ISOSPECTRAL FLOWS: THEIR HAMILTONIAN STRUCTURES, MIURA MAPS AND MASTER SYMMETRIES

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IMA Preprint Series # 623

February 1990

ISOSPECTRAL FLOWS: THEIR HAMILTONIAN STRUCTURES, MIURA MAPS AND MASTER SYMMETRIES*

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Abstract. We consider a variety of spectral problems, polynomially dependent upon the spectral parameter. When the polynomial is of degree N, there are (generically) $(N+1)$ locally defined, compatible Hamiltonian structures which have a **universal form**, involving some operator J_k. The operators J_k have a **particular form** for each specific spectral problem.

Examples include spectral dependent versions of the Schrödinger operator and its super-extensions and of generalised Zakharov–Shabat problems. Associated equations include the KdV, DWW, Ito, SIT and the Heisenberg FM equations.

A simple shift in the spectral parameter induces a transformation of the variables, corresponding to a particularly simple master symmetry. This gives a simple proof of compatibility of the Hamiltonian structures.

A remarkable sequence of Miura maps can be presented for many of these equations. The modified equations are also (multi-) Hamiltonian.

1. Introduction. Perhaps the most important property of a 'soliton' equation is that of being an isospectral flow. It is this association with a linear spectral problem which enables such equations to be solved by the inverse spectral transform (IST) and related methods. However, for the purposes of this article the associated linear spectral problem will play another important role: that of the basis for a number of systematic algebraic constructions associated with the isospectral hierarchy.

In this article we discuss three particular aspects of integrable nonlinear evolution equations:

(a) Hamiltonian Property

A very simple construction is presented with which (corresponding to a given spectral problem) it is possible to simultaneously derive:

(i) the isospectral flows and corresponding time evolutions of the eigenfunctions,

(ii) an infinite hierarchy of constants of motion,

(iii) the *locally defined* Hamiltonian structures associated with the spectral problem.

(b) Miura Maps

*To be published in the proceedings of the IMA Workshop, "Applications of Solitons", eds. P.J. Olver and D.H. Sattinger.

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The emphasis here will be on Hamiltonian Miura maps, which (often) provide co-ordinates in which a complicated Hamiltonian structure takes a particularly simple form, analogous to the canonical form in classical mechanics. For the main examples presented in this paper it is possible to use a generalised factorisation approach to construct a particularly interesting sequence of Hamiltonian Miura maps.

(c) Master Symmetries

In all the examples of spectral problem considered in this paper, a simple shift of spectral parameter induces an invertible transformation of co-ordinates, which can be used to define a particularly simple master symmetry. The main application of this is to give a simple proof of compatibility of the Hamiltonian structures of (a).

The general framework of this paper has very wide applicability. The construction (a) can be used with most (generic) classes of spectral problem. Whilst the ideas of (b) and (c) are general, the specific constructions have limited applicability.

The simplest and most often studied spectral problems are *linearly* dependent upon the spectral parameter. The isospectral hierarchies of such spectral problems (in this paper) will be bi-Hamiltonian. However, in this paper we present spectral problems which are polynomial of degree N in the spectral parameter and the isospectral hierarchies of these possess $(N+1)$ compatible, locally defined, Hamiltonian structures. The main example presented here is the energy-dependent Schrödinger operator which contains a remarkably rich set of examples of mult-Hamiltonian systems: KdV, Harry Dym, Dispersive Water Waves, Shallow Water Waves and Ito's equations are the simplest and best known. It is a simple matter to give super-extensions of all these equations.

Another important extension is the polynomial generalisation of the Zakharov-Shabat spectral problem. In the 2×2 matrix case, this includes the NLS, DNLS, Heisenberg ferromagnet and SIT equations.

In all of these examples the Hamiltonian operators take *exactly* the same algebraic form (3.5b) in terms of some operators J_i. However, for each spectral problem the operators J_i take a *different* form. Similar remarks can be made regarding the Miura maps and master symmetries of these systems.

Before presenting the results we first give a brief introduction to the Hamiltonian theory of nonlinear evolution equations, referring to [1,2] for details.

2. Hamiltonian Property. In this paper we are concerned with systems of NLEEs (in $(1+1)$-dimensions) which can be written in Hamiltonian form:

$$\mathbf{u}_t = \mathbf{B}\delta\mathcal{H}\,, \tag{2.1}$$

where $\mathbf{B}$ is a (matrix differential) Hamiltonian operator. $\delta\mathcal{H}$ is the variational derivative of function $\mathcal{H}$ and generally a vector : $\delta \equiv \delta_u \equiv (\delta_0, \cdots, \delta_{N-1})^T$, $\delta_i = \dfrac{\delta}{\delta u_i}$. In the context

of analysis and physics one would then deal with constants of motion and Poisson brackets in their integral form, respectively:

$$\mathbb{H} = \int \mathcal{H} dx \ , \ \{\mathbb{K}, \mathbb{H}\} = \int \delta\mathcal{K} \mathbf{B} \delta\mathcal{H} dx \ , \tag{2.2}$$

which would involve particular boundary conditions on the functions $u_i(x)$. To avoid any such considerations it is customary to work within the framework of differential algebras.

Roughly speaking, when calculating conserved densities and Poisson brackets we work *modulo exact x-derivatives*. The justification for this is that such exact derivatives contribute only *boundary terms* in the above definitions of $\mathbb{H}$ and $\{\mathbb{K}, \mathbb{H}\}$. Thus, with appropriate boundary conditions, these terms vanish. Furthermore, the total x-derivatives constitute the kernel of the variational derivative operator. Thus, we may consider two functions of u_i and their x-derivatives to be *equivalent* if they differ by a total x-derivative. The corresponding equivalence class (and, by an abuse of notation, any convenient representative) is written $\mathcal{H}$, as it was in (2.1) and (2.2).

Conservation Laws

Let $K[u_i]$ be a function on phase space. Then:

$$\frac{dK}{dt} = \mathbf{K}' \mathbf{u}_t = (\delta K)^T \mathbf{u}_t + \frac{dF}{dx} \ , \tag{2.3a}$$

where $\mathbf{K}'$ denotes the Fréchet derivative (operator) defined by $\mathbf{K}'[\mathbf{u}]\mathbf{v} = \dfrac{d}{d\epsilon} K[\mathbf{u} + \epsilon \mathbf{v}]_{|\epsilon=0}$. The second step corresponds to an integration by parts. Adding an exact derivative J_x to K does not change δK but does change F to $\widetilde{F} = F + J_t$. Thus we may choose any convenient representative of the equivalence class $\mathcal{K}$ of $\mathbf{K}$. Using (2.1) we find:

$$\mathcal{K}_t = \frac{\delta \mathcal{K}}{\delta u_i} \ B_{ij} \frac{\delta \mathcal{H}}{\delta u_j} + \widetilde{F}_x \ . \tag{2.3b}$$

Defining the quadratic form $\{\mathcal{K}, \mathcal{H}\}$ by:

$$\{\mathcal{K}, \mathcal{H}\} = \frac{\delta \mathcal{K}}{\delta u_i} \ B_{ij} \frac{\delta \mathcal{H}}{\delta u_j} \ , \tag{2.4}$$

it follows from (2.3) that whenever $\{\mathcal{K}, \mathcal{H}\} = 0 \ (\text{mod} \ \ Im\partial)$, we have the local conservation law:

$$\mathcal{K}_t = \mathcal{F}_x \tag{2.3c}$$

where $\mathcal{F}$ is the flux corresponding to $\mathcal{K}$.

Poisson Brackets and Hamiltonian Operators.

The quadratic form (2.4) defines a Poisson bracket if and only if it is skew-symmetric and satisfies the Jacobi identity : i.e. if for any 3 functions $\mathcal{H}, \mathcal{J}, \mathcal{K}$:

(2.5a) (i) $\{\mathcal{H}, \mathcal{J}\} + \{\mathcal{J}, \mathcal{H}\} \in Im\partial$,

(2.5b) (ii) $\{\{\mathcal{H}, \mathcal{J}\}, \mathcal{K}\} + \{\{\mathcal{K}, \mathcal{H}\}, \mathcal{J}\} + \{\{\mathcal{J}, \mathcal{K}\}, \mathcal{H}\} \in Im\ \partial$.

When (2.4) satisfies conditions (2.5a,b) then $\mathbf{B}$ is called a Hamiltonian operator (or Hamiltonian structure). Property (2.5a) is guaranteed by choosing operator $\mathbf{B}$ to be skew adjoint, while the Jacobi identity (2.5b) is a much stronger (and much more complicated) constraint (see [1, 2] for details). Thus, most skew symmetric operators are *not* Hamiltonian.

