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FOR CHANGE OF TYPE**

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SHOCKS NEAR THE SONIC LINE: A COMPARISON BETWEEN STEADY AND UNSTEADY MODELS FOR CHANGE OF TYPE*

BARBARA LEE KEYFITZ†

Abstract. We look at the structure of shocks for states near a locus where equations change type. Two basic models are considered: steady transonic flow, and models for unsteady change of type. Our result is that these two problems may be distinguished by the nature of the timelike directions and the forward light cone. This leads in a natural way to different candidates for admissible shocks in the two cases.

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1. Introduction. Models for Change of Type. Systems of quasilinear equations which are of different type at different states arise in two different ways in applications. The first is typified by the pair of conservation laws which governs steady, inviscid, irrotational, isentropic flow [2]:

$$(1.1) \quad \begin{aligned} (\rho u)_x + (\rho v)_y &= 0 \\ v_x - u_y &= 0 \end{aligned}$$

Here $w = (u, v)$ represents the velocity of a flow in the xy plane; $\rho = \rho(|w|)$ is the density, given by Bernoulli's law as a function of the speed. The first equation expresses conservation of mass; the second irrotationality of the flow. Under the assumption that the medium is an ideal gas (or some other reasonable thermodynamics), there is a speed, c_* , the *sonic speed*, such that system (1.1) is hyperbolic if $|w| > c_*$ ("supersonic flow") and nonhyperbolic (elliptic) if $|w| < c_*$ ("subsonic"). The flow changes type along the curve $u^2 + v^2 = c_*^2$, which we shall call the *sonic line* in this paper. The classification of these equations, and their relation to other models for gas flow, were studied by Courant and Friedrichs [2]. Here we are interested in the following property of *weak solutions* of (1.1):

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for every supersonic state w_0 , there is a one-parameter family of states, $w_1(\beta)$, such that the function

$$(1.2) \quad w(x, y) = \begin{cases} w_0, & x \cos \beta + y \sin \beta < 0 \\ w_1, & x \cos \beta + y \sin \beta > 0 \end{cases}$$

is a weak solution of (1.1). The set $\{w_1(\beta)\}$, called the *shock polar* through w_0 , has a self-intersection at w_0 . The portion of the curve for which $|w_1| < |w_0|$, which we shall call the *Hugoniot loop*, $H(w_0)$, contains a subinterval in the elliptic region. All weak solutions (1.2) for which $w_1 \in H(w_0)$ are said to be *admissible*. Courant and Friedrichs [2] discuss the fact that only such shocks are compressive; they also point out that there may be other restrictions, depending on the particular boundary value problem one is trying to solve. However, all compressive shocks are felt to be physically realizable in some context, and none that are expansive are ever physical. In [7], we discuss these admissibility conditions in the context of steady viscous perturbations and give more details of the calculations summarized above.

The situation described by (1.1), *transonic steady* (TS) flow, may be replicated in other physical situations: steady visco-elastic flows of *composite* type have flow regimes where a pair of characteristic speeds may change from real to complex - as if (1.1) were embedded in a larger system where all the remaining characteristic fields are of constant type [11]; change of type which may resemble this also occurs in steady flows of granular materials [12].

Change of type in systems of conservation laws modelling unsteady flows has also been observed. The list of flow models exhibiting this phenomenon includes several models for two-fluid flow [13], multi-phase porous media flow [1], and possibly some examples in granular flow [12]. Some models for unsteady visco-elastic flows exhibit change of type, but most likely not in flow regions where the models are valid. In addition, there are models for fluids undergoing phase transitions where the equations change type, but where it may be the case that additional, different physical mechanisms are necessary to model the fluid behavior. However, in two-phase nonreacting flows, such as are modelled in large-scale computations in nuclear reactor engineering, the standard model exhibits change of type in the fluid regime of interest [13]. These models have been in practical use for years.

A prototype model is given by a pair of equations for *unsteady* flow (US)

$$(1.3) \quad f(w)_x + w_t = 0$$

with $w = (u, v)$, $f = (f_1, f_2)$. In discussing the general structure of shock solutions, it is convenient to write (1.1) or (1.3) as

$$(1.4) \quad f(w)_x + g(w)_y = 0.$$

In the next section, we shall make precise what is meant by the change of type locus or *sonic line* for (1.4); and we shall prove that under some nondegeneracy conditions equivalent to genuine nonlinearity in the hyperbolic region, (1.4) admits a Hugoniot loop near the sonic line. In section three we shall review and abstract a shock classification based on a determination of the forward light cone for (1.4). This criterion, which says, in essence, that the shock surface is spacelike when viewed from the upstream direction, applies also when (1.4) is not hyperbolic at one state, and yields the well-known admissibility of compressive shocks for the transonic case, TS. For the unsteady case, US, the situation is more complicated. When we apply the same principle which leads to the standard conclusion in TS, then we can use the theory of local unfolding of vector field dynamics to show that the Hugoniot loop can be divided into three intervals, two of which represent admissible shocks. However, determination of the precise end points of these intervals has not yet been accomplished.

