}] \in G_n$ be the isotopy class. Define the braid type of $o(x, f)$, denoted $bt(x, f)$, to be the conjugacy class of $[hf_x h^{-1}]$ in G_n . By passing to the conjugacy class we have made $bt(x, f)$ independent of the choice of homeomorphism h . The set of all possible braid types on the annulus is called BT . Given a homeomorphism $f : A \rightarrow A$, define its set of braid types as $bt(f) = \{bt(x, f) : o(x, f) \text{ is a periodic orbit in } \text{Int}(A)\}$.

We define an order relation on BT as follows. If $\alpha, \beta \in BT$, $\alpha \geq \beta$ if and only if $\alpha \in bt(f)$ implies $\beta \in bt(f)$ for all homeomorphisms f . The first proposition says that this order relation is, in fact, a partial order. This is stated without proof in [Bd2]. It relies on a result of Brunovsky [Br] which says that, given $N > 0$, an isotopy between Kupka-Smale diffeomorphisms f_0 and f_1 can always be approximated by a diffeotopy with the property that all orbits of period less than N undergo only saddle node and flip bifurcations. Further, all bifurcations involving orbits of period less than N occur at a finite number of distinct parameter values.

PROPOSITION 1. *The order relation $(BT, \leq)$ is a partial order.*

Proof. It is obvious from the definition that $\alpha \leq \alpha$ and that $\alpha \leq \beta$ and $\beta \leq \gamma$ implies $\alpha \leq \gamma$. We must therefore show that $\alpha \leq \beta$ and $\beta \leq \alpha$ implies $\alpha = \beta$. For this it suffices to show that $\alpha \geq \beta$ and $\alpha \neq \beta$ implies the existence of an f with $\beta \in bt(f)$ but $\alpha \notin bt(f)$, i.e. $\beta \not\geq \alpha$.

Let f_0 and f_1 be a Kupka-Smale maps with $\alpha \notin bt(f_0)$ and $\beta \notin bt(f_0)$ and $\beta \in bt(f_1)$. Let N be larger than the period of α or β . Use Brunovsky's theorem to obtain a nice diffeotopy, f_μ , between f_0 and f_1 . Now let $\mu_0 = \inf\{\mu : \alpha \in bt(f_\mu) \text{ or } \beta \in bt(f_\mu)\}$. There must be a bifurcation occurring for f_{μ_0} . If this bifurcation is a saddle node, then the two orbits created as μ increases have the same braid type which must be β as $\alpha \geq \beta$. If the

bifurcation is a flip, then the orbit that persists through the bifurcation was present for $\mu < \mu_0$ and thus cannot have type α or β . The doubled orbit must therefore have type β . In either case, picking $\varepsilon > 0$ small enough so that no bifurcation of period less than N happens between μ_0 and $\mu_0 + 2\varepsilon$, we have $\beta \in bt(f_{\mu_0+\varepsilon})$ but $\alpha \notin bt(f_{\mu_0+\varepsilon})$. $\square$

Since a braid type is essentially an isotopy class, in order to understand this partial order one needs a good understanding of isotopy classes on surfaces. This is provided by the Thurston-Nielson Theory (see [T] or [FLP] for more details). This theory provides a “prime decomposition theorem” for isotopy classes. If a class is irreducible, it is either finite order or pseudoAnosov. In the first case there is a map ϕ in the class that satisfies $\phi^n = id$ for some n . In the second case, there is a pseudoAnosov map ϕ in the isotopy class. As the Thurston type of an isotopy class is unchanged by conjugacy, it makes sense to speak of reducible, irreducible, finite order and pseudoAnosov braid types.

Following Handel ([Hn3]) we say that a map is “pseudoAnosov relative to a finite invariant set K ” if it satisfies all the properties of a pseudoAnosov homeomorphism except the associated stable and unstable foliations may have 1-pronged singularities at points in K .