REMARK. For two Hamiltonians to Poisson commute *wrt* (2.4) means that the *rhs* of (2.4) is an exact derivative. In the analytic context the *rhs* is a boundary term which can only be 'thrown away' with an appropriate choice of boundary condition.

A system of evolution equations is said to be bi-Hamiltonian if there exist two Hamiltonian operators $\mathbf{B}_0$ and $\mathbf{B}_1$ and two Hamiltonians $\mathcal{G}$ and $\mathcal{H}$ such that

(2.6) $$\mathbf{u}_t = \mathbf{B}_0 \delta\mathcal{G} = \mathbf{B}_1 \delta\mathcal{H} .$$

It is particularly interesting if the operator $\mathbf{B}_0 + \mathbf{B}_1$ is also Hamiltonian, in which case $\mathbf{B}_0$ and $\mathbf{B}_1$ are said to be compatible (in general the sum of the Poisson brackets would fail to satisfy the Jacobi identity). The importance of compatibility is that it enables us (under certain conditions) to construct an infinite hierarchy of (Poisson commuting) Hamiltonians. This important condition was first noticed by Magri [3]. It is now possible to state a useful lemma (see [1] for a proof).

LEMMA. *If* $\mathbf{B}_0$ *and* $\mathbf{B}_1$ *are compatible Hamiltonian operators, with* $\mathbf{B}_0$ *non-degenerate, and*

(2.7a) $$\mathbf{B}_1 \delta\mathcal{G} = \mathbf{B}_0 \delta\mathcal{H} , \quad \mathbf{B}_1 \delta\mathcal{H} = \mathbf{B}_0 \mathbf{K} ,$$

then there exists a function $\mathcal{K}$ *s.t.* $\mathbf{K} = \delta\mathcal{K}$.

To prove the existence of an infinite hierarchy of Hamiltonians, $\mathcal{H}_n$, related to compatible Hamiltonian operators $\mathbf{B}_0, \mathbf{B}_1$, we need to check that two conditions hold:

(i) $\exists$ an infinite sequence of vector functions $\mathbf{K}_0, \mathbf{K}_1, \ldots$ satisfying

(2.7b) $$\mathbf{B}_1 \mathbf{K}_n = \mathbf{B}_0 \mathbf{K}_{n+1} ,$$

(ii) $\exists$ two function(al)s $\mathcal{H}_0$ and $\mathcal{H}_1$ s.t.

$$\mathbf{K}_0 = \delta\mathcal{H}_0, \quad \mathbf{K}_1 = \delta\mathcal{H}_1 .$$

It then follows from the Lemma that there exist function(al)s $\mathcal{H}_n$ s.t.

(2.7c) $$\mathbf{K}_n = \delta\mathcal{H}_n \quad \forall\ n \geq 0 .$$

REMARKS.

(a) Condition (i) is not always easy to check, although it is for our systems. Indeed, it may not even be satisfied, as shown by an example of Kupershmidt [4].

(b) Given the existence of the infinite sequence $\mathcal{H}_n$, the bi-Hamiltonian property gives a very simple proof of involutivity *wrt both* Hamiltonian structures.

For this construction, it is of no advantage for a system to be *more* than bi-Hamiltonian. However, the existence of multi-Hamiltonian structures does lead to a rich supply of (multi-) Hamiltonian modifications.

Recursion Operator.

Suppose we use (2.7b) to define an evolution parameter t_n by:

$$\mathbf{u}_{t_n} = \mathbf{B}_0\mathbf{K}_n = \mathbf{G}_n \ . \tag{2.8}$$

We define an integro-differential operator $\mathbf{R}$ by formally inverting $\mathbf{B}_0$:

$$\mathbf{R} = \mathbf{B}_1\mathbf{B}_0^{-1} \ . \tag{2.9}$$

Then

$$\mathbf{R}u_{t_n} = (\mathbf{B}_1\mathbf{B}_0^{-1})\mathbf{B}_0\mathbf{K}_n = \mathbf{B}_1\mathbf{K}_n = \mathbf{B}_0\mathbf{K}_{n+1} = \mathbf{u}_{t_{n+1}}.$$

Thus $\mathbf{R}$ maps flows onto flows. $\mathbf{R}$ is called the recursion operator since it can be used to generate the infinite sequence of flows (2.8) once we have the first. It is known [1] that $\mathbf{R}^\dagger$ satisfies the Lax type equation

$$\mathbf{R}^\dagger_{t_n} = [(-\mathbf{G}'_n)^\dagger \ , \mathbf{R}^\dagger] \ , \tag{2.10a}$$

where $\mathbf{G}'_n$ is the Fréchet derivative of the *rhs* of (2.8). This is the integrability condition of the spectral problem (squared eigenfunctions):

$$\mathbf{R}^\dagger\Phi = \lambda\Phi \tag{2.10b}$$

and the linear evolution :

$$\Phi_t = (-\mathbf{G}'_n)^\dagger\Phi \ , \tag{2.10c}$$

which is just the adjoint of the linearisation of equation (2.8). Thus the bi-Hamiltonian system (2.8) has the Lax representation (2.10). The next section is concerned with the reverse problem : given a Lax representation, what are the Hamiltonian structures (and how many of them are locally defined).

3. Hamiltonian Operators from Lax Equations. This section is concerned with aspect (a) of the introduction. Since the basic construction is the same for all our spectral problems I only present the details in the context of the energy-dependent Schrödinger operator [5–7].

Consider the second order scalar spectral problem:

$$\mathbb{L}\psi \equiv (\epsilon\partial^2 + u)\psi \equiv \sum_0^{\mathbf{N}} \lambda^i(\epsilon_i\partial^2 + u_i)\psi = 0 \ , \tag{3.1a}$$

with ϵ_i being constant and u_i functions of x.

We look for time evolutions of the wave function ψ of the form:

$$\psi_t = \mathbb{P}\psi \equiv (\frac{1}{2}P\partial + Q)\psi \ , \tag{3.1b}$$

where P and Q are functions of u_i and their x-derivatives, and of the spectral parameter λ. A simple calculation leads to

$$\mathbb{L}_t - [\mathbb{P}, \mathbb{L}] = u_t + \epsilon Q_{xx} - \frac{1}{2}Pu_x + \frac{1}{2}\epsilon(P_{xx} + 4Q_x)\partial + \epsilon P_x\partial^2 \ . \tag{3.2a}$$

Evidently, we cannot expect the usual Lax equation to hold. However, the integrability conditions of (3.1a, b) imply that $(\mathbb{L}_t - [\mathbb{P}, \mathbb{L}])\psi = 0$ for eigenfunctions of (3.1a). To match the coefficient of ∂^2 we must take:

$$\mathbb{L}_t - [\mathbb{P}, \mathbb{L}] = P_x\mathbb{L} \ . \tag{3.2b}$$

This further implies that $P_{xx} + 4Q_x = 0$, so that (3.2b) takes the remarkably simple form:

$$u_t = \left(\frac{1}{4}\epsilon\partial^3 + \frac{1}{2}(u\partial + \partial u)\right) P \equiv JP \ . \tag{3.2c}$$

REMARK. On the phase space defined by just one function u, the operator J, defined by (3.2c) is Hamiltonian, being (when $\epsilon = 1$) just the second Hamiltonian structure of the KdV equation. The operator J is the basic unit out of which all our Hamiltonian operators are built.

With ϵ and u defined by (3.1a), the operator J takes the form:

$$J = \sum_{k=0}^{\mathbf{N}} \lambda^k J_k \equiv \sum_{k=0}^{\mathbf{N}} \lambda^k \left(\frac{1}{4}\epsilon_k\partial^3 + \frac{1}{2}(u_k\partial + \partial u_k)\right) \ . \tag{3.3a}$$

Equation (3.2c) then takes the form

$$\sum_{k=0}^{\mathbf{N}} \lambda^k u_{kt} = \left(\sum_{k=0}^{N} \lambda^k J_k \right) P \ . \tag{3.3b}$$

In [7] we continued the general development to include both KdV and Harry Dym type equations. Here we just consider the KdV reduction. This simplifies some of the formulae and statements:

KdV Case. $u_{\mathbf{N}} = -1$, $\epsilon_{\mathbf{N}} = 0$, so that $J_{\mathbf{N}} = -\partial$.

