**UNDERCOMPRESSIVE SHOCKS FOR
NONSTRICTLY HYPERBOLIC CONSERVATION LAWS**

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Abstract

We study 2×2 systems of hyperbolic conservation near an umbilic point. These systems have undercompressive shock wave solutions, i.e., solutions whose viscous profiles are represented by saddle connections in an associated family of planar vector fields. Previous studies near umbilic points have assumed that the flux function is a quadratic polynomial, in which case saddle connections lie on invariant lines. We drop this assumption and study saddle connections using Golubitsky-Schaeffer equilibrium bifurcation theory and the Melnikov integral, which detects the breaking of heteroclinic orbits. The resulting information is used to construct solutions of Riemann problems.

1 Introduction

In solving Riemann problems for nonstrictly hyperbolic systems of conservation laws, and for certain equations of mixed hyperbolic and elliptic type, new types of shock waves arise. The new shock waves are called undercompressive or overcompressive, depending on whether fewer or more characteristics enter the shock than for a compressive shock satisfying Lax's entropy condition [11]. Overcompressive shocks are understood as adjacent compressive shocks with the same speed [8,13,16]. In this paper, we contribute to the understanding of undercompressive shocks and their role in solving Riemann problems.

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The context in which we work is 2×2 systems of nonstrictly hyperbolic conservation laws in one space dimension. A 2×2 system of conservation laws

$$U_t + F(U)_x = 0, \quad (1.1)$$

$U = U(x, t) \in \mathbb{R}^2$, $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, is *nonstrictly hyperbolic* if the eigenvalues $\lambda_1(U) \leq \lambda_2(U)$ of $DF(U)$ are real, but not distinct for every U . System (1.1) has an *umbilic point* at $U = U^*$ if $DF(U^*)$ is a multiple of the identity. In this paper we describe shock wave solutions of (1.1) for which the value of U on each side of the discontinuity is near an umbilic point U^* . We then show how to use these shock wave solutions to solve Riemann problems for (1.1) with initial conditions near U^* . Our analysis focuses in particular on undercompressive shocks.

The Riemann problem is a special initial value problem for (1.1) in which the initial data is piecewise constant with a single jump:

$$U(x, 0) = \begin{cases} U_L & \text{if } x < 0 \\ U_R & \text{if } x > 0. \end{cases} \quad (1.2)$$

The understanding of Riemann problems is an important part of solving initial value problems, both analytically and numerically [1-5].

Riemann problems are solved by fitting together shock wave solutions of (1.1) and another special type of solution, centered rarefaction waves. In [19], Shearer *et al.* introduced the idea of studying shocks by using equilibrium bifurcation diagrams of associated vector fields. Our approach here is to add to the equilibrium bifurcation diagrams information about heteroclinic orbits of the vector fields. The augmented equilibrium bifurcation diagrams are then used in the construction of solutions of Riemann problems.

We assume without loss of generality that the umbilic point U^* of (1.1) lies at the origin, and that $DF(0) = 0$. Writing $U = (u, v)$, after a linear change of coordinates $F(u, v)$ may be written in the form

$$F(u, v) = \nabla \left(\frac{1}{3} au^3 + bu^2v + uv^2 \right) + \text{higher order terms},$$

where ∇ means gradient (cf.[15]). In this paper, we shall take the parameter a to be -1 , so that we are in Case I ($a < 3b^2/4$) of the classification of equations with umbilic points [15]. Since we are interested in solutions of (1.1) near the origin, we rescale the problem

by defining new dependent and independent variables

$$\tilde{U} = \epsilon U, \tilde{t} = t/\epsilon, \tilde{x} = x.$$

Substituting into (1.1) and dropping the tildes, we find that equation (1.1) has the same form, but now $F(U)$ is given by

$$F(u, v) = \nabla(-\frac{1}{3}u^3 + bu^2v + uv^2) + \epsilon C(u, v) + \mathcal{O}(\epsilon^2). \quad (1.3)$$

Here, $C : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a homogeneous cubic map:

$$C(u, v) = (\bar{\mu}_1 u^3 + \bar{\mu}_2 u^2 v + \bar{\mu}_3 u v^2 + \bar{\mu}_4 v^3, \bar{\mu}_5 u^3 + \bar{\mu}_6 u^2 v + \bar{\mu}_7 u v^2 + \bar{\mu}_8 v^3) \quad (1.4)$$

Henceforth in this paper we restrict attention to equation (1.1) with $F(U)$ given by (1.3).

We now describe the relationship between shock waves and heteroclinic orbits in some detail. A *shock wave solution* of (1.1) with speed s is a piecewise constant function

$$U(x, t) = \begin{cases} U_- & \text{if } x < st \\ U_+ & \text{if } x > st \end{cases} \quad (1.5)$$

that satisfies the *Rankine-Hugoniot condition*

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0. \quad (1.6)$$

Such a shock wave is *admissible* if it possesses a *viscous profile*, i.e., a traveling wave solution

$$U = U((x - st)/\lambda) \quad (1.7)$$

of the parabolic system

$$U_t + F(U)_x = \lambda U_{xx}, \quad (1.8)$$

with boundary conditions

$$U(\pm\infty) = U_{\pm}, \quad U'(\pm\infty) = 0. \quad (1.9)$$

We shall adopt the viewpoint that only shocks with viscous profiles are physically realistic.

Substitution of (1.7) into (1.8), and one integration, using the left-hand boundary condition from (1.9), leads to the system of ordinary differential equations

$$\frac{dU}{d\xi} = F(U) - F(U_-) - s(U - U_-), \quad (1.10)$$

where $\xi = (x - st)/\lambda$, and U_- and s are parameters. One equilibrium of (1.10) is U_- . The triple (U_-, U_+, s) satisfies the Rankine-Hugoniot condition (1.6) if and only if U_+ is also an equilibrium of (1.10). In this case, the shock wave (1.5) has a viscous profile if and only if there is an orbit of (1.10) from U_- to U_+ . An orbit that goes from one equilibrium to another is called a *heteroclinic* orbit or a *connection*. We have reduced the study of admissible shock wave solutions of (1.1) to the study of heteroclinic orbits of (1.10).

