**AXIALLY SYMMETRIC JETS
OF COMPRESSIBLE FLUID**

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Abstract. This paper deals with a free boundary problem arising from a shaped-charge jet. An axially symmetric incoming jet with fixed mass flux collapses on the symmetric axis, forming a high velocity jet in one direction and a relative low velocity jet in the opposite direction. The boundary of the flow is free and has to be determined together with the velocity potential. Using a variational approach, we prove that this free boundary problem has a unique stationary solution.

Key words. Shaped-charge jet, impinging jet, collision of jets, subsonic flow, mass flux

0. Introduction. A shaped-charge is a device consisting of a cylinder containing high explosive with a conical metal-lined cavity as shown in Figure 1.

Figure 1

After the initiation of the charge, the detonation shock waves quickly cause the lining of the cavity to collapse and high velocity jet, made up of the inner portion of the liner, is formed. The high velocity jet field, described by the solid vectors in Figure 2, gives rise to velocities in the collapse point frame of reference described by the broken vector (see [11, pp. 365-6]).

Figure 2

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Figure 3

In this frame of reference, the vector field has the form described in Figure 3, where the stagnation point S is taken to be fixed on the axis of symmetry.

In this paper, we consider an idealized form of the shaped-charge jet, that is, the flow is axially symmetric with two unbounded free-surfaces, Γ_1 and Γ_2 . Also the total mass flux m of the incoming jet is assumed to be fixed; therefore, the thickness of the incoming jet must decrease to zero as the distance goes to ∞ .

There are two jet problems related to the shaped-charge problem: (1) the collision of two jets and (2) jet impinging on a wall.

Problem (1) is considered in [9] and can be obtained by reversing the velocity field of the shaped-charge jet. Problem (2) was briefly studied in Friedman's book [10, pp. 365-6] in the case of incompressible and irrotational plane (2-D) flow.

A theory explaining the formation of the shaped-charge jet was first proposed by Birkhoff et al. in [8]. More detailed descriptions of the shaped-charge jet problem and the impinging jet problem can be found in Friedman's book [11, pp. 145-155].

In this paper, we prove the existence and uniqueness of a steady solution of the compressible axially symmetric jet problem. This method also applies to the easier case of the incompressible and irrotational plane flow. Our method relies on the variational techniques developed by Alt and Caffarelli [2] and by Alt, Caffarelli, and Friedman [3-6].

This paper is organized as follows. In section 1 we formulate precisely the shaped-charge jet problem and state the existence and uniqueness results. We introduce a variational problem in section 2 and then solve it in sections 3 and 4. In section 5 we show that the jet problem and the variational problem are equivalent and therefore complete

the proof of the existence and uniqueness. In section 6 we briefly study the collision and impinging problems.

1. The Main Result. We assume that the flow is inviscid, compressible, stationary, and irrotational, and denote by $\vec{U} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ the velocity vector of the fluid and by ρ its density. Then in the flow region

$$(1.1) \quad \nabla \times \vec{U} = 0 \quad (\text{irrotational}),$$

$$(1.2) \quad \nabla \cdot (\rho \vec{U}) = 0 \quad (\text{conservation of mass}),$$

$$(1.3) \quad (\vec{U} \cdot \nabla) \vec{U} - \nabla p / \rho = 0 \quad (\text{inviscid, Euler equation}),$$

$$(1.4) \quad p = P(\rho) \quad (\text{state equation}).$$

Equations (1.1) and (1.3) imply the Bernoulli law

$$\frac{1}{2} \nabla q^2 + \nabla p / \rho = 0 \quad (\text{conservation of momentum}),$$

where $q = (\bar{u}_1^2 + \bar{u}_2^2 + \bar{u}_3^2)^{1/2}$ is the speed of the fluid or, in the integrated form,

$$(1.5) \quad \frac{1}{2} q^2 + \int_{\rho_{max}}^{\rho} \frac{1}{t} P'(t) dt = 0,$$

where ρ_{max} is the density of the fluid at the stagnation point (which is also the point of maximum pressure).

We now specialize to axially symmetric flows, taking the x -axis to be the axis of symmetry. We denote a generic point by $X = (x, y)$, where y is the distance to the x -axis. Writing

$$\vec{U} = u_1 \vec{e}_x + u_2 \vec{e}_y \quad (\vec{e}_x = (1, 0), \vec{e}_y = (0, 1)),$$

equations (1.1) and (1.2) become

$$\begin{aligned} (u_2)_x - (u_1)_y &= 0, \\ (y \rho u_1)_x + (y \rho u_2)_y &= 0. \end{aligned}$$

Consequently, there exist a velocity potential ϕ and a stream function ψ satisfying

$$(1.6) \quad u_1 = \phi_x = \frac{1}{y \rho} \psi_y \quad \text{and} \quad u_2 = \phi_y = -\frac{1}{y \rho} \psi_x,$$

and the differential equations for u_1 and u_2 transform into

$$\begin{aligned} (1.7) \quad \mathcal{L}\psi &\equiv \left(\frac{1}{y \rho} \psi_x\right)_x + \left(\frac{1}{y \rho} \psi_y\right)_y = 0, \\ \tilde{\mathcal{L}}\phi &\equiv (y \rho \phi_x)_x + (y \rho \phi_y)_y = 0. \end{aligned}$$

The *Mach number* is defined by

$$M = q/\sqrt{P'(\rho)} = \sqrt{q/(|q'|\rho)},$$

where $q = q(\rho)$ is determined by (1.5). It is easy to check (see [7]) that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are elliptic if and only if $M < 1$. The flow is called *subsonic* if $M < 1$ and *supersonic* if $M > 1$. All the flows considered in this paper will be subsonic.

We shall assume that the state function $P(\rho)$ satisfies

$$P'(\rho) > 0 \quad \text{and} \quad \rho P''(\rho) + 2P'(\rho) \geq 0 \quad \text{for all } \rho > 0;$$

then (see [6, pp. 84–6]) there exists a critical value ρ_{cr} such that the flow is subsonic if and only if

$$\rho_{cr} < \rho \leq \rho_{max}.$$

In this case, (1.5) and (1.6) lead to the relation:

$$(1.8) \quad \frac{|\nabla\psi|^2}{y^2} = 2\rho^2 \int_{\rho}^{\rho_{max}} \frac{1}{t} P'(t) dt;$$

this relation defines a strictly monotone decreasing branch $\rho = \rho(|\nabla\psi|^2/y^2)$ for

$$(1.9) \quad 0 \leq \frac{|\nabla\psi|^2}{y^2} < 2\rho_{cr}^2 \int_{\rho_{cr}}^{\rho_{max}} \frac{1}{t} P'(t) dt \equiv \lambda_{cr}^2.$$

Define a function g by

$$(1.10) \quad g(t) = 1/\rho(t) \quad \text{for } 0 \leq t < \lambda_{cr}^2.$$

Equation (1.7) for ψ then becomes

$$(1.11) \quad \mathcal{L}\psi \equiv \nabla \left(g \left(\frac{|\nabla\psi|^2}{y^2} \right) \frac{\nabla\psi}{y} \right) = 0 \quad \left(\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \right).$$

From now on, the letter $\mathcal{L}$ will always refer to the above quasi-linear operator.

We now define the jet problem. Given three constants (λ, β, m) satisfying

$$(1.12) \quad 0 < \lambda < \lambda_{cr}, \quad 0 < \beta \leq \pi/2, \quad 0 < m < +\infty,$$

we seek a five-tuple $(\psi, \Gamma_1, \Gamma_2, Q_1, Q_2)$, where Q_1 and Q_2 are two positive constants, satisfying the following conditions:

(1) For $i = 1, 2$, Γ_i is a curve having the form (see Figure 3)

$$(1.13) \quad \Gamma_i = \{(x, y) \in \mathbb{R}^{2+} \mid x = k_i(y), h_i < y < +\infty\},$$

where h_i is a positive constant and $k_i(\cdot)$ is a smooth ($C^{2+\alpha}$) function satisfying

$$(1.14) \quad (-1)^{i+1} k_i(y) \longrightarrow \infty \quad \text{as } y \rightarrow h_i^+,$$

$$(1.15) \quad k_i(y) - y \cot \beta \longrightarrow 0 \quad \text{as } y \rightarrow +\infty;$$

$$(1.16) \quad k_2(y) < k_1(y) \quad \text{for } \max\{h_1, h_2\} < y < +\infty;$$

$$(1.17) \quad \lambda \cdot 2\pi y \cdot (k_1(y) - k_2(y)) \cdot \sin \beta \longrightarrow m \quad \text{as } y \rightarrow +\infty.$$

(2) ψ is defined on a domain G bounded by Γ_1, Γ_2 , and the x -axis, satisfying

$$(1.18) \quad \begin{cases} \mathcal{L}\psi = 0 & \text{in } G, \\ \psi = 0 & \text{on } x\text{-axis}, \\ \psi = (-1)^{(i+1)} Q_i & \text{on } \Gamma_i, \quad i = 1, 2, \\ \frac{1}{y} \frac{\partial \psi}{\partial \nu} = (-1)^{(i+1)} \lambda & \text{on } \Gamma_i, \quad i = 1, 2 \quad (\nu = \text{outward normal}); \end{cases}$$

$$(1.19) \quad \frac{1}{y} \nabla \psi(x, y) \longrightarrow \lambda(\sin \beta, -\cos \beta) \quad \text{for } (x, y) \in G, \quad y \rightarrow +\infty;$$

$$(1.20) \quad \frac{1}{y} \nabla \psi(x, y) \longrightarrow \lambda(0, \pm 1) \quad \text{for } (x, y) \in G, \quad y \text{ bounded}, \quad x \rightarrow \pm\infty;$$

$$(1.21) \quad \frac{1}{y} |\nabla \psi(x, y)| \leq \lambda \quad \text{for } (x, y) \in G.$$

DEFINITION 1.1. A solution of an axially symmetric jet problem with parameters (λ, β, m) is a five-tuple $(\psi, \Gamma_1, \Gamma_2, Q_1, Q_2)$ satisfying (1.13)–(1.21).

Note that the condition (1.21) simply means that the flow is subsonic and therefore $\mathcal{L}$ is elliptic. The maximum principle can then be applied, yielding $-Q_2 < \psi < Q_1$ in G .

One can easily obtain some relations among the constants $m, \lambda, Q_1, Q_2, h_1$, and h_2 . From (1.18), the mass flux through the cross section $I_c \equiv \{(x, c) | k_2(c) < x < k_1(c)\}$ is

$$\int_{I_c} \rho \vec{U} \cdot d\vec{S} = \int_{k_2(c)}^{k_1(c)} \rho \left(-\frac{\psi_x}{y\rho} \right) (2\pi y dx) = -2\pi \psi(x, c) \Big|_{x=k_2(c)}^{x=k_1(c)} = -2\pi(Q_1 + Q_2).$$

However by (1.17) and (1.19),

$$\lim_{c \rightarrow +\infty} \int_{k_2(c)}^{k_1(c)} \left(\frac{\psi_x}{y} \right) (2\pi y dx) = m.$$

Hence,

$$(1.22) \quad 2\pi(Q_1 + Q_2) = m.$$

Integrating $\psi_y(x, y)$ from $y = 0$ to the free boundaries Γ_1 or Γ_2 , letting $|x| \rightarrow \infty$, and using the conditions (1.14), (1.18), and (1.20), one get

$$(1.23) \quad Q_i = \frac{1}{2} \lambda h_i^2, \quad i = 1, 2.$$

The main result of this paper is the following:

THEOREM 1.2. *For every (λ, β, m) satisfying (1.12), there exists a unique solution of the axially symmetric jet problem with*

$$(1.24) \quad Q_1 = \frac{m \sin^2 \frac{\beta}{2}}{2\pi}, \quad Q_2 = \frac{m \cos^2 \frac{\beta}{2}}{2\pi};$$

$$(1.25) \quad h_1 = \sqrt{\frac{m}{\lambda\pi}} \sin \frac{\beta}{2}, \quad h_2 = \sqrt{\frac{m}{\lambda\pi}} \cos \frac{\beta}{2}.$$

We conclude this section with a lemma about scaling of solutions.

LEMMA 1.3. *If $(\psi, \Gamma_1, \Gamma_2, Q_1, Q_2)$ is a solution of the jet problem with parameters (λ, β, m) , then $(\psi_s, \Gamma_1^s, \Gamma_2^s, Q_1/s^2, Q_2/s^2)$ is a solution of the jet problem with parameters $(\lambda, \beta, m/s^2)$; here s is any positive number and*

$$\begin{aligned} \psi_s(x, y) &= \psi(sx, sy)/s^2, \\ \Gamma_i^s &= \{(x, y) \mid (sx, sy) \in \Gamma_i\}, \quad i = 1, 2. \end{aligned}$$

The proof is immediate.

