**A FREE BOUNDARY PROBLEM ARISING
IN ELECTROPHOTOGRAPHY**

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A FREE BOUNDARY PROBLEM ARISING IN ELECTROPHOTOGRAPHY

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Abstract. A free boundary problem which arises in the development of a photocopy is considered. The electric potential $-u$ satisfies the equation $\Delta u = 1$ in the toner region and $\Delta u = 0$ elsewhere. On the free boundary which bounds the toner's region $\partial u / \partial n = 0$ from both sides. The electric charge which is responsible for the development occupies an interval of length ε . The free boundary problem is studied here in the cases of ε "large" and ε "small."

Key words. Free boundary problems, electrophotography, variational inequality, photocopy.

§1. The model. In the process of photocopying a document, one of the steps consists of the development of the electrical image of the document into a visible image. This is accomplished by means of a chemical substance called a toner (something like an ink). The toner material is carried by small iron balls whose motion is affected by a magnetic brush roll. In this paper we consider the question of how the toner's concentration gets distributed on the photoconductor drum so as to provide a visible image of the document. The model to be described below was communicated to us by John Spence from Eastman Kodak Company, and appears in [3; Chap. 7]; for more general information on electrophotography we refer to [1] [4] [5] [6].

As the charged surface of the photoconductor enters the development zone, i.e., the zone where the toner is deposited, visible dark images are created as pattern on the charged surface become partially covered with toner. When the pattern has moved out of the development zone, it is "fully developed" with dark images. We wish to concentrate on one dot or "cell" and determine how it becomes fully developed. We adopt a "quasi-static" approach. In Figure 1 the heavily shaded domain designates the toner image at present. In the next state another layer is added (lighter shaded domain).

FIGURE 1

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To describe the mathematical model, we need to calculate the electric field and add toner proportional to the field strength in a layer adjacent to the previous layer, and keep iterating the process. The differential equation in the entire region (excluding the bold segment on the photoconductor surface) is

$$(1.1) \quad \nabla \cdot (\kappa \vec{E}) = \frac{4\pi\rho}{\varepsilon_0}, \quad \vec{E} = -\nabla V$$

and the interface condition on the toner's boundary is

$$(1.2) \quad \vec{n} \cdot (\vec{E}_{\text{air}} - \kappa_{\text{toner}} \vec{E}_{\text{toner}}) = 0$$

where $\vec{n}$ is the normal to the interface, pointing in the direction of the air. Here κ is the dielectric constant, ρ is the density of holes, $\vec{E}$ is the electric field with potential V , and the constant ε_0 is the permittivity of the vacuum.

The next layer is determined by the rule:

$$\text{if } \vec{n} \cdot \vec{E}_{\text{air}} < 0, \text{ move out the interface,}$$

$$(1.3) \quad \text{if } \vec{n} \cdot \vec{E}_{\text{air}} > 0, \text{ move in the interface.}$$

If $\vec{n} \cdot \vec{E}_{\text{air}} = 0$ we do not move the interface since there is no force to attract or repel toner. It follows that the final position of the interface is determined by

$$(1.4) \quad \vec{n} \cdot \vec{E}_{\text{air}} = \vec{n} \cdot \kappa_{\text{toner}} \vec{E}_{\text{toner}} = 0.$$

The bold segment on the surface of the photoconductor represents surface charge. The electric field across it has a jump:

$$(1.5) \quad V_y(x, 0+) - V_y(x, 0-) = -\frac{4\pi}{\varepsilon_0} \hat{\sigma}, \quad \hat{\sigma} < 0$$

where $\hat{\sigma}$ is the surface charge. To perform the development process one applies a certain voltage $-M$, i.e.,

$$V(x, b) = -M, \quad V(x, -h) = 0$$

where $y = -h$ is the bottom level of the film, $y = 0$ is the photoconductor surface, and $y = b$ is the level (in the air region) where the voltage is applied.

Finally, we impose the condition $V_x = 0$ at the boundaries $x = \pm a$ of the cell.

In the sequel we take for simplicity

$$(1.6) \quad \kappa = 1, \quad \frac{4\pi}{\varepsilon_0} = 1 \quad , \quad \rho = \begin{cases} 1 & \text{in the toner} \\ 0 & \text{elsewhere} \end{cases}$$

and

$$(1.7) \quad \sigma(x) = \begin{cases} -\hat{\sigma} & \text{if } -\varepsilon < x < \varepsilon \quad (\hat{\sigma} < 0) \\ 0 & \text{elsewhere} \end{cases}$$

for some $0 < \varepsilon < a$. Writing $u = -V$ we can formulate the stationary problem as follows:

Set

$$\begin{aligned} \Omega^+ &= \{(x, y); |x| < a, 0 < y < b\}, \\ \Omega^- &= \{(x, y); |x| < a, -h < y < 0\}, \\ \Omega &= \{(x, y); |x| < a, -h < y < b\}, \\ I &= \{(x, 0); |x| < \varepsilon\}, \\ J &= \{(x, 0), |x| < a\}. \end{aligned}$$

For a curve Γ (to be defined below) which lies in Ω^+ with endpoints on either $x = \pm a$ (as in Figure 2(a)) or on J (as in Figure 2(b)), denote by A the subset in Ω^+ which is enclosed by Γ and parts of the x -axis and of $x = \pm a$. Denote by χ_A the characteristic function of A .

FIGURE 2

Actually we do not require that Γ is connected, so that shapes such as those described in Figure 3 are also admitted.

FIGURE 3

We wish to find a continuous curve Γ in $\overline{\Omega^+}$, C^1 inside Ω^+ , and a function u in $\overline{\Omega}$ such that

$$(1.8) \quad \Delta u = 1 \quad \text{in } A,$$

$$(1.9) \quad \Delta u = 0 \quad \text{in } \Omega \setminus \overline{A},$$

$$(1.10) \quad u \in C^1(\overline{\Omega} \setminus \overline{I})$$

$$(1.11) \quad u_y(x, 0+) - u_y(x, 0-) = -\sigma \quad \text{in } I \quad (\sigma = -\hat{\sigma} > 0)$$

and u satisfies the free boundary condition

$$(1.12) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \cap \Omega^+$$

where $\partial/\partial n$ is the outward normal to A , and the boundary conditions:

$$(1.13) \quad u(x, -h) = 0 \quad -a < x < a,$$

$$(1.14) \quad u(x, b) = M, \quad -a < x < a,$$

$$(1.15) \quad u_x(\pm a, y) = 0 \quad -h < y < b.$$

Since the data of the problem are symmetric with respect to x , we expect that the solution (Γ, u) will also be symmetric with respect to the y -axis.

The condition (1.12) on the free boundary, together with the additional requirements that u is C^1 across Γ , form a non-standard set of free boundary conditions.

In this paper we shall study the cases

- (i) $a - \varepsilon$ is small,
- (ii) ε is small.

In case (i) we shall prove that the free boundary problem can be reduced to a variational inequality which has a unique solution; the latter is actually an obstacle problem, with an obstacle which is not continuously differentiable across $y = 0$.

In case (ii) our results are not as complete although quite surprising. We show that there exist infinitely many solutions for which the toner set has two symmetric components; it is not clear whether such solutions are physical. We also show that there exist “ ε^m -approximate” solutions (for any $m > 0$) for which the toner set consists of a single component.

Case (i) is studied in Section 2–6 and Case (ii) is studied in Sections 7–10.

§2. The case $\varepsilon = a$. When $\varepsilon = a$ we expect a solution u which is independent of x . In fact, there are several such solutions.

Solution (1).

$$(2.1) \quad u(y) = \begin{cases} \lambda \left(1 + \frac{y}{h}\right) & \text{if } -h < y < 0 \\ \lambda \left(1 + \frac{y}{h}\right) - \sigma y & \text{if } 0 < y < M \end{cases}$$

where λ is determined by the condition $u(b) = M$:

$$\lambda = \frac{M + \sigma b}{1 + b/h}.$$

If $M > \sigma h$ then $u(y)$ is increasing for all y , whereas if $M < \sigma h$ then $u(y)$ is increasing for $y < 0$ but decreasing for $y > 0$. Since the solution has no toner (even if σ is large), this solution is not physical.

Solution (2).

$$(2.2) \quad u(y) = \begin{cases} \lambda \left(1 + \frac{y}{h}\right) & \text{if } y < 0 \\ \lambda \left(1 + \frac{y}{h}\right) - \sigma y + \frac{y^2}{2} & \text{if } y > 0 \end{cases}$$

where λ is determined by the condition $u(b) = M$:

$$\lambda = \frac{M + \sigma b - b^2/2}{1 + b/h}.$$

Now the toner is the entire interval $0 < y < b$, which is certainly non-physical if b is large enough.

Solution (3).

$$(2.3) \quad u(y) = \begin{cases} \lambda \left(1 + \frac{y}{h}\right) & \text{if } -h < y < 0 \\ \lambda \left(1 + \frac{y}{h}\right) - \sigma y + \frac{y^2}{2} & \text{if } 0 < y < c \\ M & \text{if } c < y < b \end{cases}$$

where λ and the free boundary $y = c$ are determined by:

$$\begin{aligned} u'(c) = 0 & : \quad \lambda = h(\sigma - c), \\ u(c) = M & : \quad c = -h \pm \sqrt{h^2 - 2(M - \sigma h)}. \end{aligned}$$

Since we must have $0 \leq c < b$, the only choice for c is

$$c = -h + \sqrt{h^2 - 2(M - \sigma h)},$$

and it must be assumed that

$$(2.4) \quad M < \sigma h ;$$

further, we must have:

$$(2.5) \quad c \equiv -h + \sqrt{h^2 - 2(M - \sigma h)} \leq b.$$

For solution (3) the toner region depends on σ and thus we may reasonably conclude that

solution (3) is physical,

although its existence requires that (2.4) and (2.5) hold.

Consider the case where (2.4) holds but not (2.5). One can verify that in this case the solution (2.2) satisfies

$$(2.6) \quad u'(b) \leq 0 ;$$

recalling that $u''(y) > 0$ if $0 < y < b$, it follows that

$$(2.7) \quad u(y) > M \quad \text{if } 0 < y < b.$$

Set

$$(2.8) \quad g(y) = \begin{cases} M \left(1 + \frac{y}{h}\right) & \text{if } -h \leq y < 0 \\ M & \text{if } 0 \leq y \leq b. \end{cases}$$

We notice that the solution (2.3) in case $c \leq b$ and the solution (2.2) in case $b < c$ both satisfy $u > M$ if $0 < y < c$ and, by linearity, $u > M(1 + y/h)$ if $-h < y < 0$. Both solutions (2.3) and (2.2) (when $b < c$) seem to be physically reasonable. Thus, we can assert that the physical solution satisfies:

$$(2.9) \quad u(y) \geq g(y).$$

It is easy to check that this physical solution also satisfies

$$-u'' \geq -\chi_{[0,b]}$$

and

$$(-u'' + \chi_{[0,b]})(u - g) = 0 \quad \text{a.e.}$$

Thus u is a solution of a variational inequality with obstacle $g(y)$. We can write the variational inequality in the form

$$(2.10) \quad \int_{-h}^b u'(\phi - u)' - \sigma(\phi - u)(0) + \int_{-h}^b \chi_{[0,b]}(\phi - u) \geq 0$$

for any ϕ Lipschitz, $\phi \geq g$, $\phi(-h) = 0$, $\phi(b) = M$.