We shall refer to a shock (1.5) as a *Lax* shock if U_- is an unstable node and U_+ is a saddle (a *slow* shock), or U_- is a saddle and U_+ is a stable node (a *fast* shock). (The connection between this interpretation of the Lax condition and the usual one relating shock speed to characteristic speeds is made by noting that the eigenvalues of an equilibrium U are eigenvalues of $-sI + DF(U)$, while the characteristic speeds are eigenvalues of $DF(U)$.) A Lax shock (1.5) is called a *compressive* shock if it is admissible, i.e., if there is a heteroclinic orbit from U_- to U_+ . Since node-to-saddle and saddle-to-node heteroclinic orbits are stable to perturbation, compressive shocks come in one-parameter families: for fixed U_- , as s varies in some interval, there exists a corresponding U_+ connected to U_- . For the equations studied in this paper, we shall find that not all Lax shocks are compressive.

An admissible shock wave is *undercompressive* if both U_- and U_+ are saddle points. The trajectory from U_- to U_+ is then a saddle-to-saddle connection. Such heteroclinic orbits are not stable to perturbation; for fixed U_- , they are expected to occur only for isolated values of s . Undercompressive shocks were not considered classically, but they arise naturally in solving the Riemann problem near an umbilic point.

Returning to the specific set of equations (1.1), (1.3) considered in this paper, let us first set $\epsilon = 0$ in (1.3). Then undercompressive shocks are present, but as s varies, the corresponding saddle-to-saddle connections fail to break as expected. Shearer *et al.* [19] noticed this phenomenon when $b = 0$, and used the undercompressive shocks to solve the Riemann problem in that case. More generally, let $U_- = (u_-, v_-)$ and fix $u_- < 0$. Then there is a function $\phi(b)$, defined for b near 0, such that if $v_- = \phi(b)$, then there is a line near the u -axis that is invariant for the quadratic differential equation (1.10) for all s . For s in an interval, this line contains two saddle points, one at (u_-, v_-) , and an orbit that joins them. Thus the connection does not break as s varies. A description of all shocks with viscous profiles when $b \neq 0$ and $\epsilon = 0$ is given in [20].

One way to make the connections break as s varies is to keep $\epsilon = 0$ in (1.9) but vary the

viscosity matrix in (1.8) away from the identity. The differential equation (1.10) remains quadratic, hence has invariant lines, but now they break as s varies. Progress has been made with this approach by Isaacson, Marchesin and Plohr [9].

In this paper we leave the viscosity term alone and study what happens when $\epsilon \neq 0$. Invariant lines are no longer present to aid the analysis. Instead our analysis is based on bifurcation theory.

Our starting point is to set $b = \epsilon = 0$ in (1.9). We then fix $u_- < 0$ and set $v_- = 0$. Then (1.10) is a family of vector fields depending on the single parameter s , for which the u -axis is invariant. For $2u_- < s < -\frac{2}{3}u_-$, there are saddles at $(u_-, 0)$ and $(-u_- - s, 0)$ that are connected by a heteroclinic orbit along the u -axis. Moreover, at $s = 2u_-$ (resp. $s = -\frac{2}{3}u_-$) the equilibrium at $(u_-, 0)$ (resp. $(-u_- - s, 0)$) undergoes a pitchfork bifurcation.

As the unfolding parameters $(v_-, b, \epsilon\bar{\mu}_1, \dots, \epsilon\bar{\mu}_8)$ vary away from 0, the bifurcation picture changes. Using Golubitsky-Schaeffer bifurcation theory [6] we determine how the equilibrium bifurcation diagram unfolds. Using the Melnikov integral [7,18] we determine how the heteroclinic orbits break.

Melnikov's integral is usually applied to problems with heteroclinic orbits joining hyperbolic saddles. In our problem, at $s = 2u_-$ and $s = -\frac{2}{3}u_-$ the heteroclinic orbit along the u -axis has at one end an equilibrium with a zero eigenvalue. The version of Melnikov's integral appropriate to this situation comes from [18].

There is a beautiful theory, due to Vegter [22], of universal unfoldings and normal forms for planar vector fields in which degenerate equilibria are connected by a heteroclinic orbit. This theory has not yet been extended to problems with a distinguished parameter (s in our problem). Vegter's theory would not apply to our problem in any event: the bifurcation diagram for $(v_-, b, \epsilon) = (0, 0, 0)$ is of infinite codimension, because of the persistence of the heteroclinic orbit connecting saddles for all s in an interval. Hence there is no universal unfolding and (presumably) no normal form. Nevertheless we are able to determine how the bifurcation diagram changes as the unfolding parameters vary, provided $\bar{\mu}_5 \neq 0$. The result of our analysis is a description of nine transition surfaces in $(v_-, b, \epsilon, \bar{\mu})$ -space that separate regions with stable bifurcation diagrams. This description is given in Section 2; proofs are postponed until Section 5.

In Section 3, we interpret the bifurcation diagrams and transition surfaces in terms of admissible shock waves. Some of the transition surfaces do not affect admissible shocks,

and so are discarded. This leads to a reduced set of bifurcation diagrams that contain information relevant to the solution of Riemann problems.

In Section 4, we use the bifurcation diagrams to construct wave curves for (1.1), (1.3). The wave curves are used to give a coordinate system on part of the plane of values of U_R . Each (U_L, U_R) in (1.2) is thereby assigned a sequence of waves that appear in the solution of the Riemann problem with that data. To simplify some of the details of this construction, it is convenient to make a transversality assumption (see Assumption 4.1). With this assumption, which we conjecture to be true, we find that for a fixed U_L in (1.2), the construction gives a unique solution of the Riemann problem. The role of Assumption 4.1 in guaranteeing uniqueness is discussed further at the end of Section 4.