In sections 2–5 we shall fix λ and β satisfying (1.12) and show that there exists an $m = m(\lambda, \beta) > 0$ such that a solution of the jet problem with parameters $(\lambda, \beta, m(\lambda, \beta))$ exists. Thus by Lemma 1.3, there exist solutions for any parameters satisfying (1.12).

In the sequel, we shall denote by the same letters C and c the various positive constants depending only on λ and β .

2. A Variational Problem. In this section, λ and β are fixed constants satisfying (1.12). We define a variational problem with two parameters Q_1 and Q_2 . Later on in section 5, we shall prove that for some suitable parameters Q_1^0 and Q_2^0 , the solution of the variational problem is actually a solution of the jet problem for some $m = m(\lambda, \beta)$.

The function $g(t)$ in (1.10) is defined only for $0 \leq t < \lambda_{cr}^2$. We now redefine g so that it coincides with the original g for $0 \leq t < (\lambda^2 + \lambda_{cr}^2)/2$ (recall that $0 < \lambda < \lambda_{cr}$) and satisfies:

$$(2.1) \quad \begin{cases} g(t) \in C^{1,1}[0, +\infty), \\ c \leq g(t) \leq C, & \forall 0 \leq t < +\infty, \\ 0 \leq g'(t) \leq C/(1+t), & \forall 0 \leq t < +\infty. \end{cases}$$

Because of (1.21), the above redefinition does not affect our discussion. We introduce

$$(2.2) \quad F(t) = \int_0^t g(t) dt \quad \text{and} \quad \Phi(t) = 2tg(t) - F(t), \quad \text{for } t > 0.$$

Figure 4

Evidently, (2.1) yields

$$(2.3) \quad \begin{cases} F(t) \in C^{2,1}[0, +\infty), & F(0) = 0, \\ c \leq F'(t) \leq C, \\ 0 \leq F''(t) \leq C/(1+t). \end{cases}$$

Let (see Figure 4) $A = (\frac{1}{\sin \beta}, 1)$, $B = (-\frac{1}{\sin \beta}, 1)$,

$$\begin{aligned} l_1 &= \{(x, 1) \mid \frac{1}{\sin \beta} \leq x < +\infty\}, & l_2 &= \{(x, 1) \mid -\infty < x \leq -\frac{1}{\sin \beta}\}, \\ \Omega_1 &= \{(x, y) \mid y > 1\}, \\ \Omega_2 &= \{(x, y) \mid 0 < y < 1, x - (y-1)\cot \beta > \frac{1}{\sin \beta}\}, \\ \Omega_3 &= \{(x, y) \mid 0 < y < 1, x - (y-1)\cot \beta < -\frac{1}{\sin \beta}\}, \quad \text{and} \\ \Omega_4 &= \{(x, y) \mid 0 < y < 1, |x - (y-1)\cot \beta| < \frac{1}{\sin \beta}\}. \end{aligned}$$

Define a vector field $\vec{e}(x, y)$ over $\mathcal{D} \equiv \mathbf{R}^{2+} \setminus (l_1 \cup l_2)$ by

$$\vec{e}(x, y) \equiv \begin{cases} (\sin \beta, -\cos \beta) & \text{in } \Omega_1 \cup \overline{\Omega_4}, \\ \frac{(-y, x + \cot \beta - \frac{1}{\sin \beta})}{\sqrt{y^2 + (x + \cot \beta - \frac{1}{\sin \beta})^2}} & \text{in } \Omega_2, \\ \frac{(-y, x + \cot \beta + \frac{1}{\sin \beta})}{\sqrt{y^2 + (x + \cot \beta + \frac{1}{\sin \beta})^2}} & \text{in } \Omega_3. \end{cases}$$

It is easy to see that $\vec{e}(x, y) \in C^{0,1}(\mathcal{D})$ and

$$(2.4) \quad \operatorname{div} \vec{e}(x, y) \equiv (e_1(x, y))_x + (e_2(x, y))_y = 0 \quad \text{in } \mathcal{D},$$

$$(2.5) \quad \vec{e}(x, y) \longrightarrow (0, \pm 1) \quad \text{as } x \rightarrow \pm\infty, 0 < y < 1.$$

Let

$$(2.6) \quad G(y, \vec{e}, v) = \left\{ y \left[F \left(\frac{|\nabla v|^2}{y^2} \right) + \Phi(\lambda^2) \right] - 2\lambda g(\lambda^2) \nabla v \cdot \vec{e} \right\} \chi_{\{-Q_2 < v < Q_1\}}, \quad y > 0,$$

where $\chi_{\mathcal{A}}$ is the characteristic function of a set $\mathcal{A}$. Define a functional

$$(2.7) \quad J(v) = \iint_{\mathbb{R}^{2+}} G(y, \vec{e}(x, y), v(x, y)) \, dx \, dy,$$

where $v(\cdot)$ is taken in the function space

$$(2.8) \quad \mathcal{K}_{Q_1, Q_2} \equiv \{v \in H_{loc}^1(\mathbb{R}^{2+}) \mid v = 0 \text{ on } x\text{-axis}, v = Q_1 \text{ on } l_1, v = -Q_2 \text{ on } l_2\}.$$

Consider the minimization problem:

$$(2.9) \quad \text{Find } u \in \mathcal{K}_{Q_1, Q_2} \text{ such that } J(u) = \min_{v \in \mathcal{K}_{Q_1, Q_2}} J(v).$$

LEMMA 2.1. Suppose that $u_{Q_1, Q_2} \in \mathcal{K}_{Q_1, Q_2}$ is a minimizer of the variational problem (2.9), then $u_{Q_1, Q_2} \in C^\alpha(\mathcal{D})$ for any $0 < \alpha < 1$, $-Q_2 \leq u_{Q_1, Q_2} \leq Q_1$ in $\mathbb{R}^{2+}$, and the free boundaries $\partial\{u_{Q_1, Q_2} < Q_1\}$ and $\partial\{u_{Q_1, Q_2} > -Q_2\}$ are smooth ($C^{2+\alpha}$) in $\mathcal{D}$; moreover

$$\begin{cases} \mathcal{L}u_{Q_1, Q_2} = 0 & \text{in } G \equiv \mathcal{D} \cap \{-Q_2 < u_{Q_1, Q_2} < Q_1\}, \\ \frac{1}{y} \frac{\partial u_{Q_1, Q_2}}{\partial \nu} = \lambda & \text{on } \mathcal{D} \cap \partial\{u_{Q_1, Q_2} < Q_1\}, \\ \frac{1}{y} \frac{\partial u_{Q_1, Q_2}}{\partial \nu} = -\lambda & \text{on } \mathcal{D} \cap \partial\{u_{Q_1, Q_2} > -Q_2\}. \end{cases}$$

Proof. The Hölder continuity of the solution follows from [4, p. 433]. This implies that the sets $\{u = -Q_2\}$ and $\{u = Q_1\}$ are closed and locally separated. By (2.4), for any point $p \in \mathcal{D}$, $J(v)$ has the form:

$$\begin{aligned} J(v) = & \iint_{\{|X-p| < r\}} y \left[F \left(\frac{|\nabla v|^2}{y^2} \right) + \Phi(\lambda^2) \cdot \chi_{\{-Q_2 < v < Q_1\}} \right] \\ & + \iint_{\{|X-p| > r\} \cap \mathbb{R}^{2+}} G(y, \vec{e}, v) - 2\lambda g(\lambda^2) \int_{\{|X-p|=r\}} v \vec{e} \cdot \vec{\nu} \, ds, \end{aligned}$$

for any $r > 0$ small enough. To this new form of J , we can now apply Theorem 6.3 in [5, p. 43] in $\{|X - p| < r\}$, for each free boundary separately; here we need the assumption (2.3).

THEOREM 2.2. If

$$(2.10) \quad 0 \leq Q_1, Q_2 \leq \lambda/2,$$

then the variational problem (2.9) has a solution and every solution u_{Q_1, Q_2} of this variational problem satisfies

$$(2.11) \quad \max \left\{ -\frac{\lambda}{2} y^2, -Q_2 \right\} \leq u_{Q_1, Q_2} \leq \min \left\{ \frac{\lambda}{2} y^2, Q_1 \right\}.$$

Proof. Note that $G(y, \vec{e}, v)$ can be written as (see [6, p. 127])

$$G = y \left\{ \left[F\left(\frac{|\nabla v|^2}{y^2}\right) - F(\lambda^2) - F'(\lambda^2) \left(\frac{|\nabla v|^2}{y^2} - \lambda^2\right) \right] + g(\lambda^2) \left| \frac{\nabla v}{y} - \lambda \vec{e} \right|^2 \right\} \chi_{\{-Q_2 < v < Q_1\}}.$$

It follows by (2.3) that if $(x, y) \in \{-Q_2 < v < Q_1\}$, then

$$(2.12) \quad g(\lambda^2) \left| \frac{\nabla v}{y} - \lambda \vec{e} \right|^2 y \leq G(y, \vec{e}, v) \leq C \left| \frac{\nabla v}{y} - \lambda \vec{e} \right|^2 y.$$

Thus $J(v) \geq 0$, $\forall v \in \mathcal{K}_{Q_1, Q_2}$. For the existence, it suffices to show that there is a $u_0 \in \mathcal{K}_{Q_1, Q_2}$ such that $J(u_0) < +\infty$; indeed any weak limit of a minimizing sequence can be shown to be a minimizer (see [10, p. 272]).

Let u_0 be a uniformly Lipschitz continuous function in $\overline{R^{2+}}$ satisfying

$$u_0(x, y) = \begin{cases} w_0(x, y, \frac{1}{\sin \beta}, 1) & \text{in } \Omega_1, \\ \min \left\{ \frac{1}{2} \lambda y^2, Q_1 \right\} & \text{in } \Omega_2, \\ \max \left\{ -\frac{1}{2} \lambda y^2, Q_2 \right\} & \text{in } \Omega_3, \end{cases}$$

where

$$(2.13) \quad w_0(x, y, \xi, \eta) \equiv \max \left\{ Q_1 - \lambda y \max \{ (y - \eta) \cos \beta - (x - \xi) \sin \beta, 0 \}, -Q_2 \right\}$$

is a Lipschitz continuous function taking the values Q_1 in $\{(x - \xi) \geq (y - \eta) \cot \beta\}$ and $-Q_2$ in $\{(x - \xi) \leq (y - \eta) \cot \beta - (Q_1 + Q_2)/(\lambda y \sin \beta)\}$.

Since $\{-Q_2 < w_0 < Q_1\} = \{0 < (y - \eta) \cos \beta - (x - \xi) \sin \beta < (Q_1 + Q_2)/(\lambda y)\}$, we see from (2.12) that for any $y_0 \geq 1$,

$$(2.14) \quad \begin{aligned} \iint_{\{y \geq y_0\}} G(y, \vec{e}, w_0) &\leq C \iint_{\{y > y_0\}} y \left| \frac{\nabla w_0}{y} - \lambda \vec{e} \right|^2 \chi_{\{-Q_2 < w_0 < Q_1\}} \\ &= C \iint_{\{y > y_0\}} y \frac{\lambda^2 |(x - \xi) \sin \beta - (y - \eta) \cos \beta|^2}{y^2} \chi_{\{-Q_2 < w_0 < Q_1\}} \\ &\leq C(Q_1 + Q_2)^3 / y_0^3. \end{aligned}$$

Later on we shall use the above estimate for $y_0 > 1$, but now we need it only for $y_0 = 1$. Similarly estimating the integral of G over Ω_2 and Ω_3 , we conclude that $J(u_0) < +\infty$.

The proof of (2.11) is the same as the proof in [10, pp 303–305].

From now on, we assume that all the Q_1 's and Q_2 's satisfy (2.10).

For the uniqueness of the solution of the variational problem, we need:

LEMMA 2.3. For any solution u_{Q_1, Q_2} of the variational problem (2.9), let

$$(2.15) \quad b_1 = \inf \{ c \mid u_{Q_1, Q_2}(x, 1) = Q_1, \forall x \in [c, +\infty) \} \quad \text{and}$$

$$(2.16) \quad b_2 = \sup \{ c \mid u_{Q_1, Q_2}(x, 1) = -Q_2, \forall x \in (-\infty, c] \}.$$

Then

- (1) $\frac{Q_1}{Q_2} > \tan^2 \frac{\beta}{2}$ implies that $b_1 = \frac{1}{\sin \beta}$;
- (2) $\frac{Q_1}{Q_2} < \tan^2 \frac{\beta}{2}$ implies that $b_2 = -\frac{1}{\sin \beta}$; and
- (3) $\frac{Q_1}{Q_2} = \tan^2 \frac{\beta}{2}$ implies that for all $b \in [b_1 - \frac{1}{\sin \beta}, b_2 + \frac{1}{\sin \beta}]$, the function

$$(2.17) \quad u_b(x, y) = u_{Q_1, Q_2}(x + b, y)$$

is also a solution of the variational problem (2.9).