So far we have assumed that $M < \sigma h$ and have identified the physical solution as the solution of a variational inequality.

Of course the variational inequality has a solution also if $M > \sigma h$. It is not hard to see that in this case the solution is given by

$$u = g(y)$$

and therefore it does not satisfy the physical condition

$$(2.11) \quad u'(0+) - u'(0-) = -\sigma.$$

Note that since the obstacle $g(y)$ is not differentiable at $y = 0$, the general theory of variational inequalities [2; Chap. 1] does not tell us anything about the differentiability of u at $y = 0$. Thus it is not surprising that (2.11) may fail to hold.

We summarize:

If $M < \sigma h$ then the variational inequality (2.10) provides the unique physical solution. If $M > \sigma h$ then the solution of the variational inequality is not physical.

If $M > \sigma h$ then solution (1) has no toner and solution (2) has full toner region. The former solution seems reasonably physical.

In the rest of the paper we shall assume that (2.4) holds.

§3. The variational inequality. From now on we assume that $\varepsilon < a$. Motivated by the discussion of the previous section we shall now study the variational inequality which extends (2.10) to the case $\varepsilon < a$. Later on we shall prove that if $a - \varepsilon$ is small enough then the solution of the variational inequality is a solution of (1.8)–(1.15).

We can write the variational inequality in the form:

$$(3.1) \quad \begin{aligned} -\Delta u + \chi_E &\geq 0, \quad u \geq g, \\ (-\Delta u + \chi_E)(u - g) &= 0 \quad \text{a.e. in } \Omega \setminus \bar{I} \end{aligned}$$

where

$$\chi_E(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y \leq 0 \end{cases}$$

with boundary conditions

$$(3.2) \quad \begin{aligned} u &= 0 & \text{on } y = -h, & |x| < a, \\ u &= M & \text{on } y = b, & |x| < a, \\ u_x &= 0 & \text{on } |x| = a, & -h < y < b \end{aligned}$$

and jump relation at $y = 0$:

$$(3.3) \quad u_y(x, 0+) - u_y(x, 0-) = -\sigma \quad \text{on } I;$$

g is the function defined in (2.8).

It will be convenient to introduce also a weak formulation of (3.1)–(3.3). Let

$$K = \left\{ \phi \in H^1(\Omega); \phi \geq g, \phi = 0 \quad \text{on } y = -h, \right. \\ \left. \phi = M \quad \text{on } y = b \right\}.$$

From (3.1) we find that a.e. in $\Omega \setminus \bar{I}$

$$(-\Delta u + \chi_E)(\phi - g) \geq 0 \quad \forall \phi \in K.$$

Taking the integral over $\Omega^+ \cup \Omega^-$ and then integrating by parts and using (3.3), we arrive at the variational inequality

$$(3.4) \quad \iint_{\Omega} \nabla u \cdot \nabla(\phi - u) - \int_{-\varepsilon}^{\varepsilon} \sigma(\phi - u)(x, 0) dx + \iint_{\Omega^+} (\phi - u) \geq 0 \quad \forall \phi \in K.$$

THEOREM 3.1. *There exists a unique solution $u \in K$ of the variational inequality (3.4).*

To prove existence we approximate g and $\sigma\chi_I$ by smooth functions g_δ and $\tilde{\sigma}_\mu(x)$ such that $g_\delta(-h) = 0$, $g_\delta(b) = M$, $0 \leq g'_\delta(y) \leq C$ and $g_\delta \rightarrow g$ uniformly as $\delta \rightarrow 0$, and $\tilde{\sigma}_\mu(x) = \tilde{\sigma}_\mu(-x)$, $\tilde{\sigma}_\mu(x) = 0$ if $-a \leq x \leq -\varepsilon - \mu$, $\tilde{\sigma}_\mu(x) = \sigma$ if $-\varepsilon \leq x \leq 0$, $\tilde{\sigma}'_\mu(x) \geq 0$ and $\tilde{\sigma}'_\mu(x) \leq C/\mu$ if $-a \leq x \leq 0$

Let $\beta_\lambda(t)$ be smooth functions in t for all $0 < \lambda < 1$ such that $\beta_\lambda(t) = 0$ if $t > 0$, $\beta'_\lambda(t) \geq 0$, and $\beta_\lambda(t) \rightarrow -\infty$ if $t < 0$, $\lambda \rightarrow 0$.

Consider the problem

$$(3.5) \quad \begin{aligned} -\Delta u + \chi_E + \beta_\lambda(u - g_\delta) &= 0 \quad \text{in } \Omega \setminus \bar{I}, \\ u_y(x, 0+) - u_y(x, 0-) &= -\tilde{\sigma}_\mu(x), \quad |x| < a, \\ u &\in C(\bar{\Omega}) \text{ and it satisfies the boundary condition (3.2).} \end{aligned}$$

One can show by standard arguments that this problem has a unique solution

LEMMA 3.2. *There is a positive constant C independent of λ, δ, μ such that*

$$|u| \leq C.$$

Proof. Let

$$u^*(y) = \begin{cases} A & \text{if } y > 0 \\ A \left(1 + \frac{y}{h}\right) & \text{if } y < 0 \end{cases}$$

where $A > \max\{M, h \cdot \max_{|x| \leq a} \tilde{\sigma}_\mu(x)\}$. Then

$$-\Delta u^* + \beta_\lambda(u^* - g_\delta) = 0 \quad \text{in } \Omega^+ \cup \Omega^-$$

if δ and λ are sufficiently small. Also

$$u_y^*(0+) - u_y^*(0-) = -\frac{A}{h} < -\tilde{\sigma}_\mu(x).$$

Applying the maximum principle to $u^* - u$ we find that $u^* - u \geq 0$ in $\bar{\Omega}$, so that $u \leq C$.

Next, from (3.5) and the fact that $\beta_\lambda \leq 0$ we get

$$-\Delta u + \chi_E \geq 0,$$

and u satisfies the transmission and boundary conditions as in (3.5). Let U be the solution of

$$-\Delta U + \chi_E = 0 \quad \text{in } \Omega,$$

satisfying the same boundary conditions as u . Since

$$U_y(x, 0+) - U_y(x, 0-) = 0 > -\tilde{\sigma}_\mu(x),$$

$u \geq U$ by the maximum principle, and therefore $u \geq -C$.

LEMMA 3.3. *There holds:*

$$\begin{aligned} u_x &\geq 0 \quad \text{if} \quad -a \leq x \leq 0, \\ u_x &\leq 0 \quad \text{if} \quad 0 \leq x \leq a. \end{aligned}$$

Proof. By uniqueness, $u(x, y) = u(-x, y)$, so that

$$u_x(x, y) = -u_x(-x, y).$$

Therefore

$$u_x = 0 \quad \text{on} \quad x = 0.$$

Since

$$\begin{aligned} -\Delta u_x + \beta'_\lambda u_x &= 0 \quad \text{in} \quad \Omega^+ \cup \Omega^-, \\ u_x &= 0 \quad \text{on} \quad x = \pm a, \quad y = -h, \quad y = b \end{aligned}$$

and

$$\begin{aligned} (u_x)_y(x, 0+) - (u_x)_y(x, 0-) &= -\tilde{\sigma}'_\mu(x) \leq 0 \\ &\quad \text{for} \quad -a \leq x \leq 0, \end{aligned}$$

the maximum principle applied to u_x in $-a \leq x \leq 0$ yields $u_x \geq 0$.

LEMMA 3.4. *The penalty term β_λ satisfies:*

$$-C_\delta \leq \beta_\lambda(u - g_\delta) \leq 0$$

where C_δ is a constant independent of λ, μ .

Proof. By Lemma 3.3, the negative minimum of $\beta_\lambda(u - g_\delta)$ must be attained on $x = -a$, say at $(-a, y_0)$. But then also $u - g_\delta$ takes negative minimum at $(-a, y_0)$.

We claim that

$$(3.6) \quad \beta_\lambda(u(-a, y_0) - g_\delta(y_0)) \geq g''_\delta(y_0) - 1.$$

Indeed, if this is not true then, in some Ω -neighborhood of $(-a, y_0)$,

$$\begin{aligned} -\Delta(u - g_\delta) &= -\Delta u + g''_\delta(y) \\ &= -\chi_E - \beta_\lambda(u - g_\delta) + g''_\delta(y) > 0. \end{aligned}$$

Therefore, by the strong maximum principle,

$$u_x(-a, y_0) = (u - g_\delta)_x(-a, y_0) > 0$$

which is a contradiction to the third equation in (3.2). (This argument works whether $y_0 = 0$ or $\neq 0$, since u is in C^1 across $y = 0$ near $(-a, 0)$.)

Having proved (3.6), we deduce that

$$\beta_\lambda(u - g_\delta) \geq \min g''_\delta(y) - 1 \geq -C_\delta,$$

and the lemma follows.

LEMMA 3.5. *There holds:*

$$|u_y| \leq C$$

where C is a constant independent of λ, δ, μ

Proof. For any function $v(x, y)$, set $v^T(x, y) = v(x, -y)$. Let

$$z(x, y) = u(x, y) - u(x, -y) - \frac{1}{2} y^2 + Cy, \quad 0 < y < \tilde{h}$$

where $\tilde{h} = \min(h, b)$. Then

$$\begin{aligned} z &= 0 \quad \text{on} \quad y = 0, \\ z_x &= 0 \quad \text{on} \quad |x| = a, \quad 0 < y < \tilde{h} \end{aligned}$$

and

$$z(x, \tilde{h}) = u(x, \tilde{h}) - u(x, -\tilde{h}) - \frac{1}{2} \tilde{h}^2 + C\tilde{h} \geq 0$$

if C is large. Also

$$\begin{aligned} \Delta z &= \Delta u - \Delta u^T - 1 = 1 + \beta_\lambda(u - g_\delta) - \beta_\lambda(u^T - g_\delta^T) - 1 \\ &= [\beta_\lambda(u - g_\delta) - \beta_\lambda(u^T - g_\delta + \frac{1}{2} y^2 - Cy)] + [\beta_\lambda(u^T - g_\delta + \frac{1}{2} y^2 - Cy) - \beta_\lambda(u^T - g_\delta^T)]. \end{aligned}$$

The expression in the first brackets on the right-hand side is equal to $\tilde{c}z$ where $\tilde{c} \geq 0$ and the expression in the second brackets is ≤ 0 if C is large enough (recall that $g_\delta(\cdot)$ is monotone increasing). Therefore

$$\Delta z - \tilde{c}z \leq 0.$$

We can now apply the maximum principle to deduce that $z(x, y) \geq 0$ if $0 \leq y \leq \tilde{h}$. Hence

$$u_y^+ + u_y^- + C \geq 0 \quad \text{on} \quad y = 0$$

where $u_y^\pm(x, y) = u_y(x, y \pm 0)$.