2 Bifurcation Results

We first review some terminology and facts from ordinary differential equations.

Let U_o be an equilibrium of the differential equation $\dot{U} = H(U)$, $U \in \mathbb{R}^2$, and let the eigenvalues of $DH(U_o)$ be λ_1 and λ_2 . The equilibrium U_o is *hyperbolic* if $\text{Re}(\lambda_i) \neq 0$, $i = 1, 2$. Suppose both λ_i are real; then U_o is a *saddle* if $\lambda_1\lambda_2 < 0$ and a *node* if $\lambda_1\lambda_2 > 0$. If exactly one $\lambda_i = 0$, then U_o is *semihyperbolic*.

Let λ_i be an eigenvalue of a saddle or the nonzero eigenvalue of a semihyperbolic equilibrium, and let V_i be a corresponding eigenvector of $DH(U_o)$. There is a unique invariant curve through U_o tangent to V_i , called the *stable* (resp. *unstable*) *manifold* of U_o if $\lambda_i < 0$ (resp. $\lambda_i > 0$). Let U_o be a node with eigenvalues $\lambda_1 < \lambda_2 < 0$ (resp. $\lambda_1 > \lambda_2 > 0$), with corresponding eigenvectors V_1 and V_2 . There is a unique invariant curve through U_o tangent to V_1 , called the *strong stable* (resp. *strong unstable*) manifold of U_o .

Let U_o be a semihyperbolic equilibrium and let V be an eigenvector of $DH(U_o)$ for the eigenvalue 0. There is an invariant curve through U_o tangent to V called the *center manifold* of U_o ; it need not be unique. Let the differential equation on the center manifold be $\dot{x} = a_2x^2 + a_3x^3 + \dots$ and let λ be the nonzero eigenvalue at U_o . U_o is a *saddle-node* if $a_2 \neq 0$; it is a *weak saddle* if $a_2 = 0$ and $\lambda a_3 < 0$.

A solution curve that tends to U_o as $t \rightarrow \infty$, from which nearby solutions diverge as $t \rightarrow \infty$, is called a *separatrix*. One may replace $t \rightarrow \infty$ by $t \rightarrow -\infty$ in this definition. At a saddle, each branch of the stable and unstable manifolds is a separatrix; at a weak saddle with a negative eigenvalue, each branch of the stable and center manifolds is a separatrix; at a saddle-node with a negative eigenvalue, each branch of the stable manifold and one branch of the center manifold is a separatrix. See Figure 1. Similar observations apply to weak saddles and saddle-nodes with a positive eigenvalue. A *separatrix connection* is a solution curve that is a separatrix as $t \rightarrow \infty$ and as $t \rightarrow -\infty$.

Let $\dot{U} = H(U, \lambda)$ be a one-parameter family of differential equations in the plane. Suppose that for some λ_o there is a semihyperbolic equilibrium at U_o . The type of bifurcation at (U_o, λ_o) is determined by the differential equation on the parameter-dependent center manifold at (U_o, λ_o) . We shall encounter the following bifurcations; the representations given below are to lowest order in appropriate coordinates.

$$\text{Saddle-node bifurcation:} \quad \dot{x} = \pm\lambda \pm x^2$$

$$\text{Transcritical bifurcation:} \quad \dot{x} = \pm x^2 \pm \lambda x$$

$$\text{Hysteresis bifurcation:} \quad \dot{x} = \pm \lambda \pm x^3$$

$$\text{Pitchfork bifurcation:} \quad \dot{x} = \pm x^3 \pm \lambda x$$

Saddle-node bifurcations occur stably in one-parameter families. Transcritical bifurcations occur stably in families in which there is a known “trivial solution” ($x = 0$ in appropriate coordinates). Pitchfork bifurcations occur stably in families with $\mathbb{Z}_2$ -symmetry. If a saddle-node or transcritical bifurcation occurs at (U_o, λ_o) , then U_o is always a saddle-node. In this paper, whenever a hysteresis or pitchfork bifurcation occurs at (U_o, λ_o) , U_o is a weak saddle.

Motivated by Section 1, we let

$$U = (u, v) \in \mathbb{R}^2;$$

$$\mu \in \mathbb{R}^8;$$

$$M = (v_-, b, \mu) \in \mathbb{R}^{10};$$

$$F(U, M) = (F_1(U, M), F_2(U, M)), \quad \text{where } F \text{ is } C^\infty \text{ and}$$

$$F_1(U, M) = -u^2 + 2buv + v^2 + \mu_1 u^3 + \mu_2 u^2 v + \mu_3 uv^2 + \mu_4 v^3 + \mathcal{O}(|\mu|^2),$$

$$F_2(U, M) = bu^2 + 2uv + \mu_5 u^3 + \mu_6 u^2 v + \mu_7 uv^2 + \mu_8 v^3 + \mathcal{O}(|\mu|^2);$$

$$U_- = (u_-, v_-), \quad u_- \text{ a fixed negative number;}$$

$$s \in \mathbb{R}.$$

We consider the differential equation

$$\dot{U} = G(U, s, M) = (G_1(U, s, M), G_2(U, s, M)) = F(U, M) - F(U_-, M) - s(U - U_-). \quad (2.1)$$

We think of s as a bifurcation parameter, so that (2.1) is a ten-parameter family of bifurcation problems on $\mathbb{R}^2$.

Let $\delta > 0$ be small. It may have to be decreased several times in the course of our arguments. Until the last paragraph of this section, we shall restrict s to lie in the interval $\mathcal{I} = [2u_- - \delta, -\frac{2}{3}u_- + \delta]$.