REMARK 2.4. (i) Note that u_b is a translation of u and the flow region $\{-Q_2 < u_b < Q_1\}$ does not intersect the lines l_1 and l_2 . However if $Q_1/Q_2 \neq \tan^2 \frac{\beta}{2}$, this translate u_b ($b \neq 0$) is not a minimizer (as will be seen from the proof below). (ii) In the case that $Q_1/Q_2 = \tan^2 \frac{\beta}{2}$, we define the *normalized solution* as the one whose right free boundary touches the left end point of l_1 (i.e. the one in (2.17) with $b = b_1 - \frac{1}{\sin \beta}$).

Proof. By the definitions of b_1 and b_2 in (2.15) and (2.16), u_b defined by (2.17) for any $b \in [b_1 - \frac{1}{\sin \beta}, b_2 + \frac{1}{\sin \beta}]$ is in $\mathcal{K}_{Q_1, Q_2}$. For any $M \gg 1$, set $\mathcal{D}_M = \{(x, y) \mid 0 < y < M, |x| < M\}$ and $\mathcal{D}_M^b = \{(x, y) \mid 0 < y < M, |x + b| < M\}$. Then

$$\begin{aligned} & \iint_{\mathcal{D}_M} G(y, \vec{e}, u_{Q_1, Q_2}) - \iint_{\mathcal{D}_M^b} G(y, \vec{e}, u_b) \\ &= -2\lambda g(\lambda^2) \left[\int_{\partial(\mathcal{D}_M \setminus (l_1 \cup l_2))} u_{Q_1, Q_2} \vec{e} \cdot \vec{\nu} \, ds - \int_{\partial(\mathcal{D}_M^b \setminus (l_1 \cup l_2))} u_b \vec{e} \cdot \vec{\nu} \, ds \right] \\ &= -2\lambda g(\lambda^2) \left\{ \left[Q_2 \int_{-M-b}^{-M} + Q_1 \int_{M-b}^M \right] [e_2(x, 1-0) - e_2(x, 1+0)] \, dx \right. \\ & \quad + \int_0^1 u_{Q_1, Q_2}(M, y) [e_1(M, y) - e_1(M-b, y)] \, dy \\ & \quad \left. - \int_0^1 u_{Q_1, Q_2}(-M, y) [e_1(-M, y) - e_1(-M-b, y)] \, dy \right\}. \end{aligned}$$

Letting $M \rightarrow +\infty$, we obtain that

$$0 \geq J(u_{Q_1, Q_2}) - J(u_b) = -2\lambda g(\lambda^2) [Q_2(-1 + \cos \beta) + Q_1(1 + \cos \beta)] b.$$

That is, the expression in the bracket has the same sign as b . But from this we can easily deduce the assertions of the lemma.

Now we establish the uniqueness for the variational problem (2.9).

THEOREM 2.5. *Let u_1 and u_2 be two solutions of the variational problem (2.9) with parameters Q_1 and Q_2 ; normalized if $Q_1/Q_2 = \tan^2 \frac{\beta}{2}$. Then $u_1 = u_2$.*

Proof. For any $\epsilon > 0$, define

$$u_i^{\pm\epsilon}(x, y) = u_i(x \pm \epsilon, y), \quad i = 1, 2,$$

$$v_1 = \min\{u_1^\epsilon, u_2\}, \quad \text{and} \quad v_2 = \max\{u_1, u_2^{-\epsilon}\}.$$

Proceeding as in [6, pp. 97-98], we see that v_1 is a minimizer of (2.9). From Lemma 2.3 and Remark 2.4 (ii), we know that there exists a point $p \in \mathcal{D}$ such that $-Q_2 < v_1(p) = u_2(p) < Q_1$. Noting that $v_1 \leq u_2$, we can then apply the maximum principle (see [10, pp. 318-320]), thereby deducing that $v_1 = u_2$. That is,

$$(2.18) \quad u_1(x + \epsilon, y) \geq u_2(x, y).$$

Letting $\epsilon \searrow 0$, we get $u_1 \geq u_2$. Similarly, we can get $u_2 \geq u_1$ and therefore $u_1 = u_2$.

THEOREM 2.6. *If u_{Q_1, Q_2} is a minimizer of the variational problem (2.9) with parameters Q_1 and Q_2 , then $(u_{Q_1, Q_2})_x \geq 0$ in $\mathcal{D}$. Consequently, there exist two functions $k_{1, Q_1, Q_2}(y) < k_{2, Q_1, Q_2}(y)$ valued in $\mathbb{R}^1 \cup \{+\infty\} \cup \{-\infty\}$ satisfying*

$$\begin{aligned} u_{Q_1, Q_2} &= Q_1 && \text{on } \{(x, y) \in \mathbb{R}^{2+} \mid k_{1, Q_1, Q_2}(y) \leq x \leq +\infty\}, \\ u_{Q_1, Q_2} &= -Q_2 && \text{on } \{(x, y) \in \mathbb{R}^{2+} \mid -\infty \leq x \leq k_{2, Q_1, Q_2}(y)\}, \\ u_{Q_1, Q_2} &\in (-Q_2, Q_1) && \text{in } \{(x, y) \in \mathbb{R}^{2+} \mid k_{2, Q_1, Q_2}(y) < x < k_{1, Q_1, Q_2}(y)\}. \end{aligned}$$

The lemma follows from (2.18) by letting $u_1 = u_2 = u_{Q_1, Q_2}$.

In next two sections, we shall study the properties of the functions $k_{1, Q_1, Q_2}(y)$ and $k_{2, Q_1, Q_2}(y)$.

3. Free Boundaries of the Variational Solution. In the sequel, $u = u_{Q_1, Q_2}$ is the solution of the variational problem (2.9) and $k_1 = k_{1, Q_1, Q_2}$ and $k_2 = k_{2, Q_1, Q_2}$ are the functions appearing in Theorem 2.6.

LEMMA 3.1. *There exists a positive constant C such that for any ball $B_r(X_0) \subset \mathbb{R}^{2+} \setminus l_2$ (or $B_r(X_0) \subset \mathbb{R}^{2+} \setminus l_1$) with $X_0 = (x_0, y_0)$, $r < y_0/2$, the inequality*

$$(3.1) \quad \frac{1}{r} \int_{\partial B_r(X_0)} (u + Q_2) \geq Cy_0 \quad \left(\text{or } \frac{1}{r} \int_{\partial B_r(X_0)} (Q_1 - u) \geq Cy_0 \right)$$

implies

$$u > -Q_2 \quad (\text{ or } u < Q_1) \quad \text{in } B_r(X_0).$$

The proof is the same as in Lemma 15.3 of [10, p. 349], provided that we use the Harnack inequality for the operator $\mathcal{L}$ established in [5, pp. 9-10].

REMARK 3.2. As in Remark 3.1 of [10, p. 277], the assumption $B_r(X_0) \subset \mathbb{R}^{2+} \setminus l_2$ can be relaxed. More precisely, for any $B_r(X_0) \subset \mathbb{R}^{2+} \setminus B$ with $X_0 = (x_0, y_0)$, $r < y_0/2$, the inequality

$$\frac{1}{r} \int_{\partial B_r(X_0) \cap \{y < 1 - \epsilon r\}} (u + Q_2) \geq C_\epsilon y_0 \quad \left(\text{or } \frac{1}{r} \int_{\partial B_r(X_0) \cap \{y > 1 + \epsilon r\}} (u + Q_2) \geq C_\epsilon y_0 \right)$$

implies

$$u > -Q_2 \text{ in } B_{(1-\epsilon)r}(X_0) \cap \{y < 1 - \epsilon r\} \quad (\text{or } B_{(1-\epsilon)r} \cap \{y > 1 + \epsilon r\})$$

for all $\epsilon < 1/2$. A similar statement holds for $Q_1 - u$.

LEMMA 3.3. (“**Bounded Gradient Lemma**”) For any $\delta > 0$, there exists a constant C_δ such that

$$(3.2) \quad \sup_{(x,y) \in \mathbb{R}^{2+} \setminus (B_\delta(A) \cup B_\delta(B))} \frac{|\nabla u(x, y)|}{y} \leq C_\delta.$$

This can be proved by the same method as in Corollary 15.4 of [10, p. 350] and [6, pp. 113–4], provided that we use Lemma 3.1, Remark 3.2, and the elliptic estimates for the operator $\mathcal{L}$ established in [5, pp. 6–7].

REMARK 3.4. Actually in the proof ([10, p. 350]), the constant C_δ in the above lemma depends also on the L^∞ bound of the solution; however since this bound is $\max\{Q_1, Q_2\}$, we conclude from (2.10) that C_δ depends only on λ and δ .

LEMMA 3.5. (“**Nondegeneracy Lemma**”) For any $p > 1$ and any $0 < k < 1$, there exists a constant $C_{k,p}$ such that for any ball $B_r(X_0) \subset \{u < Q_1\}$ (or $B_r(X_0) \subset \{u > -Q_2\}$) with $X_0 = (x_0, y_0)$, $r < y_0/2$, the inequality

$$(3.3) \quad \frac{1}{r} \left(\int_{\partial B_r(X_0)} (u + Q_2)^p \right)^{1/p} \leq C_{k,p} y_0 \quad \left(\text{or } \frac{1}{r} \left(\int_{\partial B_r(X_0)} (Q_1 - u)^p \right)^{1/p} \leq C_{k,p} y_0 \right)$$

implies

$$u = -Q_2 \quad (\text{or } u = Q_1) \quad \text{in } B_{kr}(X_0).$$

The proof is given in [5, pp. 12–13]; Remark 3.4 applies here too.

LEMMA 3.6. (“**Nonoscillation Lemma**”) Let $\mathcal{O}$ be a domain in $\mathcal{D} \cap \{-Q_2 < u < Q_1\}$ bounded by two disjoint arcs γ_1, γ_2 belonging to $\partial\{u < Q_1\}$ (or $\partial\{u > -Q_2\}$), and by $\{x = x_1\}, \{x = x_2\}$. Suppose that γ_i lies in $\{x_1 < x < x_2\}$ with endpoints (x_1, α_i) and (x_2, β_i) , and that $\text{dist}(\mathcal{O}, (A \cup B)) \geq \delta > 0$. Then

$$(3.4) \quad |x_2 - x_1| \leq C_\delta \max\{|\alpha_2 - \alpha_1|, |\beta_2 - \beta_1|\},$$

for some constant C_δ depending only on λ and δ .

Proof. Integrating $\mathcal{L}u = 0$ in $\mathcal{O}$ and using the divergence theorem, we get

$$\int_{\gamma_1 \cup \gamma_2} g \left(\frac{|\nabla u|^2}{y^2} \right) \frac{1}{y} \frac{\partial u}{\partial \nu} ds + \int_{\partial \mathcal{O} \cap (\{x=\alpha_1\} \cup \{x=\alpha_2\})} g \left(\frac{|\nabla u|^2}{y^2} \right) \frac{1}{y} \frac{\partial u}{\partial \nu} ds = 0.$$

Using Lemma 2.1 to calculate the first term and the Lipschitz continuity of u to estimate the second term, we obtain (3.4).

THEOREM 3.7. *Let Ω be any compact subset of $\overline{\mathbb{R}^{2+}}$. Then*

$$(3.5) \quad \lim_{p \rightarrow +\infty} u(x+p, y) = \min \left\{ \frac{\lambda}{2} y^2, Q_1 \right\} \quad \text{uniformly in } \Omega,$$

$$(3.6) \quad \lim_{p \rightarrow -\infty} u(x+p, y) = \max \left\{ -\frac{\lambda}{2} y^2, -Q_2 \right\} \quad \text{uniformly in } \Omega.$$

Proof. Proceeding analogously to [10, pp. 353–5], we set

$$u_p(x, y) = u(x+p, y) \quad \text{and} \quad \vec{e}_p(x, y) = \vec{e}(x+p, y).$$

Since $J(u) < +\infty$, by (2.12),

$$I_{p,\Omega} \equiv \iint_{\Omega} \left| \frac{\nabla u_p}{y} - \lambda \vec{e}_p \chi_{\{-Q_2 < u_p < Q_1\}} \right|^2 \longrightarrow 0 \quad \text{as } p \rightarrow \pm\infty.$$

Hence for any sequence $p_n \rightarrow +\infty$, there exists a subsequence $p_{n_k} \rightarrow +\infty$ such that $u_{p_{n_k}} \rightarrow u_0$ weakly in $H^1(\Omega)$ and strongly in $C^{\frac{1}{2}}(\Omega)$ (recall Lemma 3.3) for some function u_0 . Clearly, $u_0 \in H^1(\Omega)$ and

$$\iint_{\Omega} \left| \frac{\nabla u_0}{y} - \lambda \vec{e}_0 \chi_{\{-Q_2 < u_0 < Q_1\}} \right|^2 \leq \lim_{p_{n_k} \rightarrow +\infty} I_{p_{n_k}, \Omega} = 0,$$

where

$$\vec{e}_0(x, y) = \begin{cases} (\sin \beta, -\cos \beta) & \text{in } 1 < y < +\infty, \\ (0, 1) & \text{in } 0 < y < 1. \end{cases}$$

Consequently,

$$\nabla u_0(x, y) = \lambda \vec{e}_0 y \chi_{\{-Q_2 < u_0 < Q_1\}}.$$

Noting that $u_0(x, 0) = 0$, we deduce that $u_0(x, y) = \max\{\lambda y^2/2, Q_1\}$. Since the sequence p_n is arbitrary, (3.5) follows. The proof of (3.6) is similar.