There are several features of pseudoAnosov maps that will be important here. Each pseudoAnosov map has a Markov partition by topological rectangles and the action in each rectangle is linear ([FLP] exposé 10). Thus pseudoAnosov maps have finitely many periodic orbits of each period and each point is nonwandering under iteration. The most important feature for us will involve the unremovability of periodic orbits as in Asimov and Franks ([AF]). It will be convenient to adapt this notion to isotopies rel a finite set.

Let S be a surface, $f_0 : S \rightarrow S$ a homeomorphism, K a finite invariant set and $\phi(x, f_0)$ a period n -periodic orbit. This periodic orbit is *unremovable (rel K)* if whenever f_1 is another homeomorphism that is isotopic to f_0 (rel K) then there exists an isotopy (rel K), f_μ , and a continuous arc $\gamma(\mu)$ (with $\gamma(\mu) \cap K = \phi$) and so that for all μ , $\gamma(\mu)$ is a period n point for f_μ and $\gamma(0) = x$. The point $\gamma(1)$ is said to be *connected by isotopy* to x .

We shall need the following minor extension of a theorem of Asimov and Franks. They did not consider the C^0 -case or isotopies rel a finite set. We mainly need to point out a few extensions of their arguments.

PROPOSITION 2. *If S is a surface and $\phi : S \rightarrow S$ a pseudoAnosov map then all its periodic orbits not on ∂S are unremovable. If $K \subseteq S$ is a finite set and $\psi : S \rightarrow S$ is pseudoAnosov rel K then all its periodic orbits in $S - \{\partial \cup K\}$ are unremovable rel K .*

Proof. By a result of Gerber and Katok we may, after a C^0 conjugation, assume that ϕ and ψ are C^∞ -diffeomorphisms. The main fact we shall need is that on surfaces, C^∞ -diffeomorphisms are dense in the space of homeomorphisms with the C^0 -topology. This was proved by Munkres ([M], theorem 6.3), Whitehead ([W], corollary 1.18) and Smale. (These authors main interest was the result for 3-manifolds. The tools for the surface

result seem to be much older.) Thus Kupka-Smale diffeomorphisms are also dense in this space. Therefore, the argument in paragraphs 9-13 of the proof of the main theorem of [A-F] goes through without changes as it only uses C^0 -convergence. This establishes the first part of the proposition.

For the second part, assume that g is a homeomorphism with $g \cong \psi \text{ rel } K$. For each n , pick a C^∞ -diffeomorphism so that $d(g, \bar{g}_n) < 1/n$ in the C^0 -topology. Pick N large enough so that for $n > N$ we can find a C^∞ -diffeomorphism h_n with $h_n \circ \bar{g}_n(K) = K$ and $\|h_n\| < 2/n$ in the C^2 -topology. Let $g_n = h_n \circ \bar{g}_n$ and note that for large enough n , $g \simeq q_n \text{ rel } K$.

Use the projective action of the derivative of g_n at each point of K to define a C^∞ -diffeomorphism $g'_n : S' \rightarrow S'$ where S' is obtained from S by replacing each point of K with a boundary circle (for more details see, for example, [Hn4]). Similarly, construct a C^∞ -pseudoAnosov diffeomorphism $\psi' : S' \rightarrow S'$.

Now let $o(x', \psi')$ be a periodic orbit with period q which is not in $\partial S'$. By the first assertion of the proposition, for all n there exists a periodic orbit $o(y'_n, g'_n)$ that is connected by isotopy to $o(x', \psi')$. Let $o(x, \psi)$ and $o(y_n, g_n)$ be the corresponding periodic orbits of the unprimed maps. Passing to a subsequence if necessary let y be such that $y_{n_i} \rightarrow y$. Once again, using the argument from [A-F] used above, $o(x, \psi)$ is connected by isotopy to $o(y, g)$. Further, a slight alteration of the proof of Lemma 2.2 in [A-F] insures that this isotopy can be done rel K and that the arc $\gamma(\mu)$ satisfies $\gamma([0, 1]) \cap K = \emptyset$. Thus to finish, we must check that $y \notin K$.