For the bifurcation problem $\dot{U} = G(U, s, 0)$, $s \in \mathcal{I}$, we readily find (see Figure 2):

- (1) There is symmetry about the u -axis. As a result, the u -axis is invariant for each s .

- (2) There are equilibria on the u -axis at $(u_-, 0)$ and $(-u_- - s, 0)$, and $u_- < -u_- - s$.
- (3) The equilibrium at $(u_-, 0)$ has eigenvectors $(1, 0)$ and $(0, 1)$. For $s \in \mathcal{I}$, the eigenvalue corresponding to $(1, 0)$ is positive; that corresponding to $(0, 1)$ is positive for $s < 2u_-$, 0 for $s = 2u_-$, negative for $s > 2u_-$.
- (4) The equilibrium at $(-u_- - s, 0)$ also has eigenvectors $(1, 0)$ and $(0, 1)$. For $s \in \mathcal{I}$, the eigenvalue corresponding to $(1, 0)$ is negative; that corresponding to $(0, 1)$ is positive for $s < -\frac{2}{3}u_-$, 0 for $s = -\frac{2}{3}u_-$, negative for $s > -\frac{2}{3}u_-$.

As sketched in Section 1, the bifurcation problem $\dot{U} = G(U, s, 0)$ has two interesting features.

- (1) Calculation shows that the equilibrium bifurcations at $s = 2u_-$ and $s = -\frac{2}{3}u_-$ are pitchfork bifurcations. Thus for $s = 2u_-$ (resp. $s = -\frac{2}{3}u_-$), the equilibrium at $(u_-, 0)$ (resp. $(-u_- - s, 0)$) is a weak saddle.
- (2) For $2u_- \leq s \leq -\frac{2}{3}u_-$, the portion of the u -axis between $(u_-, 0)$ and $(-u_- - s, 0)$ is a separatrix connection. For $s = 2u_-$ it runs from a weak saddle to a saddle; for $2u_- < s < -\frac{2}{3}u_-$ it connects two saddles; and for $s = -\frac{2}{3}u_-$ it runs from a saddle to a weak saddle.

We shall use the word *smooth* to mean C^k for some large k . Within the class of smooth vector fields having symmetry about the u -axis, the bifurcation problem $\dot{U} = G(U, s, 0)$ is stable. However, the symmetry is broken by the unfolding (2.1). Within the larger class of all smooth vector fields, as mentioned in Section 1, the bifurcation problem $\dot{U} = G(U, s, 0)$ is of infinite codimension.

We remark that for $2u_- - \delta \leq s < 2u_-$, the connection between the node at $(u_-, 0)$ and the saddle at $(-u_- - s, 0)$ is along the strong unstable manifold of the former. See Figure 2. The fact that such a connection persists for all s in an interval is also of infinite codimension. A similar situation occurs for $-\frac{2}{3}u_- < s \leq -\frac{2}{3}u_- + \delta$. The occurrence of this type of connection between a node and a saddle is not significant either for topological equivalence of vector fields or for shocks. However, we find that it is sometimes helpful to the intuition to consider such connections, which can metamorphose into separatrices as parameters vary. Since our analysis does provide information about such connections, we shall discuss them whenever it seems helpful.

Let $\mathcal{U}$ be a small neighborhood of $[u_-, -3u_-] \times \{0\}$ in the uv -plane. We consider (2.1) with $U \in \mathcal{U}$, $s \in \mathcal{I}$. We shall call the bifurcation problem $\dot{U} = G(U, s, M)$, M fixed, *stable* on $\mathcal{U} \times \mathcal{I}$ provided:

- (I) The only equilibrium bifurcations that occur are transcritical bifurcations at $(u, v) = (u_-, v_-)$ or saddle-node bifurcations at $(u, v) \neq (u_-, v_-)$.
- (II) The only separatrix connections that occur are between saddles, and they break in a nondegenerate manner as s varies.
- (III) For each fixed s at most one bifurcation of type (I) or (II) occurs.
- (IV) No bifurcation of type (I) or (II) occurs on the boundary of $\mathcal{I}$, and no equilibrium or separatrix connection meets the boundary of $\mathcal{U}$.

The reason for allowing transcritical bifurcations at $(u, v) = (u_-, v_-)$ is that for each fixed $M = (v_-, b, \mu)$ there is an equilibrium of (2.1) at (u_-, v_-) for every s . We note that $\dot{U} = G(U, s, 0)$ violates (I), (II), and (III). We may, however, assume δ and $\mathcal{U}$ are chosen so that $\dot{U} = G(U, s, 0)$ satisfies (IV). Then (IV) is necessarily satisfied for all M near 0, so we shall not discuss it further.

Let us set $\mu = 0$. As noted in Section 1, there is a curve

$$v_- = \phi(b) = -\frac{1}{3}u_-b + \dots \quad (2.2)$$

such that for each small b there is a line near the u -axis that is invariant under the flow of

$$\dot{U} = G(U, s, \phi(b), b, 0) \quad (2.3)$$

for every s . For $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$, this line contains two saddles and a separatrix connection that joins them. Thus each bifurcation problem (2.3) is itself of infinite codimension.

In M -space there are various *transition surfaces* on which one of conditions (I)–(III) is violated. It turns out that only the following nine surfaces occur in our analysis. Each surface is the closure of the set of parameter values for which the described condition occurs.

Surfaces on which one of conditions (I)–(III) is violated near $s = -\frac{2}{3}u_-$:

$\mathcal{B}$: A transcritical bifurcation occurs near $s = -\frac{2}{3}u_-$ at an equilibrium near $(-\frac{1}{3}u_-, 0)$.

$\mathcal{H}$: A hysteresis bifurcation occurs near $s = -\frac{2}{3}u_-$ at an equilibrium near $(-\frac{1}{3}u_-, 0)$.

$\mathcal{E}$: For some s near $-\frac{2}{3}u_-$, there is a separatrix connection from a saddle at (u_-, v_-) to a semihyperbolic equilibrium near $(-\frac{1}{3}u_-, 0)$.