2. Systems of Conservation Laws near the Sonic Line. We write (1.4) in quasilinear form

$$(2.1) \quad P(w, \partial)w \equiv A(w)\partial_x w + B(w)\partial_y w = 0$$

where

$$(2.2) \quad A(w) = (a_{ij}) = df \quad B(w) = (b_{ij}) = dg$$

and form the linearized symbol (at a constant state w)

$$(2.3) \quad P(w, \zeta) = A(w)\xi + B(w)\eta$$

with $\zeta = (\xi, \eta) \neq 0$. We may write $\zeta = \rho(\cos \beta, \sin \beta)$ where

$$\rho \in \mathbb{R} \setminus \{0\} \text{ and } \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

(We will justify this choice later). It will be sufficient to consider unit vectors. The *characteristic covectors* or “characteristics” (by an abuse of notation which is sufficiently confusing that we shall avoid it) are the solutions to

$$(2.4) \quad \det P(w, \zeta) \equiv \det(A(w)\cos \beta + B(w)\sin \beta) = 0.$$

System (2.1) is *strictly hyperbolic* at a state w if (2.4) has real distinct solutions $\beta_1 < \beta_2$, and *nonhyperbolic* if the roots of (2.4) are not real; we let $\mathcal{H} \subset \mathbb{R}^2$ be the set of states w where (2.1) is strictly hyperbolic, $\mathcal{E}$ the set of nonhyperbolic states. Define $\mathcal{B}$, the *sonic line* (by analogy with steady transonic flow) as the locus of states w where (2.4) has equal roots, β .

We introduce the notation B^* for the adjunct (transpose of the matrix of cofactors) of B and $q(w) = \text{tr}(B^*A)$. Note that $\det B^* = \det B$. Let

$$(2.5) \quad D(w) = q^2 - 4 \det(B^*A).$$

The following proposition, which is a slight generalization of Proposition 2.1 of [5], characterizes the sonic line in a nondegenerate case.

PROPOSITION 2.1. Assume $f, g \in C^2$ and that there is a point w_0 where $D = 0$ and $\nabla_w D \neq 0$. Then in a neighborhood $\mathcal{O}(w_0)$, $\mathcal{B} = \{w | D(w) = 0\}$ is a C^1 curve separating states $w \in \mathcal{H}$ from states $w \in \mathcal{E}$. Furthermore $\text{Rank } P(w, \zeta) \geq 1$ everywhere, including $\mathcal{B}$; at a point $w \in \mathcal{B}$, at least one of $\det A$ or $\det B$ is nonzero, and P has a nonzero eigenvalue there if and only if the other has precisely one nonzero eigenvalue.

Proof. Define

$$(2.6) \quad p(w, \beta) = \det P(w, \zeta) = \det A \cos^2 \beta + q \cos \beta \sin \beta + \det B \sin^2 \beta.$$

This has real roots, β_i , if $D(w) \geq 0$. Now $w \in \mathcal{B}$ if (2.6) has a double root; since (2.6) is a smooth function of β , this implies

$$(2.7) \quad \frac{\partial p}{\partial \beta}(w, \beta) = 0.$$

By a calculation, this is equivalent to

$$(2.8) \quad D(w) = 0.$$

The nondegeneracy condition $\nabla_w D \neq 0$ guarantees, by the implicit function theorem, that there is a open ball $\mathcal{O} \in \mathbb{R}^2$ centered at w_0 in which solutions of (2.8) form a connected C^1 curve, and also that $D > 0$ on one side, $\mathcal{H}$, and $D < 0$ on the other, $\mathcal{E}$. Since

$$(2.9) \quad \nabla_w D = 2q \nabla_w q - 4 \det A \nabla(\det B^*) - 4 \det B^* \nabla(\det A),$$

and $\det(B^* A) = 0 \Leftrightarrow q = 0$ when (2.8) holds, we see that $\nabla D \neq 0$ only if at least one of $\det A, \det B$ is nonzero. Hence on $\mathcal{B}$, (2.6) is a nontrivial form and β is well defined:

$$\tan \beta = -(\det A / \det B)^{1/2}.$$

Now let $C = B^* A$, so $D = (\text{tr } C)^2 - 4 \det C$ and, with $C = (c_{ij})$

$$(2.10) \quad \frac{1}{2} \nabla D = (c_{11} - c_{22}) \nabla c_{11} + (c_{22} - c_{11}) \nabla c_{22} + 2c_{12} \nabla c_{21} + 2c_{21} \nabla c_{12}.$$

If we suppose, without loss of generality, that $\det B(w_0) \neq 0$, then at the double root of (2.6),

$$B^* P(w, \zeta) = C \cos \beta + (\det B) I \sin \beta = \cos \beta (C - \lambda I)$$

where $\lambda = -(\det B) \tan \beta = (\det C)^{1/2} = \frac{1}{2} \text{tr } C$. We note that $\det B \neq 0$ if and only if $\cos \beta \neq 0$ so that $P = 0$ if and only if

$$N = C - \left(\frac{1}{2} \text{tr } C \right) I = \frac{1}{2} \begin{pmatrix} c_{11} - c_{22} & 2c_{12} \\ 2c_{21} & c_{22} - c_{11} \end{pmatrix}$$

is zero. But if every component of N is zero, then, from (2.10), ∇D is also zero, contrary to hypothesis. This shows that $\text{Rank } P \geq 1$ everywhere. Finally, still assuming $\det B \neq 0$, we find

$$(2.11) \quad P(w, \zeta) = \frac{\cos \beta}{|\det B|} BN$$

Now, the last statement of the theorem concerns whether $P(w_0, \zeta)$ is nilpotent: clearly $P^2 = 0 \Leftrightarrow (BN)^2 = 0$ and since B is invertible, we have $P = 0$ if and only if $NBN = 0$. Now $N = B^*A - \frac{1}{2}\text{tr}(B^*A)I = B^*A - (\det B^*A)^{1/2}I$, and so $NBN = N(NB + BB^*A - B^*AB) = |B|NA - NB^*AB$ and $NB^*A = N^2 + \sqrt{|AB|}N = \sqrt{|AB|}N$, so