To see this, assume that $y \in K$. We claim that this implies that y and x are in the same Nielsen class as fixed points of $(\psi')^q$. This is a contradiction as it implies that the interior fixed point y is in the same Nielsen class as a boundary fixed point under $(\psi')^q$. This cannot happen in a pseudoAnosov map.

Now to prove the claim, let $H(x, t)$ be the isotopy that connects x as a fixed point under ψ^q to y as a fixed point of g^q . Then $H(x, 0) = \psi^q(x)$, $H(x, 1) = g^q(x)$ and for all t , $H(\gamma(t), t) = \gamma(t)$ and $H(y, t) = y$ since $y \in K$. Inside $S \times [0, 1]$ consider the set $B = \{(\gamma(t), s) : t \leq 1 - s\}$. The set B is a topological disk whose boundary consists of three arcs $\alpha_1 = \{(\gamma(t), 0) : t \in [0, 1]\}$, $\alpha_2 = \{y\} \times [0, 1]$ and $\alpha_3 = \{(\gamma(t), t) : t \in [0, 1]\}$. Now $H(\alpha_1) = \psi^q(\gamma([0, 1]))$, $H(\alpha_2) = y$ and $H(\alpha_3) = \gamma([0, 1])$. Thus the arcs $\psi^q(\gamma)$ and γ bound the continuous image of a topological disk. They are thus isotopic and therefore x and y are in the same ψ^q -Nielsen class. $\square$

A braid type, β , is said to be of type (p, q) if one (and hence all) periodic orbits, $o(x, f)$, with $bt(x, f) = \beta$ are of type (p, q) . If β is a pseudoAnosov (p, q) -braid type we say that the map ϕ with lift $\tilde{\phi}$ represents β if $\phi : A \rightarrow A$ has a periodic orbit, $o(x)$, with $bt(x, \phi) = \beta$ and $T^{-p}\tilde{\phi}^q(\tilde{x}) = \tilde{x}$ for any lift $\tilde{x}$ of x and ϕ is pseudoAnosov rel $o(x)$. Handel has shown ([Hn4]) that $\rho(\tilde{\phi})$ is a closed interval (closely related results were proved by Fried, [Fr1], Lemma 3 and [Fr2], Theorem 4). We may thus define the rotation interval

of β as $\rho i(\beta) = \rho(\tilde{\phi})$. Since any two isotopic pseudoAnosov maps are conjugate ([FLP] exposé 12), this definition is independent of the choice of ϕ , however, it does depend on the choice of $\tilde{\phi}$. More precisely, it depends on the fact that we have treated β as a (p, q) -braid type and ignored the fact that β is also a $(p + k, q)$ -braid type for any $k \in L$. To avoid this ambiguity we shall always assume that the p and q in the notation “ (p, q) -braid type” and “ (p, q) -periodic orbit” satisfy $0 < p/q < 1$.

Braid types will be used to define the notion of a topologically monotone periodic orbit. If $\tilde{T}_{p/q} : \tilde{A} \rightarrow \tilde{A}$ is defined by $z \mapsto z + (p/q, 0)$, let $T_{p/q} : A \rightarrow A$ be the projection. Define $\alpha_{p/q} \in BT$ to be the braid type of a periodic orbit of $T_{p/q}$. A periodic orbit is topologically monotone if $bt(x, f) = \alpha_{p/q}$. One can check that if f is monotone twist, a periodic orbit is topological monotone if and only if it is Birkhoff in the usual sense ([H2]). For the braid type $\alpha_{p/q}$, we define $\rho i(\alpha_{p/q}) = p/q$.

The next proposition collects together some useful properties of the objects we have defined. Parts (1) and (2) are contained in [Bd3] and were also known to Smillie. Part (1) is a simple exercise in plane topology and the definition of reducibility. Part (2) is a consequence of a theorem of Brouwer ([Bw]), Kerekjarto ([Kj]) and Eilenberg ([E]) which says that any homeomorphism, g , of the disk that satisfies $g^q = Id$, is in fact conjugate to $T_{p/q}$ for some p . Part (4) is a consequence Proposition 2 and part (3) follows from [Hn 3].