THEOREM 3.8. Let h_1 and h_2 be positive constants satisfying (1.23). Then

$$(3.7)^{(i)} \quad |k_i(y)| < +\infty \quad \text{for } h_i < y < +\infty,$$

$$(3.8) \quad (-1)^{i+1} k_i(y) = \infty \quad \text{for } 0 < y < h_i;$$

(ii) when $|x|$ is large enough, the free boundaries $x = k_1(y)$ and $x = k_2(y)$ in $\{y < 1\}$ can be written in the form $y = m(x)$ with $x m'(x) \leq 0$ and

$$|m'(x)| + |m''(x)| \longrightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

moreover

$$(3.9) \quad (\nabla u - \lambda y(0, \pm 1) \chi_{\{0 < y < m(x)\}}) \longrightarrow 0 \quad \text{as } x \rightarrow \pm\infty, \quad \text{uniformly for } y \in [0, 1];$$

(iii) $k_i(y) \in C^{2+\alpha}(h_i, 1) \cup C^{2+\alpha}(1, +\infty)$ and the following limits exist:

$$(3.10) \quad \begin{aligned} \lim_{y \rightarrow 1\pm} k_i(y) &\equiv k_i(1\pm), \\ \lim_{y \rightarrow h_i^+} (-1)^{i+1} k_i(y) &= \infty. \end{aligned}$$

As a special case, we have:

COROLLARY 3.9. If $Q_1 = \lambda/2$, then $k_1(1-) = +\infty$; if $Q_2 = \lambda/2$, then $k_2(1-) = -\infty$.

Proof of Theorem 3.8. The estimate (3.7) follows from the nondegeneracy lemma and the uniform convergence in Theorem 3.7. The assertion (3.8) follows from (2.11).

Note that for any $X_0 = (x_0, h_1)$ with x_0 large enough (by Lemma 3.1, this implies that $u > -Q_2$ in $B_r(X_0)$), u satisfies the following “flatness condition” in $B_r(X_0)$, $0 < r < \min\{h_1/2, (1 - h_1)/2\}$:

$$\begin{aligned} u &= Q_1 & \text{in } B_r(X_0) \cap \{y - h_1 > \sigma r\}, \\ u &< Q_1 & \text{in } B_r(X_0) \cap \{y - h_1 < -\sigma r\}, \end{aligned}$$

where $0 < \sigma < 1$ can be made arbitrary small provided that x_0 is large enough. The assertion (ii) then follows from the fact (Theorem 8.1 of [2, p. 142]) that the flatness condition implies the smoothness of the free boundary in $B_{cr}(X_0)$ for some small positive constant c . The case $x_0 \rightarrow -\infty$ is similar.

The assertion (iii) follows from Lemmas 9.4–9.7 of [10, pp. 313–314] (here we use the nonoscillation lemma), provided that we use the fact (see [6, p.100]) that the only solution of the following Cauchy problem

$$(3.11) \quad \begin{cases} \mathcal{L}\psi = 0 & \text{in } (x_1, x_2) \times (y_1, y_2), \\ \psi = Q & \text{on } y = y_1 \quad (\text{or } y = y_2), \\ \frac{1}{y} \frac{\partial \psi}{\partial y} = \lambda & \text{on } y = y_1 \quad (\text{or } y = y_2) \end{cases}$$

is

$$(3.12) \quad \psi = Q + \frac{\lambda}{2}(y^2 - y_1^2) \quad (\text{or } \psi = Q + \frac{\lambda}{2}(y^2 - y_2^2)).$$

To study the limit behavior of the solution as $y \rightarrow +\infty$, we need:

LEMMA 3.10. *There exists a constant C such that for all $y_0 > 2$,*

$$(3.13) \quad \int_{\{y > y_0\}} G(y, \vec{e}, u) \leq C/y_0^3.$$

Proof. The idea of the proof is to construct a function v which coincides with u in $\{y < y_1\}$ and coincides with w_0 (defined in (2.13)) in $\{y > y_1 + 1/y_1\}$, and to use the inequality $J(u) \leq J(v)$; here y_1 is a constant which needs to be carefully chosen to make ∇v small in $\{y_1 < y < y_1 + 1/y_1\}$.

From (2.12),

$$I(y_0) \equiv \iint_{\{y_0/2 < y < y_0\}} G(y, \vec{e}, u) \geq g(\lambda^2) \int_{y_0/2}^{y_0} \frac{1}{y} dy \int_{k_2(y)}^{k_1(y)} |\nabla u - \lambda y \vec{e}|^2 dx.$$

Applying the mean value theorem to the right hand side, we find that for some $y_1 \in [y_0/2, y_0]$,

$$(3.14) \quad \int_{k_2(y_1)}^{k_1(y_1)} |\nabla u(x, y_1) - \lambda y_1 \vec{e}(x, y_1)|^2 dx \leq \frac{2I(y_0)}{g(\lambda^2)}.$$

Consequently,

$$(3.15) \quad \begin{aligned} |(Q_1 + Q_2) - \lambda y_1 \sin \beta [k_1(y_1) - k_2(y_1)]| &= \left| \int_{k_2(y_1)}^{k_1(y_1)} (u_x(x, y_1) - \lambda y_1 \sin \beta) dx \right| \\ &\leq \sqrt{\frac{2I(y_0)}{g(\lambda^2)} [k_1(y_1) - k_2(y_1)]}. \end{aligned}$$

Since the right hand side is bounded by $C + \lambda y_1 \sin \beta [k_1(y_1) - k_2(y_1)]/2$ and $y_0/2 < y_1 < y_0$, we get $k_1(y_1) - k_2(y_1) \leq C/y_0$. Substituting it back into the right hand side of (3.15), we get

$$(3.16) \quad |(k_1(y_1) - k_2(y_1)) - \frac{Q_1 + Q_2}{\lambda y_1 \sin \beta}| \leq C \sqrt{\frac{I(y_0)}{y_0^3}}.$$

We now construct v :

$$v(x, y) = \begin{cases} u(x, y) & \text{in } \{y < y_1\}, \\ \zeta(y) u(x - (y - y_1) \cot \beta, y_1) + (1 - \zeta(y)) w_0(x, y, k_1(y_1), y_1) & \text{in } \{y \geq y_1\}, \end{cases}$$

where $\zeta(y) = \max\{0, \frac{y_2 - y}{y_2 - y_1}\}$.

Note that

$$\begin{aligned}
\Delta_w &\equiv \{(x, y) \mid y_1 < y < y_2, -Q_2 < w_0(x, y, k_1(y_1), y_1) < Q_1\} \\
&= \{(x, y) \mid y_1 < y < y_2, 0 < [k_1(y_1) + (y - y_1)\cot\beta] - x < \frac{Q_1 + Q_2}{\lambda y \sin\beta}\}; \\
\Delta_u &\equiv \{(x, y) \mid y_1 < y < y_2, -Q_2 < u(x - (y - y_1)\cot\beta, y_1) < Q_1\} \\
&= \{(x, y) \mid y_1 < y < y_2, 0 < [k_1(y_1) + (y - y_1)\cot\beta] - x < k_1(y_1) - k_2(y_1)\}; \\
\Delta_v &\equiv \{(x, y) \mid y_1 < y < y_2, -Q_2 < v(x, y) < Q_1\} = \Delta_w \cup \Delta_u.
\end{aligned}$$

Note also that if $(x, y) \in \Delta_w \cap \Delta_u$, i.e. if $y_1 < y < y_2$ and $0 < (y - y_1)\cot\beta - x < L$ with $L \equiv \min\{k_1(y_1) - k_2(y_1), \frac{Q_1 + Q_2}{\lambda y_1 \sin\beta}\}$, then

$$\begin{aligned}
&|u(x - (y - y_1)\cot\beta, y_1) - w_0(x, y, k_1(y_1), y_1)| \\
&= \left| \int_{k_1(y_1)}^{x - (y - y_1)\cot\beta} (u_x(z, y_1) - \lambda y_1 \sin\beta) dz \right. \\
(3.17) \quad &\quad \left. - \int_{k_1(y_1) + (y - y_1)\cot\beta}^x (w_{0,x}(z, y, k_1(y_1), y_1) - \lambda y_1 \sin\beta) dz \right| \\
&\leq \sqrt{\frac{2I(y_0)}{g(\lambda^2)}} \cdot \sqrt{L} + \lambda(y_2 - y_1) \sin\beta \cdot L.
\end{aligned}$$

When (x, y) is to the left of $\Delta_w \cap \Delta_u$, one of the two functions $u(x - (y - y_1)\cot\beta, y_1)$ and w_0 is equal to $-Q_2$; the monotonicity of u and w_0 in x then implies that (3.17) also holds at (x, y) . Similarly, it holds for any point (x, y) to the right of $\Delta_w \cap \Delta_u$.

Using (2.12), we deduce that

$$\begin{aligned}
\iint_{\Delta_v} G(y, \vec{e}, v) &\leq \frac{C}{y_1} \iint_{\Delta_v} |\nabla v - \lambda y \vec{e}|^2 \\
&\leq \frac{4C}{y_1} \left[\iint_{\Delta_w} (1 - \zeta)^2 |\nabla w_0 - \lambda y \vec{e}|^2 + \iint_{\Delta_u} \zeta^2 |\nabla u(x - (y - y_1)\cot\beta, y_1) - \lambda y \vec{e}|^2 \right. \\
&\quad \left. + \iint_{\Delta_v} |u(x - (y - y_1)\cot\beta, y_1) - w_0(x, y, k_1(y_1), y_1)|^2 |\nabla \zeta|^2 + \iint_{\Delta_v \setminus (\Delta_w \cap \Delta_u)} |\lambda y \vec{e}|^2 \right].
\end{aligned}$$

Take $y_2 = y_1 + 1/y_1$. Estimating the first term by the explicit form of w_0 in (2.13), the second term by the inequality (3.14), the third term by (3.17), and the last term by (3.16), we then obtain

$$\iint_{\Delta_v} G(y, \vec{e}, v) \leq C \left[\frac{1}{y_0^3} + \frac{I(y_0)}{y_0^2} + \sqrt{\frac{I(y_0)}{y_0^3}} \right].$$

Clearly $v \in \mathcal{K}_{Q_1, Q_2}$, so $J(u) \leq J(v)$. Using (2.14), we thus conclude that

$$I(2y_0) \leq \iint_{\{y_0 < y\}} G(y, \vec{e}, u) \leq \frac{C}{y_0^3} + \frac{1}{16} I(y_0).$$

The lemma now follows by mathematical induction on $y_0 = 2^i, i = 1, 2, \dots$.

THEOREM 3.11. When $s \rightarrow +\infty$, the following holds:

$$(3.18) \quad s(k_1(s) - k_2(s)) \longrightarrow \frac{Q_1 + Q_2}{\lambda \sin \beta};$$

$$(3.19) \quad s(k_i(s + \frac{y}{s}) - k_i(s)) \longrightarrow y \cot \beta, \quad \forall y \text{ bounded, } i = 1, 2;$$

$$(3.20) \quad \left(\frac{\nabla u}{s} - \lambda \vec{e} \chi_{\{-Q_2 < u < Q_1\}} \right)(x, s) \longrightarrow 0 \quad \text{uniformly in } x;$$

$$(3.21) \quad k_i(s) = s \cot \beta + o(\ln s).$$

Proof. Setting $u_s(x, y) = u(k_1(s) + x/s, s + y/s)$, using the estimate (3.13), and applying the method used in the proof of Theorem 3.7, we obtain (3.18)–(3.20).