$$NBN = N \left\{ |B|A - \sqrt{|AB|}B \right\}.$$

If $\det A = 0$, then $N = B^*A$ so $NBN = |B|B^*A^2$ and this is zero precisely when $A^2 = 0$, i.e. A is also nilpotent. If $\det A \neq 0$, then $NBN = N \left\{ A - \sqrt{\frac{|A|}{|B|}}N \right\} = \sqrt{abs|A|}N \left\{ \tilde{A} - \tilde{B} \right\}$ where $\det \tilde{A} = \det \tilde{B} = 1$ and, since $P = BN$, $\det |\tilde{A} - \tilde{B}| = 0$. Since $\tilde{A}$ and $\tilde{B}$ have a common eigenvector $\tilde{r}$, we can write them in a basis $\{\tilde{r}, \tilde{s}\}$ as

$$\tilde{A} = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} 1 & 0 \\ \gamma & \delta \end{pmatrix} :$$

and now $\beta = \delta = 1$; hence $\tilde{A} - \tilde{B}$ is nilpotent. Thus we have proved the final assertion of the proposition, and we have also shown the useful fact that P has a nonzero eigenvalue only in the case that one of A or B is singular. $\square$

From now on we will work in the ball $\mathcal{O}$ given by the proof of Proposition 2.1. It is not necessary to restrict $\mathcal{O}$ to be a ball, but we omit details of the extension to other geometries. We also note that the matter of P being nilpotent is not invariant under reasonable notions of equivalence, such as interchanging the order of writing the equations. The appropriate definition of equivalence will be given after we discuss shocks.

Requiring that (1.2) be a weak solution of (1.4) results in the equation

$$(2.12) \quad V(w_1, w_0, \beta) \equiv (f(w_1) - f(w_0)) \cos \beta + (g(w_1) - g(w_0)) \sin \beta = 0.$$

We consider this as a bifurcation equation with distinguished parameter β : for a fixed w_0 in $\mathcal{O}$, we are interested in the zero-set $w_1 = w_1(\beta)$ of $V(w_1, w_0, \beta)$. This set (the bifurcation diagram) is unchanged if V is multiplied on the left by an invertible matrix, and equivalent bifurcation diagrams result under coordinate changes $\beta \mapsto \beta'(\beta)$, $w \mapsto w'(w, \beta)$. The mapping V has the property $V(w_0, w_0, \beta) \equiv 0$ and so do equivalent mappings if both w_0 and w_1 are transformed by the same function. This type of contact equivalence is called t -equivalence in [3, p 129]. We use the singularity theory approach of [3] to prove the following theorem which extends [6].

THEOREM 2.2. *Under the further nondegeneracy condition*

$$(2.13) \quad l_0 \cdot (d^2 f \cdot r_0 r_0 \cos \beta_0 + d^2 g \cdot r_0 r_0 \sin \beta_0) \neq 0$$

at (w_0, β_0) , where $w_0 \in \mathcal{B}$ and β_0 is the solution of (2.4) and $l_0 P = P r_0 = 0$, there is a ball $\mathcal{O} \subset \mathbb{R}^2$, $w_0 \in \mathcal{O}$, in which the bifurcation problem

$$(2.14) \quad V(w, w_0, \beta) = 0$$

is t -equivalent to the one-state-variable problem

$$(2.15) \quad h(x, \mu) = \varepsilon_1 x^2 + \varepsilon_2 x \mu^2 = x(\varepsilon_1 x + \varepsilon_2 \mu^2)$$

near $x = \mu = 0$. Here we may take $\mu = \beta - \beta_0$ and $x = w \cdot r_0$; ε_1 has the sign of the expression in (2.13) and ε_2 is also non-zero. Problem (2.15) has t -codimension one; an unfolding is given by

$$(2.16) \quad h_a(x, \mu) = x(\varepsilon_1 x + \varepsilon_2 \mu^2 + a)$$

where we may take $a = w_0 \cdot r_0$, $w_0 \notin \mathcal{B}$. The problem and its unfolding are shown in Figure 2.1, a-c.

Proof. We summarize the calculation. Fix $w_0 \in \mathcal{B}$, and suppose $\det B(w_0) \neq 0$. It can be shown that (2.13) implies that r_0 is not tangent to $\mathcal{B}$; we choose a direction for (r_0, l_0) so that r_0 is directed towards $\mathcal{H}$ and (2.13) is positive. We write

$$w = w_0 + x r_0 + W(x, \beta) r_0^\perp$$

and let

$$(2.17) \quad h(x, \beta) = l_0 V(w_0 + x r_0 + W(x, \beta) r_0^\perp, w_0, \beta)$$

where W is determined from $l_0^\perp V \equiv 0$. Performing the steps of the Liapunov Schmidt reduction recursively yields

$$h_x(0, \beta_0) = W_x(0, \beta_0) = 0$$

and

$$h_{xx}(0, \beta_0) = l_0 d^2 V(w_0, w_0, \beta_0) \cdot r_0 r_0 = \varepsilon_1 \neq 0.$$