To construct the 'polynomial' time evolutions we first seek a solution of:

$$J\mathcal{P} = 0 \ , \ \mathcal{P} = \sum_{k=0}^{\infty} P_k \lambda^{-k} \ , \tag{3.4a}$$

written explicitly as :

$$J_0 P_{k-\mathbf{N}} + J_1 P_{k-\mathbf{N}+1} + \cdots + J_{\mathbf{N}} P_k = 0 \ , \quad \forall \ k \geq 0 \ . \tag{3.4b}$$

A polynomial expansion $P_{(m)}$ is then defined by:

$$P_{(m)} = (\lambda^m \mathcal{P})_+ = \sum_{k=0}^{m} P_{m-k} \lambda^k \quad . \tag{3.4c}$$

Upon substitution of $P_{(m)}$ into (3.3c) (with t_m parametrising the corresponding evolution) the coefficients of λ^k, $k \geq N$, are identically zero, whilst the remaining ones give the equations of motion for $u_0, \ldots, u_{\mathbf{N}-1}$:

$$\begin{pmatrix} u_0 \\ \cdot \\ \cdot \\ \cdot \\ u_{\mathbf{N}-1} \end{pmatrix} = \begin{pmatrix} 0 & & & & \cdot & J_0 \\ & & & \cdot & & \cdot \\ & & \cdot & & & \cdot \\ & \cdot & & & & \cdot \\ \cdot & & & & & \cdot \\ J_0 & \cdot & \cdot & \cdot & \cdot & J_{N-1} \end{pmatrix} \begin{pmatrix} P_{m-N+1} \\ \cdot \\ \cdot \\ \cdot \\ P_m \end{pmatrix} . \tag{3.4d}$$

It is a remarkable fact that the scalar recursion relation (3.4b) can be written as an $N \times N$ matrix equation in exactly $(N+1)$ different ways:

$$\mathbf{B}_n \mathbf{P}^{(k-1)} = \mathbf{B}_{n-1} \mathbf{P}^{(k)} \ , \qquad n = 1, \ldots, N, \tag{3.5a}$$

where $\mathbf{P}^{(k)} = (P_{k-N+1}, \ldots, P_k)^T$ and the matrix differential operators $\mathbf{B}_n$ are determined by the following requirement: $\mathbf{B}_n$ is skew adjoint and the n^{th} row of each matrix equation (3.5a) is just (3.4b), the remaining ones being identities. Explicitly, $\mathbf{B}_n$ are:

(3.5b)
$$\mathbf{B}_n = \left(\begin{array}{ccc|ccc} & & J_0 & & & \\ & \iddots & \vdots & & & \\ 0 & & \vdots & & 0 & \\ & \iddots & \vdots & & & \\ & & \vdots & & & \\ J_0 & \cdots\cdots & J_{N-1} & & & \\ \hline & & & -J_{n+1} & \cdots & -J_N \\ & & & \vdots & \iddots & \\ & 0 & & \vdots & \iddots & 0 \\ & & & -J_N & & \end{array}\right)$$

and satisfy the formal relation $\mathbf{B}_n = \mathbf{R}\mathbf{B}_{n-1}$ where :

(3.5c)
$$\mathbf{R} = \mathbf{B}_1\mathbf{B}_0^{-1} = \left(\begin{array}{ccc|c} 0 & \cdots\cdots & 0 & -J_0J_N^{-1} \\ \hline 1 & & & \\ & \ddots & & -J_1J_N^{-1} \\ & & 0 & \\ & \ddots & & \vdots \\ & & & \\ 0 & & \ddots & \\ & & 1 & -J_{N-1}J_N^{-1} \end{array}\right) .$$

In [7] we prove 3 basic facts:

(1) The operators $\mathbf{B}_n$ are each Hamiltonian and, furthermore, are mutually compatible.

(2) The recursion relation (3.4b) can be solved for all k, subject to the condition $\epsilon_{\mathbf{N}} = 0$.

(3) The vectors $\mathbf{P}^{(k)}$ given by (3.5a) are variational derivatives of a sequence of function(al)s $\mathcal{H}_k$ (the Hamiltonians).

Then it follows from (3.5a) that the equations of motion (3.4d) can be written in Hamiltonian form in $(N+1)$ distinct ways:

(3.6)
$$\mathbf{u}_{t_m} = \mathbf{B}_{\mathbf{N}}\delta\mathcal{H}_m = \cdots = \mathbf{B}_0\delta\mathcal{H}_{m+\mathbf{N}} \ .$$

We refer to [7] for the details. Here, we present one example.

Example. Dispersive water waves

We illustrate the above construction by the simplest nontrivial example, $N = 2$. The resulting hierarchy is tri-Hamiltonian. Performing the invertible change of variables:

$$q = u_0 + \frac{1}{4}u_1^2 - \frac{1}{2}u_{1x} \ , \ r = u_1 \ , \tag{3.7a}$$

changes the second order flow of (3.1a) into the standard DWW form [8] :

$$q_{t_1} = \frac{1}{2}(-q_x + 2qr)_x \ , \ r_{t_1} = \frac{1}{2}(r_x + 2q + r^2)_x \ . \tag{3.7b}$$

The 3 Hamiltonian operators then take the form:

$$\overline{B}_0 = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \ , \quad \overline{\mathbf{B}}_1 = \frac{1}{2}\begin{pmatrix} q\partial + \partial q & -\partial^2 + r\partial \\ \partial^2 + \partial r & 2\partial \end{pmatrix} \ ,$$

$$\overline{\mathbf{B}}_2 = \frac{1}{4}\begin{pmatrix} (r-\partial)(q\partial + \partial q) + (q\partial + \partial q)(r + \partial) & (r-\partial)^2\partial + 2(q\partial + \partial q) \\ \partial(r+\partial)^2 + 2(q\partial + \partial q) & 2(r\partial + \partial r) \end{pmatrix} . \tag{3.8}$$

Numerous other examples can be found in [5]. In particular, with $N = 2$ and setting $\epsilon = \lambda$, $u_0 = \frac{1}{4}r^2$, $u_1 = q$, gives a tri-Hamiltonian hierarchy which contains Ito's equation:

$$\begin{aligned} q_t &= q_{xxx} + 6qq_x + 2rr_x \\ r_t &= 2(qr)_x \ . \end{aligned} \tag{3.9}$$

REMARK. When $\epsilon = 0$ the spectral problem (3.1a) is no longer valid. However, the Hamiltonian structures and multi-Hamiltonian hierarchies do survive this reduction, leading to 'dispersionless' versions of our equations. In particular the dispersive water wave equations reduce to the shallow water-wave equations of Riemann.

Other Spectral Problems

Applying the above construction to other polynomial spectral problems results in analogous results. We have the same *locally defined*, compatible Hamiltonian operators (3.5b) so that the isospectral flows are multi-Hamiltonian of the form (3.6). The only difference is that the operators J_k take a different form. Whenever the details can be found elsewhere, I shall just present the spectral problem, together with the corresponding operators J_k.

Super-Schrödinger

We generalise (3.1a) by writing:

$$(\epsilon\partial^2 + u)\psi + \eta\varphi = 0, \quad \epsilon\partial\varphi + \eta\psi = 0 \ , \tag{3.10a}$$

with

$$\epsilon = \sum_0^{N-1} \epsilon_i \lambda^i, \quad u = \sum_0^N u_i \lambda^i, \quad \eta = \sum_0^N \eta_i \lambda^i \ ,$$

where ϵ_i are even constants, u_i and η_i are respectively even and odd function of x. In this case J_k are 2×2 matrices:

$$J_k = \begin{pmatrix} \epsilon_k \partial^3 + 2u_k \partial + 2\partial u_k & 2\eta_k \partial + \partial \eta_k \\ 2\partial \eta_k + \eta_k \partial & \epsilon_k \partial^2 + u_k \end{pmatrix} \ . \tag{3.10b}$$

This is just a copy of the second Hamiltonian structure of Kupershmidt's sKdV equation [9]. The isospectral flows of (3.10a) are just super-extensions of those of (3.1a). The simplest example is Kupershmidt's sKdV equation.

$$u_t = (u_{xx} + 3u^2 + 12\eta\eta_x)_x \ , \ \eta_t = 4\eta_{xxx} + 3u_x\eta + 6u\eta_x \ . \tag{3.10c}$$

In [10] we also present sHD, sDWW and sIto equations.