For (3.21), we set $v_s(x, y) = u(k_1(s) + x \cos \beta - y \sin \beta/s, s + x \sin \beta + y \cos \beta/s)$. As the above, in any bounded domain $\Omega \in \mathbb{R}^2$,

$$(3.22) \quad v_s \longrightarrow v_0 \equiv \max \{Q_1 - \max\{\lambda y, 0\}, -Q_2\} \quad \text{weakly in } H^1(\Omega).$$

Recalling Lemma 3.3, we see that $\{v_s : 1 < s < \infty\}$ is a bounded set in the Banach space

$$\mathcal{W}(\Omega) \equiv \{v \in H^1(\Omega) \mid v_y \in L^\infty(\Omega)\}$$

with the norm

$$\|v\|_{\mathcal{W}(\Omega)} \equiv \|v\|_{H^1(\Omega)} + \|v_y\|_{L^\infty(\Omega)}.$$

One can show that $\mathcal{W}(\Omega) \subset C^{\frac{1}{3}}(\Omega)$. Hence, the convergence in (3.22) is uniform in $C^{\frac{1}{4}}(\Omega)$ and therefore

$$\partial\{-Q_2 < v_s < Q_1\} \subset \{(x, y) \mid y \leq o(1) \text{ or } y \geq \frac{Q_1 + Q_2}{\lambda} + o(1)\}.$$

In combination with (3.18), this implies that

$$\partial\{-Q_2 < v_s < Q_1\} = \{(x, y) \mid y = o(1) \text{ or } y = \frac{Q_1 + Q_2}{\lambda} + o(1)\}.$$

According to the definitions of v_s , $k_1(s)$, and $k_2(s)$, the free boundaries of v_s are given by

$$\{(x, y) \mid k_1(s) + x \cos \beta - \frac{y}{s} \sin \beta = k_i(s + x \sin \beta + \frac{y}{s} \cos \beta), \ i = 1 \text{ or } 2\}.$$

Hence, replacing y by $o(1)$, we get

$$k_1(s) + x \cos \beta - o(1/s) = k_1(s + x \sin \beta + o(1/s));$$

similarly, replacing y by $(Q_1 + Q_2)/\lambda + o(1)$, we get

$$k_1(s) + x \cos \beta - \frac{Q_1 + Q_2 + o(1)}{\lambda s} \sin \beta = k_2(s + x \sin \beta + \frac{Q_1 + Q_2 + o(1)}{\lambda s} \cos \beta).$$

Setting $z_1 = x \sin \beta + o(1/s)$ and $z_2 = x \sin \beta + \frac{Q_1 + Q_2}{\lambda s} \cos \beta + o(1/s)$, we then obtain

$$\begin{aligned} k_1(s + z_1) &= k_1(s) + z_1 \cot \beta + o(1/s), \\ k_2(s + z_2) &= k_1(s) - \frac{Q_1 + Q_2}{\lambda s \sin \beta} + z_2 \cot \beta + o(1/s). \end{aligned}$$

Taking $z_1 = 1$ and $z_2 = 1$, and summing over $s = n, n + 1, \dots, 2n$, we find that

$$k_i(2n) = k_i(n) + n \cot \beta + o(1), \quad i = 1, 2.$$

Again taking $n = 2^j$ and summing over $j = 1, 2, \dots$, we get (3.21).

So far we have proved that for any pair of parameters (Q_1, Q_2) satisfying (2.10), the solution of the variational problem (2.9) has all the properties required by the solution of the jet problem except the continuous fit at $y = 1$. In next section, we shall prove that for an appropriate pair of parameters (Q_1^0, Q_2^0) , the free boundaries of $u_{Q_1^0, Q_2^0}$ fit continuously, i.e.

$$k_{i, Q_1^0, Q_2^0}(1+) = k_{i, Q_1^0, Q_2^0}(1-) = \frac{(-1)^{i+1}}{\sin \beta} \quad \text{for } i = 1, 2.$$

4. Continuous fit. In order to establish continuous fit, we need:

LEMMA 4.1. *If for $i = 1, 2$, $Q_i^n \rightarrow Q_i$, as $n \rightarrow \infty$, then*

$$(4.1) \quad u_{Q_1^n, Q_2^n} \longrightarrow u_{Q_1, Q_2} \quad \text{uniformly in every compact subset of } \mathbb{R}^{2+},$$

$$(4.2) \quad k_{i, Q_1^n, Q_2^n}(y) \longrightarrow k_{i, Q_1, Q_2}(y) \quad \text{for } h_{i, Q_1, Q_2} < y \leq 1-, 1+ \leq y < +\infty, i = 1, 2.$$

Proof. Since any limit of the minimizers $u_{Q_1^n, Q_2^n}$ is a minimizer of (2.9) with the pair of parameters (Q_1, Q_2) and the minimizer is unique (note that a limit of normalized minimizers is also a normalized minimizer), we get (4.1). Consequently, from Lemma 3.1 and Lemma 3.5 we obtain (4.2) for $y \neq 1\pm$, and

$$\begin{aligned} \limsup_{n \rightarrow +\infty} k_{1, Q_1^n, Q_2^n}(1\pm) &\leq k_{1, Q_1, Q_2}(1\pm), \\ \liminf_{n \rightarrow +\infty} k_{2, Q_1^n, Q_2^n}(1\pm) &\geq k_{2, Q_1, Q_2}(1\pm). \end{aligned}$$

If (4.2) does not hold at $y = 1\pm$, say

$$\liminf_{n \rightarrow +\infty} k_{1, Q_1^n, Q_2^n}(1-) \leq k_{1, Q_1, Q_2}(1-) - \alpha$$

for some $\alpha > 0$, then using the theorem of convergence of free boundaries (see [10, pp. 288–292]), we deduce that

$$\frac{\partial u_{Q_1, Q_2}}{\partial y}(x, 1-) = \lambda \quad \text{for } k_{1, Q_1, Q_2} - \alpha < x < k_{1, Q_1, Q_2}.$$

It follows from (3.11) and (3.12) that $u_{Q_1, Q_2} = Q_1 + \lambda(y^2 - 1)/2$, a contradiction.

We shall also need the following comparison lemma:

LEMMA 4.2. If $Q_1 > 0, Q_2 < \lambda/2$, then for any $0 < \epsilon \leq \min(Q_1, \lambda/2 - Q_2)$,

$$(4.3) \quad u_{Q_1-\epsilon, Q_2+\epsilon} + \epsilon \geq u_{Q_1, Q_2} \quad \text{in } \mathbb{R}^{2+},$$

$$(4.4) \quad k_{i, Q_1-\epsilon, Q_2+\epsilon}(y) < k_{i, Q_1, Q_2}(y) \quad \text{for } h_{i, Q_1, Q_2} < y \leq 1-, \quad 1+ \leq y < +\infty, \quad i = 1, 2.$$

Proof. It is easy to check that $u_{\epsilon, Q_1, Q_2} \equiv u_{Q_1-\epsilon, Q_2+\epsilon} + \epsilon$ is a minimizer of the following variational problem:

$$\text{Find } u \in \mathcal{K}_{\epsilon, Q_1, Q_2} \text{ such that } J(u) = \min_{v \in \mathcal{K}_{\epsilon, Q_1, Q_2}} J(v)$$

with

$$\mathcal{K}_{\epsilon, Q_1, Q_2} = \left\{ v \in H_{loc}^1(\mathbb{R}^{2+}) \mid v = Q_1 \text{ on } l_1, \quad v = -Q_2 \text{ on } l_2, \quad \text{and } v = \epsilon \text{ on } x\text{-axis} \right\}.$$

The inequality (4.3) thus follows from the same proof as the one for Theorem 2.5 or as in [10, pp. 318–320]. Consequently,

$$k_{i, Q_1-\epsilon, Q_2+\epsilon}(y) \leq k_{i, Q_1, Q_2}(y), \quad \forall 0 < y \leq 1-, \quad 1+ \leq y < +\infty, \quad i = 1, 2.$$

Furthermore, using Lemma 2.1 and applying the strong maximum principle, we can get the strict inequality (4.4) (see [10, p. 321]) for $y \neq 1\pm$; for $y = 1\pm$, we get the same assertion if we use the smooth fit theorem 6.1 of [6, pp. 103–111] and Corollary 11.4 of [10, p. 334].

LEMMA 4.3. There exists a small constant $c_0 > 0$ such that

$$(4.5) \quad Q_1 + Q_2 < c_0$$

implies that

$$(4.6) \quad \min \{ k_{1, Q_1, Q_2}(1\pm), -k_{2, Q_1, Q_2}(1\pm) \} < \frac{1}{\sin \beta},$$

i.e. either $\partial\{u_{Q_1, Q_2} < Q_1\}$ does not intersect l_1 or $\partial\{u_{Q_1, Q_2} > -Q_2\}$ does not intersect l_2 .

Proof. Since for $0 < r < \frac{1}{2}$,

$$\frac{1}{r} \left(\int_{\partial B_r(A_1)} (Q_1 - u)^2 \right)^{\frac{1}{2}} \leq \frac{Q_1 + Q_2}{r} \quad (A_1 = (k_{1, Q_1, Q_2}(1-), 1)),$$

the nondegeneracy lemma implies that $B_{(Q_1+Q_2)/c'}(A_1)$ intersects $\{u = -Q_2\}$ for some small positive constant c' independent of Q_1 and Q_2 . Similarly, $B_{(Q_1+Q_2)/c'}(B_1)$ ($B_1 = (k_{2, Q_1, Q_2}(1-), 1)$) intersects $\{u = Q_1\}$. Assume that

$$(4.7) \quad Q_1 + Q_2 < \frac{c'}{2 \sin \beta}.$$

By the bounded gradient lemma, $\nabla u/y$ is bounded by a constant depending only on λ . Hence,

$$\sup_{K_3} |\nabla l| \leq 2 \left(\sup_{\partial K_3 \cap \{y=\alpha_2\}} |w_{0,x}| + \sup_{\partial K_3 \cap \{x=-\frac{1}{2\sin\beta}\}} |u_y| \right) \leq C.$$

It follows that

$$\iint_{K_3} G(y, \vec{e}, v) \leq \sup_{K_3} G(y, \vec{e}, v) \iint_{K_3} 1 \leq C(Q_1 + Q_2)^2.$$

Combining this with (2.14), we get

$$(4.9) \quad \iint_{K_2+K_3} G(y, e, v) \leq C(Q_1 + Q_2)^2$$

for some constant C independent of Q_1 and Q_2 . On the other hand, by (2.12),

$$\begin{aligned} \iint_{K_2+K_3} G(y, \vec{e}, u) &\geq g(\lambda^2) \iint_{\mathbf{O} \cap \{-Q_2 < u < Q_1\}} y \left| \frac{\nabla u}{y} - \lambda \vec{e} \right|^2 \\ &= g(\lambda^2) \iint_{\mathbf{O} \cap \{-Q_2 < u < Q_1\}} \left\{ \left[\frac{|\nabla u|^2}{y} + \lambda^2 y + 2\lambda \cos \beta \frac{\partial u}{\partial y} \right] - 2\lambda \sin \beta \frac{\partial u}{\partial x} \right\}. \end{aligned}$$

Using Cauchy's inequality and noting that $\nabla u = 0$ a.e. in $\{u = -Q_2 \text{ or } Q_1\}$, we deduce that

$$(4.10) \quad \iint_{K_2+K_3} G(y, \vec{e}, u) \geq g(\lambda^2) \sin^2 \beta \iint_{\mathbf{O}} \frac{|\nabla u|^2}{y} - 2\lambda g(\lambda^2) \sin \beta \iint_{\mathbf{O}} \frac{\partial u}{\partial x}.$$

Due to Lemma 2.1, the first integral on the right hand side of (4.10) can be estimated by

$$\begin{aligned} \iint_{\mathbf{O}} \frac{|\nabla u|^2}{y} &\geq \left(\sup_{\mathbf{O}} g \left(\frac{|\nabla u|^2}{y^2} \right) \right)^{-1} \iint_{\mathbf{O}} \left(g \left(\frac{|\nabla u|^2}{y^2} \right) \frac{\nabla u}{y} \right) \cdot \nabla u \\ &= \left(\sup_{\mathbf{O}} g \left(\frac{|\nabla u|^2}{y^2} \right) \right)^{-1} \int_{\partial(\mathbf{O} \cap \{-Q_2 < u < Q_1\})} g \left(\frac{|\nabla u|^2}{y^2} \right) \frac{1}{y} \frac{\partial u}{\partial \nu} u \, dl \\ (4.11) \quad &\geq \left(\sup_{\mathbf{O}} g \left(\frac{|\nabla u|^2}{y^2} \right) \right)^{-1} \left\{ g(\lambda^2) \lambda \left[\int_{\overline{\mathbf{O}} \cap \partial\{u < Q_1\}} Q_1 \, dl + \int_{\overline{\mathbf{O}} \cap \partial\{u > -Q_2\}} Q_2 \, dl \right] \right. \\ &\quad \left. - \sup_{\mathbf{O}} \left| g \left(\frac{|\nabla u|^2}{y^2} \right) \frac{\nabla u}{y} \right| \left[\int_{\overline{\mathbf{O}} \cap \{x = -\frac{1}{2\sin\beta}\}} |u| \, dl + \int_{\overline{\mathbf{O}} \cap \{x = \frac{1}{2\sin\beta}\}} |u| \, dl \right] \right\} \\ &\geq c(Q_1 + Q_2) - C(Q_1 + Q_2)^2. \end{aligned}$$