Also $h_\beta = 0$ and $W_\beta = 0$ and

$$(2.18) \quad h_{x\beta}(0, \beta_0) = l_0 P_\beta(w_0, \beta_0) r_0;$$

using properties of P :

$$Pr_0 = Ar_0 \cos \beta + Br_0 \sin \beta = 0,$$

so, with $Br_0 = y$, $Ar_0 = -y \tan \beta$ and $P_\beta r_0 = y \sec \beta$, $l_0 P_\beta r_0 = l_0 Br_0 \sec \beta$. Now, $l_0 B$ is the left eigenvector of N (in (2.11)); hence it is orthogonal to the right eigenvector r_0 . Hence in (2.18), $h_{x\beta} = 0$. Calculating $(l_0^\perp V)_{x\beta} = 0$:

$$l_0^\perp P_\beta r_0 + (l_0^\perp Pr_0^\perp) W_{x\beta} = 0$$

and neither matrix vector product is zero, since

$$l_0^\perp P_\beta r_0 = a_2 (l_0 B)^\perp r_0 \sec \beta,$$

where $l_0^\perp B = a_1 l_0 B + a_2 (l_0 B)^\perp$ and $a_2 \neq 0$ since B is invertible; since $(l_0 B)^\perp$ is orthogonal to the left eigenvector of N it is not orthogonal to r_0 . Finally,

$$l_0^\perp Pr_0^\perp = \frac{\cos \beta}{|\det B|} l_0^\perp B \alpha r_0 = \frac{a_2 \alpha \cos \beta}{|\det B|} t^\perp r_0 \neq 0.$$

Now $h_{\beta\beta} = W_{\beta\beta} = 0$ as a consequence of t -equivalence, and similarly $h_{\beta\beta\beta} = W_{\beta\beta\beta} = 0$. The final calculation is,

$$(2.19) \quad h_{x\beta\beta} = l_0 P_{\beta\beta} r_0 + 2W_{x\beta} l_0 P_\beta r_0^\perp = 2W_{x\beta} l_0 P_\beta r_0^\perp,$$

for $l_0 P_{\beta\beta} r_0 = -l_0 Pr_0 = 0$, and thus $h_{x\beta\beta} \neq 0$ by the calculations above. These defining and nondegeneracy conditions determine the singularity to be equivalent to (2.15); by our normalization, we may take $\varepsilon_1 = 1$. In the unsteady and transonic cases where we have done the calculation, the sign of ε_2 is -1 ; this means that the nontrivial solutions of (2.15) have $x > 0$ and hence $w \in \mathcal{H}$. If we unfold the singularity (2.15), we may do so by perturbing w_0 away from $\mathcal{B}$, say to $w_0 = ar_0 + w_0^*$, where the calculations above are performed at a fixed point $w_0^* \in \mathcal{B}$. Now we obtain the normal form

$$h_a(x, \beta) = x(x - \mu^2 + \varepsilon a),$$

where $\varepsilon > 0$ in both the unsteady and transonic cases. Again the case $a > 0$ corresponds to $w_0 \in \mathcal{H}$.

This concludes the proof. $\square$

The qualitative shape of the Hugoniot locus and its representation as a bifurcation diagram are shown in Figures 2.1 and 2.2.

We remark that to each triple $\{w_1, w_0, \beta\}$ there correspond two different shock configurations: (1.2) is one and

$$(2.20) \quad w(x, y) = \begin{cases} w_1, & x \cos \beta + y \sin \beta < 0 \\ w_0, & x \cos \beta + y \sin \beta > 0 \end{cases}$$

(its mirror image) is the other.

2.1 The Hugoniot locus in state space.

(a) $w_0 \in \mathcal{B}$

(b) $w_0 \in \mathcal{H}$

(c) $w_0 \in \mathcal{E}$

2.2 The bifurcation diagram for the Hugoniot locus.

(a) $h(x, \beta) = 0$

(b) $h_a, a > 0$

(c) $h_a, a < 0$

Our final use of the local theory in this section will be to obtain the relation between the shock angle β , given by (2.12), and the local characteristic angles $\beta_1(w) < \beta_2(w)$ obtained by solving (2.4) at w_0 or w_1 , when these points are in $\mathcal{H}$. We observe the following in the case $w_0 \in \mathcal{H}$.

2.3 The characteristic angles along the loop.

PROPOSITION 2.3. *At the points $w_1 = w_0$, corresponding to $\mu = \pm\sqrt{a}$, we have $\beta = \beta_1(w_0)$ or $\beta_2(w_0)$; thus $\beta_1 = \beta_0 - \sqrt{a}$ and $\beta_2 = \beta_0 + \sqrt{a}$, approximately. The genuine nonlinearity of the system in $\mathcal{O} \cap \mathcal{H}$, (a consequence of (2.13)), and the fact that the loop crosses $\mathcal{B}$, show that the angles $\beta_i(w_1)$, calculated as functions of β along the locus, have the structure shown in Figure 2.3. For $\beta_1(w_0) \leq \beta < \beta_1^*$, $w_1 \in \mathcal{H}$ and $\beta(w_0, w_1) < \beta_1(w_1) < \beta_2(w_1)$; in the interval $\beta_1^* < \beta_2^*$, $w_1 \in \mathcal{E}$ and there are no real solutions to the characteristic equation. There is a second point β_2^* such that $w_1(\beta_2^*) \in \mathcal{B}$ and for $\beta \in (\beta_0^*, \beta_2(w_0))$, $\beta_1(w_1) < \beta_2(w_1) < \beta(w_0, w_1)$.*

This section has generalized the results in [6] and localized the picture in [7]: our main conclusion is that the nondegeneracy conditions of Proposition 2.1 and Theorem 2.2 are sufficient to ensure a shock polar or Hugoniot loop near the sonic line and to establish the direction of variation of the characteristic covectors along the loop. We now show how this local behavior is related to shock admissibility.