Combining this with

$$u_y^+ - u_y^- = -\tilde{\sigma}_\mu$$

we get

$$(3.7) \quad \begin{aligned} u_y^+ &\geq -C \quad \text{on} \quad y = 0, \\ u_y^- &\geq -C \quad \text{on} \quad y = 0. \end{aligned}$$

Similarly, working with

$$z(x, y) = u(x, y) - u(x, -y) - \frac{1}{2} y^2 - Cy, \quad 0 < y < \tilde{h},$$

we find that $\Delta z - \tilde{c}z \geq 0$ if C is large enough (here we use: $g_\delta - g_\delta^T \leq \text{const. } y, y > 0$)
Therefore $z(x, y) \leq 0$, which leads to

$$(3.8) \quad \begin{aligned} u_y^+ &\leq C \quad \text{on } y = 0, \\ u_y^- &\leq C \quad \text{on } y = 0. \end{aligned}$$

By comparison (using Lemma 3.2) we find that

$$\pm u \leq Ch \left(1 + \frac{y}{h}\right) \quad \text{in } -h < y < 0.$$

Therefore

$$(3.9) \quad |u_y| \leq C \quad \text{on } y = -h.$$

Similarly, in $0 < y < b$, the function u lies between the two function

$$\frac{1}{2} (y - b)^2 + M \pm C(b - y)$$

if C is large enough. Hence

$$(3.10) \quad |u_y| \leq C \quad \text{on } y = b.$$

The function u_y satisfies

$$-\Delta u_y + \beta'_\lambda(u_y - g'_\delta) = 0 \quad \text{in } \Omega^+ \cup \Omega^-.$$

Since $g_{\delta,y} = 0$ in Ω^+ ,

$$-\Delta u_y + \beta'_\lambda u_y = 0 \quad \text{in } \Omega^+.$$

Recalling (3.7), (3.8), (3.10) and $(u_y)_x = 0$ on $x = \pm a$, the maximum principle yields

$$|u_y| \leq C \quad \text{in } \Omega^+.$$

In Ω^- we have $g_{\delta,y} \geq 0$ so that

$$-\Delta u_y + \beta'_\lambda u_y = \beta'_\lambda g_{\delta,y} \geq 0$$

and therefore, by the maximum principle and (3.7), (3.9),

$$u_y \geq -C \quad \text{in } \Omega^-.$$

Finally,

$$-\Delta(u_y - C) + \beta'_\lambda(u_y - C) = \beta'_\lambda(g_{\delta,y} - C) \leq 0 \quad \text{in } \Omega^-$$

if C is large enough, and we can conclude by the previous argument that $u_y - C$ is bounded from above, i.e., $u_y \leq C$ with another constant C .

LEMMA 3.6. *There holds:*

$$|u_x| \leq C_\mu \quad \text{in} \quad \Omega^+ \cup \Omega^-$$

where C_μ is a positive constant independent of λ, δ .

Proof. We have

$$\begin{aligned} -\Delta u_x + \beta'_\lambda u_x &= 0 \quad \text{in} \quad \Omega^+ \cup \Omega^-, \\ u_x &= 0 \quad \text{on} \quad x = \pm a, \quad y = -h \quad \text{and} \quad y = b, \end{aligned}$$

and

$$(u_x)_y^+ - (u_x)_y^- = -\tilde{\sigma}'_\mu(x) \quad \text{on} \quad y = 0.$$

Take $A > \max_{|x| \leq a} |\tilde{\sigma}'_\mu(x)|$ and

$$u^*(y) = \begin{cases} \frac{Ah}{b+h} (b-y) & , \quad 0 < y < b \\ \frac{Abh}{b+h} \left(1 + \frac{y}{h}\right) & , \quad -h < y < 0. \end{cases}$$

Then

$$-\Delta u^* + \beta'_\lambda u^* \geq 0 \quad \text{in} \quad \Omega^+ \cup \Omega^-$$

and

$$[u_y^*]_{0-}^{0+} = -\frac{Ah}{b+h} - \frac{Ab}{b+h} = -A < -\tilde{\sigma}'_\mu(x).$$

Therefore, by comparison,

$$u_x \leq u^* \leq C \quad \text{in} \quad \Omega^+ \cup \Omega^-.$$

Similarly one can prove that $u_x \geq -C$.

Proof of existence. We shall now prove the existence part of Theorem 3.1. Denote by K_δ the convex set K corresponding to g_δ . Then the solution u of (3.1), (3.2) satisfies:

$$\iint_{\Omega^+ \cup \Omega^-} (-\Delta u + \chi_E + \beta_\lambda(u - g_\delta))(\phi - u) = 0 \quad \forall \phi \in K_\delta.$$

By integration by parts and the transmission condition at $y = 0$, we obtain

$$\begin{aligned} & \iint_{\Omega} \nabla u \cdot \nabla(\phi - u) - \int_{\{y=0\} \cap \{|x| < a\}} \tilde{\sigma}_\mu(x)(\phi - u)(x, 0) dx \\ & + \iint_{\Omega^+} (\phi - u) + \iint_{\Omega} \beta_\lambda(u - g_\delta)(\phi - g_\delta + g_\delta - u) = 0. \end{aligned}$$

Since $\beta_\lambda(u - g_\delta)(u - g_\delta) \geq 0$ and $\beta_\lambda(u - g_\delta)(\phi - g_\delta) \leq 0$, it follows that

$$(3.11) \quad \begin{aligned} & \iint_{\Omega} \nabla u_{\lambda, \delta, \mu} \cdot (\nabla(\phi - u_{\lambda, \delta, \mu})) - \int_{\{y=0\} \cap \{|x| < a\}} \tilde{\sigma}_\mu(x)(\phi - u_{\lambda, \delta, \mu})(x, 0) dx \\ & + \iint_{\Omega^+} (\phi - u_{\lambda, \delta, \mu}) \geq 0 \quad \forall \phi \in K_\delta \end{aligned}$$

where, for clarity, we designated u by $u_{\lambda, \delta, \mu}$

By Lemmas 3.5, 3.6, we can take a sequence $\lambda \rightarrow 0$ so that

$$u_{\lambda, \delta, \mu} \rightarrow u_{\delta, \mu}$$

uniformly in $\overline{\Omega}$ and weakly in $W^{1,p}(\Omega)$ for any $p > 1$. By Lemma 3.4 it also follows that $u_{\delta, \mu} \geq g_\delta$. Taking $\lambda \rightarrow 0$ in (3.11) we find that $u_{\delta, \mu}$ is a solution to the variational inequality

$$(3.12) \quad \begin{aligned} & \iint_{\Omega} \nabla u_{\delta, \mu} \cdot \nabla(\phi - u_{\delta, \mu}) - \int_{\{y=0\} \cap \{|x| \leq a\}} \tilde{\sigma}_\mu(x)(\phi - u_{\delta, \mu}) \\ & + \iint_{\Omega^+} (\phi - u_{\delta, \mu}) \geq 0 \quad \forall \phi \in K_\delta, \quad u_{\delta, \mu} \in K_\delta. \end{aligned}$$

By Lemmas 3.5, 3.6 there is a sequence $\delta \rightarrow 0$ such that

$$u_{\delta, \mu} \rightarrow u_\mu$$

uniformly in $\overline{\Omega}$ and weakly in $W^{1,p}(\Omega)$ for any $p > 1$. Taking $\phi \in \bigcap_{0 < \delta < \delta_0} K_\delta$ in (3.12) and letting $\delta \rightarrow 0$, we find that u_μ solves the variational inequality

$$(3.13) \quad \iint_{\Omega} \nabla u_\mu \cdot \nabla(\phi - u_\mu) - \int_{\{y=0\} \cap \{|x| \leq a\}} \tilde{\sigma}_\mu(\phi - u_\mu) + \iint_{\Omega^+} (\phi - u_\mu) \geq 0.$$

Since any ϕ in K can be approximated by ϕ in $\bigcap_{0 < \delta \leq \delta_0} K_\delta$, it follows that (3.13) holds for all $\phi \in K$. It is also clear that $u_\mu \geq g$, so that $u_\mu \in K$.

Since, by Lemma 3.2, $|u_\mu| \leq C$ where C is a constant independent of μ , we can deduce from (3.13) that

$$\iint_{\Omega} |\nabla u_\mu|^2 \leq C.$$

Consequently

$$u_\mu \rightarrow u \quad \text{weakly in } H^1(\Omega)$$

for some sequence $\mu \rightarrow 0$. Taking $\mu \rightarrow 0$ in (3.13) we find that u is a solution of the variational inequality (3.4).

REMARK 3.1. The proof of existence extends to the case where

$$(3.14) \quad u_y(x, 0+) - u_y(x, 0-) = -\sigma(x)$$

where $\sigma(x)$ is any piecewise continuous function.

The uniqueness assertion in Theorem 3.1 is contained in the following more general monotonicity result.

THEOREM 3.7. *Let u_1, u_2 be two solutions of the variational inequality corresponding to $\sigma_1(x)$ and $\sigma_2(x)$ (cf. Remark 3.1). If $\sigma_1 \geq \sigma_2$ then $u_1 \geq u_2$.*

Proof. We have

$$\iint_{\Omega} \nabla u_1 \cdot \nabla(\phi - u_1) - \int_{\{y=0\}} \sigma_1(\phi - u_1) + \iint_{\Omega^+} (\phi - u_1) \geq 0.$$

Taking $\phi = \max(u_1, u_2)$ so that $\phi - u_1 = (u_2 - u_1)^+$, we get

$$(3.15) \quad \iint_{\Omega} \nabla u_1 \cdot \nabla(u_2 - u_1)^+ - \int_{\{y=0\}} \sigma_1(u_2 - u_1)^+ + \iint_{\Omega^+} (u_2 - u_1)^+ \geq 0.$$

Similarly, taking $\phi = \min(u_1, u_2)$ (so that $\phi - u_2 = -(u_2 - u_1)^+$) in the variational inequality for u_2 , we get

$$- \iint_{\Omega} \nabla u_2 \cdot \nabla(u_2 - u_1)^+ + \int_{\{y=0\}} \sigma_2(u_2 - u_1)^+ - \iint_{\Omega^+} (u_2 - u_1)^+ \geq 0.$$

Adding this to (3.15) results in

$$- \iint_{\Omega} \nabla(u_2 - u_1) \cdot \nabla(u_2 - u_1)^+ + \int_{\{y=0\}} (\sigma_2 - \sigma_1)(u_2 - u_1)^+ \geq 0.$$

Since $\sigma_2 - \sigma_1 \leq 0$, it follows that $\nabla(u_2 - u_1)^+ = 0$ and therefore $u_1 \geq u_2$.