The second integral on the right hand side of (4.10) can be estimated by

$$(4.12) \quad \iint_{\mathbf{O}} \frac{\partial u}{\partial x} = \int_{\alpha_1}^{\beta_2} \left[u\left(\frac{1}{2\sin\beta}, y\right) - u\left(-\frac{1}{2\sin\beta}, y\right) \right] dy \leq C(Q_1 + Q_2)^2.$$

Putting (4.10), (4.11), and (4.12) together, we obtain

$$\iint_{K_2+K_3} G(y, \vec{e}, u) \geq c(Q_1 + Q_2) - C(Q_1 + Q_2)^2$$

for some positive constants c and C independent of Q_1 and Q_2 . This inequality, together with (4.8) and (4.9), yields

$$c(Q_1 + Q_2) - C(Q_1 + Q_2)^2 \leq 0,$$

i.e.

$$(Q_1 + Q_2) \geq \frac{c}{C}.$$

Similarly, we can get the above inequality for the case (b). Combining this with (4.7), we conclude that if (4.6) is not true, then $(Q_1 + Q_2)$ cannot be smaller than $\min\{\frac{c}{C}, \frac{c'}{2\sin\beta}\}$. This implies the assertion of the lemma.

THEOREM 4.4. (Continuous Fit) *There exists one and only one pair (Q_1^0, Q_2^0) such that*

$$(4.13) \quad \begin{cases} k_{1, Q_1^0, Q_2^0}(1\pm) = \frac{1}{\sin\beta}, \\ k_{2, Q_1^0, Q_2^0}(1\pm) = -\frac{1}{\sin\beta}. \end{cases}$$

Moreover the free boundaries Γ_{1, Q_1^0, Q_2^0} and Γ_{2, Q_1^0, Q_2^0} are smooth ($C^{2+\alpha}$) in $\mathbb{R}^{2+}$ and

$$(4.14) \quad \frac{Q_1^0}{Q_2^0} = \tan^2 \frac{\beta}{2}.$$

Proof. Let

$$\Sigma = \{(Q_1, Q_2) \mid 0 \leq Q_1, Q_2 \leq \lambda/2, k_{1, Q_1, Q_2}(1\pm) \geq \frac{1}{\sin\beta}, k_{2, Q_1, Q_2}(1\pm) \leq -\frac{1}{\sin\beta}\}.$$

By Corollary 3.9 and Lemma 4.1, Σ is nonempty and closed. Set

$$Q = \inf\{Q_1 + Q_2 \mid (Q_1, Q_2) \in \Sigma\}$$

and denote by (Q_1^0, Q_2^0) a pair in Σ , satisfying $Q_1^0 + Q_2^0 = Q$. By Lemma 4.3 and Corollary 3.9, $Q > 0$, $Q_1^0 < \lambda/2$, and $Q_2^0 < \lambda/2$.

Without loss of generality, we may assume that $Q_1^0 > 0$ and $Q_2^0 > 0$. In fact, suppose for example that Q_2^0 is zero. Then consider another pair $(Q_1^0 - \epsilon, \epsilon)$ with ϵ being a small

positive constant satisfying $(Q_1 - \epsilon)/\epsilon > \tan^2 \frac{\beta}{2}$, i.e. $0 < \epsilon < Q_1^0 \cos^2 \frac{\beta}{2}$. Lemma 2.3 case (1) shows that

$$k_{1, Q_1^0 - \epsilon, \epsilon}(1 \pm) \geq \frac{1}{\sin \beta}$$

and Lemma 4.2 implies that

$$k_{2, Q_1^0 - \epsilon, \epsilon}(1 \pm) < k_{2, Q_1^0, 0} \leq \frac{1}{\sin \beta}.$$

That is to say, the new pair $(Q_1^0 - \epsilon, \epsilon) \in \Sigma$ and its two parameters are positive.

Now we want to prove that for such a pair (Q_1^0, Q_2^0) (with $Q_1^0 > 0, Q_2^0 > 0$) the assertions of the theorem hold. Indeed, suppose for contradiction that (4.13) is not true, say

$$k_{1, Q_1^0, Q_2^0}(1-) > \frac{1}{\sin \beta}.$$

By Lemma 4.1, k_{i, Q_1, Q_2} is continuous with respect to Q_1 and Q_2 . Hence we have

$$k_{1, Q_1^0 - \epsilon, Q_2^0 + \epsilon}(1-) > \frac{1}{\sin \beta}$$

for all $\epsilon > 0$ small enough. On the other hand, from Lemma 4.2,

$$k_{2, Q_1^0 - \epsilon, Q_2^0 + \epsilon}(1-) < k_{2, Q_1^0, Q_2^0}(1-) \leq -\frac{1}{\sin \beta}.$$

Using Lemma 4.1 again, we also have

$$k_{1, Q_1^0 - \epsilon, Q_2^0 + \epsilon - \eta}(1-) > \frac{1}{\sin \beta} \quad \text{and} \quad k_{2, Q_1^0 - \epsilon, Q_2^0 + \epsilon - \eta}(1-) < -\frac{1}{\sin \beta}$$

for $\eta > 0$ small enough. Note from Lemma 2.1 that

$$\min \{k_{1, Q_1, Q_2}(1-), k_{1, Q_1, Q_2}(1+)\} < \frac{1}{\sin \beta} \quad \text{implies} \quad k_{1, Q_1, Q_2}(1-) = k_{1, Q_1, Q_2}(1+)$$

and

$$\max \{k_{2, Q_1, Q_2}(1-), k_{2, Q_1, Q_2}(1+)\} > -\frac{1}{\sin \beta} \quad \text{implies} \quad k_{2, Q_1, Q_2}(1-) = k_{2, Q_1, Q_2}(1+);$$

therefore $(Q_1^0 - \epsilon, Q_2^0 + \epsilon - \eta) \in \Sigma$, contradicting the definition of Q . Hence, (4.13) holds.

We next establish the relation (4.14). By Lemma 4.2,

$$(4.15) \quad \begin{aligned} k_{1, Q_1^0 - \epsilon, Q_2^0 + \epsilon}(1-) &< k_{1, Q_1^0, Q_2^0}(1-) = \frac{1}{\sin \beta}, \\ k_{2, Q_1^0 + \epsilon, Q_2^0 - \epsilon}(1-) &> k_{2, Q_1^0, Q_2^0}(1-) = -\frac{1}{\sin \beta}, \end{aligned}$$

so that by Lemma 2.3,

$$\frac{Q_1^0 - \epsilon}{Q_2^0 + \epsilon} \leq \tan^2 \frac{\beta}{2} \quad \text{and} \quad \frac{Q_1^0 + \epsilon}{Q_2^0 - \epsilon} \geq \tan^2 \frac{\beta}{2},$$

and the relation (4.14) follows by letting $\epsilon \rightarrow 0$.

We proceed to establish the smooth fit. By inequality (4.15), $\Gamma_{1, Q_1^0 - \epsilon, Q_2^0 + \epsilon}$ does not touch l_1 and therefore is smooth.

We claim that the restriction of u_ϵ ($\equiv u_{Q_1^0 - \epsilon, Q_2^0 + \epsilon} + \epsilon$) to $B_r(A)$ is a minimizer of the functional

$$(4.16) \quad \tilde{J}(v) = \iint_{B_r(A)} y \left[F \left(\frac{|\nabla v|^2}{y^2} \right) + \Phi(\lambda^2) \cdot \chi_{\{v < Q_1^0\}} \right]$$

with respect to the function space

$$K_\epsilon = \{v \in H^1(B_r(A)) \mid v = u_{Q_1^0 - \epsilon, Q_2^0 + \epsilon} + \epsilon \text{ on } \partial B_r(A)\}$$

if r is small enough. Indeed, recalling Theorem 6.3 in [5, p. 43] and chapter 3 of [10], we know that the above minimization problem has a unique minimizer $\tilde{u}_\epsilon$ whose free boundary is smooth; furthermore $\tilde{u}_{\epsilon, x} \geq 0$. Let

$$\tau \equiv \inf\{c > 0 \mid u_\epsilon(x - c, y) \leq \tilde{u}_\epsilon(x, y), \forall (x, y) \in B_r(A)\}.$$

By the monotonicity (in x) of u_ϵ and $\tilde{u}_\epsilon$, and the strong maximum principle, we find that $\tau = 0$ (for more details, cf. the end of section 5 below). Thus $u_\epsilon \leq \tilde{u}_\epsilon$ in $B_r(A)$. Similarly we can establish the reverse inequality. Therefore $u_\epsilon = \tilde{u}_\epsilon$.

Letting $\epsilon \rightarrow 0$ and using the argument in the proof of Lemma 4.1, we find that the restriction of $u_{Q_1^0, Q_2^0}$ to $B_r(A)$ is a minimizer of the functional (4.16) with respect to the function space

$$K_0 = \{v \in H^1(B_r(A)) \mid v = u_{Q_1^0, Q_2^0} \text{ on } \partial B_r(A)\}.$$

Hence applying Theorem 6.3 in [5, p. 43], we conclude that $\Gamma_{Q_1^0, Q_2^0}$ is smooth in a neighborhood of the point A (smooth fit). Similarly we can show that Γ_{2, Q_1^0, Q_2^0} is smooth in a neighborhood of the point B . Hence, by Lemma 2.1, the free boundaries Γ_{1, Q_1^0, Q_2^0} and Γ_{2, Q_1^0, Q_2^0} are smooth.

For the uniqueness of the parameters, suppose that there is another pair (Q'_1, Q'_2) which has the same properties as (Q_1^0, Q_2^0) . By (4.14), we can assume that $Q_1^0 > Q'_1$, $Q_2^0 > Q'_2$. Going through the same proof as the one for Lemma 4.2, we deduce that

$$u_{Q'_1, Q'_2} \leq u_{Q_1^0, Q_2^0} + (Q_2^0 - Q'_2).$$

It follows that

$$-\lambda = \frac{\partial u_{Q'_1, Q'_2}}{\partial \nu}(-\frac{1}{\sin \beta}, 1) < \frac{\partial u_{Q_1^0, Q_2^0}}{\partial \nu}(-\frac{1}{\sin \beta}, 1) = -\lambda,$$

a contradiction.

5. Completion Proof of Theorem 1.2. Firstly, we want to show that the five-tuple $(u_{Q_1^0, Q_2^0}, \Gamma_{1, Q_1^0, Q_2^0}, \Gamma_{2, Q_1^0, Q_2^0}, Q_1^0, Q_2^0)$ is a solution of the jet problem. Since actually all the assertions (1.13)–(1.21) except (1.15) and (1.21) have already been proved, it only remains to prove (1.15) and (1.21). In the following, the subscript (Q_1^0, Q_2^0) is omitted.

To prove (1.21), we need lemma 7.2 of [6]:

LEMMA 5.1. *On any stream line $\psi = \text{const}$, which bounds fluid from one side,*

$$(5.1) \quad \frac{\partial q}{\partial \nu} + kq = 0,$$

where $q = g(\frac{|\nabla \psi|^2}{y^2})\frac{|\nabla \psi|}{y}$ is the velocity of the fluid, ν is the outward normal, and k is the curvature of the streamline, taken positive when the stream line is concave to the fluid.

Set $M = \sup_G \frac{|\nabla u|}{y}$ and let $\{X_m\}_{m=1}^{m=\infty}$ be a sequence such that $\frac{|\nabla u|}{y}(X_m)$ tends to M . The argument in [6, pp. 113–115] (here we need Lemma 5.1) shows that $\{X_m\}_{m=1}^{m=\infty}$ cannot have limit points on the x -axis as well as in G . Therefore either X_m has an accumulation point $X = (x, y)$ in $\Gamma_1 \cup \Gamma_2$ or $|X_m| \rightarrow +\infty$. In the first case, Lemma 2.1 shows that $M = \frac{|\nabla u|}{y}(X) = \lambda$. In the second case, the same assertion $M = \lambda$ follows from Theorem 3.8 and Theorem 3.11. We thus obtain (1.21).