3. Spacelike curves and timelike conormals. We recall, using the notation of section 2, that the characteristic covectors $\zeta = (\cos \beta, \sin \beta)$ are normals (or conormals) to the *characteristic surfaces* $\phi_i(x, y) = 0$, defined by $\det P(w, \nabla_x \phi) = 0$. Linearizing around a constant state, we write the corresponding surfaces as

$$\phi_i(x, y) \equiv x \cos \beta_i + y \sin \beta_i = 0$$

3.1 (a) Characteristic surfaces

(b) Characteristic conormals

Now, for a constant $w_0 \in \mathcal{H}$, the linearized system corresponding to (2.1), $P(w_0 \partial)w$, is a constant-coefficient hyperbolic operator; it admits a fundamental solution with support in any one of the four quadrants separated by S_1 and S_2 in Figure 3.1 (a). Selecting one of these quadrants as the *forward light cone* (for example, C_2^+ in Figure 3.1 (a)), determines a complementary set of *spacelike curves* with the property that their conormals lie in the open cone Γ_2 of *timelike conormals* in the dual space. Near the sonic line, β_1 is near β_2 and the two possible conormal cones are qualitatively different. We define

$$\Gamma_1 = \{\zeta = \rho(\cos \theta, \sin \theta), \beta_1 < \theta < \beta_2, \rho \in \mathbb{R} \setminus \{0\}\}$$

$$\Gamma_2 = \{\zeta = \rho(\cos \theta, \sin \theta), -\frac{\pi}{2} < \theta < \beta_1 \text{ or } \beta_2 < \theta \leq \frac{\pi}{2}, \rho \in \mathbb{R} \setminus \{0\}\};$$

then Γ_1 is squeezed to a narrow cone for w near $\mathcal{B}$, and is empty if $w \in \mathcal{B}$, while Γ_2 contains many directions if w is near $\mathcal{B}$ and consists of two open half-planes if $w \in \mathcal{B}$. Corresponding to Γ_1 , the light cone, $C_1 = C_1^+ \cup C_1^-$, is, by contrast, wide, while C_2 , corresponding to Γ_2 , is narrow. (To include the case $\beta_0 = \frac{\pi}{2}$, or to give a global smooth definition of Γ , we can extend this definition in the obvious way.)

We make a couple of remarks here. First, the fact that a system like (2.1) does not have a uniquely defined light cone is an anomaly occurring only in the case of two independent variables: for systems in three or more variables, there is only one cone - C_1 or C_2 - supporting a fundamental solution, and there is a corresponding choice for the timelike conormals. Near the sonic line for a system which changes type, there will still be only two qualitatively different types of behavior: squeezing or expansion of the conormal cone and the opposite behavior for supports. For an abstract equation like (2.1), either choice is a priori possible: we shall see, soon, that Γ_1 is the appropriate choice for transonic flow and Γ_2 for the unsteady model equations we have studied, of the type (1.3). However, there is no a priori reason that steady equations could not have a conormal cone like Γ_2 , and there are possibly examples, in unsteady granular flow models for instance, where Γ_1 is appropriate for an unsteady problem. Here we are interested in both choices, and we will consider both.

A second comment concerns the choice of a forward light cone: C always has two components, one of which is designated C^+ , and the choice of Γ_1 or Γ_2 does not suffice to determine C^+ . This ambiguity is present also for more general systems: it expresses the fact that (2.1) remains hyperbolic under various changes of variables, $x \mapsto -x$ or $t \mapsto -t$, for instance, for an evolution equation, or $x \mapsto -x$ and $u \mapsto -u$ for the transonic equations. As is well-known, classical solutions of (2.1) are also invariant under these coordinate reversals, but admissible weak solutions are not, and the choice of C^+ , from the two connected components of a given C , is bound up with other criteria for admissibility. For example, when (1.4) is strictly hyperbolic, the standard admissibility criteria for a weak solution (1.2) containing a single shock can be simply expressed in these terms. We define the shock, $\mathcal{S} = \{x \cos \beta + y \sin \beta = 0\}$, to be *spacelike with respect to a state w_i* if it forms a spacelike curve with respect to the conormal cone Γ determined by the state w_i . We say further that w_i is an *upstream state* for a spacelike shock $\mathcal{S}$ if the forward light cone C^+ , from any point (x, y) where $w = w_i$, intersects $\mathcal{S}$, and the backward cone does not. Classical shocks (1.2) are admissible when precisely one of w_0 and w_1 is an upstream state. It is easy to verify that for unsteady systems of the form (1.3) this is equivalent to the Lax condition if the choice of C^+ is the classical forward cone: $\{t > 0, \lambda_1 t < x < \lambda_2 t\}$. For the steady transonic flow or transonic small disturbance equations, C^+ is the connected component of the complement of $\mathcal{S}_1 \cup \mathcal{S}_2$ which contains the flow vector $w = (u, v)$. This motivates the following definition:

DEFINITION 3.1. Suppose given a determination $C^+(w)$ for all $w \in \mathbb{R}^2$. A shock $\{w_1, w_0, \beta\}$ satisfies the C^+ criterion if there is precisely one upstream state.

We have already commented on the following classical observations.