PROPOSITION 3. *Let $\beta \in BT$ be a (p, q) -braid type with p and q positive and relatively prime then*

- (1) β is irreducible.
- (2) $\beta = \alpha_{p/q}$ if and only if β is finite order.
- (3) $\beta \in bt(f)$ implies $\rho i(\beta) \subset \rho(f)$.
- (4) If β is pseudoAnosov and ϕ represents β then $bt(\phi) = \{\sigma \in BT : \sigma \leq \beta\}$.

Section 3. The main result on $(BT, \leq)$ is

THEOREM 1. *If β is a (p, q) -braid type then $\beta \geq \alpha_{p/q}$. Further, if $\beta \neq \alpha_{p/q}$ then $I(p/q) \subset \rho i(\beta)$.*

The main theorem is an easy consequence of Theorem 1, the definition of $\leq$ and Proposition 3. For the proof of Theorem 1 we shall need the following lemma. Its proof came from a discussion with John Franks.

LEMMA 1. *If $f : \tilde{A} \rightarrow \tilde{A}$ is an orientation and boundary preserving homeomorphism with $fT = Tf$ and $0 \notin \text{convex hull}(\rho(f))$ then $\tilde{A}/f$ is an annulus.*

Proof. A theorem of Franks ([F1]) states that if f has a nonwandering point it has a fixed point. Thus $0 \notin \rho(f)$ implies that given $x \in \tilde{A}$ there exists an $\varepsilon > 0$ so that for all $n \neq 0$, $f^n(B_\varepsilon(x)) \cap B_\varepsilon(x) = \emptyset$. This gives charts for $\tilde{A}/f$.

The main work lies in showing that $\tilde{A}/f$ is Hausdorff. For this we must show that given x and $y \notin o(x, f)$, there exists $\varepsilon > 0$ so that $f^n(B_\varepsilon(x)) \cap f^m(B_\varepsilon(y)) = \emptyset$ for all m and n . Equivalently, $B_\varepsilon(y) \cap f^n(B_\varepsilon(x)) = \emptyset$ for all n . We assume that $\rho(f) \subset (0, \infty)$ (the other case being similar) and claim there exists an M such that for all z and $|n| > M$,

$$|\pi_x(f^n(z)) - \pi_x(z)| > 2.$$

This claim easily implies that for any $\varepsilon < 1/2$ and $|j| > M$, $B_\varepsilon(u) \cap f^j(B_\varepsilon(x)) = \emptyset$. While for $|j| \leq M$, continuity and the fact that $x \notin o(y)$ imply that there exists an $\varepsilon < 1/2$ with $f^j(B_\varepsilon(x)) \cap B_\varepsilon(y) = \emptyset$. Thus granted the claim we are done.

To prove the claim, note that $\rho(f) \subset (0, \infty)$ implies that there exists a $\delta > 0$ so that for any $z \in \tilde{A}$ there is an $N'(z)$ so that

$$\frac{\pi_x(f^n(z)) - \pi_x(z)}{n} > \delta$$

for all $n \geq N'(z)$. Thus using continuity, there exists $\varepsilon(z)$ and $N(z)$ so that $d(y, z) < \varepsilon(z)$ implies

$$\pi_x(f^{N(z)}(y)) - \pi_x(y) \stackrel{\text{def}}{=} D(y, f^{N(z)}) > 2.5$$

Thus using compactness, there are z_i, ε_i and N_i for $i = 1, \dots, k$ so that $\cup B_{\varepsilon_i}(z_i)$ covers $[0, 1] \times [0, 1] \subseteq \tilde{A}$ and $z \in B_{\varepsilon_i}(z_i)$ implies that $D(z, f^{N_i}) > 2$. Let $B_{\varepsilon_i}(z_i) = B_i$.