COROLLARY 5.2. *The free boundaries are convex to the fluid.*

Proof. Using (1.18) and (1.21), and applying the maximum principle on q (see [12, pp. 270–271]), we deduce that

$$\frac{\partial q}{\partial \nu} > 0 \quad \text{on } \Gamma_1 \cup \Gamma_2.$$

The assertion of the corollary then follows from Lemma 5.1.

By virtue of the convexity of the free boundaries, the estimate (3.21) can be strengthened to (1.15):

THEOREM 5.3. *There exists a constant b , such that for $i = 1, 2$,*

$$(5.2) \quad \lim_{y \rightarrow +\infty} (k_i(y) - y \cot \beta) = b.$$

Consequently, the five-tuple $(u(x+b, y), \Gamma_1^b, \Gamma_2^b, Q_1^0, Q_2^0)$ is a solution of the jet problem with parameters $(\lambda, \beta, m(\lambda, \beta))$, where

$$\Gamma_i^b \equiv \{(x, y) \in \mathbb{R}^{2+} \mid x = k_i(y) - b, h_i < y < +\infty\}, \quad i = 1, 2,$$

and

$$(5.3) \quad m(\lambda, \beta) = 2\pi(Q_1^0 + Q_2^0).$$

The proof follows directly from Theorem 3.11 and Corollary 5.2.

We now establish the relations (1.24) and (1.25). In view of (1.22) and (1.23), we only need to prove the following lemma:

LEMMA 5.4. *The constants h_1 and h_2 appearing in (1.13) satisfy $\pi\lambda(h_2^2 - h_1^2) = m \cos \beta$.*

Proof. For the solution constructed above (in Theorem 5.3), this lemma can be easily obtained from (4.14). However, since we shall use the relation (1.24) for the proof of the uniqueness of the solutions, we need a proof which is independent of the construction of solutions. The proof given below is based on the conservation of momentum in the x -direction.

For $M \gg 1$, set $I_1^M = \{(M, y) \mid 0 < y < m(M)\}$, $I_2^M = \{(-M, y) \mid 0 < y < m(-M)\}$, and $I_3^M = \{(x, M) \mid k_2(M) < x < k_1(M)\}$, where the function m is defined in Theorem 3.8. Let Ω_M be the domain in $\mathbb{R}^{2+}$ bounded by I_1^M , Γ_1 , I_3^M , Γ_2 , I_2^M , and the x -axis. Then

$$\begin{aligned} \iint_{\Omega_M} y \left(\Phi \left(\frac{|\nabla \psi|^2}{y^2} \right) \right)_x &= \int_{\partial \Omega_M} y \left[\Phi \left(\frac{|\nabla \psi|^2}{y^2} \right) - \Phi(\lambda^2) \right] \nu_x dl \quad (\text{by divergence theorem}) \\ &= \int_{I_1^M + I_2^M + I_3^M} y \left(\Phi \left(\frac{|\nabla \psi|^2}{y^2} \right) - \Phi(\lambda^2) \right) \nu_x dl \quad (\text{by (1.18)}) \\ &\longrightarrow 0, \quad \text{as } M \rightarrow +\infty \quad (\text{by (1.19) and (1.20)}). \end{aligned}$$

On the other hand, using definition (2.2), we deduce that

$$\begin{aligned} \iint_{\Omega_M} y \left(\Phi \left(\frac{|\nabla \psi|^2}{y^2} \right) \right)_x &= \iint_{\Omega_M} y \left(2 \frac{|\nabla \psi|^2}{y^2} g' + g \right) \left(\frac{|\nabla \psi|^2}{y^2} \right)_x \\ &= \iint_{\Omega_M} 2 \frac{\psi_x^2}{y} g_x + 2 \frac{\psi_y^2}{y} g_x + 2g\psi_x \left(\frac{\psi_x}{y} \right)_x + g \left(\frac{\psi_y^2}{y} \right)_x \\ &= \iint_{\Omega} 2\psi_x \left(g \frac{\psi_x}{y} \right)_x + 2 \left(g \frac{\psi_y^2}{y} \right)_x - g \left(\frac{\psi_y^2}{y} \right)_x \\ &= \iint_{\Omega_M} 2\psi_x \mathcal{L}\psi - 2\psi_x \left(g \frac{\psi_y}{y} \right)_y + 2 \left(g \frac{\psi_y^2}{y} \right)_x - 2g \frac{\psi_y \psi_{xy}}{y} \end{aligned}$$

$$\begin{aligned}
&= 2 \iint_{\Omega^M} \left(g \frac{\psi_y^2}{y} \right)_x - \left(g \frac{\psi_x \psi_y}{y} \right)_y \\
&= 2 \int_{\partial\Omega_M} \left(g \frac{\psi_y^2}{y} \nu_x - g \frac{\psi_x \psi_y}{y} \nu_y \right) dl \\
&= 2 \int_{\partial\Omega_M} g \frac{\psi_y}{y} \frac{\partial\psi}{\partial l} dl \\
&= 2 \int_{I_1^M} g \frac{\psi_y^2}{y} dy - 2 \int_{I_2^M} g \frac{\psi_y^2}{y} dy - 2 \int_{I_3^M} g \frac{\psi_y \psi_x}{y} dx \\
&\longrightarrow g(\lambda^2) \lambda^2 h_1^2 - g(\lambda^2) \lambda^2 h_2^2 + 2g(\lambda^2) \lambda \sin \beta \cos \beta \frac{m}{2\pi \sin \beta}
\end{aligned}$$

as $M \rightarrow +\infty$, where the last step is obtained by (1.19), (1.20), and (1.17). Equating the above two limits, the lemma easily follows.

Finally, we prove the uniqueness. We achieve this by showing that every solution $(\psi, \Gamma_1, \Gamma_2, Q_1, Q_2)$ of the jet problem coincides with the solution of the variational problem (2.9). By Lemma 1.3, it suffices to consider the special case $m = m(\lambda, \beta)$ given by (5.3). In this case, we deduce from (1.24), that $Q_1 = Q_1^0$ and $Q_2 = Q_2^0$.

Let b be the constant in (5.2). Denote $\psi(x - b, y)$ by $\psi_b(x, y)$ and the free boundaries of ψ_b by

$$\Gamma_{i, \psi_b} : x = k_{i, \psi_b}, \quad i = 1, 2.$$

It then suffices to show that $\psi_b = u$.

Let $u_\epsilon = \epsilon + u_{Q_1^0 - \epsilon, Q_2^0 + \epsilon}$ be the function in Lemma 4.2. From that lemma we know that

$$(5.4) \quad k_{i, u_\epsilon}(y) < k_{i, u}(y) \quad \text{for } \max\{h_{i, \epsilon}, h_1\} < y \leq 1-, 1+ \leq y < +\infty, \quad i = 1, 2,$$

where

$$(5.5) \quad h_{1, \epsilon} = \sqrt{\frac{2(Q_1^0 - \epsilon)}{\lambda}} \quad \text{and} \quad h_{2, \epsilon} = \sqrt{\frac{2(Q_2^0 + \epsilon)}{\lambda}}.$$

Noting that $h_{1, \epsilon} < h_1$ and $h_{2, \epsilon} > h_2$, we deduce that there exists a constant $0 < c < +\infty$ such that

$$(5.6) \quad k_{i, u_\epsilon}(y) \leq k_{i, \psi_b}(y) + c \quad \text{for } \max\{h_i, h_{i, \epsilon}\} < y < +\infty, \quad i = 1, 2.$$

Define for every $\epsilon > 0$,

$$c_\epsilon \equiv \inf \{ c > 0 \mid c \text{ satisfies (5.6)} \}.$$

Clearly, c_ϵ also satisfies (5.6) due to the continuity of the solutions. By the maximum principle,

$$(5.7) \quad u_\epsilon(x + c_\epsilon, y) \geq \psi_b(x, y) \quad \text{in } \mathbb{R}^{2+}.$$

Now we want to show that $c_\epsilon = 0$ for all $\epsilon > 0$. Let $\{y_n\}_{n=1}^{n=\infty}$ be a sequence satisfying

$$\text{either } k_{1,\psi_b}(y_n) + c_\epsilon - k_{1,u_\epsilon}(y_n) \searrow 0 \quad \text{or} \quad k_{2,\psi_b}(y_n) + c_\epsilon - k_{2,u_\epsilon}(y_n) \searrow 0.$$

If $c_\epsilon > 0$ for some $\epsilon > 0$, then from (1.15), (5.2), (5.4), and (5.5), it follows that $\{y_n\}_{n=1}^{n=\infty}$ is a bounded sequence and therefore has a point of accumulation y_ϵ , and then

$$(5.8) \quad k_{i,\psi_b}(y_\epsilon) + c_\epsilon = k_{i,u_\epsilon}(y_\epsilon) \quad \text{for } i = 1 \text{ or } i = 2.$$

We claim that necessarily

$$(5.9) \quad y_\epsilon = 1 \quad \text{and} \quad i = 2.$$

In fact, if $y_\epsilon \neq 1$, we can apply the strong maximum principle to $u_\epsilon(x + c_\epsilon, y) - \psi_b(x, y)$ at either $x = k_{1,\psi_b}(y_\epsilon)$ or $x = k_{2,\psi_b}(y_\epsilon)$ to get

$$\begin{aligned} \text{either } \lambda &= \frac{1}{y_\epsilon} \frac{\partial u_\epsilon}{\partial \nu}(x + c_\epsilon, y_\epsilon) < \frac{1}{y_\epsilon} \frac{\partial \psi_b}{\partial \nu}(x, y_\epsilon) = \lambda \\ \text{or } -\lambda &= \frac{1}{y_\epsilon} \frac{\partial u_\epsilon}{\partial \nu}(x + c_\epsilon, y_\epsilon) < \frac{1}{y_\epsilon} \frac{\partial \psi_b}{\partial \nu}(x, y_\epsilon) = -\lambda, \end{aligned}$$

a contradiction. Similarly, if $i = 1$, we can also get the same contradiction because in this case the maximum principle is still applicable at the point $(k_{1,\psi_b}(1), 1)$ (recall that k_{1,u_ϵ} is smooth). Therefore from (5.8), (5.9), and the second case of Lemma 2.3, we get

$$c_\epsilon = k_{2,u_\epsilon}(1) - k_{2,\psi_b}(1) = -\frac{1}{\sin \beta} - k_{2,\psi_b}(1) \equiv c_0$$

and by (5.7),

$$u(x + c_0, y) \geq \psi_b(x, y).$$

Note that $u(x + c_0, y) \not\equiv \psi_b(x, y)$ because of (1.15), (5.2), and $c_0 = c_\epsilon > 0$. We now apply the strong maximum principle to $u(x + c_0, y) - \psi_b(x, y)$ at the point $(k_{2,\psi_b}(1), 1)$ to get

$$-\lambda = \frac{\partial u}{\partial \nu}\left(-\frac{1}{\sin \beta}, 1\right) < \frac{\partial \psi_b}{\partial \nu}(k_{2,\psi_b}(1), 1) = -\lambda,$$

which is a contradiction. In conclusion, we must have $c_\epsilon = 0$ for all $\epsilon > 0$.

Letting $\epsilon \searrow 0$, we then obtain from (5.7), that $u(x, y) \geq \psi_b$. Similarly we can get the reverse inequality. Thus $u = \psi_b$.

REMARK 5.5. Actually, in defining the variational problem (2.9), we can make a more general choice for the rays l_1 and l_2 . In particular, for any positive numbers a and b we can take $l_1 = \{(x, b) \mid x \geq a\}$ and $l_2 = \{(x, b) \mid x \leq -a\}$ in (2.8) (of course, we have to modify the direction function $\vec{e}$). In this case, the solution is a scaling of the solution $u_{Q_1^0, Q_2^0}$ established in Theorem 5.3.

6. Collision and Impinging of Jets. The problem of the collision of two inviscid, compressible, irrotational, and axially symmetric jet flows (see Figure 6) is a nonstationary problem. We shall show how it can be reduced to a stationary problem which has been solved in the previous sections.