PROPOSITION 3.2. *Let $C^+(w) = \{t > 0, \lambda_1(w)t < x < \lambda_2(w)t\}$ for a strictly hyperbolic genuinely nonlinear system (1.3). Then a shock satisfies the C^+ criterion if and only if it satisfies the Lax geometric entropy criterion.*

PROPOSITION 3.3. *Let $C^+(w)$ be the component of $\mathbb{R}^2 \setminus \{\mathcal{S}_1 \cup \mathcal{S}_2\}$ containing $w = (u, v)$ if w is a supersonic state of (1.1), and let $C^+(w)$ be the open half-plane $C^+(w_0)$, $w_0 \in \mathcal{B}$, otherwise. Then the shocks that satisfy the C^+ criterion are precisely the compressive shocks.*

(The extension of C^+ to $w \in \mathcal{E}$ is arbitrary: no state in $\mathcal{E}$ is ever upstream.)

We note that for hyperbolic but nonstrictly hyperbolic systems, an overcompressive shock may be upstream with respect to both states, while a noncompressive shock never satisfies the C^+ criterion. These observations, for a hyperbolic system, are not at all new. Definition 3.1 is an extremely special case of Majda's multidimensional shock stability condition [9], which he proves equivalent to Lax's geometric condition. In fact, for a system that is not genuinely nonlinear, where Lax's condition is not sufficient, the C^+ -criterion is not either. This definition, however, allows an extension to systems without a physical time variable, such as (1.1), and the possibility of extension (as in Proposition 3.3) to systems that change type.

Definition 3.1 may be useful in a couple of ways: it throws a light on the contrast between transonic and unsteady change of type by showing that these two systems differ already in the strictly hyperbolic region in their choice of C^+ . And it is closely connected with the construction of viscous profiles, as we will show in the next section. It is limited in some obvious ways: it does not admit a simple extension to systems of more than two equations except for extreme shocks. And it does not stand alone as a physical or mathematical admissibility criterion but merely suggests a method for proving a stability theorem.

Finally, the definition is not complete without some words on the selection of $C^+(w)$ for $w \in \mathcal{O}$: for smooth systems, the mapping $w \mapsto C^+(w)$ is smooth in $\mathcal{H}$, and $w \mapsto C_1^+(w)$ can be extended smoothly to $\mathcal{O}$, as indicated in Proposition 3.3. However, the only continuous extension of C_2^+ into $\mathcal{E}$ is $C_2^+(w) = \emptyset$ (the empty set) for $w \in \mathcal{E}$, and this gives $\Gamma_2 = \mathbb{R}^2 \setminus \{0\}$, according to which all shocks are spacelike. There is not a clear generalization of “upstream”; however, if we replace C^+ by $\overline{C^+}$ (the closure of C^+) then clearly the choice C_2^+ results, for $w_0 \in \mathcal{B}$, in a cone that degenerates to a line along which the fundamental solution is more singular. For $w_0 \in \mathcal{E}$, the support of the fundamental solution is $\mathbb{R}^2$, but one can choose a fundamental solution with certain growth properties. For the moment, fix any such specification continuously in $\mathcal{E}$. We have the following.

THEOREM 3.4. *Consider the states $\{w_1, w_0, \beta\}$ for (1.4), parameterized by β , in $\mathcal{O}$, with $w_0 \in \mathcal{H}$. Then*

a) with the specification of the conormal cone $\Gamma_1(w)$, the shock is spacelike with respect to w_0 for all pairs $\{w_1, w_0\}$ on the loop. In particular, there exists a smooth specification of $C_1^+(w)$ such that w_0 is the upstream state for all pairs.

b) for the conormal cone $\Gamma_2(w)$, the shock is spacelike with respect to w_1 for $\beta \in (\beta_1(w_0), \beta_1^*) \cup (\beta_2^*, \beta_2(w_0))$, and there is a locally smooth specification of $C_2^+(w)$ such that w_1 is the upstream state for all such pairs. However, there is no smooth extension of $C_2^+(w)$ to $w \in \mathcal{E}$ such that w_1 is the upstream state on the entire loop.

Proof. Recall that $\mathcal{S}$ is spacelike for a state w_i and cone Γ_j if its normal $\zeta = (\cos \beta, \sin \beta)$ is in $\Gamma_j(w_i)$. For $\Gamma_1(w_0)$, $\mathcal{S}$ is clearly spacelike since $\beta_1(w_0) < \beta < \beta_2(w_0)$ along the entire loop. From the definition, w_0 is either an upstream or a downstream state, and there is a choice of C_1^+ that makes w_0 the upstream state. Since $\mathcal{S}$ is never characteristic for w_0 , this determination of C_1^+ is smooth. Furthermore, $\mathcal{S}$ is not spacelike with respect to w_1 for $w_1 \in \mathcal{H}$, and under the natural choice of Γ_1 in $\mathcal{E}(\Gamma_1 = \phi)$, $\mathcal{S}$ is not spacelike for $w_1 \in \mathcal{E}$ either.

3.2 The geometry of the light cones

(b) For Γ_2 , we have, immediately, that $\mathcal{S}$ is never spacelike with respect to w_0 . On $(\beta_1(w_0), \beta_1^*)$ and $(\beta_2^*, \beta_2(w_0))$, $\mathcal{S}$ is spacelike with respect to $\Gamma_2(w_1)$. To see that there is no

smooth determination of $C_2^+(w)$ for $w \in \mathcal{E}$ that makes w_2 an upstream state everywhere, we make a geometric/topological argument (See Figure 3.2). For $w_1 \in \mathcal{H}$, $C_2^+(w_1) \subset C_2^+(w_0)$, because of the monotonicity of characteristic speeds. Now $\mathcal{S}$ always lies in one or other component of $C_2(w_0) \setminus C_2(w_1)$, by the first statement after (b). However, for β near $\beta_1(w_0)$, $\mathcal{S}$ is almost characteristic and is near the $\mathcal{S}_1$ characteristic; near $\beta_2(w_0)$, $\mathcal{S}$ is near the $\mathcal{S}_2$ characteristic. Thus $\mathcal{S}$ cannot be exterior to $C_2(w_1)$ for all w_1 on the loop for any determination of C_2 . This argument does not even depend on the choice of which component is $+$, but says $\mathcal{S}$ will actually fail to be spacelike. This has important implications, some of which we discuss in the next section.