§4. Regularity of the solution. The solution of the variational inequality satisfies: $|u| \leq C$ (Lemma 3.2),

$$|u_y| \leq C \quad (\text{Lemma 3.5}),$$

and

$$\iint_{\Omega} |\nabla u|^2 \leq C.$$

From the last two estimates one can deduce, by a standard imbedding argument, that

$$(4.1) \quad u \in C^{1/3}(\overline{\Omega})$$

Since $u \geq g$, $-\Delta u \geq 0$ in Ω^- and g is harmonic in Ω^- , the strong maximum principle shows that

$$(4.2) \quad \text{either } u \equiv g \text{ in } \Omega^- \text{ or } u > g \text{ in } \Omega^-.$$

The obstacle $g(y)$ is smooth except on $y = 0$ where its derivative has a jump. By the general theory of variational inequalities [2], the solution is Lipschitz continuous if the obstacle is Lipschitz continuous, and the solution is in $C^{1,1}$ if the obstacle is in $C^{1,1}$. Thus away from $\{y = 0\}$ the solution u is in $C^{1,1}$. At $y = 0$ the solution may not be continuously differentiable due to the jump of $g'(y)$ at $y = 0$ and also possibly due also to the transmission condition on I .

In this section we prove:

THEOREM 4.1. *The derivative u_x is bounded away from I , that is, u_x is bounded in every compact subset of $\overline{\Omega} \setminus \overline{I}$.*

Proof. We consider the solution $u = u_{\lambda,\delta,\mu}$ of (3.4) and choose β_λ such that

$$(4.3) \quad \begin{aligned} \beta_\lambda(t) &= \beta\left(\frac{t}{\lambda}\right), \\ \beta(t) &= \begin{cases} 0 & \text{if } t \geq 0 \\ t+1 & \text{if } t \leq -2 \end{cases} \end{aligned}$$

and $0 \leq \beta'(t) \leq 1$, $\beta''(t) \leq 0$ if $-2 \leq t \leq 0$.

Setting $u = u_{\lambda,\delta,\mu}$ we have, by Lemma 3.2,

$$u - g_\delta \geq -C - g_\delta \geq -C - M \equiv -M^*.$$

Therefore, if $u - g_\delta \leq -2\lambda$

$$\begin{aligned} M^* \beta'_\lambda(u - g_\delta) + \beta_\lambda(u - g_\delta) &\geq \frac{M^*}{\lambda} + \beta_\lambda(-M^*) \\ &= \frac{M^*}{\lambda} - \frac{M^*}{\lambda} + 1 = 1. \end{aligned}$$

If $-2\lambda < u - g_\delta < 0$ then $M^* > 2\lambda > -(u - g_\delta)$ for λ small, so that

$$(4.4) \quad \begin{aligned} M^* \beta'_\lambda(u - g_\delta) + \beta_\lambda(u - g_\delta) &\geq -(u - g_\delta) \beta'_\lambda(u - g_\delta) + \beta_\lambda(u - g_\delta) \\ &\geq \beta_\lambda(0) = 0 \end{aligned}$$

since $\beta'' \leq 0$. Since finally the left-hand side of (4.4) vanishes when $u - g_\delta > 0$, we conclude that in all cases

$$(4.5) \quad M^* \beta'_\lambda(u - g_\delta) + \beta_\lambda(u - g_\delta) \geq 0.$$

Take a cut-off function $\zeta(x, y)$ of I , that is,

$$\begin{aligned} \zeta &\in C^\infty(\overline{\Omega}), \\ \zeta &= 0 \quad \text{in a neighborhood of } \overline{I}, \\ 0 &\leq \zeta \leq 1, \\ \zeta &= 1 \quad \text{outside a neighborhood of } \overline{I}. \end{aligned}$$

Consider the function

$$w = \zeta^2 u_x^2 + \Lambda(u + c)^2, \quad \Lambda > 0, \quad c > 0.$$

Then

$$\begin{aligned} \Delta w &= 2|\nabla \zeta|^2 u_x^2 + 2\zeta(\Delta \zeta) u_x^2 + 8\zeta u_x \nabla \zeta \cdot \nabla u_x \\ &\quad + 2\zeta^2(u_{xx}^2 + u_{xy}^2) + 2\zeta^2 u_x \Delta u_x + 2\Lambda|\nabla u|^2 + 2\Lambda(u + c)\Delta u \\ &= J_1 + J_2 + \cdots + J_7. \end{aligned}$$

Clearly

$$\begin{aligned} J_1 &\geq 0, \\ J_3 + J_4 + \frac{1}{3} J_6 &\geq 0 \quad \text{if } \Lambda \text{ is large, say } \Lambda \geq \Lambda_0, \\ J_2 + \frac{1}{3} J_6 &\geq 0 \quad \text{if } \Lambda \text{ is large, say } \Lambda \geq \Lambda_0. \end{aligned}$$

Next, by

$$(4.6) \quad -\Delta u_x + \beta'_\lambda u_x = 0$$

and (4.5),

$$J_5 = 2\zeta^2 u_x^2 \beta'_\lambda \geq -2\zeta^2 u_x^2 \beta_\lambda / M^*.$$

Since

$$J_7 = 2\Lambda(u + c)(\chi_E + \beta_\lambda) \geq 2\Lambda(u + c)\beta_\lambda,$$

we conclude that

$$(4.7) \quad J_5 + J_7 \geq \left(\frac{2\zeta^2 u_x^2}{M^*} - 2\Lambda(u + c) \right) (-\beta_\lambda), \quad (-\beta_\lambda) \geq 0.$$

To prove the theorem it suffices to show that w is bounded by a constant C independent of λ, δ, μ . To do this we fix $\Lambda = \Lambda_0$. Then $|2\Lambda(u + c)| \leq C$, by Lemma 3.2. We wish to estimate w at its point of maximum (x_0, y_0) . We may clearly assume that

$$\frac{2\zeta^2 u_x^2}{M^*} > 2\Lambda(u + c) \quad \text{at} \quad (x_0, y_0),$$

for otherwise $w(x_0, y_0) \leq C$. Then, by the previous inequalities on the J_i ,

$$\Delta w \geq \frac{1}{3} J_6 > 0$$

in some Ω -neighborhood of (x_0, y_0) , which is a contradiction, unless $(x_0, y_0) \in \partial\Omega$. Since in the latter case $w \leq C$, the proof is complete.

§5. Properties of the free boundary. By Lemma 3.3, the solution u of the variational inequality (3.4) satisfies

$$u_x \geq 0 \quad \text{if} \quad -a \leq x < 0.$$

It follows that

$$\{u > M\} \cap (\overline{\Omega^+} \setminus \{|x| = a\}) = \{(x, y) \in \overline{\Omega^+}, |x| < f(y), 0 \leq y \leq b\}.$$

for some function $f(y)$, $0 \leq f(y) \leq a$.

Define

$$\Sigma = \{y \in (0, b); \quad 0 < |f(y)| < a\}$$

and let

$$\Gamma^\pm \equiv \{(x, y); \quad x = \pm f(y), \quad y \in \overline{\Sigma}\}.$$

Then $\Gamma = \Gamma^+ \cup \Gamma^-$ is the free boundary.

The set

$$\{y \in (0, b); \quad |f(y)| > 0\}$$

is open and must consist of a single interval $(0 < y < b_0)$ where $b_0 \leq b$. Indeed, otherwise this set contains at least one maximal interval $(b_1 < y < b_2)$ with $b_1 > 0$. Since $f(b_1) = 0$, $u(x, b_1) \equiv M$. For the same reason $u(x, b_2) \equiv M$ if $b_2 < b$, whereas (by the boundary conditions) $u(x, b_2) \equiv M$ if $b_2 = b$. We now apply the maximum principle to u in the subregion of Ω^+

$$\Omega_0 = \{u(x, y) > M, \quad b_1 < y < b_2\}$$

noting that

$$\Delta u = 1 \quad \text{in} \quad \Omega_0$$

and $u_x = 0$ on $\partial\Omega_0 \cap \{x = \pm a\}$. We deduce that $u < M$ in Ω_0 , which is a contradiction.

THEOREM 5.1. *The function $f(y)$ is continuous for $0 \leq y \leq b_0$.*

Proof. Set $F(y) = -f(y)$ and consider a section Γ_0 of the free boundary Γ^- given by $x = F(y)$, $y_0 \leq y \leq y_1$ such that $F(y_0) = F(y_1) \equiv \beta$ and $-a < F(y) < \beta$ for $y_0 < y < y_1$. Consider the region Ω_0 bounded by Γ_0 and $x = \beta$. Let

$$v(x) = \begin{cases} \frac{1}{2}(x - \alpha)^2 & \text{if } \alpha \leq x < \beta \\ 0 & \text{if } -a < x < \alpha. \end{cases}$$

for some $-a < \alpha < \beta$. Since $|u_y| \leq C$,

$$u(\beta, y) - M \leq C(y_1 - y_0) = v(\beta)$$

if

$$(5.1) \quad \frac{1}{2} (\beta - \alpha)^2 = C(y_1 - y_0).$$

We can then compare $u - M$ with v in Ω_0 and deduce that $u - M \leq v$. In particular, $u - M = 0$ if $x < \alpha$, so that

$$(5.2) \quad |F(y) - F(y_0)| \leq \beta - \alpha \leq C\sqrt{y_1 - y_0}$$

provided α satisfies (5.1) and $-a < \alpha$.

We next establish (5.2) in case $F(y) = -a$ for some values of y in (y_0, y_1) . We set $x_0 = (-a + \beta)/2$ and take as a comparison function

$$v(x) = \frac{(x - x_0)^2}{2l^2} C(y_1 - y_0) \quad \text{where } l = \beta - (-a).$$

Again $u(\beta, y) - M < v(\beta)$; further, if

$$(5.3) \quad l^2 > C(y_1 - y_0)$$

then $\Delta v < 1$. Noting that $-v_x(-a, y) > 0$ we can apply the maximum principle to $u - M - v$ and conclude that $u - M \leq v$ in the region Ω_0 bounded by $x = \beta$, $x = -a$ and the free boundary for $y_0 < y < y_1$. Since $v(x_0) = 0$ we conclude that

$$u - M = u_x = 0 \quad \text{at } x = x_0$$

and, by unique harmonic continuation,

$$u(x, y) = M + \frac{1}{2} (x - x_0)^2 \quad \text{in } \Omega_0,$$

which is a contradiction to $u_x(-a, y) = 0$. This contradiction implies that the inequality (5.3) is not true, so that (5.2) again must be satisfied.

The estimate (5.2) which now holds regardless of whether the free boundary in $\{y_0 < y < y_1\} \cap \{x < 0\}$ meets $x = -a$ or not, implies that the region $\{u > M\}$ does not have ‘‘fingers,’’ or, that the free boundary is not oscillatory. Since further the free boundary cannot contain a straight line, we deduce that the function $f(y)$ is continuous.