Non-standard Lax Operators In [8] Kupershmidt introduced some special integro-differential Lax operators, which he termed "non-standard". These can be written as purely differential operators, but with λ-dependent coefficients. The construction of this section can be used to give a much simpler derivation of Kupershmidt's Hamiltonian operators. The simplest example of this type of Lax operator is second order:

$$\mathbb{L} = \epsilon \partial^2 + r\partial + q \ . \tag{3.11a}$$

The choice $r = r_0 - \lambda, \quad q = q_0, \epsilon = 1$ corresponds to Kupershmidt's non-standard Lax representation of the DWW equations. More generally we may set [2]:

$$r = \sum_0^{N-1} r_i \lambda^i - \lambda^N, \qquad q = \sum_0^{N-1} q_i \lambda^i \ , \quad \epsilon = \sum_0^{N-1} \epsilon_i \lambda^i, \tag{3.11b}$$

in which case we obtain the $(N+1)$ Hamiltonian operators (3.5b) with:

$$J_k = \frac{1}{2} \begin{bmatrix} q_k \partial + \partial q_k & -\epsilon_k \partial^2 + r_k \partial \\ \epsilon_k \partial^2 + \partial r_k & 2\epsilon_k \partial \end{bmatrix} \ . \tag{3.11c}$$

REMARK. Since (3.11a) can be gauge transformed onto (3.1a) with $2N$ components, we should really have $(2N+1)$ Hamiltonian structures. The remaining structures can be obtained through the action of the recursion operator. When $N = 1$, this gauge transformation gives rise to the change of variables (3.7a).

Generalised Zakharov–Shabat Spectral Problem

Here we consider the spectral problem:

$$\epsilon\psi_x = U\psi, \qquad \epsilon = \sum_0^{N-1} \epsilon_i \lambda^i \ , \qquad U = \sum_0^N U_i \lambda^i \ , \tag{3.12a}$$

where ϵ_i are scalar constants and the potential functions U_i are elements of some matrix Lie algebra $\mathfrak{g}$. If the wave functions evolve according to:

$$\psi_{t_m} = P_{(m)}\psi, \quad P_{(m)} = \sum_0^m V_{m-i}\lambda^i \ , \tag{3.12b}$$

then U and $P_{(m)}$ satisfy the integrability conditions:

$$U_{t_m} = \epsilon P_{(m)x} - [U, P_{(m)}] \equiv JP_{(m)} \equiv \left(\sum_{k=0}^N J_k \lambda^k \right) P_{(m)}, \tag{3.12c}$$

where J_k is the $(\dim \mathfrak{g}) \times (\dim \mathfrak{g})$ matrix:

$$J_k = \epsilon_k \partial - adU_k \ . \tag{3.12d}$$

As in the Schrödinger operator case we have a certain amount of gauge freedom. The following two choices are particularly convenient and are analogous, respectively, to the KdV and Harry Dym choices discussed at the beginning of this section:

(i) $U_n = A$, a constant, diagonal matrix ,

(ii) $U_0 = 0$.

Choice (i) includes the NLS, DNLS and (sharp line limit) SIT equations, whilst (ii) includes the Heisenberg ferromagnet equations. We (mainly) consider choice (i) below.

When $\mathfrak{g} = \mathfrak{sl}(2)$, let U_i and V_i be defined by:

$$U_i = \begin{pmatrix} w_i & q_i \\ r_i & -w_i \end{pmatrix}, \qquad V_i = \begin{pmatrix} V_i^0 & V_i^+ \\ V_i^- & -V_i^0 \end{pmatrix} \ , \tag{3.13a}$$

which define vectors:

$$\mathbf{u}_i = (q_i, r_i, w_i)^T, \qquad \mathbf{v}_i = (V_i^-, V_i^+, 2V_i^0)^T \ , \tag{3.13b}$$

so that (3.12c) take the form (3.6) with:

$$J_k = \begin{pmatrix} 0 & \epsilon_k\partial - 2w_k & q_k \\ \epsilon_k\partial + 2w_k & 0 & -r_k \\ -q_k & r_k & \frac{1}{2}\epsilon_k\partial \end{pmatrix}, \quad J_N = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.13c}$$

and

$$P^{(m)} = (\mathbf{v}_{m-N+1}, \ldots, \mathbf{v}_m)^T = \delta \mathcal{H}_m \ . \tag{3.13d}$$

REMARKS.

(a) It is assumed here that the matrices U_i are 'generic'. For instance, setting $w_{N-1} = 0$ constitutes a reduction which is Hamiltonian *wrt* $\mathbf{B}_\circ, \ldots, \mathbf{B}_{N-1}$, but *not wrt* $\mathbf{B}_N$.

(b) This Hamiltonian structure is just the Lie-Poisson bracket modified by the cocycle $\epsilon\partial$.

Example. $N = 1, \quad \epsilon = 1$

This hierarchy is a bi-Hamiltonian generalisation of the usual ZS/AKNS hierarchy and is discussed in [11].

Example. $N = 2, \quad \epsilon = \lambda$

This is a tri-Hamiltonian hierarchy which includes the (sharp line limit) SIT and NLS equations as reductions, respectively bi- and mono-Hamiltonian. The SIT system corresponds to the reduction $w_1 = 0$. As remarked earlier, w_1 is a Casimir of $\mathbf{B}_0$ and $\mathbf{B}_1$, but *not* of $\mathbf{B}_2$. On the resulting 5 dimensional phase space, the two surviving Hamiltonian structures take the form [12]:

$$\mathbf{B}_0 = \begin{pmatrix} 0 & -\partial & -q_1 & 0 & 2 \\ \partial & 0 & r_1 & -2 & 0 \\ q_1 & -r_1 & -\frac{1}{2}\partial & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \mathbf{B}_1 = \begin{pmatrix} 0 & -2w_0 & q_0 & 0 & 0 \\ 2w_0 & 0 & -r_0 & 0 & 0 \\ -q_0 & r_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix} \tag{3.14}$$

REMARK. Since $\mathbf{B}_0$ is degenerate in the unreduced case, we cannot invert it to form the recursion operator (3.5c). However, the $\mathbf{B}_0$ of (3.14) is invertible so that a recursion operator does exist in this case.

Example. Case (ii), $N = 1, \ \epsilon = 1$

The best known system which fits into this case is that of the Heisenberg ferromagnet [11]. With $U_0 = 0$, the Hamiltonian structures take the following form:

$$\mathbf{B}_0 = J_0 = \partial \ , \ \mathbf{B}_1 = -J_1 = adU_1.$$

The symplectic leaves of $\mathbf{B}_1$ are the level surfaces of $\det U_1$.

REMARK. By setting $\epsilon = 0$, equations (3.12c) reduce to ODEs, which include (as reductions) the stationary flows of integrable PDEs, such as the quartic potentials of [13].

4. Miura Maps. Miura presented his famous transformation over 20 years ago [14]. He showed that if:

$$u = -v_x - v^2 \tag{4.1a}$$

and v satisfies the MKdV equation (4.1b) then u satisfies the KdV equation (4.1c). The property of most interest for this paper is that (4.1a) can be used to construct the second Hamiltonian structure of the KdV equation out of the single Hamiltonian structure of the MKdV equation (see [15, 16]). The MKdV equation can be written in Hamiltonian form

$$v_t = v_{xxx} - 6v^2 v_x = (-\partial)\frac{\delta \widetilde{\mathcal{H}}}{\delta v} \ , \ \widetilde{\mathcal{H}} = \frac{1}{2}(v_x^2 + v^4) \ . \tag{4.1b}$$

If we denote (4.1a) by $u = M[v]$, the Fréchet derivative M' of M is given by $(-\partial - 2v)$. Given any functional $\mathcal{H}[u]$ we define $\widetilde{\mathcal{H}}[v]$ by $\widetilde{\mathcal{H}}[v] = \mathcal{H} \circ M[v]$ (mod $Im\partial$). It is then an easy matter to show that, as a consequence of (4.1b).

$$u_t = M'(-\partial)(M')^\dagger \delta_u \mathcal{H} = (\partial^3 + 4u\partial + 2u_x)\delta_u \mathcal{H} = u_{xxx} + 6uu_x \tag{4.1c}$$

when $\mathcal{H} = \frac{1}{2}u^2$. For an arbitrary differential mapping $u = M[v]$ this process would take us out of the differential algebra setting, since the differential operator $M'(-\partial)(M')^\dagger$, which has coefficients given in terms of v and its derivatives, would not normally be *locally defined* in terms of just u and its derivatives.

REMARKS.

(a) This remarkable property enables us to deduce that (4.1a) is a map between hierarchies rather than just between the KdV and MKdV equations.

(b) The Hamiltonian nature of the third order differential operator (4.1c) follows from that of $(-\partial)$ through the formula $M'(-\partial)(M')^\dagger$.

In a more general algebraic setting, let $\mathbf{u} = (u_0, \ldots, u_{N-1})^T$ and $\mathbf{v} = (v_0, \ldots, v_{N-1})^T$ be the (respectively) unmodified and modified variables. Then

DEFINITION. The mapping $\mathbf{u} = M[\mathbf{v}]$, is a Miura map for Hamiltonian operator $\widetilde{\mathbf{B}}$ (acting in the v space) if:

(i) M is not invertible

(ii) $\mathbf{B} = \mathbf{M}'\widetilde{\mathbf{B}}(\mathbf{M}')^\dagger\Big|_{u=M[v]}$ is locally defined in terms of u and its derivatives.