4. The C^+ criterion, viscous profiles and vectorfield dynamics. In this section we will outline the relation between a definition of admissibility based on the C^+ criterion and the construction of travelling wave solutions, or viscous profiles, for an associated perturbed system.

A perturbation of (1.4) by higher-derivative terms would take the form

$$(4.1) \quad f(w)_x + g(w)_y = \varepsilon \{ \partial_x(Dw_x + Ew_y) + \partial_y(Fw_x + Gw_y) \}$$

where D, E, F, G are 2×2 matrices which, typically, might depend on w , and ε measures the strength of the perturbation. In specific applications, introduction of such terms might be motivated by considerations of viscosity or other dissipative or dispersive mechanisms. The idea is well known for unsteady systems where, if y represents time, the perturbation is $(Dw_x)_x$ and (4.1) should have a uniformly parabolic character [10]. For a discussion of the physically motivated viscosity terms in the transonic case, see [7]. We will consider the general case for (4.1) elsewhere, observing here only that a parabolic character is what is desired, in order that solutions of (4.1) will converge to admissible weak solutions: the directions of rapid decay can be related to the light cone and conormal cone for (2.1). Here we will motivate the general case and illustrate the C^+ criterion by considering self-similar travelling wave solutions of (4.1) with similarity parameter

$$(4.2) \quad \chi = (x \cos \beta + y \sin \beta) / \varepsilon.$$

Solutions approaching the shock (1.2) for $\varepsilon \rightarrow 0$ satisfy

$$(4.3) \quad w(\chi) \rightarrow w_0 \text{ as } \chi \rightarrow -\infty, w \rightarrow w_1 \text{ as } \chi \rightarrow \infty, w'(\pm\infty) = 0.$$

The solution whose limit is the “mirror image” shock (2.20) satisfy

$$(4.4) \quad w \rightarrow w_1 \text{ as } \chi \rightarrow -\infty, w \rightarrow w_0 \text{ as } \chi \rightarrow +\infty.$$

Substituting $w(\chi)$ in (4.1), integrating once and using (4.3) or (4.4) results in

$$(4.5) \quad \begin{aligned} & (D \cos^2 \beta + (E + F) \cos \beta \sin \beta + G \sin^2 \beta) w' \\ & = (f(w) - f(w_0)) \cos \beta + (g(w) - g(w_0)) \sin \beta. \end{aligned}$$

We abbreviate this as

$$(4.6) \quad M(w, \beta)w' = V(w, w_0, \beta),$$

where M is a 2×2 matrix that measures the effect of the higher order terms and V is the mapping introduced and studied in section two.

Remark. We have already introduced the notion of the shock $\mathcal{S}$ as a spacelike surface, at least with respect to one state w_i . Now χ , defined in (4.2), which measures orthogonal distance from this surface, might be thought of as representing a “stream coordinate” in physical space; in that case $\chi \rightarrow -\infty$ will represent “upstream” and $\chi \rightarrow +\infty$ “downstream” limits, and these intuitive notions may be helpful. Since we are using the convention that w_0 is the state that is always in $\mathcal{H}$, we need different boundary conditions, (4.3) or (4.4), to cover the possibilities - both of which arise - that it be the upstream or downstream state. However, χ does not necessarily represent “time”. For example, in unsteady systems, (1.3), the characteristic speed is $s = -\tan \beta$ and $\chi = \frac{\cos \beta}{\varepsilon}(x - st)$; since $\cos \beta > 0$ in our normalization, χ increases with increasing t only if $s < 0$. This emphasizes the fact that “timelike” is properly defined in the cotangent space.

We now consider separately the cases TS and US . We have noted that the state w_0 satisfies the C_1^+ criterion for all w_1 on its Hugoniot loop. That is, w_0 is an upstream state. Furthermore, for states $w_1 \in \mathcal{O}$ beyond the loop, w_1 is the upstream state. That is, if there is a solution to (4.6) satisfying (4.3) or (4.4) according as w_1 is in the loop or not, then (4.6) will be consistent with a well-known physical admissibility criterion, namely, admissibility of compressive shocks. We remark that the scalar dynamical system

$$(4.7) \quad \dot{x} = x(x - \mu^2 + a)$$

completely describes these dynamics; here $h = x(x - \mu^2 + a)$ is the function obtained in Theorem 2.2 and both the cases $a > 0 (w_0 \in \mathcal{H})$ and $a < 0 (w_0 \in \mathcal{E})$ are covered. See Figure 4.1. Now the full dynamics of (4.6) will be either one- or two- dimensional. (In [7] we showed that the addition of physical viscosity results in a one-dimensional system.) We may speak of M , or of the perturbation (4.1), as *consistent with TS dynamics in $\mathcal{O}$* if (4.6) is vectorfield equivalent to (4.7). In [7] we showed that a system obtained by adding physical viscosity to (1.1) was consistent with TS dynamics. The general result is

4.1 The reduced vectorfield dynamics in the TS case.

THEOREM 4.1. *Suppose M in (4.6) is either of rank 1 or of rank 2 uniformly in $\mathcal{O}$. Then in either case the reduction in Theorem 2.2 can be extended to (4.6). If M has rank 2, this will be a centermanifold reduction of*

$$w' = M^{-1}V(w)$$

and will reduce to (4.7) if the eigenvalues of $d(M^{-1}V)$ at w_0^ have the appropriate signs. If M has rank 1, then the dynamics of (4.6) are already one dimensional and will coincide with (4.7) for an open set of rank 1 matrices.*

A proof is sketched in [7].