REMARK 5.1. From the general theory of variational inequalities [2; Chapter 2] it follows that $f(y)$ is analytic function on any point $y \in (0, b_0)$ where $0 < f(y) < a$.

So far we have not used the assumption $M < \sigma h$. We shall now use it to prove that the free boundary stays away from I .

THEOREM 5.2. *If $M < \sigma h$ then $f(0) \geq \varepsilon$.*

Proof. Setting $F(y) = -f(y)$ we shall work in the region $\{-a < x < 0\}$ and prove that $F(0) \leq -\varepsilon$. Suppose

$$(5.4) \quad F(0) > -\varepsilon.$$

Then there is a ball $B_r(x_0, y_0)$ with $-\varepsilon < x_0 - r < x_0 + r < F(0)$ such that $r < \min(h, b)$ and

$$(5.5) \quad u = M \quad \text{in} \quad B_r(x_0, 0) \cap \{y > 0\}.$$

Take $\phi = u + \delta\zeta$, $\zeta \geq 0$, $\text{supp } \zeta \subset B_r(x_0, 0)$, and δ a positive number. Then $\phi \in K$ and by (3.4)

$$(5.6) \quad - \iint_{\Omega} \nabla u \cdot \nabla \zeta + \int_I \sigma \zeta - \iint_{\Omega^+} \zeta \leq 0.$$

Set $u^- = u|_{\Omega^-}$. Then $\Delta u^- = 0$ and, by (5.5), $u^-(x, 0) = M$ if $|x - x_0| < r$. It follows that $u^-(x, y)$ is smooth up to $y = 0$, if $|x - x_0| < r$. Since further

$$u^-(x, 0) = M, \quad u^-(x, y) \geq g(y) = M \left(1 + \frac{y}{h}\right),$$

we have

$$(5.7) \quad \frac{\partial u^-}{\partial y}(x, 0) \leq \frac{M}{h} \quad \text{if} \quad |x - x_0| < r.$$

We now compute, using (5.5), (5.7),

$$- \iint_{\Omega} \nabla u \cdot \nabla \zeta = - \iint_{\Omega^-} \nabla u^- \cdot \nabla \zeta = - \int_{\{y=0\}} \zeta \frac{\partial u^-}{\partial y} \geq - \int_{\{y=0\}} \zeta \frac{M}{h}.$$

Substituting this into (5.6), we get

$$\int_{\{y=0\}} \left(\sigma - \frac{M}{h}\right) \zeta \leq \iint_{\Omega^+} \zeta.$$

Taking

$$\begin{aligned} \zeta &= (\beta^2 - (x - x_0)^2)^2 (\alpha^2 - y^2)^2 \quad \text{if} \quad |x - x_0| < \beta, \quad |y| < \alpha, \\ \zeta &= 0 \quad \text{otherwise,} \end{aligned}$$

and letting $\alpha \rightarrow 0$, we get $\sigma \leq M/h$, which is a contradiction.

COROLLARY 5.3. *If $M < \sigma h$ then $u > g$ in Ω^- .*

Indeed, if the assertion is not true then, by (4.2), $u - g \equiv 0$ in Ω^- . Since by Theorem 5.2, the free boundary stays away from I , we then get

$$u_y^+(x, 0) = -\sigma + g'(0) = -\sigma + \frac{M}{h} < 0$$

if $|x| < \varepsilon$. Consequently $u^+(0, y) < M$ for y positive and small, a contradiction.

§6. When does the variational inequality solution solve the original problem (1.8)–(1.15)? We continue to assume that $M < \sigma h$. Then the free boundary stays away from I ; consequently the variational inequality becomes equality in a neighborhood of I . This implies that the transmission condition (1.11) holds. By the same argument, if $f(0) > \varepsilon$ then u is C^1 across the intervals

$$\{(x, 0); -f(0) < x < -\varepsilon\}, \quad \{(x, 0); \varepsilon < x < f(0)\}$$

since $\Delta u = \chi_E$ in a neighborhood of these intervals.

Thus the solution u of the variational inequality solves the systems (1.8)–(1.15) with the possible exception of (1.10):

Since the obstacle g is not C^1 across $y = 0$, it is not a priori clear whether u is in C^1 across the intervals

$$\begin{aligned} T_1 &= \{(x, 0); -a < x < -f(0)\}, \\ T_2 &= \{(x, 0); f(0) < x < a\}. \end{aligned}$$

Of course, if $f(0) = a$ then these intervals are empty and u does satisfy (1.10).

We shall now prove:

THEOREM 6.1. *If $f(0) < a$ then*

$$(6.1) \quad u_y(x, 0+) - u_y(x, 0-) > 0 \quad \text{if } -a < x < -f(0) \text{ or } f(0) < x < a.$$

COROLLARY 6.2. *The solution of the variational inequality is a solution of the toner problems (1.8)–(1.15) if and only if $f(0) = a$.*

The case $f(0) = a$ is depicted in Figures 2(a) and 3(a).

Proof of Theorem 6.1. Set

$$u^+ = u|_{\Omega^+}, \quad u^- = u|_{\Omega^-}.$$

Let $-a < x_* < -f(0)$. Then, for small $r > 0$, $u \equiv M$ in $B_r(x_*, 0)$. Taking $\varphi = u + \delta\zeta$ in (3.4) where $\zeta \geq 0$, $\text{supp } \zeta \subset B_r(x, 0)$ and δ is a positive number, we get

$$-\iint_{\Omega^-} \nabla u \cdot \nabla \zeta - \iint_{\Omega^+} \zeta \leq 0,$$

or

$$\int_{\{y=0\}} u_y^- \zeta - \iint_{\Omega^+} \zeta \leq 0.$$

Since ζ is arbitrary, it follows that

$$(6.2) \quad u_y^-(x, 0) \leq 0;$$

observe that the function u^- satisfies

$$\begin{aligned} \Delta u^- &= 0 \quad \text{in } B_- = B_r(x_*, 0) \cap \Omega^-, \\ u^-(x, 0) &= M, \quad |x - x_*| < r, \end{aligned}$$

and it is therefore smooth up to $y = 0$.

Since $u^-(x, 0) = M$,

$$(6.3) \quad u_x^-(x, 0) = 0 \quad \text{if } |x - x_*| < r.$$

Recalling that $u_x \leq 0$ if $-a < x < 0$, we see that u_x^- takes its maximum in $\overline{B_-}$ on $y = 0$. Hence, by the maximum principle, $(u_x^-)_y(x, 0) > 0$. This means that

$$\frac{d}{dx} u_y^-(x, 0) > 0.$$

Recalling (6.2) we conclude that

$$u_y^-(x, 0) < u_y^-(x_* + r, 0) \leq 0$$

for $|x - x_*| < r$ and, as x_* is arbitrary, $u_y^-(x, 0) < 0$ if $-a < x < -f(0)$. Since $u^+ \equiv M$ in $B_r(x_*, 0)$, we also have $u_y^+(x, 0) = 0$, and (6.1) follows.

We finally establish:

THEOREM 6.3. *If $M < \sigma h$ and $a - \varepsilon$ is sufficiently small then $f(0) = a$ and thus the solution of the variational inequality solves the toner problem (1.8)–(1.13).*

Proof. As $\varepsilon \rightarrow a$, u_ε converges uniformly to a function u^* which satisfies the variational inequality

$$\iint_{\Omega} \nabla u^* \cdot \nabla (\phi - u) - \int_{-a}^a \sigma(\phi - u^*)(x, 0) dx + \iint_{\Omega^+} (\phi - u^*) \geq 0.$$

By uniqueness

$$u^* = u(y)$$

where $u(y)$ is given by (2.3) or (2.2), and this implies in particular that

$$u^*(x, 0) = u(0) > M, \quad -a \leq x \leq a.$$

From the uniform convergence $u_\varepsilon \rightarrow u^*$ we then conclude that $f(0) = a$ if $a - \varepsilon$ is small enough.

§7. The case $\varepsilon = 0$. The rest of the paper is concerned with the case where ε is small. We shall assume that $b \geq h$, which is physically reasonable. When $\varepsilon = 0$ there is a unique solution of (1.8)–(1.15) with no toner, that is, with $A = \phi$, namely $u = (y + h)M/(b + h)$. In this section we show that there is no solution with nonempty toner region:

THEOREM 7.1. *If $\varepsilon = 0$ and $b \geq h$ then there is no solution of (1.8)–(1.15) with nonempty A .*

The solution is understood to be symmetric with respect to the y -axis, so that

$$(7.1) \quad u_x(0, y) = 0,$$

and with $\partial A \cap \{x > 0\}$ a differentiable graph in the y -direction. The latter property implies that if $\vec{n} = (n_x, n_y)$ is the outward normal to A then

$$(7.2) \quad n_x > 0 \quad \text{if} \quad 0 < x < a.$$

Since χ_A is monotone decreasing in x in $\{x > 0\}$ (with jump across ∂A),

$$(7.3) \quad \Delta u_x \leq 0 \quad \text{in} \quad \{0 < x < a\}$$

in the distribution sense.

Proof. Set

$$w = \frac{y + h}{b + h} M - u.$$

Then

$$\Delta w = \begin{cases} -1 & \text{in } A \\ 0 & \text{in } \Omega \setminus A \end{cases}$$

and $w(x, -h) = w(x, b) = 0$, $w_x(\pm a, y) = 0$. By the maximum principle $w \geq 0$ in Ω and

$$(7.4) \quad w_y(x, b) < 0.$$

Since $\Delta w_x = -\Delta u_x \geq 0$ by (7.3), and $w_x(0, y) = -u_x(0, y) = 0$ by (7.1), the maximum principle applied to w_x in $\Omega \cap \{x > 0\}$ gives

$$(7.5) \quad w_x < 0 \quad \text{if} \quad 0 < x < a.$$

The function

$$z(x, y) = w(x, y) - w(x, -y)$$

is defined in $\Omega_0 = \Omega^+ \cap \{y < h\}$ (since $b \geq h$) and $z(x, h) = w(x, h) \geq 0$. By the maximum principle, $z \geq 0$ in Ω_0 and $z_y(x, 0) > 0$; thus

$$(7.6) \quad w_y(x, 0) > 0.$$

Next we consider $\partial w / \partial n$ on $\partial A \cap \{y > 0\}$:

$$(7.7) \quad w_x n_x + w_y n_y = \frac{\partial w}{\partial n} = \frac{M}{b+h} n_y - \frac{\partial u}{\partial n} = \frac{M}{b+h} n_y$$

since $\partial u / \partial n = 0$. If $n_y = 0$ then necessarily $w_x n_x = 0$, and since $w_x \neq 0$ by (7.5), we get $n_x = 0$ which is a contradiction (since $\vec{n} \neq 0$). Having proved that $n_y \neq 0$ along $\partial A \cap \{x > 0\}$, it is clear that

$$n_y > 0 \quad \text{along} \quad \partial A \cap \{x > 0\}$$

Consequently, by (7.7),

$$(7.8) \quad \frac{\partial w}{\partial n} > 0 \quad \text{along} \quad \partial A \cap \{x > 0\}.$$

From (7.4), (7.6), (7.8) we find that

$$\int_{\partial(\Omega^+ \setminus A)} \frac{\partial w}{\partial n} \neq 0$$

which is a contradiction since $\Delta w = 0$ in $\Omega^+ \setminus A$.