$\mathcal{F}$: For some s near $-\frac{2}{3}u_-$, there is a separatrix connection from a saddle at (u_-, v_-) to a saddle near $(-\frac{1}{3}u_-, 0)$, and at the same s there is a semihyperbolic equilibrium near $(-\frac{1}{3}u_-, 0)$.

Surfaces on which one of conditions (I)–(III) is violated near $s = 2u_-$:

$\mathcal{P}$: A pitchfork bifurcation occurs near $s = 2u_-$ at (u_-, v_-) .

$\mathcal{E}_1$: For some s near $2u_-$, there is a separatrix connection from a semihyperbolic equilibrium at (u_-, v_-) to a saddle near $(-3u_-, 0)$.

$\mathcal{E}_2$: For some s near $2u_-$, there is a separatrix connection from a semihyperbolic equilibrium near, but not at, (u_-, v_-) to a saddle near $(-3u_-, 0)$.

$\mathcal{F}_1$: For some s near $2u_-$, there is a separatrix connection from a saddle at (u_-, v_-) to a saddle near $(-3u_-, 0)$, and at the same s there is a semihyperbolic equilibrium near (u_-, v_-) .

$\mathcal{F}_2$: For some s near $2u_-$, there is a separatrix connection from a saddle near, but not at, (u_-, v_-) to a saddle near $(-3u_-, 0)$, and at the same s there is a semihyperbolic equilibrium at (u_-, v_-) .

Our analysis will avoid the plane $\mu_5 = 0$ (except the origin) in M -space. Thus we write $\mu = \epsilon\bar{\mu}$ where $\bar{\mu}_5 = 1$.

Theorem 2.1 *There is a continuous positive function $\rho(\bar{\mu})$, defined for $\{\bar{\mu} : \bar{\mu}_5 = 1\}$, such that if $\sup\{|v_-|, |b|, |\epsilon|\} < \rho(\bar{\mu})$, then the bifurcation problem (2.1) with $\mu = \epsilon\bar{\mu}$ satisfies conditions (I)–(IV) on $\mathcal{U} \times \mathcal{I}$ unless $(v_-, b, \epsilon\bar{\mu})$ belongs to one of the nine transition surfaces defined above. Each surface is given by a smooth function of the form $v_- = v_-(b, \bar{\mu}, \epsilon)$.*

In order to describe how these nine surfaces fit together, we shall first consider the bifurcation problem (2.1) near $s = -\frac{2}{3}u_-$ and near $s = 2u_-$.

Near $s = -\frac{2}{3}u_-$, we replace the parameters v_- and b by new parameters α and β , the parameters that appear in the standard unfolding of the pitchfork bifurcation [6]:

$$x^3 - \lambda x + \alpha + \beta x^2.$$

The signs of x^3 and λx are the appropriate ones here. To first order,

$$\begin{aligned}\alpha &= \frac{4}{9}\sqrt{-2u_-}v_- + \frac{4}{27}u_-\sqrt{-2u_-}b + \frac{14}{81}u_-^2\sqrt{-2u_-}\mu_5 + \dots, \\ \beta &= \frac{3}{2\sqrt{-2u_-}}v_- + \frac{1}{12}\sqrt{-2u_-}b - \frac{1}{18}u_-\sqrt{-2u_-}\mu_2 - \frac{1}{72}u_-\sqrt{-2u_-}\mu_5 - \frac{1}{9}u_-\sqrt{-2u_-}\mu_7 + \dots.\end{aligned}$$

Theorem 2.2 *There is a continuous positive function $\rho(\bar{\mu})$, defined for $\{\bar{\mu} : \bar{\mu}_5 = 1\}$, such that if $0 < \epsilon < \rho(\bar{\mu})$, the transition surfaces $\mathcal{B}$, $\mathcal{H}$, $\mathcal{E}$, and $\mathcal{F}$ meet $\{(v_-, b, \mu) : |v_-| < \rho(\bar{\mu}), |b| < \rho(\bar{\mu}), \mu = \epsilon\bar{\mu}\}$ in smooth curves situated as in Figure 3. In particular, $\mathcal{B}$ is the β -axis and $\mathcal{H}$ is the curve $\alpha = \beta^3/27$. There are six points of intersection of the various curves, each given by smooth functions of the form $\alpha = \alpha(\bar{\mu}, \epsilon)$, $\beta = \beta(\bar{\mu}, \epsilon)$ that extend smoothly to $\epsilon = 0$. These points and their β -coordinates are (with $c = -\frac{2}{27}u_-\sqrt{-2u_-}$):*

P_1 , transversal intersection of $\mathcal{H}$ and $\mathcal{E}$, $\beta = 6c\epsilon + \mathcal{O}(\epsilon^2)$;

P_2 , tangential intersection of $\mathcal{B}$ and $\mathcal{H}$, $\beta = 0$;

P_3 , transversal intersection of $\mathcal{H}$ and $\mathcal{F}$, $\beta = -\frac{3}{4}c\epsilon + \mathcal{O}(\epsilon^2)$;

P_4 , tangential intersection of $\mathcal{B}$ and $\mathcal{F}$, $\beta = -c\epsilon + \mathcal{O}(\epsilon^2)$;

P_5 , transversal intersection of $\mathcal{B}$ and $\mathcal{E}$, $\beta = -2c\epsilon + \mathcal{O}(\epsilon^2)$;

P_6 , tangential intersection of $\mathcal{H}$, $\mathcal{E}$, and $\mathcal{F}$, $\beta = -3c\epsilon + \mathcal{O}(\epsilon^2)$.

The two curves meeting at P_4 and the three meeting at P_6 are separated at second order.