By contrast, when the unsteady case, (1.3), is perturbed by addition of a viscosity type term, one does not anticipate a simple addition of one-dimensional dynamics to the bifurcation equations (2.16). We give some heuristic arguments before summarizing a theorem that describes a special case.

First, we have noted that in the US case, where w_1 is the upstream direction in the loop near w_0 , we cannot consistently choose w_1 to be upstream over the whole loop. Thus we do not expect to be able to reduce the dynamics to $-\dot{x} = x(x - \mu^2 + a)$ (which

is Figure 4.1 with the arrows reversed) for any M which would be consistent with the parabolic nature of the perturbed system. In fact, could we do so, w_1 would be a source in the 1-dimensional dynamics and w_0 a sink ; for a nonsingular M , then, w_1 would be a saddle in the 2-dimensional dynamics (before the center manifold reduction) and w_0 a sink. However, for equation (1.3), where one flux function is the identity mapping, it can be checked that the eigenvalues of dV are complex in the region $\mathcal{E}$ and that it is impossible for $M^{-1}V$ to be a saddle for w_1 on the entire loop; if M is singular, it is still impossible to do this smoothly. Finally, we quote a theorem from [6] which describes one possible situation in this case.

THEOREM 4.2. *Consider*

$$(4.8) \quad u_t + f(u)_x = \varepsilon M u_{xx}$$

at $(w_0^*, \beta_0) \in \mathcal{B}$. If (2.13) holds:

$$a = l_0 \cdot d^2 f \cdot r_0 r_0 \neq 0$$

and also the condition

$$b = l_0 \cdot d^2 f \cdot r_0 l_0 + r_0 \cdot d^2 f \cdot r_0 r_0 \neq 0$$

then (4.6) is equivalent, at $w_0 = w_1 = w_0^*$ and $M = I$, to the codimension two Takens-Bogdanov vectorfield

$$(4.9) \quad \begin{cases} \dot{x} = y \\ \dot{y} = ax^2 - bxy. \end{cases}$$

Furthermore, if $w_0 \in \mathcal{H}$ and $\{w_1, w_0, \beta\}$ satisfies the Hugoniot relation in $\mathcal{O}$, and M is near I , a complete unfolding of (4.9), up to t -equivalence, is obtained:

$$(4.10) \quad \begin{aligned} \dot{x} &= y \\ \dot{y} &= ax^2 - bxy + \mu_1 x + \mu_2 y. \end{aligned}$$

The unfolding of the singularity is described in [8], though not in normal form; unfolding of the normal form under equivalence (not t -equivalence) is given in [4], and applied to (4.8) in [6]. Neither of the treatments emphasizes the distinguished parameter, β . In the unfolding, we may consider $(\mu_1(\beta), \mu_2(\beta))$ in (4.10) as a path through the unfolding space: the path begins and ends on the μ_2 axis, where the critical points coincide, corresponding to the ends of the loop, and $\mu_1 < 0$ gives a path with $w_0 \in \mathcal{H}$ and $\beta \in (\beta_1(w_0), \beta_2(w_0))$ such

as we have been considering. The sketch in Figure 4.2 shows representative vectorfields in the three qualitatively different segments of the loop. Comparison of characteristic and shock speeds here shows that w_1 is the upstream state in the two intervals where connecting orbits exist (and the similarity parameter χ in (4.2) need not approach the upstream state at $-\infty$). This is further evidence of the impossibility of any admissibility condition that includes all states in this case. However, we note that the points where the curve $(\mu_1(\beta), \mu_2(\beta))$ meets B_h (locus of Hopf bifurcation) and B_{sc} (homoclinic bifurcation) are in the interior of $\mathcal{E}$ for M near I . Thus there are some unsteady shocks joining points in regions of different types that are admissible under the viscous profile criterion.

4.2 Vectorfield dynamics in the US case.

In this section, we have not given a complete discussion of the relation between the C^+ criterion and viscous perturbation, but several points have emerged:

1. The question of “suitability” of viscous perturbations is related to stability of the associated linearized parabolic system, as observed by Majda and Pego [10]; since that stability question is studied by examining the forward light cone and comparing it to evolution of the parabolic system, we note that here, too, the choice of “suitable” viscosities is related to a “suitable” definition of $C^+(w)$. In particular, the perturbation must make the system “parabolic” and not “elliptic”.

2. Travelling wave solutions of the parabolic system, because they are self-similar, reduce the study of the PDE to an ODE in the similarity variable. Locally we are interested

in rest points of the vectorfield; these are given, up to some notion of equivalence, by the development in section 2: the steady state theory identifies the rest points (w_0 and w_1), and the C^+ -criterion specifies something about their type (repellers for upstream limits, attractors for downstream).

3. The two prototype systems we have studied, TS and US, exemplify two qualitatively different kinds of behavior: in the TS case, there is *no* vectorfield bifurcation along the loop, and a one-dimensional theory is adequate. In the US case, it is necessary to consider a fully two-dimensional vectorfield unfolding.

4. These observations give a framework in which it may be possible to perform rigorous stability analysis of TS and US shocks.

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