This definition is adopted from [16].

REMARK. The Hamiltonian nature of the operator $\mathbf{B}$ follows from that of $\widetilde{\mathbf{B}}$ provided Miura map $\mathbf{u} = M[\mathbf{v}]$ is nondegenerate (injective) (see [16] for a more detailed discussion of this).

Factorisation of Differential Operators

The relationship of Miura maps to the factorisation of differential operators is discussed in [15–18]. The map (4.1a) can be obtained from the Schrödinger operator by the following identification:

$$\mathbb{L} = \partial^2 + u = (\partial + v)(\partial - v) \ . \tag{4.3a}$$

The spectral problem $\mathbb{L}\psi_1 = \lambda\psi_1$ for the KdV equation can be used to define that for the MKdV equation :

$$(\partial - v)\psi_1 = \lambda\psi_2 \ , \qquad (\partial + v)\psi_2 = \psi_1. \tag{4.3b}$$

This notion is easily extended to higher order Lax operators [15, 16, 18].

We now generalise the factorisation approach described above to the case of the energy dependent Schrödinger operator (3.1). It is not enough to just choose v to be a polynomial in λ in order to obtain u as a polynomial. We replace the factorisation (4.3a) by a quadratic form. We present N modifications corresponding to a sequence of N such quadratic forms.

We denote the modified variables by $\mathbf{v} = (v_0, \ldots, v_N)^T$.

Define.

$$l_k = \alpha_k\partial + v_k \ , \quad \alpha_k \text{ constants}, k = 0, \ldots, N, \ l = (l_0, \ldots, l_N). \tag{4.4}$$

Let $\wedge$ be any constant, λ-dependent, $(N+1) \times (N+1)$ matrix, and use this to define a λ-dependent second order differential operator by the quadratic from $l \wedge (-l^\dagger)$, the λ-dependence being derived purely from that of $\wedge$. Equating this to our operator $\mathbb{L}$ of (3.1) gives rise to a map between functions v_i and u_i. Different choices of $\wedge$ give rise to different maps. Once again we restrict ourselves to the KdV case, referring to [7] for the general discussion. In this case $\alpha_N = 0$ and $v_N = -1$.

The following quantities occur frequently below:

Define.

$$\mathcal{V}_{kn} = -\alpha_k v_{nx} - \alpha_n v_{kx} - 2v_k v_n \ , \tag{4.5}$$

and

(4.6) $$\Lambda_r = \left(\begin{array}{c|c} \begin{matrix} 1 & \cdots & \lambda^{r-1} \\ \vdots & \iddots & \\ \lambda^{r-1} & & \end{matrix} & \begin{matrix} & & \\ & 0 & \\ & & \end{matrix} \\ \hline \begin{matrix} & & \\ & 0 & \\ & & \end{matrix} & \begin{matrix} \lambda^r & & \\ \vdots & \iddots & \\ \lambda^N & \cdots & \lambda^{2N-r} \end{matrix} \end{array}\right), r = 0, \ldots, N.$$

Making use of the formula $l_k l_n^\dagger + l_n l_k^\dagger = -2\alpha_k \alpha_n \partial^2 - \mathcal{V}_{kn}$ one can easily see that the identification:

(4.7) $$\mathsf{L} = l \wedge_r (-l^\dagger)$$

gives rise to the equations:

(4.8a) $$\epsilon_k = \sum_{i=0}^{k} \alpha_{k-i}\alpha_i \quad , \quad k = 0, \ldots, r-1$$

(4.8b) $$\epsilon_k = \sum_{i=1}^{N-k-1} \alpha_{k+i}\, \alpha_{N-i} \quad , \quad k = r, \ldots, N-1$$

(4.9a) $$u_k = \frac{1}{2} \sum_{i=0}^{k} \mathcal{V}_{i,k-i} \quad , \quad k = 0, \ldots, r-1$$

(4.9b) $$u_k = 2v_k + \frac{1}{2} \sum_{i=1}^{N-k-1} \mathcal{V}_{k+i,N-i} \quad , \quad k = r, \ldots, N-1 \, .$$

REMARK. The formulae (4.8) are not a priori consistent. For instance, for $r > 1$, there is no choice of α_k which would give $\epsilon_0 = 0, \quad \epsilon_1 = 1$. Thus, Ito's equation is ruled out of consideration here. Such inconsistencies are, however, exceptional.

When these formulae are consistent, equations (4.9) define a differential mapping from v_i to u_i, sometimes invertible, sometimes not. In fact, (4.9b) always defines an invertible map between $v_r, \ldots, v_{N-1}$ and $u_r, \ldots, u_{N-1}$. The invertibility of M thus rests upon the map (4.9a) between $v_0, \ldots, v_{r-1}$ and $u_0, \ldots, u_{r-1}$, which is invertible if and only if $\epsilon_0 = 0$.

Subject only to the consistency of (4.8a,b) we have the following important proposition.

PROPOSITION 1. Under the change of variables $\mathbf{u} = \mathbf{M}_r[\mathbf{v}]$ defined by (4.9a, b), the Hamiltonian operator $\mathbf{B}_r$, given by (3.5b), is the image:

$$\mathbf{B}_r = \mathbf{M}_r' \widetilde{\mathbf{B}}_r (\mathbf{M}_r')^\dagger \Big|_{u=\mathbf{M}_r[\mathbf{v}^{(r)}]} \tag{4.10a}$$

of the constant, first order Hamiltonian operator $\widetilde{\mathbf{B}}_r$:

$$\widetilde{\mathbf{B}}_r = \frac{1}{4} \left(\begin{array}{ccc|ccc} & & -\partial & & & \\ & 0 & & & & \\ & & & & 0 & \\ & & 0 & & & \\ -\partial & & & & & \\ \hline & & & & & \partial \\ & & & & 0 & \\ & 0 & & & & \\ & & & & & 0 \\ & & & & \partial & \end{array} \right) , \tag{4.10b}$$

where the diagonal blocks are respectively $r \times r$ and $(N-r) \times (N-r)$.

Here we have used $\mathbf{v}^{(r)}$ to denote the modified variables corresponding to the map $\mathbf{u} = M_r[\mathbf{v}^{(r)}]$.

Proof. The Fréchet derivative of the mapping M_r is given by

$$\mathbf{M}_r' = \left(\begin{array}{ccc|ccc} m_0 & & 0 & & & \\ \cdot & \cdot & & & & \\ \cdot & & \cdot & & 0 & \\ \cdot & & & \cdot & & \\ m_{r-1} & \cdot & \cdot & \cdot & m_0 & \\ \hline & & & & & m_{r+1} \\ & & & & & \cdot \\ & & & & & \cdot \\ & & & & & \cdot \\ & 0 & & & & \cdot \\ & & & & 0 & \cdot \\ & & & & & m_N \end{array} \right) , \tag{4.11a}$$

where

$$m_k = -\alpha_k \partial - 2v_k \ , \qquad k = 0, \ldots, N \ , \tag{4.11b}$$

(giving $m_{\mathbf{N}} = 2$ when $\alpha_{\mathbf{N}} = 0,\ v_{\mathbf{N}} = -1$). To obtain (4.10a) we use:

$$m_k \partial m_n^\dagger + m_n \partial m_k^\dagger = -2(\alpha_k \alpha_n \partial^3 + \mathcal{V}_{kn}\partial + \partial \mathcal{V}_{kn}) \ . \tag{4.11c}$$

The formulae (4.9a, b) then give the result.

REMARK. Using (4.11c) one can easily check that the factorisation (4.7) has its counterpart on the level of the third order operator J : $J = m(-\frac{1}{4}\partial) \wedge m^\dagger$, where $m = (m_0, \ldots, m_N)$. The factorisation of J is, in fact, the only one that survives the super extension given in [10] and below.

We now concentrate on the case $\epsilon_0 \neq 0$, so that the map (4.9a, b) is a genuine Miura map (for $r > 0$). For clarity, we choose the most interesting case [7] of $\epsilon_0 = 1,\ \epsilon_i = 0$ for $i \geq 1$.