In Figure 3 there are twelve open connected regions, numbered 1–12, in which the bifurcation diagram is stable near $s = -\frac{2}{3}u_-$, i.e., conditions (I)–(IV) are satisfied for $-\frac{2}{3}u_- - \delta \leq s \leq -\frac{2}{3}u_- + \delta$. The bifurcation diagrams in the twelve regions for s near $-\frac{2}{3}u_-$ are indicated in Figure 4. Each numbered picture shows an equilibrium bifurcation diagram for equilibria near $(u, v) = (-\frac{1}{3}u_-, 0)$ and s near $-\frac{2}{3}u_-$. The dots indicate saddles

to which the saddle at (u_-, v_-) is connected, and nodes to which the saddle at (u_-, v_-) is connected along their strong stable manifold. The letters refer to the phase portraits at the bottom of Figure 4.

Remark 2.1 As α decreases, the s -coordinate of the dot in pictures 1 and 10 decreases past $s = -\frac{2}{3}u_- - \delta$; as α increases, the s -coordinate of the dot in pictures 4 and 6 increases past $s = -\frac{2}{3}u_- + \delta$. These transitions occur along curves $\mathcal{A}_-$ and $\mathcal{A}_+$ respectively, which are not shown in Figure 3. $\mathcal{A}_-$ (resp. $\mathcal{A}_+$) is located to the left (resp. right) of $\mathcal{B}$, $\mathcal{E}$, and $\mathcal{F}$, but crosses $\mathcal{H}$ transversally.

As $\epsilon \rightarrow 0$ the curves $\mathcal{E}$ and $\mathcal{F}$ converge to $\mathcal{B}$. When $\epsilon = 0$ the curves $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{B}$ coincide with the curve (2.2).

Near $s = 2u_-$, we replace the parameter v_- by the new parameter β that appears in the standard unfolding of the pitchfork bifurcation with known trivial solution [6]:

$$-x^3 - \lambda x + \beta x^2.$$

The signs of x^3 and λx are appropriate here. To first order,

$$\beta = \sqrt{-2u_-} \left(\frac{3}{2u_-} v_- + \frac{3}{2} b + \frac{1}{2} u_- \mu_2 + \frac{3}{4} u_- \mu_5 + u_- \mu_7 \right) + \dots$$

Theorem 2.3 *There is a continuous positive function $\rho(\bar{\mu})$, defined for $\{\bar{\mu} : \bar{\mu}_5 = 1\}$, such that if $0 < \epsilon < \rho(\bar{\mu})$, the transition surfaces $\mathcal{P}$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{F}_1$, and $\mathcal{F}_2$ meet $\{(v_-, b, \mu) : |v_-| < \rho(\bar{\mu}), |b| < \rho(\bar{\mu}), \mu = \epsilon \bar{\mu}\}$ in smooth curves situated as in Figure 5. In particular, $\mathcal{P}$ is the b -axis. All five curves pass through the point Q given by $\beta = 0$, $b = b(\bar{\mu}, \epsilon) = \frac{1}{2}u_-(1 - \bar{\mu}_2 - 2\bar{\mu}_7)\epsilon + \mathcal{O}(\epsilon^2)$, where $b(\bar{\mu}, \epsilon)$ is smooth even at $\epsilon = 0$. To first order, we have:*

$$\mathcal{E}_1 \text{ and } \mathcal{F}_1: b = \frac{1}{\sqrt{-2u_-}} \beta + \frac{1}{2}u_-(1 - \bar{\mu}_2 - 2\bar{\mu}_7)\epsilon + \dots,$$

$$\mathcal{E}_2: b = -\frac{1}{2\sqrt{-2u_-}} \beta + \frac{1}{2}u_-(1 - \bar{\mu}_2 - 2\bar{\mu}_7)\epsilon + \dots,$$

$$\mathcal{F}_2: b = -\frac{2}{\sqrt{-2u_-}} \beta + \frac{1}{2}u_-(1 - \bar{\mu}_2 - 2\bar{\mu}_7)\epsilon + \dots.$$

$\mathcal{E}_1$ and $\mathcal{F}_1$ are separated at second order at Q .

In Figure 5 there are ten open connected regions, numbered 1–10, in which the bifurcation diagram is stable near $s = 2u_-$, i.e., conditions (I)–(IV) are satisfied for $2u_- - \delta \leq$

$s \leq 2u_- + \delta$. The bifurcation diagrams in the ten regions for s near $2u_-$ are shown in Figure 6. Each numbered picture shows an equilibrium bifurcation diagram for equilibria near $(u, v) = (u_-, v_-)$ and s near $2u_-$. In the diagrams, the family of trivial equilibria (u_-, v_-) appears as a horizontal line. The dots indicate saddles connected to the saddle near $(-3u_-, 0)$, and nodes connected to this saddle along their strong unstable manifold. The letters A, B, G, H, I, J refer to the phase portraits at the bottom of Figure 6. The subscripts l, m, u indicate whether the lower, middle, or upper of the three equilibria at the left is the distinguished equilibrium (u_-, v_-) .

Remark 2.2 As b decreases, the s -coordinate of the dot on the s -axis in pictures 1, 2, 3 and 10 decreases past $s = 2u_- - \delta$; as b increases, the s -coordinate of the dot on the s -axis in pictures 5, 6, 7 and 8 increases past $s = 2u_- + \delta$. These transitions occur along curves $\mathcal{A}_{1-}$ and $\mathcal{A}_{1+}$ respectively, which are not shown in Figure 5. $\mathcal{A}_{1-}$ (resp. $\mathcal{A}_{1+}$) lies to the left (resp. right) of $\mathcal{E}_1$ and $\mathcal{F}_1$ but crosses $\mathcal{P}$, $\mathcal{E}_2$, and $\mathcal{F}_2$ transversally.

At Q there is a pitchfork bifurcation with a separatrix connection from the weak saddle to the saddle near $(-3u_-, 0)$. We remark that at the transitions across $\mathcal{E}_1$, two dots simultaneously pass through the transcritical bifurcation point.