§8. Asymptotic behavior as $\varepsilon \rightarrow 0$. In this section we assume that for every small $\varepsilon > 0$ there exists a solution $u = u_\varepsilon$ of (1.8)–(1.15) and we denote the toner region by A_ε . Set

$$(8.1) \quad \tilde{u}_\varepsilon(x, y) = \frac{1}{\varepsilon} [u_\varepsilon(\varepsilon x, \varepsilon y) - u_\varepsilon(0, 0)],$$

$$(8.2) \quad \tilde{v}(x, y) = -\frac{\sigma}{2\pi} \int_{-1}^1 \log \sqrt{(x-\xi)^2 + y^2} d\xi + \frac{My}{b+h} - \frac{\sigma}{\pi}.$$

THEOREM 8.1. *If $\text{meas } A_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ then*

$$(8.3) \quad \tilde{u}_\varepsilon(x, y) \rightarrow \tilde{v}(x, y) \quad \text{as } \varepsilon \rightarrow 0$$

uniformly in compact subsets of $\mathbb{R}^2$.

Proof. Set

$$(8.4) \quad v_\varepsilon(x, y) = - \int_{-\varepsilon}^{\varepsilon} \frac{\sigma}{2\pi} \log \sqrt{(x - \xi)^2 + y^2} d\xi.$$

$$w_\varepsilon = u_\varepsilon - v_\varepsilon - \frac{M}{b+h} (y+h).$$

Then $w_{\varepsilon,y}$ is continuous across $y = 0$,

$$\begin{aligned} w_\varepsilon(x, -h) &= -v_\varepsilon(x, -h), & |x| < a, \\ w_\varepsilon(x, b) &= -v_\varepsilon(x, b), & |x| < a, \\ w_{\varepsilon,x}(\pm a, y) &= -v_{\varepsilon,x}(\pm a, y), & -h < y < b \end{aligned}$$

and the right-hand sides clearly converge to zero in C^2 as $\varepsilon \rightarrow 0$. Since also

$$\int_{\Omega} |\Delta w_\varepsilon|^p = |\text{meas } A_\varepsilon| \rightarrow 0 \quad \text{if } \varepsilon \rightarrow 0,$$

it follows by L^p -estimates that

$$\|w_\varepsilon\|_{W^{2,p}(\Omega)} \rightarrow 0 \quad \forall p > 1;$$

in particular

$$(8.5) \quad \|w_\varepsilon\|_{C^{1,\alpha}(\Omega)} \rightarrow 0 \quad \text{if } \varepsilon \rightarrow 0 \quad \forall 0 < \alpha < 1$$

and thus

$$(8.6) \quad \sup_{\Omega} |u_{\varepsilon,x} - v_{\varepsilon,x}| + \sup_{\Omega} |u_{\varepsilon,y} - v_{\varepsilon,y} - \frac{M}{b+h}| \rightarrow 0 \quad \text{if } \varepsilon \rightarrow 0.$$

We can write $\tilde{u}_\varepsilon(x, y)$ in the form

$$\begin{aligned} \tilde{u}_\varepsilon(x, y) &= \frac{1}{\varepsilon} [u_\varepsilon(\varepsilon x, \varepsilon y) - u_\varepsilon(0, \varepsilon y)] + \frac{1}{\varepsilon} [u_\varepsilon(0, \varepsilon y) - u_\varepsilon(0, 0)] \\ &= \frac{1}{\varepsilon} \int_0^{\varepsilon x} u_{\varepsilon,x}(\xi, \varepsilon y) d\xi + \frac{1}{\varepsilon} \int_0^{\varepsilon y} u_{\varepsilon,y}(0, \eta) d\eta \\ &= \frac{1}{\varepsilon} \int_0^{\varepsilon x} v_{\varepsilon,x}(\xi, \varepsilon y) d\xi + \varepsilon \int_0^{\varepsilon y} \left[v_{\varepsilon,y}(0, \eta) + \frac{M}{b+h} \right] d\eta + \gamma_\varepsilon \end{aligned}$$

where, by (8.6),

$$(8.7) \quad \gamma_\varepsilon \rightarrow 0 \quad \text{uniformly in compact subsets.}$$

Next we can write

$$(8.8) \quad \begin{aligned} \tilde{u}_\varepsilon(x, y) - \gamma_\varepsilon &= \frac{1}{\varepsilon} [v_\varepsilon(\varepsilon x, \varepsilon y) - v_\varepsilon(0, 0)] + \frac{1}{\varepsilon} \frac{M}{b+h} \varepsilon y \\ &= - \left\{ \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \frac{\sigma}{2\pi} \left[\log \sqrt{(\varepsilon x - \xi)^2 + \varepsilon^2 y^2} - \log \sqrt{\xi^2} \right] d\xi \right\} + \frac{M}{b+h} y \end{aligned}$$

and substituting $\xi \rightarrow \varepsilon \xi$ we get

$$-\frac{\sigma}{4\pi} \int_{-1}^1 \{ \log[(x - \xi)^2 + y^2] - \log \xi^2 \} d\xi + \frac{My}{b+h}$$

which is the right-hand side of (8.2). Hence (8.8) and (8.7) yield the assertion (8.3).

COROLLARY 8.2. *Under the assumption of Theorem 8.1, for any compact subset $K \subset \mathbb{R}^2$ and for any $p > 1$*

$$(8.9) \quad \|\tilde{u}_\varepsilon - \tilde{v}\|_{W^{2,p}(K)} \rightarrow 0 \quad \text{if } \varepsilon \rightarrow 0.$$

Indeed, this follows from (8.3) and the estimate $|\Delta(\tilde{u}_\varepsilon - \tilde{v})| \leq \varepsilon$, by L^p estimates.

We shall need later on the following easily verified formulas:

$$(8.10) \quad \tilde{v}_x = \frac{\sigma}{4\pi} \log \frac{(1-x)^2 + y^2}{(1+x)^2 + y^2} \begin{cases} < 0 & \text{if } x > 0 \\ > 0 & \text{if } x < 0, \end{cases}$$

$$(8.11) \quad \begin{aligned} \tilde{v}_y &= -\frac{\sigma}{2\pi} \int_{-1}^1 \frac{y}{(x - \xi)^2 + y^2} d\xi + \frac{M}{b+h} \\ &= -\frac{\sigma}{2\pi} \left(\arctan \frac{1-x}{y} + \arctan \frac{1+x}{y} \right) + \frac{M}{b+h} \end{aligned}$$

so that, as $y \rightarrow 0+$

$$(8.12) \quad \tilde{v}_y \rightarrow \begin{cases} \frac{M}{b+h} & \text{if } |x| > 1, \\ -\frac{\sigma}{2} + \frac{M}{b+h} & \text{if } |x| < 1 \\ -\frac{\sigma}{4} + \frac{M}{b+h} & \text{if } |x| = 1. \end{cases}$$

The equation $\tilde{v}_y(0, y) = 0$ has a positive solution $y = y_0$ if and only if

$$\arctan \frac{1}{y_0} = \frac{\pi}{\sigma} \frac{M}{b+h} .$$

A necessary and sufficient condition for a solution y_0 to exist is that

$$\frac{\pi}{\sigma} \frac{M}{b+h} < \frac{\pi}{2} .$$

As in §7 we shall assume that

$$(8.13) \quad b \geq h , \quad M < \sigma h$$

and then the preceding inequality holds. It follows that

$$(8.14) \quad \tilde{v}_y(0, y_0) = 0 \quad \text{has a unique positive solution } y_0 = \cot \left(\frac{\pi}{\sigma} \frac{M}{b+h} \right) .$$

Formally,

$$(8.15) \quad \frac{1}{\varepsilon} \partial A_\varepsilon \text{ converges to a curve } \Gamma_0 : y = \varphi_0(x) \text{ on which } \partial \tilde{v} / \partial n = 0$$

Denote by $z(x, y)$ the harmonic conjugate of $\tilde{v}$, that is,

$$(8.16) \quad z_x = -\tilde{v}_y , \quad z_y = \tilde{v}_x .$$

Clearly

$$(8.17) \quad z(x, \varphi_0(x)) = \text{const.} ,$$

so that

$$(8.18) \quad \varphi'_0(x) = - \frac{z_x(x, \varphi_0(x))}{z_y(x, \varphi_0(x))} .$$

Since z is harmonic, if $\nabla z(x_1, y_1) \neq 0$ then the set $\{z(x, y) = z(x_1, y_1)\}$ in a neighborhood of (x_1, y_1) consists of a unique analytic curve. When $(x_1, y_1) = (0, y_1)$ and $y_1 \neq y_0$,

$$\nabla z(0, y_1) = (z_x(0, y_1), 0) \neq 0$$

by (8.14). Hence, near $(0, y_1)$, the set $\{z(x, y) = z(0, y_1)\}$ is an interval on the y -axis, provided $y_1 \neq y_0$. If we accept (8.15), then we are led to the conclusion that φ_0 must satisfy:

$$(8.19) \quad \varphi_0(0) = y_0 .$$

THEOREM 8.2. *If (8.13) holds then the curve $y = \varphi_0(x)$ determined by (8.17), (8.19) satisfies:*

$$(8.20) \quad \varphi_0(x) > 0 \quad \text{if} \quad |x| < \bar{x}_0, \quad \bar{x}_0 = \frac{b+h}{M} \frac{\sigma}{2} > 1,$$

$$(8.21) \quad \varphi_0(\pm \bar{x}_0) = 0;$$

$\varphi_0(x)$ is analytic for $-\bar{x}_0 \leq x \leq \bar{x}_0$, $\varphi_0(x) = \varphi_0(-x)$, and

$$(8.22) \quad \varphi_0'(x) < 0 \quad \text{if} \quad 0 < x \leq \bar{x}_0,$$

$$(8.23) \quad \varphi_0''(x) < 0 \quad \text{if} \quad 0 \leq x \leq \bar{x}_0.$$

Proof. Solving the Cauchy–Riemann equations (8.16) we find that up to an additive constant,

$$(8.24) \quad z(x, y) = \frac{\sigma}{4\pi} \left\{ y \log \frac{(1-x)^2 + y^2}{(1+x)^2 + y^2} - 2(1-x) \arctan \frac{1-x}{y} + 2(1+x) \arctan \frac{1+x}{y} \right\} - \frac{M}{b+h} x.$$

Recalling (8.17) and noting that $z(0, y) \equiv 0$, we conclude that

$$z(x, \varphi_0(x)) = 0.$$

In particular, there is a unique positive $\bar{x}_0$ such that $z(\bar{x}_0, 0) = 0$, given by (8.20) (here we use (8.13)).