Miura Maps

Let $\alpha_0 = 1, \quad \alpha_i = 0$ for $i \geq 1$, so that $\epsilon_0 = 1, \quad \epsilon_i = 0$ for $i \geq 1$. The map $\mathbf{u} = \mathbf{M}_0[\mathbf{v}]$, corresponding to Λ_0, is invertible, whilst those corresponding to all other $\Lambda_r (r > 0)$ are genuine Miura maps. In fact (4.9b) defines an invertible map whilst (4.9a) is the genuine Miura part. Thus the upper block of Λ_r is the important part when discussing genuine Miura maps. We therefore consider the map $M_{\mathbf{N}}$ corresponding to $\Lambda_{\mathbf{N}}$. In this case the Miura map $\mathbf{u} = \mathbf{M}_{\mathbf{N}}[\mathbf{v}]$ is given purely by (4.9a). The Fréchet derivative (4.11a) is then:

$$M'_{\mathbf{N}} = \begin{pmatrix} m_0 & & & & 0 \\ \cdot & \cdot & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ m_{\mathbf{N}-1} & \cdot & \cdot & \cdot & m_0 \end{pmatrix} , \tag{4.12a}$$

with $m_0 = -\partial - 2v_0$, $\quad m_i = -2v_i$, $\quad i \geq 1$, and the constant coefficient operator:

$$\widetilde{\mathbf{B}}_{\mathbf{N}} = \frac{1}{4} \begin{pmatrix} & & & & -\partial \\ & & & & \cdot \\ 0 & & & \cdot & \\ & & \cdot & & \\ & \cdot & & & 0 \\ \cdot & & & & \\ -\partial & & & & \end{pmatrix} \tag{4.12b}$$

is mapped onto $\mathbf{B}_{\mathbf{N}}$ of (3.5b). It is easy to see that the pre-image of $\mathbf{B}_r$, for $r < N$, is non-local.

The Miura map $\mathbf{u} = M_{\mathbf{N}}[\mathbf{v}^{(N)}]$ can be decomposed into N primitive ones. *Define* a sequence of maps $\mathbf{u}^{(k)} = M_N^{k+1}[\mathbf{u}^{(k+1)}]$ by:

$$u_k^{(k)} = \frac{1}{2}\sum_{i=0}^{k} \mathcal{U}_{i,k-i}^{(k+1)} , \qquad u_i^{(k)} = u_i^{(k+1)} \qquad \text{for} \quad i \neq k , \tag{4.13a}$$

where $\mathcal{U}_{ij}^{(k)}$ is given by (4.5) but with v_i replaced by $u_i^{(k)}$. We can write $\mathbf{u} \equiv \mathbf{u}^{(0)} = M_{\mathbf{N}}[\mathbf{u}^{(\mathbf{N})}] \equiv M_{\mathbf{N}}[\mathbf{v}^{(\mathbf{N})}]$ as the composition of these maps :

$$\mathbf{u}^{(0)} = M_{\mathbf{N}}[\mathbf{u}^{(\mathbf{N})}] = M_N^1 \circ M_N^2 \circ \cdots \circ M_N^N[\mathbf{u}^{(N)}] .$$

The Fréchet derivative $\mathbf{M}'_{\mathbf{N}}$ is thus the product of N Fréchet derivatives: $\mathbf{M}'_{\mathbf{N}} = (\mathbf{M}_N^1)' \times \cdots \times (\mathbf{M}_{\mathbf{N}}^{\mathbf{N}})'$, where :

$$(\mathbf{M}_{\mathbf{N}}^k)' = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \ddots & & & & & & \\ 0 & & & \ddots & & 0 & & & \\ & & & & 1 & & & & \\ m_{k-1} & \cdots & \cdots & \cdots & & m_0 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & 0 & & & & & & \ddots \\ & & & & & & & & & 1 \end{pmatrix} , \tag{4.13b}$$

with $m_{k-1}, \ldots, m_0$ on the k^{th} row. It is a very simple calculation to see that the product of these matrices is just (4.12a).

REMARK. Each of these maps is non-invertible and injective.

Starting with $\mathbf{B}_{\mathbf{N}}^{\mathbf{N}} = \widetilde{\mathbf{B}}_{\mathbf{N}}$ of (4.12a) define $\mathbf{B}_{\mathbf{N}}^k$ inductively by :

$$\mathbf{B}_{\mathbf{N}}^{k-1} = (M_{\mathbf{N}}^k)' \mathbf{B}_{\mathbf{N}}^k ((M_{\mathbf{N}}^k)')^\dagger \Big|_{\mathbf{u}^{(k-1)} = M_{\mathbf{N}}^k[\mathbf{u}^{(k)}]} , \qquad k = N, \ldots, 1 . \tag{4.13c}$$

$\mathbf{B}_{\mathbf{N}}^0$ is just our original $\mathbf{B}_{\mathbf{N}}$ of (3.5b).

Direct calculation shows that, as indicated by the notation, each $\mathbf{B}_N^k$ is *locally* defined in terms of the variables $\mathbf{u}^{(k)}$. Thus, each of the maps $M_{\mathbf{N}}^k$ is a genuine, Hamiltonian Miura map.

Let $\mathbf{M}^{(r)} = \mathbf{M}_{\mathbf{N}}^1 \circ \mathbf{M}_{\mathbf{N}}^2 \circ \cdots \circ \mathbf{M}_{\mathbf{N}}^r$. Explicitly, this has the form

$$u_k^{(0)} = \frac{1}{2}\sum_{i=0}^{k} \mathcal{U}_{i,k-i}^{(r)}, \qquad , \qquad k = 0, \ldots, r-1 , \tag{4.14a}$$

$$u_k^{(0)} = u_k^{(r)} \qquad , \qquad k = r, \ldots, N-1 . \tag{4.14b}$$

$M^{(r)}$ and M_r differ by an invertible map [7]. Thus we have:

PROPOSITION 2. There exists local Hamiltonian operators $\mathbf{B}_k^r$ such that $(M^{(r)})'\mathbf{B}_k^r$ $((M^{(r)})')^\dagger = \mathbf{B}_k^0 \equiv \mathbf{B}_k$ for $k = r, \ldots, N$. These constitute $(N - r + 1)$ compatible Hamiltonian structures for the r^{th} modification. The sequence of modified Hamiltonians is defined by $\mathcal{H}_n^r = \mathcal{H}_n \circ M^{(r)}$ and the r^{th} modified hierarchy is written as:

$$\mathbf{u}_{t_n}^{(r)} = \mathbf{B}_{\mathbf{N}-k}^r \delta \mathcal{H}_{n+k}^r \,, \qquad k = 0, \ldots, N - r, \qquad n = 0, 1 \ldots \tag{4.15}$$

REMARK. The operator $\mathbf{B}_r^r$ takes the form:

$$B_r^r = \left(\begin{array}{ccc|ccc} & & & & -\frac{1}{4}\partial & \\ & 0 & & & & \\ & \cdot & & & 0 & \\ & \cdot & & & & \\ -\frac{1}{4}\partial & & 0 & & & \\ \hline & & & -J_{r+1} & \cdots & -J_N \\ & & & \cdot & & \cdot \\ & & & \cdot & \cdot & \\ & 0 & & \cdot & \cdot & 0 \\ & & & -J_N & & \end{array}\right) \tag{4.16}$$

and is related to (4.10b) through an invertible transformation.

We can represent these modifications and their Hamiltonian structures schematically as follows:

$$\mathbf{u} \equiv \mathbf{u}^{(0)} \xleftarrow{M_N^1} \mathbf{u}^{(1)} \xleftarrow{M_N^2} \cdots \xleftarrow{M_N^r} \mathbf{u}^{(r)} \xleftarrow{M_N^{r+1}} \cdots \xleftarrow{M_N^N} \mathbf{u}^{(N)} \equiv \mathbf{v}^{(N)}$$

$$\mathbf{B}_N \equiv \mathbf{B}_N^0 \longleftarrow \mathbf{B}_N^1 \cdots\cdots\cdots \longleftarrow \mathbf{B}_N^r \longleftarrow \cdots \longleftarrow B_N^N$$

$$\vdots$$

$$\mathbf{B}_r \equiv \mathbf{B}_r^0 \longleftarrow \mathbf{B}_r^1 \cdots\cdots\cdots \longleftarrow \mathbf{B}_r^r$$

$$\vdots$$

$$\mathbf{B}_0 \equiv \mathbf{B}_0^0$$

FIGURE 1

Modified Spectral Problem

Generalising the derivation of (4.3b) we can use the factorisation (4.7) to obtain the spectral problem corresponding to each of our modifications (4.14). The modification leads to :

$$(\partial + u_0^{(1)})(\partial - u_0^{(1)})\psi_1 + (\lambda u_1^{(1)} + \cdots + \lambda^{N-1} u_{N-1}^{(1)})\psi_1 = \lambda^N \psi_1 \ . \tag{4.17a}$$

Defining ψ_2 by :

$$(\partial - u_0^{(1)})\psi_1 = \lambda \psi_2 \ , \tag{4.17b}$$

we find :

$$(\partial + u_0^{(1)})\psi_2 = (-u_1^{(1)} - \cdots - \lambda^{N-2} u_{N-1}^{(1)} + \lambda^{N-1})\psi_1 \ . \tag{4.17c}$$

Equations (4.17b, c) constitute a 2×2 matrix spectral problem for the first modification. The spectral problems for the remaining modifications are obtained from this one in succession by a series of substitutions and gauge transformations. This is illustrated by the example given below.