Now let $\bar{N} = \max\{N_i\}$ and $C = \sup\{D(z, f^j) : 0 \leq j \leq \bar{N} \text{ and } z \in \tilde{A}\}$. Since $Tf = fT$ and A is compact, C is finite. Pick $m > \max\{C, 2\}$ and let $M = m\bar{N}$.

Given z and $n > M$, let i_0 be so that $z \in B_{i_0}$ and let $z_1 = f^{N_{i_0}}(z)$. Next let i_1 be so that z_1 is contained in an integer translate of B_{i_1} and let $z_2 = f^{N_{i_1}}(z_1)$. Continue until i_k which satisfies

$$\sum_{\ell=1}^{k-1} N_{i_\ell} \leq n \text{ but } \sum_{j=1}^k N_{i_j} \geq n.$$

Define j by $j = n - \sum_{j=1}^{k-1} N_{i_j}$. We then have

$$\begin{aligned} D(z, f^n) &= D(z, f^{N_{i_0}}) + \dots + D(z_{k-1}, f^{N_{i_{k-1}}}) \\ &\quad + D(z_k, f^j) > 2m - C > 2 \end{aligned}$$

which proves the claim.

Thus $\tilde{A}/f$ is a smooth compact surface. Its easy to check that its first homology is $\mathbb{Z}$ and so its an annulus. $\square$

Proof of Theorem 1. We prove the second assertion first. By virtue of proposition 3 (1) and (2), β is pseudoAnosov. Let ϕ and $\tilde{\phi}$ represent β with $bt(x, \phi) = \beta$. If we let $I(p/q) = [m/n, k/\ell]$ then since $\rho(\beta)$ is a closed interval, it suffices to show that $m/n, k/\ell \in \rho(\tilde{\phi})$.

We proceed by contradiction and assume that $m/n \notin \rho(\tilde{\phi})$, the case $k/\ell \notin \rho(\tilde{\phi})$ is similar. By Lemma 1, $\tilde{A}/\tilde{\phi}^n T^{-m}$ is an annulus which we denote B . Let $X \subset \tilde{A}$ be the total lift of $o(x, \phi)$. Because $o(x, \phi)$ is of type (p/q) , the elements of X can be labeled as $\{\tilde{x}_i\}$ which satisfy $\tilde{\phi}(\tilde{x}_i) = \tilde{x}_{i+p}$ and $T(\tilde{x}_i) = \tilde{x}_{i+q}$. Since m/n is an endpoint of the Farey interval of $p/q, mq - np = 1$ and so $\tilde{\phi}^n T^{-m}(x_i) = x_{i-1}$. Thus if $\pi : \tilde{A} \rightarrow B$ is the projection, $\pi(X)$ is a single point.

Because T and $\tilde{\phi}$ commute, $T^{-p}\tilde{\phi}^q$ induces a map on B , which will be denoted ψ . Now using an argument of Fried ([Fr3], pp. 561–562), ψ is flow equivalent to ϕ . Thus by [Fr4] (Lemma on page 261), ψ is pseudoAnosov rel the point $\pi(X)$. Alternatively, one can imitate the proof of Fried's lemma to produce the measured foliations for ψ directly. In any case, we have a contradiction as there are no pseudoAnosov maps rel a single point on the annulus.

To prove the first assertion of the theorem, we may assume $\beta \neq \alpha_{p/q}$ and thus it is pseudoAnosov. Let $\phi, \tilde{\phi}$ and x continue to be as in the first paragraph of the proof. By the second assertion of the theorem which we have just proved, p/q is in the interior of $\rho(\tilde{\phi})$. Further, as noted in section 2, ϕ has finitely many periodic orbits of each period and each point of A is nonwandering under ϕ . Under these hypotheses, theorem 3.3 of [F2] implies that ϕ has at least two (p, q) -periodic orbits in the interior of A with nonzero Lefschetz index. One of these orbits might be $o(x)$ (actually not, see remarks after the proof) and we denote the second by $o(y)$.