When $\epsilon = 0$, the curves $\mathcal{E}_1$ and $\mathcal{F}_1$ coincide with the curve (2.2).

We now indicate how all nine of our surfaces fit together, first for $\bar{\mu}_2 + 2\bar{\mu}_7 < 0$. We return to v_-b -coordinates.

Theorem 2.4 *There is a continuous positive function $\rho(\bar{\mu})$, defined for $\{\bar{\mu} : \bar{\mu}_5 = 1 \text{ and } \bar{\mu}_2 + 2\bar{\mu}_7 < 0\}$ such that if $0 < \epsilon < \rho(\bar{\mu})$, the nine transition surfaces meet $\{(v_-, b, \mu) : |v_-| < \rho(\bar{\mu}), |b| < \rho(\bar{\mu}), \bar{\mu} : \epsilon\bar{\mu}\}$ in smooth curves situated as in Figure 7, except in one respect discussed below. To first order we have:*

$$\begin{aligned} P_i : v_- &= \left(\frac{2}{27}a_i - \frac{1}{9}\right)u_-^2\epsilon - \frac{1}{18}u_-^2(\bar{\mu}_2 + 2\bar{\mu}_7)\epsilon + \mathcal{O}(\epsilon^2), \\ b &= \left(-\frac{2}{9}a_i - \frac{5}{6}\right)u_- \epsilon + \frac{1}{6}u_- (\bar{\mu}_2 + 2\bar{\mu}_7)\epsilon + \mathcal{O}(\epsilon^2), \end{aligned}$$

where $a_1 = 6$, $a_2 = 0$, $a_3 = -\frac{3}{4}$, $a_4 = -1$, $a_5 = -2$, $a_6 = -3$;

$$\begin{aligned} Q : v_- &= -u_-^2\epsilon + \frac{1}{6}u_-^2(\bar{\mu}_2 + 2\bar{\mu}_7)\epsilon + \mathcal{O}(\epsilon^2), \\ b &= \frac{1}{2}u_- \epsilon - \frac{1}{2}u_- (\bar{\mu}_2 + 2\bar{\mu}_7)\epsilon + \mathcal{O}(\epsilon^2). \end{aligned}$$

The intersections of $\mathcal{P}$, $\mathcal{E}_2$, and $\mathcal{F}_2$ with $\mathcal{B}$, $\mathcal{H}$, $\mathcal{E}$, and $\mathcal{F}$ are transverse. Like P_i and Q , these intersections are given by smooth functions of the form $v_- = v_-(\bar{\mu}, \epsilon)$, $b = b(\bar{\mu}, \epsilon)$ that extend smoothly to $\epsilon = 0$ and are 0 at $\epsilon = 0$.

The respect in which Figure 7 is misleading is the following. At the left side of the picture, $\mathcal{H}$ continues to fall and meets $\mathcal{F}_1$ and $\mathcal{E}_1$. We have not proved anything more about these intersections. In any event, this part of the diagram turns out not to be important for shocks. Ignoring this difficulty, in Figure 7 there are thirty-three connected open regions in which the bifurcation diagram is stable on $2u_- - \delta \leq s \leq -\frac{2}{3}u_- + \delta$. These regions may be labeled i - j , where, roughly speaking, the number i , $1 \leq i \leq 10$, represents the bifurcation diagram near $s = 2u_-$, and the number j , $1 \leq j \leq 12$, represents the bifurcation diagram near $s = -\frac{2}{3}u_-$. Let $\mathcal{C}$ be the following curve: Starting from negative b , follow $\mathcal{F}$ to P_6 , then $\mathcal{H}$ to P_3 , then $\mathcal{F}$. We distinguish the following groups of regions:

Above $\mathcal{F}_1$: $i = 1, 2, 3, 4, 9, 10$, $j = 1$;

Between $\mathcal{F}_1$ and $\mathcal{C}$: $i = 5, 6, 7, 8$, $j = 1, 10$;

Below $\mathcal{C}$: $i = 6, 7, 8$, $j = 6, 11, 12$; $i = 5$, $j \neq 1, 10$.

In Figure 8 we draw the bifurcation diagram on $2u_- - \delta \leq s \leq -\frac{2}{3}u_- + \delta$ for one region from each group (regions 1-1, 5-1, and 5-2). The diagrams show equilibria near (u_-, v_-) above, with the trivial equilibria (u_-, v_-) lying on a horizontal line, and the other family of equilibria below. Separatrix connections between saddles, and connections between a saddle and a node along the strong stable or unstable manifold of the latter, are again shown by dots. The letters refer to the phase portraits at the bottom of Figures 4 and 6.

Proposition 2.1 *The rules for amalgamating the diagrams i and j to produce the diagram for region i - j are as follows:*

Above $\mathcal{F}_1$: In diagram j , omit interval A and the first dot.

Between $\mathcal{F}_1$ and $\mathcal{C}$: Identify the AB transitions in diagrams i and j . (There is one such transition.)

Below $\mathcal{C}$: In diagram i , omit the last dot and interval B .

The proofs of Theorem 2.4 and Proposition 2.1 will require some analysis of the interval $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$. On the other hand, the following result is easily proved by looking at the thirty-three diagrams.

Proposition 2.2 *Each of the thirty-three diagrams has two dots, exactly one of which represents a connection involving (u_-, v_-) .*

Remark 2.3 Proposition 2.2 is misleading in two respects. By Remark 2.1, as b increases, the s -coordinate of the connection involving (u_-, v_-) in regions 6-6, 7-6, 8-6, 5-6, and 5-4 increases past $s = -\frac{2}{3}u_- + \delta$ as $\mathcal{A}_+$ is crossed. By Remark 2.2, as b decreases, that in regions 1-5, 2-5, 3-5, and 10-5 decreases past $s = 2u_- - \delta$ as $\mathcal{A}_{1-}$ is crossed. Since all these connections are between a saddle and a node, they are of no real importance.