Example. Dispersive water waves

We illustrate the above construction by our previous example of the DWW equations. The first Miura map, written in the q, r coordinates is:

$$q = -w_{0x} - \frac{1}{2} w_{1x} - w_0^2 + \frac{1}{4} w_1^2 \quad , \quad r = w_1 \ , \tag{4.18}$$

and is easily seen to be equivalent to the Kupershmidt's first modification of DWW hierarchy [8]. The second Miura map :

$$w_0 = v_0 \ , \ w_1 = -v_{1x} - 2v_0 v_1 \ , \tag{4.19}$$

however, is not equivalent to either of Kupershmidt's second modifications. The first nontrivial flow is:

$$\begin{aligned} v_{0t_1} &= (\frac{1}{4} v_{1xx} + \frac{1}{2} v_1 v_{0x} - v_0^2 v_1)_x \ , \\ v_{1t_1} &= (v_0 - v_0 v_1^2 - \frac{1}{2} v_1 v_{1x})_x \ . \end{aligned} \tag{4.20}$$

The spectral problem for the first modification is given by (4.17), which, in this case, takes the form :

$$\begin{pmatrix} \psi_1 \\ \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} w_0 & \lambda \\ \\ \lambda - w_1 & -w_0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \\ \psi_2 \end{pmatrix} \ . \tag{4.21a}$$

Writing (4.21a) in the variables (v_0, v_1) (using (4.19)) and gauge transforming with $T = \begin{pmatrix} 1 & 0 \\ -v_1 & 1 \end{pmatrix}$ we obtain the spectral problem for the second modification :

$$\begin{pmatrix} \phi_1 \\ \\ \phi_2 \end{pmatrix}_x = \begin{pmatrix} v_0 + \lambda v_1 & \lambda \\ \\ \lambda(1 - v_1^2) & -v_0 - \lambda v_1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \\ \phi_2 \end{pmatrix} . \tag{4.21b}$$

Super-Extension

An analogous sequence of Miura maps exists for the super-extension (3.10a). To obtain these we factorise our basic operator $J = \sum_0^N J_k \lambda^k$ as

$$J = (m_0, \ldots, m_N)(-D) \wedge \begin{pmatrix} m_0^{s\dagger} \\ \vdots \\ m_N^{s\dagger} \end{pmatrix} \tag{4.22a}$$

where $\wedge$ is a symmetric (λ-dependent) $(N+1) \times (N+1)$ matrix, $D = \begin{pmatrix} \partial & 0 \\ \\ 0 & 1 \end{pmatrix}$ and:

$$m_k = \begin{pmatrix} -\alpha_k \partial - 2v_k & -\theta_k \partial + \theta_{kx} \\ \\ -\theta_k & -\alpha_k \partial - v_k \end{pmatrix}, \quad m_k^{s\dagger} = \begin{pmatrix} \alpha_k \partial - v_k & -\theta_k \\ \\ -\partial\theta_k - \theta_{kx} & \alpha_k \partial - v_k \end{pmatrix} \tag{4.22b}$$

are copies of the Fréchet derivative of the elementary Miura map:

$$u = -\alpha v_x - v^2 - \theta\theta_x, \qquad \eta = -\alpha\theta_x - v\theta \tag{4.22c}$$

given by Kupershmidth [9] for his sKdV equation.

The remaining formulae are the same as in the even case, but with m_k given by (4.22b) and many of the ∂'s replaced by D's. The diagram of Figure 1 is not changed. The details can be found in [10].

REMARK. In order to factorise the linear 'operator' of (3.10a) we must introduce odd space-time variables and thus enter the realm of sypersymmetry.

Other Spectral Problems

The above factorisation approach can be applied to other differential operators such as (3.11a) [2] and higher order Lax operators [15, 18, 19]. However, other methods [20] have to be used for Zakharov-Shabat like spectral problems and it is still an open question whether or not a sequence of *Hamiltonian* Miura maps can be obtained for (3.12a).

5. Master Symmetries. The isospectral flows of any of our spectral problems mutually commute and thus give rise to an Abelian algebra of (Hamiltonian) vector fields:

$$\mathbf{u}_{t_m} = \mathbf{X}_m \ . \tag{5.1}$$

This algebra can be embedded in a larger graded Lie algebra satisfying relations of the type:

$$[\mathbf{X}_m, \mathbf{X}_n] = 0 \ , \tag{5.2a}$$

$$[\mathbf{V}_m, \mathbf{X}_n] = (n+1)\mathbf{X}_{n+m} \ , \tag{5.2b}$$

$$[\mathbf{V}_m, \mathbf{V}_n] = (n-m)\mathbf{V}_{n+m} \ , \tag{5.2c}$$

the precise coefficients depending upon labelling and normalisations. Analogous relations hold when differentiating the corresponding Hamiltonians in the direction of the vector fields $\mathbf{V}_m$. Since $\mathbf{X}_n$ and $\mathbf{X}_m$ commute, the entire sequence of vector fields $\mathbf{X}_n$ are considered as symmetries for the flow (5.1). The new elements $\mathbf{V}_m$ are often called master symmetries since relation (5.2b) shows that they enable us to generate a sequence of symmetries from a given one. This is achieved through commutation relations in contrast to the action of the recursion operator. On the other hand, the recursion operator can be used to define both sequences of $\mathbf{X}_n$ and $\mathbf{V}_n$ from their 'starting values' $\mathbf{X}_0$ and $\mathbf{V}_0$:

$$\mathbf{X}_n = \mathbf{R}^n \mathbf{X}_0 \ , \quad \mathbf{V}_n = \mathbf{R}^n \mathbf{V}_0 \ . \tag{5.3}$$

While the sequence $\mathbf{X}_n$ is locally defined, only the first few of the $\mathbf{V}_n$ are local. These are both uniquely defined if we set all the constants of integration to zero. In this article I shall not further discuss the general properties of master symmetries, turning now to specific examples.

The spectral problems of section 3 depend polynomially upon the spectral parameter λ. A simple shift $\lambda \to \lambda + s$ induces a simple, invertible transformation of the potential functions which is generated by a particularly simple master symmetry vector field ($\mathbf{V}_{-1}$ in the above notation). For simplicity we consider only the energy dependent Schrödinger operator (3.1a), but the construction can immediately be applied to the other cases.

Let $\mathbb{L}(\lambda)$ denote the operator of (3.1a):

$$\mathbb{L}(\lambda) = \sum_0^N \lambda^i (\epsilon_i \partial^2 + u_i) \ . \tag{5.4a}$$

Corresponding to the shift $\lambda \to \lambda + s$, we have:

$$\mathbb{L}(\lambda + s) = \mathbb{L}(\lambda) + s\frac{\partial \mathbb{L}}{\partial \lambda} + \cdots = \overline{\mathbb{L}} \ , \tag{5.4b}$$

which is an operator of the same kind, depending upon new constants $\overline{\epsilon}_i$ and new potential functions $\overline{u}_i$. Let us denote by $\mathbf{B}_i(s)$ $(\equiv \overline{B}_i)$ the operator $\mathbf{B}_i$ when its coefficients depend upon the barred variables (with $\mathbf{B}_i(0) = \mathbf{B}_i$) (and similarly for $J_i(s)$). Let $\mathcal{J}(s)$ denote the Jacobian of the *inverse* of (5.4b). If $\mathbf{V}$ denotes the infinitesimal generator of the transformation (5.4b), then the Lie derivative $\mathcal{L}_{\mathrm{v}}\mathbf{B}_i$ is defined by:

$$\mathcal{L}_{\mathrm{v}}\mathbf{B}_i = \frac{d}{ds}\,\mathcal{J}(s)\mathbf{B}_i(s)\mathcal{J}^T(s)\Big|_{s=0} \tag{5.5}$$

and is proportional to $\mathbf{B}_{i-1}$ as will be seen in the example below. Furthermore, the quantity $\mathcal{J}(s)\ \mathbf{B}_i(s)\ \mathcal{J}^T(s)$ is an s-dependent Hamiltonian operator which is a linear combination of $\mathbf{B}_i(0),\ \mathbf{B}_{i-1}(0), \ldots, \mathbf{B}_0(0)$.