Since z is harmonic and $z_y \neq 0$ if $x \neq 0$, the level line $y = \varphi_0(x)$ is analytic curve for all $x \neq 0$.

At $(0, y_0)$, $\nabla z = 0$. However $\nabla^2 z(0, y_0) \neq 0$ (in fact $z_{xy}(0, y) \neq 0$ if $y > 0$). It follows that there are two analytic level lines passing through $(0, y_0)$, the first one is the y -axis and the second one is a subarc of $y = \varphi_0(x)$. Further, $\varphi_0'(0) = 0$, and $\varphi_0(x)$ is analytic also at $x = 0$.

It remains to prove (8.22) and (8.23).

Denote by $y = h(x)$ the level line of $z_x = 0$ passing through $(0, y_0)$. In view of (8.11), (8.16), $y = h(x)$ is determined by

$$\arctan \frac{1-x}{y} + \arctan \frac{1+x}{y} = \frac{2\pi}{\sigma} \frac{M}{b+h}$$

or, after taking “tan” on both sides,

$$(8.25) \quad \frac{2y}{y^2 - (1-x^2)} = \tan \frac{2\pi}{\sigma} \frac{M}{b+h} = \frac{1}{\alpha}$$

where $\alpha = \cot \frac{2\pi}{\sigma} \frac{M}{b+h}$ is some real number. Thus the curve $y = h(x)$, for $y > 0$, is given by

$$y^2 - 2y\alpha + x^2 = 0.$$

If $\alpha \leq 0$ then

$$h(x) = \alpha + \sqrt{1 + \alpha^2 - x^2},$$

for $0 \leq x \leq 1$, and $h(1) = 0$. If $\alpha \geq 0$ then the level line $z_x = 0$ has two branches. The first one is $y = h(x)$ where $h(x)$ is defined as above for $0 \leq x \leq \sqrt{1 + \alpha^2}$, and the second branch, for $y < \alpha$, is defined by $h(x) = \alpha - \sqrt{1 + \alpha^2 - x^2}$ for $1 \leq x \leq \sqrt{1 + \alpha^2}$; but we shall only consider the first branch.

Notice that

$$z_x = -\tilde{v}_y < 0 \quad \text{above } y = h(x).$$

Hence if we can show that

$$(8.26) \quad \varphi_0(x) \geq h(x) \quad \text{for } 0 < x < \alpha^*$$

where

$$\alpha^* = \begin{cases} 1 & \text{if } \alpha \leq 0 \\ \sqrt{1 + \alpha^2} & \text{if } \alpha \geq 0 \end{cases}$$

then, by (8.18), $\varphi'_0(x) \leq 0$ if $0 < x \leq \bar{x}_0$.

To prove (8.26) we first show that, for any $\delta > 0$,

$$(8.27) \quad \varphi_0(x) + \delta > h(x) \quad \text{if } 0 \leq x \leq \alpha^* - C\delta.$$

Clearly this is true for x near 0. If the assertion is not true, then there exists a smallest value $x = \tilde{x}$ such that $0 < \tilde{x} < \alpha^* - C\delta$ and

$$\begin{aligned} \varphi_0(x) + \delta &> h(x), \quad 0 \leq x < \tilde{x}, \\ \varphi_0(\tilde{x}) + \delta &= h(x_0). \end{aligned}$$

Hence $\varphi'_0(\tilde{x}) \leq h'(\tilde{x}) < 0$. However, since $(\tilde{x}, \varphi_0(\tilde{x})) \in \{z_x < 0\}$, we must have $\varphi'_0(\tilde{x}) > 0$ (by (8.18)), which is a contradiction.

Letting $\delta \rightarrow 0$ in (8.27), (8.26) follows.

If for a particular $x \in (0, \alpha^*)$ we have $h(x) < \varphi_0(x)$ then, by (8.18), $\varphi'_0(x) < 0$. On the other hand if $h(x) = \varphi_0(x)$ then $\varphi'_0(x) \leq h'(x) < 0$. Thus $\varphi'_0(x) < 0$ whenever $0 < x < \alpha^*$. In case $\alpha^* > 1$ (i.e., $\alpha > 0$), the curve $y = \varphi_0(x)$ clearly lies in $\{z_x < 0\}$ for $\alpha^* \leq x < \bar{x}_0$ so that $\varphi'_0(x) < 0$ also for $\alpha^* \leq x < \bar{x}_0$. The same conclusion is obvious, of course, in case $\alpha \leq 0$. This then completes the proof of (8.22) (upon recalling (8.18) and (8.16), (8.10)).

To prove (8.23) we differentiate $z(x, \varphi_0(x)) = 0$ twice to get

$$(8.28) \quad z_{xx} + 2z_{xy}\varphi'_0 + z_{yy}(\varphi'_0)^2 + z_y\varphi''_0 = 0.$$

By direct calculation,

$$(8.29) \quad z_{xx} = -z_{yy} = \frac{\sigma}{2\pi} \frac{-4xy}{[y^2 + (1-x)^2][y^2 + (1+x)^2]},$$

$$(8.30) \quad z_{xy} = \frac{\sigma}{2\pi} \frac{-2y^2 - 2(1-x^2)}{[y^2 + (1-x)^2][y^2 + (1+x)^2]}.$$

Dividing (8.28) by x , letting $x \rightarrow 0$ and using (8.29), (8.30) we find that

$$\varphi''_0(0) = -\frac{2}{3} \frac{y_0}{1+y_0^2} < 0.$$

If the assertion (8.23) is not true then

$$(8.31) \quad \varphi''_0(x) < 0 \quad \text{if} \quad 0 \leq x < b_0, \quad \varphi''_0(b_0) = 0$$

for some $0 < b_0 \leq \bar{x}_0$. Substituting the second derivatives of z from (8.29), (8.30) into (8.28) we find that

$$\begin{aligned} f(x) &\equiv -xy + (-y^2 - (1-x^2))\varphi'_0 + xy(\varphi'_0)^2 \\ &= \frac{2\pi}{4\sigma} [y^2 + (1-x)^2][y^2 + (1+x)^2](-z_y\varphi''_0) \quad (y = \varphi_0(x)). \end{aligned}$$

Since $z_y < 0$ for $0 < x < \bar{x}_0$ it follows from (8.31) that

$$f(x) < 0 \quad \text{if} \quad 0 < x < b_0, \quad f(b_0) = 0;$$

consequently

$$(8.32) \quad f'(b_0) \geq 0.$$

On the other hand, from the definition of $f(x)$ and $\varphi''_0(b_0) = 0$, we find, after some manipulations that

$$f'(b_0) = (1 + (\varphi'_0)^2)(-\varphi_0 + x\varphi'_0) \quad \text{at} \quad x = b_0$$

and the right-hand side is negative, by (8.22), which is a contradiction to (8.32). Thus (8.23) is proved.

§9. Existence of infinitely many solutions. In this section we prove that for small ε there exist infinitely many smooth solutions of (1.8)–(1.15) whose free boundary consist of two components. The first component lies in $\{x > 0\}$ and has the form

$$(9.1) \quad \begin{aligned} \Gamma_1 : y = \gamma_1(x), \quad x_0 \leq x \leq x_1, \quad \gamma_1(x_0) = \gamma_1(x_1) = 0, \quad 0 < x_0 < 1 < x_1, \\ \gamma_1(x) > 0 \quad \text{if} \quad x_0 < x < x_1, \end{aligned}$$

and the second component Γ_2 is the reflection of Γ_1 with respect to the y -axis:

$$(9.2) \quad \Gamma_2 : y = \gamma_2(x), \quad \gamma_2(x) = \gamma_1(-x), \quad -x_1 \leq x \leq -x_0 .$$

Denote by A_i the domain bounded by Γ_i and the x -axis and set

$$(9.3) \quad \begin{aligned} \Gamma &= \Gamma_1 \cup \Gamma_2 , \\ A &= A_1 \cup A_2 . \end{aligned}$$

THEOREM 9.1. *For any $0 < x^* < x^{**} < 1$ there exists an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and for any $x^* \leq x_0 \leq x^{**}$ there is a smooth solution of (1.8)–(1.15) with Γ and A described by (9.1)–(9.3).*

Proof. Let Γ_{x_0} denote the level curve of z ,

$$\{(x, y); \quad z(x, y) = z(x_0, 0)\},$$

passing through $(x_0, 0)$ and given by

$$(9.4) \quad \begin{aligned} \Gamma_0 : y = \gamma_0(x), \quad x_0 \leq x \leq x_2 , \\ \gamma_0(x_0) = \gamma_0(\bar{x}_1) = 0 , \\ \gamma_0(x) > 0 \quad \text{if} \quad x_0 < x < \bar{x}_1 , \\ \gamma_0(x) < 0 \quad \text{if} \quad \bar{x}_1 < x \leq x_2 . \end{aligned}$$

By the results of §8, $\gamma_0(x)$ is analytic and (see Figure 4) $\gamma_0'(x) > 0$ if $x_0 \leq x < \hat{x}$, and

$$(9.5) \quad \gamma_0(1) > 0, \quad \bar{x}_1 > 1.$$

FIGURE 4

Introduce the set

$$(9.6) \quad M = \left\{ \gamma(x); \gamma \in C^{0,1}[x_0, x_2], \quad \gamma(x_0) = 0, \quad \gamma(x_2) \leq 0 \right. \\ \left. \|\gamma\|_{W^{1,\infty}(x_0, x_2)} \leq N_0, \quad \gamma(x) > 0 \quad \text{if} \quad x_0 \leq x \leq 1, \right. \\ \left. \gamma(1) \geq c_0 > 0 \right\}$$

where N_0, c_0 are positive constants to be determined.

For any $\gamma \in M$ introduce the region $A_{1,\gamma} = \{(x, y); 0 < y < \gamma(x), x_0 < x < \tilde{x}_1\}$ where $\tilde{x}_1$ is the smallest positive zero of $\gamma(x)$. Let $A_{2,\gamma}$ denote the reflection of $A_{1,\gamma}$ with respect to the y -axis and set $A_\gamma = A_{1,\gamma} \cup A_{2,\gamma}$, and

$$A_{\gamma/\varepsilon} = \left\{ (x, y); \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) \in A_\gamma \right\}.$$

Let u_γ be the solution of (1.8)–(1.11), (1.13)–(1.15) with A replaced by $A_{\gamma/\varepsilon}$, and set

$$(9.7) \quad u_{\gamma,\varepsilon} = \frac{1}{\varepsilon} [u_\gamma(\varepsilon x, \varepsilon y) - u_\gamma(0, 0)].$$

As in Section 8, by L^p estimates with any $p > 2$, we can establish that

$$(9.8) \quad \|u_{\gamma,\varepsilon} - \tilde{v}\|_{C^1(K)} \leq C\varepsilon^{2/p} \quad (\forall p > 2)$$

where K is any compact subset in $\mathbf{R}^2$; we shall take

$$K = \{(x, y); x_0 \leq x \leq x_2, |y| \leq C^*\}$$

where $C^* = 2 \max_{x_0 \leq x \leq x_2} |\gamma_0(x)|$.