Figure 6

Denote by $\vec{U}$ the velocity vector of the fluid and by ρ its density. In the flow region,

$$(6.1) \quad \nabla \times \vec{U} = 0 \quad (\text{irrotational}),$$

$$(6.2) \quad \rho_t + \nabla \cdot (\rho \vec{U}) = 0 \quad (\text{conservation of mass}),$$

$$(6.3) \quad \vec{U}_t - (\vec{U} \cdot \nabla) \vec{U} + \nabla p / \rho = 0 \quad (\text{Euler equation}),$$

$$(6.4) \quad p = P(\rho) \quad (\text{state equation}).$$

From (6.1), it follows that $U = \nabla \phi$; ϕ is called the velocity potential. Let $q = \|\vec{U}\|$ be the speed of the fluid. From (6.3) and (6.4), we obtain Bernoulli's law:

$$\phi_t + \frac{1}{2} q^2 + \int_{\rho_{max}}^{\rho} \frac{1}{t} P(t) dt = \lambda_B \quad (\lambda_B \text{ is a constant}).$$

Assume that the flow region $\Omega(t)$ is surrounded by air. This implies that the pressure p is a constant on the interface of the fluid and air. By (6.4), the density ρ is also a constant on the interface; we denote this constant by ρ_∞ . Denote by $\Gamma_1(t)$ the connected left-half

of the interface and $\Gamma_2(t)$ the connected right-half of the interface. Bernoulli's law then implies that

$$(6.5) \quad \phi_t + \frac{1}{2}q^2 + \int_{\rho_{max}}^{\rho_{\infty}} \frac{1}{t}P(t) dt = \lambda_B \quad \text{on } \Gamma_1(t) \cup \Gamma_2(t), \quad \forall t \in (0, +\infty).$$

Note that the equation of continuity in the fluid means that the velocity of the fluid on the interface $\Gamma_1(t) \cup \Gamma_2(t)$ in the normal direction coincides with the velocity of the boundary of the fluid. Hence, if $\Gamma_1 \cup \Gamma_2$ has the form $\{X \in \mathbb{R}^3 \mid \Psi(X, t) = 0 \text{ with } \nabla \Psi(X, t) \neq 0\}$ for some C^2 function Ψ , then

$$(6.6) \quad \vec{U} \cdot \nabla \Psi = -\Psi_t \quad \text{on } \Gamma_1(t) \cup \Gamma_2(t), \quad \forall t \in (0, +\infty).$$

Now we consider the axially symmetric collision problem. As usual, we denote by the x -axis the axis of symmetry, by $X = (x, y)$ the generic point, where y is the distance to the x -axis.

Assume that the fluxes of the two jets are m_+ and m_- , and their velocities are v_+ and v_- ; for definiteness, we assume that $v_- > 0 > v_+$. Then

$$(6.7) \quad \lim_{N \rightarrow \infty} \Omega(t) \cap \{x = \pm N\} = \{0 \leq y < h_{\pm}\}, \quad \forall t \in (0, +\infty),$$

$$(6.8) \quad \lim_{N \rightarrow \infty} \rho(\pm N, y, t) = \rho_{\infty}, \quad 0 \leq y < h_{\pm}, \quad \forall t \in (0, +\infty),$$

$$(6.9) \quad \lim_{N \rightarrow \infty} \vec{U}(\pm N, y, t) = (v_{\pm})\vec{e}_x, \quad 0 \leq y < h_{\pm}, \quad \forall t \in (0, +\infty),$$

where $h_{\pm} \equiv \sqrt{m_{\pm}/(\pi v_{\pm})}$ and $\vec{e}_x = (1, 0)$.

We define a *solution of the axially symmetric collision problem* as a pair $(\vec{U}(X, t), \rho(X, t))$ which satisfies the equations (6.1)–(6.9) with some initial conditions.

Consider two cases: (1) $|v_+| \neq |v_-|$ and (2) $|v_-| = |v_+|$.

In the first case, we deduce from (6.5) that $\phi_t(\cdot, t)$ is not a constant function. Hence, for any initial conditions, there is no stationary solution for the collision problem.

In the second case, we find that the collision problem admits a stationary solution given by

$$(6.10) \quad \rho(X) = 1/g\left(\frac{|\nabla \psi(X)|^2}{y^2}\right) \quad \text{and} \quad \vec{U}(X) = \frac{1}{y\rho(X)}(\psi_y(X), -\psi_x(X)),$$

where $g(\cdot)$ is the function given by (1.8)–(1.10) with ρ_{max} being the solution of

$$\frac{1}{2}v_-^2 + \int_{\rho_{max}}^{\rho_{\infty}} \frac{1}{t}P(t) dt = 0$$

and $\psi(X)$ (together with Γ_1, Γ_2, Q_1 , and Q_2) is a solution (established in Theorem 1.2) of the jet problem with parameters $\lambda = (v_- - v_+)\rho_\infty/2$, $\beta = 2 \tan^{-1} \sqrt{m_+/m_-}$, and $m = m_+ + m_-$.

It is easy to see that the transformation

$$\begin{cases} \vec{V}(x, y, t) = \vec{U}(x + vt, y) - v\vec{e}_x, \\ \tilde{\rho}(x, y, t) = \rho(x + vt, y, t), \\ \tilde{\Psi}(x, y, t) = \Psi(x + vt, y, t), \\ \tilde{\phi}(x, y, t) = \phi(x + vt, y, t) - \frac{1}{2}v^2t, \\ \tilde{\psi}(x, y, t) = \psi(x + vt, y, t) \end{cases}$$

with $v = (v_- + v_+)/2$ transforms the first case into the second case. In fact, equation (6.9) becomes

$$\lim_{N \rightarrow \infty} \vec{V}(\pm N, y, t) = \pm(v_+ - v_-)\vec{e}_x,$$

while all the other equations (6.1)–(6.8) remain the same.

REMARK 6.1. So far we have solved the collision problem for the initial condition given by (6.10). It would be interesting to consider the collision problem (6.1)–(6.9) with general initial conditions.

The variational method described in sections 2–5 also applies to the axially symmetric impinging problem (see figure 7).

Figure 7

Denote by l_0 the positive x -axis which is the axis of symmetry, by N the wall $\{(x, y) \in \mathbb{R}^2 \mid x = K(y), 0 \leq y < +\infty\}$ on which the jet impinges from the right side, and by Ω the domain $\{(x, y) \in \mathbb{R}^2 \mid x \geq K(y), 0 < y < +\infty\}$. We assume that $K \in C^2[0, +\infty)$.

$K(0) = 0$, and $\lim_{y \rightarrow +\infty} K'(y) = \cot \beta$ for $0 \leq \beta \leq \pi/2$. Also we assume that the wall is concave to the fluid, i.e.

$$(6.10) \quad K''(y) \geq 0 \quad \forall 0 \leq y < +\infty.$$

The impinging problem with parameters λ and Q , $\lambda \in (0, \lambda_{cr})$ and $Q \in (0, +\infty)$, is to find a pair (ψ, Γ) satisfying the following conditions:

(1) Γ is a curve having the form

$$\Gamma \equiv \{(x, y) \in \Omega \mid x = k(y), h < y < +\infty\},$$

where $h = \sqrt{2Q/\lambda}$, $k \in C^{2+\alpha}(h, +\infty)$, $k(y) > K(y)$ for $h < y < +\infty$, and $\lim_{y \rightarrow h^+} k(y) = +\infty$;

(2) ψ is a function defined in the domain G bounded by N , l_0 , and Γ , and satisfies

$$\begin{cases} \mathcal{L}\psi = 0 & \text{in } G, \\ \psi = 0 & \text{on } N \cup l_0, \\ \psi = Q & \text{on } \Gamma, \\ \frac{\partial \psi}{\partial \nu} = \lambda & \text{on } \Gamma. \end{cases}$$

THEOREM 6.2. *For any parameters $\lambda \in (0, \lambda_{cr})$ and $Q \in (0, +\infty)$, there exists a unique solution for the impinging jet problem.*

Proof. Let (a, b) be a point in Ω . Introduce the line segment $l_{a,b} \equiv \{(x, b) \mid x \geq a\}$ and the function space

$$\mathcal{K}_{a,b,Q} \equiv \{v \in H_{loc}^1(\Omega) \mid v = 0 \text{ on } N \cup l_0, \quad v = Q \text{ on } l_{a,b}\}.$$

Define

$$\vec{e}(x, y) \equiv \begin{cases} (\sin \beta, -\cos \beta), & \text{if } y > b \quad \text{or} \quad x < (y - b) \cot \beta + a, \\ \frac{(-y, x - a + b \cot \beta)}{\sqrt{y^2 + (x - a + b \cot \beta)^2}} & \text{if } 0 < y < b \quad \text{and} \quad x > (y - b) \cot \beta + a; \end{cases}$$

$$G(y, \vec{e}, v) \equiv \left\{ y \left[F\left(\frac{|\nabla v|^2}{y^2}\right) + \Phi(\lambda^2) \right] - 2\lambda g(\lambda^2) \nabla v \cdot \vec{e} \right\} \chi_{\{v < Q\}}, \quad y > 0;$$

and

$$J(v) \equiv \iint_{\Omega} G(y, \vec{e}, v) \, dx dy.$$

We define the variational problem corresponding to the impinging problem as follows:

$$\text{Find } u \in \mathcal{K}_{a,b,Q} \text{ such that } J(u) = \min_{v \in \mathcal{K}_{a,b,Q}} J(v).$$

Using the method developed in sections 2–4, one can show that there exists a function $Q = Q(\lambda, a, b)$ defined in $(0, +\infty) \times \Omega$ such that the solution of the variational problem with parameters $(\lambda, a, b, Q(\lambda, a, b))$ is a solution of the impinging problem with parameters $(\lambda, Q(\lambda, a, b))$ and the free boundary Γ of the solution passes through the point (a, b) . Therefore one can prove Theorem 6.2 for all the parameters

$$(\lambda, Q) \in \{(\lambda, Q) \mid 0 < \lambda < +\infty \text{ and } Q = Q(\lambda, a, b) \text{ for some } (a, b) \in \Omega\}$$

by the method used in section 5. Hence the proof can be completed if we can show that for every $\lambda \in (0, +\infty)$, $\{Q(\lambda, a, b) \mid (a, b) \in \Omega\} = (0, +\infty)$; however this is a consequence of the following lemma:

LEMMA 6.3. *The function $Q(\lambda, \cdot)$ is continuous in Ω and satisfies*

$$(6.11) \quad Cb \operatorname{dist}\{(a, b), N \cup l_0\} \leq Q(\lambda, a, b) \leq \lambda b(a - k(b)),$$

where C is a constant depending only on λ .

Proof. The continuity of the function $Q(\lambda, a, b)$ follows from an argument similar to that used in the proof of Lemma 4.1.

As we did in section 5, we can also show that $\sup_G (|\nabla \psi|/y) = \lambda$; therefore

$$Q(\lambda, a, b) = \psi(a, b) = \int_{K(b)}^a \psi_x(x, b) dx \leq \lambda b(a - K(b)).$$

Since the point (a, b) is on the free boundary when $Q = Q(a, b)$, by the nondegeneracy lemma, we know that for any ball $B_r((a, b))$ in Ω with $0 < r \leq \frac{1}{2}b$,

$$Cb < \frac{1}{r} \left(\int_{\partial B_r((a, b))} (Q - \psi)^2 \right)^{\frac{1}{2}} \leq \frac{Q}{r},$$

that is, $Q(\lambda, a, b) \geq Cbr$. Taking $r = \frac{1}{2} \operatorname{dist}\{(a, b), N \cup l_0\}$, we get the other half inequality in (6.11). The lemma thus follows.

REMARK 6.4. If the wall N is not concave to the flow region, i.e. (6.10) does not hold, then the velocity of the fluid q may possibly attain its maximum on N and in that case the flow might not be subsonic. However, as in [6], one can show that if λ is small enough,

the flow is subsonic and the solution of the impinging problem can be obtained from the variational problem defined above.

REMARK 6.5. W. Lücken developed a numerical method (see [13]) to solve the following minimization problem: Given a bounded domain $\Omega \in \mathbb{R}^2$ and a nonnegative function $u_0 \in C^{0,1}(\overline{\Omega})$, find $u \in \mathcal{K} \equiv \{v \in H^1(\Omega) \mid v = u_0 \text{ on } \partial\Omega\}$, such that

$$\iint_{\Omega \cap \{u > 0\}} |\nabla u|^2 + Q = \min_{v \in \mathcal{K}} \iint_{\Omega \cap \{v > 0\}} |\nabla v|^2 + Q.$$

His method can be easily modified to apply to the variational problems studied in this paper.

Theoretical results on the numerical approximation for the above minimization problem was recently given by Aguilera [1].

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621	J. David Logan	Wave propagation in a qualitative model of combustion under equilibrium conditions
622	M.L. Zeeman	Hopf bifurcations in competitive three-dimensional Lotka-Volterra Systems
623	Allan P. Fordy	Isospectral flows: their Hamiltonian structures, Miura maps and master symmetries
624	Daniel D. Joseph, John Nelson, Michael Renardy, and Yuriko Renardy	Two-Dimensional cusped interfaces