This is a direct greneralisation of the well known example of the KdV hierarchy:

Example. KdV

$$\mathbb{L} = \partial^2 + u - \lambda, \quad \lambda \to \lambda + s \Rightarrow \overline{u} = u - s\ ,$$

$$\mathbf{B}_1(s) = \partial^3 + 4\overline{u}\partial + 2\overline{u}_x = \partial^3 + 4u\partial + 2u_x - 4s\partial = \mathbf{B}_1 - 4s\mathbf{B}_0\ . \tag{5.6}$$

This example is *too* simple since $\mathcal{J}(s) \equiv 1$, which is thus rendered superfluous.

Example. Two component case

The two component example is nontrivial. With:

$$\mathbb{L}(\lambda) = (\epsilon_0 + \epsilon_1\lambda)\partial^2 + u_0 + u_1\lambda - \lambda^2\ , \tag{5.7}$$

the shift $\lambda \to \lambda + s$ induces the transformation:

$$\overline{\epsilon}_0 = \epsilon_0 + \epsilon_1 s, \quad \overline{\epsilon}_1 = \epsilon_1, \tag{5.8a}$$

$$\overline{u}_0 = u_0 + u_1 s - s^2, \quad \overline{u}_1 = u_1 - 2s\ , \tag{5.8b}$$

with infinitesimal generator:

$$\mathbf{V} = \epsilon_1\frac{\partial}{\partial\epsilon_0} + u_1\frac{\partial}{\partial u_0} - 2\frac{\partial}{\partial u_1} + prl\quad , \tag{5.8c}$$

where prl denotes the usual prolongation of the vector field.

It follows that:

$$J_0(s) = J_0 + sJ_1 + s^2J_2, \quad J_1(s) = J_1 + 2sJ_2, \quad J_2(s) = J_2\ , \tag{5.9a}$$

so that, with $\mathcal{J}(s) = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$, we have:

$$\mathcal{J}(s)\mathbf{B}_2(s)\mathcal{J}^T(s) = \mathbf{B}_2 - 2s\mathbf{B}_1 + s^2\mathbf{B}_0\ ,$$

$$\mathcal{J}(s)\mathbf{B}_1(s)\mathcal{J}^T(s) = \mathbf{B}_1 - s\mathbf{B}_0, \quad \mathcal{J}(s)\mathbf{B}_0(s)\mathcal{J}^T(s) = \mathbf{B}_0\ , \tag{5.9b}$$

which implies:

$$\mathcal{L}_{\mathrm{v}}\mathbf{B}_2 = -2\mathbf{B}_1, \quad \mathcal{L}_{\mathrm{v}}\mathbf{B}_1 = -\mathbf{B}_0, \quad \mathcal{L}_{\mathrm{v}}\mathbf{B}_0 = 0\ . \tag{5.9c}$$

REMARK. These formulae show, very simply, that the Hamiltonian operators $\mathbf{B}_i$ are compatible.

The first 4 Hamiltonians are given by [5]:

$$\mathcal{H}_0 = 2u_1, \quad \mathcal{H}_1 = 2u_0 + \frac{1}{2}u_1^2, \quad \mathcal{H}_2 = u_0u_1 + \frac{1}{4}u_1^3 - \frac{1}{8}\epsilon_1 u_{1x}^2 \ ,$$

(5.10a)

$$\mathcal{H}_3 = \frac{1}{2}u_0^2 + \frac{3}{4}u_0u_1^2 + \frac{5}{32}u_1^4 + \frac{1}{4}\epsilon_1 u_0 u_{1xx} + \frac{1}{32}\epsilon_1^2 u_{1xx}^2 - \frac{5}{16}\epsilon_1 u_1 u_{1x}^2 - \frac{1}{8}\epsilon_0 u_{1x}^2 \ ,$$

and satisfy:

(5.10b) $$\mathbf{V}(\mathcal{H}_0) = -2, \ \mathbf{V}(\mathcal{H}_1) = 0, \ \mathbf{V}(\mathcal{H}_2) = -\mathcal{H}_1, \ \mathbf{V}(\mathcal{H}_3) = -2\mathcal{H}_2 + (\frac{1}{4}\epsilon_1 u_1 u_{1x})_x \ .$$

We generally have:

(5.10c) $$\mathbf{V}(\mathcal{H}_n) = (1-n)\mathcal{H}_{n-1} \ (\text{mod } Ker\ B_0\delta), \ n \geq 1 \ .$$

REMARK. The transformation (5.8) acts on an enlarged space, including the constants ϵ_i as well as the functions u_i. While some reductions (such as $\epsilon \equiv \epsilon_0$) are invariant under this transformation, others (such as $\epsilon \equiv \epsilon_1\lambda$) are not. We thus always consider the general case in these calculations. (To see the role played by the $\epsilon_1 \dfrac{\partial}{\partial \epsilon_0}$ term in $\mathbf{V}$, it is instructive to calculate $\mathbf{V}(\mathcal{H}_3)$).

Comparing (5.10) with (5.2) we see that the above $\mathbf{V}$ plays the role of $\mathbf{V}_{-1}$. For the case $\epsilon \equiv \epsilon_0$ it is possible to use this as the starting point to define $\mathbf{V}_n$ by:

(5.11) $$\mathbf{V}_n = \mathbf{R}^{n+1}\mathbf{V}_{-1} = \mathbf{R}^n\mathbf{V}_0 \ .$$

$\mathbf{V}_0$ just corresponds to the scaling symmetry [1] and is thus locally defined. Generally, $\mathbf{V}_{-1}$ and $\mathbf{V}_0$ are the only local master symmetries. However, when $N = 2$, $\mathbf{V}_1$ is also locally defined:

$$J_0 = \frac{1}{4}\partial^3 + u_0\partial + \frac{1}{2}u_{0x} \ , \quad J_1 = u_1\partial + \frac{1}{2}u_{1x} \ , \ J_2 = -\partial \ ,$$

(5.12a)

$$\mathbf{R} = \begin{pmatrix} 0 & J_0\partial^{-1} \\ 1 & J_1\partial^{-1} \end{pmatrix}, \quad \mathbf{V}_{-1} = u_1\frac{\partial}{\partial u_0} - 2\frac{\partial}{\partial u_1} \quad \text{or} \quad \mathbf{V}_{-1} = (u_1, -2)^T \ .$$

Thus:

$$\mathbf{R}\mathbf{V}_{-1} = \begin{pmatrix} -2u_0 - xu_{0x} \\ -u_1 - xu_{1x} \end{pmatrix} = \mathbf{V}_0 \ ,$$

(5.12b)

$$\mathbf{R}\mathbf{V}_0 = -\begin{pmatrix} \frac{1}{4}(xu_{1xxx} + 3u_{1xx}) + u_0u_1 + xu_0u_{1x} + \frac{1}{2}xu_1u_{0x} \\ 2u_0 + xu_{0x} + u_1^2 + \frac{3}{2}xu_1u_{1x} \end{pmatrix} = \mathbf{V}_1 \ .$$

Only in this case is the second component of $\mathbf{V}_0$ an exact derivative.

6. Conclusions. In this paper we have discussed the systematic construction of isospectral flows, Hamiltonian structures, Miura maps and master symmetries in the context of some fairly general classes of 'energy dependent' spectral problems. An important feature is the universality of algebraic form of the Hamiltonian structures, Miura maps and so on. The spectral problems discussed contain many interesting examples, some known, many new. Special cases of energy dependent spectral problems are also discussed in [21–23].

It is also possible to apply these ideas to higher order Lax operators. The simplest such example is a 4 component, tri-Hamiltonian extension of the Boussinesq hierarchy, which will be presented elsewhere [19]. The degree of the λ-polynomial is limited here since the second Gelfand-Dikii Hamiltonian structure is *quadratic* in the potential functions.

It is possible to give the Hamiltonian structures (3.5b) an r-matrix interpretation [24, 25], but such an algebraic interpretation of our Miura maps is still an open question.

Recently, Dorfman [26] and Wilson [27] have presented what they respectively call Dirac and quasi-Hamiltonian structures related to the KdV equation through a Miura map. Analogous results can be obtained for the KdV like flows of (3.1a), and are related to an interesting generalisation of the Schwarzian derivative [28].

Acknowledgements. The scope of this paper is somewhat larger than that of the lecture I gave in the workshop. The additional items (Zakharov-Shabat extensions and master symmetries) are results obtained during and shortly after my visit to the IMA. I am grateful to the IMA for their hospitality and for providing a stimulating atmosphere in which to work. I thank both the IMA and the Royal Society for their financial support of my visit.

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