If ε is small enough then (9.8) implies that

$$(9.9) \quad \frac{\partial}{\partial x} u_{\gamma,\varepsilon} \leq -c_1 < 0 \quad \text{in } K.$$

Define a mapping T by $\eta = T\gamma$ where

$$(9.10) \quad \eta(x_0) = 0, \quad \eta'(x) = \frac{\frac{\partial}{\partial y} u_{\gamma,\varepsilon}(x, \gamma(x))}{\frac{\partial}{\partial x} u_{\gamma,\varepsilon}(x, \gamma(x))}, \quad x_0 \leq x \leq x_2.$$

From (9.9) we deduce that

$$(9.11) \quad |\eta|_{W^{1,\infty}(x_0, x_2)} \leq N_0$$

if N_0 is initially chosen large enough; further,

$$(9.12) \quad |\eta|_{C^{1,\alpha}[x_0, x_2]} \leq C(N_0), \quad 0 < \alpha < 1;$$

here we required the fact that $\gamma(1) \geq c_0 > 0$ in order to ensure that the denominator in (9.10) does not become unbounded.

Because of (9.8), in addition to (9.11), the function η satisfies all other requirements needed to belong to M , provided ε is small enough. Thus $\eta \in M$, and T maps M into itself.

T is clearly continuous and, by (9.12), TM is a compact subset of M . Consequently, by the Schauder fixed point theorems, T has a fixed point γ ; the corresponding u_γ is a solution of (1.8)–(1.13).

§10. Construction of ε^m -approximate solution. In this section we prove that there exist approximate solutions to (1.8)–(1.13); the approximation improves faster than ε^m as ε decreases to zero, where m is any positive number. Unlike the solutions constructed in §9, the set $\{\Delta u = 1\}$ for the present approximate solution has a single component.

We shall work with a modification of the class M defined in (9.6):

$$(10.1) \quad M_\varepsilon = \left\{ \gamma(x); \gamma \in C^{0,1}[x_\varepsilon^*, x_2], \gamma(x_\varepsilon^*) = y_\varepsilon^*, \gamma(x_2) \leq 0, \right. \\ \left. \|\gamma\|_{W^{1,\infty}(x_\varepsilon^*, x_2)} \leq N_0, \gamma(x) > 0 \text{ if } x_\varepsilon^* \leq x \leq 1, \right. \\ \left. \gamma(1) \geq c_0 > 0 \right\}$$

where

$$(10.2) \quad x_\varepsilon^* = C\varepsilon^m, \quad |y_\varepsilon^* - y_0| \leq C\varepsilon;$$

here m is any positive number and C is a constant depending on m .

Denote by $\tilde{x}_1$ the smallest zero of $\gamma(x)$.

For any $\gamma \in M_\varepsilon$ we denote by $\hat{\gamma}$ the function defined by $\hat{\gamma}(x) = \gamma(-x)$ and denote by A_γ the domain in

$$\{-\tilde{x}_1 < x < \tilde{x}_1, y > 0\}$$

bounded by $y = \gamma$, $y = \hat{\gamma}$, the interval $\{(x, y_\varepsilon^*); |x| \leq x_\varepsilon^*\}$ and the x -axis.

As in §9, we denote by u_γ the solutions of (1.8)–(1.11), (1.13)–(1.15) with A replaced by $A_{\gamma/\varepsilon}$, and define $u_{\gamma,\varepsilon}$ as in (9.7).

LEMMA 10.1. For any $0 < \delta < N < \infty$,

$$(10.3) \quad \frac{\partial}{\partial x} u_{\gamma,\varepsilon} \leq -cx \quad \text{if } x \geq x_\varepsilon^*, \quad \delta \leq y \leq N$$

for some $c > 0$, provided ε is small enough.

Proof. Let w be the solution to

$$\Delta w = \begin{cases} \varepsilon & \text{in } A_\gamma \\ 0 & \text{in } R_N \setminus A_\gamma, \end{cases} \\ w = 0 \quad \text{on } \partial R_N$$

where R_N is the rectangle $\{|x| < 2N, |y| < 2N\}$, N large. By elliptic estimates

$$\left| \frac{w}{\varepsilon} \right|_{C^{1,\alpha}(R_N)} \leq C_N.$$

Since $w_x(0, y) = 0$, it follows that

$$(10.4) \quad w_x(x, y) \leq C\varepsilon x^\alpha \quad \text{in } R_N \cap \{x > 0\}.$$

Clearly

$$\Delta(u_{\gamma,\varepsilon} - \tilde{v} - w) = 0 \quad \text{in } R_N,$$

and, by (9.8) and interior Schauder estimates,

$$\|u_{\gamma,\varepsilon} - \tilde{v} - w\|_{C^2(R_N)} \leq C\varepsilon^{2/p} \quad (\forall p > 2).$$

Noting that

$$\frac{\partial}{\partial x} (u_{\gamma,\varepsilon} - \tilde{v} - w) = 0 \quad \text{at } x = 0,$$

we deduce that

$$\frac{\partial}{\partial x} (u_{\gamma,\varepsilon} - \tilde{v} - w) \leq C\varepsilon^{2/p}x \quad \text{in } R_{N/2} \cap \{x > 0\}.$$

Combining this with (10.4) and noting (by (8.10)) that

$$\tilde{v}_x \leq -c_1x < 0 \quad \text{if } \delta < y < N, \quad x > 0,$$

we get

$$\frac{\partial}{\partial x} u_{\gamma,\varepsilon} \leq C\varepsilon^{2/p}x - c_1x + C\varepsilon x^\alpha \leq -\frac{1}{2}c_1x$$

if $x \geq C\varepsilon^{1/(1-\alpha)}$. Since α can be chosen arbitrarily close to 1, the assertion (10.3) follows.

By (9.8)

$$(10.5) \quad \frac{\partial}{\partial x} u_{\gamma,\varepsilon} \leq -c < 0$$

in any compact set where $\tilde{v}_x \leq -2c$ (if ε is small enough). Combining this with Lemma 10.1 we get that, for any $\gamma \in M_\varepsilon$,

$$(10.6) \quad \frac{\partial}{\partial x} u_{\gamma,\varepsilon}(x, \gamma(x)) \leq -cx < 0 \quad \text{if } x_\varepsilon^* \leq x \leq x_2.$$

Now take a small rectangle $R : |y - y_0| < \sigma_0, 0 < x < \sigma$ (σ, σ_0 independent of ε). Then, by (9.8),

$$\begin{aligned} \frac{\partial}{\partial y} u_{\gamma,\varepsilon} &< 0 \quad \text{on } y = y_0 + \sigma_0, \quad |x| \leq \sigma, \\ \frac{\partial}{\partial y} u_{\gamma,\varepsilon} &> 0 \quad \text{on } y = y_0 - \sigma_0, \quad |x| \leq \sigma, \end{aligned}$$

provided σ/σ_0 is small enough and ε is sufficiently small. It follows (using (10.6)) that the curve $y = \eta(x)$ defined by (9.10) cannot exit $R \cap \{x > 0\}$ at $y = y_0 + \sigma_0$ (where $\eta' < 0$) or at $y = y_0 - \sigma_0$ (where $\eta' > 0$). Thus $y = \eta(x)$ must satisfy:

$$|\eta(x) - y_0| < \sigma_0 \quad \text{if } x_\varepsilon^* \leq x \leq \sigma.$$

Consequently,

$$\eta(\sigma) \in [y_0 - \sigma_0, y_0 + \sigma_0],$$

and by (9.8), for all small ε ,

$$\eta(x) > 0 \quad \text{if} \quad \sigma \leq x < 1, \quad \eta(1) \geq c_0 > 0$$

(if c_0 is small enough). We can now proceed as in §9 to prove that T maps M_ε into a compact subset, so that it has a fixed point.

We denote the fixed point by γ , and introduce a curve

$$(10.7) \quad \begin{aligned} \Gamma : \quad y &\equiv \gamma_0(x), \quad -\tilde{x}_1 \leq x \leq \tilde{x}_1 \\ \gamma_0(x) &= \begin{cases} \gamma(x) & \text{if } x_\varepsilon^* \leq x \leq \tilde{x}_1 \\ \gamma(-x) & \text{if } -\tilde{x}_1 \leq x \leq -x_\varepsilon^* \\ y_\varepsilon^* & \text{if } -x_\varepsilon^* \leq x \leq x_\varepsilon^* \end{cases} \end{aligned}$$

where $\tilde{x}_1$ is the smallest positive zero of $\gamma(x)$. Recall that

$$(10.8) \quad \begin{aligned} \gamma(x) &> 0 \quad \text{if } x_\varepsilon^* \leq x < \tilde{x}_1, \quad \gamma(\tilde{x}_1) = 0, \\ \gamma(x) &\text{ is analytic if } x_\varepsilon^* \leq x \leq \tilde{x}_1. \end{aligned}$$

The solution $u = u_\gamma$ corresponding to this fixed point satisfies:

$$(10.9) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma \cap \{|x| > x_\varepsilon^*\}.$$

We summarize:

THEOREM 10.2. *For any $m > 0$, let x_ε^* and y_ε^* satisfy (10.2) for some C depending on m . Then, if ε is small enough, there exists a solution (u, Γ) of (1.8)–(1.11), (1.13)–(1.15) and (10.9).*

The pair (u, Γ) is an “ ε^m -approximate” solution to the original problem (1.8)–(1.13).

REMARK 10.1. It is not difficult to show that as $\varepsilon \rightarrow 0$ the free boundary Γ will converge to the curve $y = \varphi_0(x)$, uniformly in x , $x \geq c > 0$, for any $c > 0$.

REMARK 10.2. If we could prove that

$$(10.10) \quad \frac{\partial}{\partial x} u_{\gamma, \varepsilon} \leq 0 \quad \text{if } x > 0,$$

then, defining $\eta = T_\tau \gamma$ ($\tau > 0$) by

$$\eta'(x) = \frac{\frac{\partial}{\partial y} u_{\gamma, \varepsilon}(x, \gamma(x))}{\frac{\partial}{\partial x} u_{\gamma, \varepsilon}(x, \gamma(x)) + \tau},$$

we can show, by the method of proof of Theorem 10.2, that T_τ has a fixed point γ_τ , and $\gamma_\tau \rightarrow \gamma$ as $\tau \rightarrow 0$; the solution corresponding to γ satisfies (1.8)–(1.13), and thus it is a solution to the toner problem. It is however not clear whether (10.10) is in effect true.

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