Since $o(y)$ is a periodic orbit of a pseudoAnosov map by proposition 2 it is unremovable under isotopy rel $o(x)$. On the other hand, an argument using Brunovsky's theorem similar to the proof of Proposition 1 shows that for each braid type γ , there is a diffeomorphism of the annulus with only one periodic orbit of braid type γ . Thus, in particular, $bt(x, \phi) \neq bt(y, \phi)$. Summarizing, we have shown that if β is a pseudoAnosov (p, q) -braid type then there is another (p, q) -braid type $\gamma \neq \beta$ with $\gamma \leq \beta$.

Now let $S = \{\sigma \in BT : \sigma \text{ is of type } (p, q) \text{ and } \sigma \leq \beta\}$. As noted in section 2, pseudoAnosov maps have only finitely many periodic orbits of each period. Thus using Proposition 3 (4), S is finite. Let $\alpha \in S$ be minimal in S i.e. it satisfies $\gamma \leq \alpha$ implies $\gamma = \alpha$ for all $\gamma \in S$. Using the result of the last paragraph, α is not pseudoAnosov so by Proposition 3 (1) and (2), it is finite order and $\alpha = \alpha_{p/q}$ $\square$

Remarks.

1) With reference to the 4th paragraph of the proof, we can apply the Euler-Poincaré formula to the invariant foliation of ϕ (after blowing down the boundary circles to points). This yields $\sum 2 - \bar{p} = 2\chi(S^2) = 4$ where the sum is over the singularities of the foliation and $\bar{p}$ is the number of prongs in the singularity. ([FLP] pg. 75). Since $o(x)$ is a periodic orbit, all the singularities at points of $o(x)$ must have the same number of prongs. There must be only one prong to get $\sum 2 - \bar{p} = 4$. On the other hand, the Lefschetz index of a one pronged singularity that is fixed by a pseudoAnosov map must be zero. Thus both

the periodic orbits given by Franks theorem are different from $o(x)$.

2) The proof given above of the second assertion of Theorem 1 was inspired by the alternative characterization of flow equivalence given on page 561 of [Fr3]. One can also obtain the result directly using flow equivalence. Briefly, the fact that $m/n \notin \rho(\tilde{\phi})$ implies that the cohomology class $u \in H^1(M_\phi)$ given by $u(av_1 + bv_2) = na - mb$ is strictly positive on the homology directions of ϕ . (Here M_ϕ is the suspension manifold of ϕ and $H_1(M_\phi)$ has generators v_1 in the annulus direction and v_2 in the flow direction.)

This implies (Theorem D of [Fr1]) the existence of a cross section to the suspension flow that represents u . The value of the cohomology class just represents the number of intersections of a homology class with the cross section. Thus, the fact that $mq - np = 1$ means that the suspension of $o(x)$ hits the cross section in just one point. The return map to this cross section is conjugate to the map ψ in the proof.

3) The inclusion $I(p/q) \subset \rho_i(\beta)$ is, in general, the best one can do. There is a braid type with $I(p/q) = \rho_i(\beta)$. Define $\beta_{p/q}^+$ to be the braid type obtained by rigidly rotating by p/q and then doing a Dehn twist around two adjacent points on an orbit. (See figure 1, the arcs between points are included to indicate the action of the complement of the orbit). The braid type $\beta_{p/q}^+$ is the same as $ot(m/n, k/\ell)$ given in [Bd2] (where $I(p/q) = [m/n, k/\ell]$). For this braid type one can compute an invariant train track and compute that $\rho(\phi)I(p/q)$ directly. If $\beta_{p/q}^-$ is the braid type obtained by doing the inverse Dehn twist along the curve, I conjecture that β is the first pseudoAnosov (p/q) -braid type. More precisely, if γ is a (p, q) -braid type and $\gamma \neq \alpha_{p/q}$ then $\gamma \geq \beta_{p/q}^+$ or $\gamma \geq \beta_{p/q}^-$.

Figure 1. The braid type $\beta_{2/5}^+$

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