In Figure 9 we draw the bifurcation diagram of $\dot{U} = G(U, s, v_-, b, 0)$, to which the diagram of Figure 7 tends as $\epsilon \rightarrow 0$. The heavy lines indicate connections involving (u_-, v_-) that persist for all s . The dots, which represent connections that do not involve (u_-, v_-) , are associated with a bifurcation of invariant lines for the quadratic system $\dot{U} = G(U, s, v_-, b, 0)$.

We now discuss how Figure 7 changes as $\bar{\mu}_2 + 2\bar{\mu}_7$ increases past 0.

Fix $\epsilon > 0$. As $\bar{\mu}_2 + 2\bar{\mu}_7$ increases, the points P_i and Q move with velocities

$$\begin{aligned} \frac{dP_i}{d(\bar{\mu}_2 + 2\bar{\mu}_7)} &= \frac{1}{6}\epsilon(u_-, -\frac{1}{3}u_-) + \mathcal{O}(\epsilon^2), \\ \frac{dQ}{d(\bar{\mu}_2 + 2\bar{\mu}_7)} &= -\frac{1}{2}\epsilon(u_-, -\frac{1}{3}u_-) + \mathcal{O}(\epsilon^2). \end{aligned}$$

Now all nine curves have slopes independent of $\bar{\mu}$ to order ϵ , and the vector $(u_-, -\frac{1}{3}u_-)$ is parallel to $\mathcal{B}$, $\mathcal{H}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{E}_1$, and $\mathcal{F}_1$ to order ϵ . Thus as $\bar{\mu}_2 + 2\bar{\mu}_7$ increases, the points P_i in Figure 7 slide parallel to $\mathcal{E}_1$ and $\mathcal{F}_1$; relative to Q , they slide to the left, so they occasionally meet $\mathcal{P}$, $\mathcal{E}_2$, and $\mathcal{F}_2$.

It is clear from Figure 7 that each of $\mathcal{P}$, $\mathcal{E}_2$, and $\mathcal{F}_2$ meets the P_i in the order $P_6, \dots, P_1$ as $\bar{\mu}_2 + 2\bar{\mu}_7$ increases. However, whether, for example, $\mathcal{P}$ meets P_5 before or after $\mathcal{F}_2$ meets P_6 must be computed. The results of the computation are given in the following theorem.

Theorem 2.5 *There is a continuous function $\rho(\bar{\mu})$, defined for $\{\bar{\mu} : \bar{\mu}_5 = 1\}$, such that if $0 < \epsilon < \rho(\bar{\mu})$, then in $\{(v_-, b, \mu) : |v_-| < \rho(\bar{\mu}), |b| < \rho(\bar{\mu}), \mu = \epsilon\bar{\mu}\}$ the curves $\mathcal{B}$, $\mathcal{E}$, and $\mathcal{F}$ lie above the curves $\mathcal{E}_1$ and $\mathcal{F}_1$, and $\mathcal{B}$, $\mathcal{H}$, $\mathcal{E}$, and $\mathcal{F}$ meet $\mathcal{P}$, $\mathcal{E}_2$, and $\mathcal{F}_2$ transversally. The intersections occur at one of the points P_i if and only if $(\bar{\mu}, \epsilon)$ lies on one of eighteen*

smooth surfaces. In fact to order ϵ ,

P_6	meets	$\mathcal{P}$	if and only if	$\bar{\mu}_2 + 2\bar{\mu}_7 = \mathcal{O}(\epsilon)$.
P_5	"	$\mathcal{P}$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \frac{1}{3} + \mathcal{O}(\epsilon)$.
P_6	"	$\mathcal{F}_2$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \frac{1}{3} + \mathcal{O}(\epsilon)$.
P_4	"	$\mathcal{P}$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \frac{2}{3} + \mathcal{O}(\epsilon)$.
P_5	"	$\mathcal{F}_2$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \frac{2}{3} + \mathcal{O}(\epsilon)$.
P_3	"	$\mathcal{P}$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \frac{3}{4} + \mathcal{O}(\epsilon)$.
P_6	"	$\mathcal{E}_2$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \sim .82 + \mathcal{O}(\epsilon)$.
P_4	"	$\mathcal{F}_2$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = 1 + \mathcal{O}(\epsilon)$.
P_2	"	$\mathcal{P}$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = 1 + \mathcal{O}(\epsilon)$.
P_3	"	$\mathcal{F}_2$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \frac{13}{12} + \mathcal{O}(\epsilon)$.
P_5	"	$\mathcal{E}_2$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \sim 1.23 + \mathcal{O}(\epsilon)$.
P_2	"	$\mathcal{F}_2$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \frac{4}{3} + \mathcal{O}(\epsilon)$.
P_4	"	$\mathcal{E}_2$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \sim 1.64 + \mathcal{O}(\epsilon)$.
P_3	"	$\mathcal{E}_2$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \sim 1.74 + \mathcal{O}(\epsilon)$.
P_2	"	$\mathcal{E}_2$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \sim 2.05 + \mathcal{O}(\epsilon)$.
P_1	"	$\mathcal{P}$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = 3 + \mathcal{O}(\epsilon)$.
P_1	"	$\mathcal{F}_2$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \frac{10}{3} + \mathcal{O}(\epsilon)$.
P_1	"	$\mathcal{E}_2$	"	$\bar{\mu}_2 + 2\bar{\mu}_7 = \sim 4.51 + \mathcal{O}(\epsilon)$.

We make two comments about Theorem 2.5. First, as in Theorem 2.4, we have not specified how $\mathcal{H}$ meets $\mathcal{E}_1$ and $\mathcal{F}_1$. Second, as $\bar{\mu}_2 + 2\bar{\mu}_7$ increases, the crossings of the P_i and the curves $\mathcal{P}$, $\mathcal{E}_2$, and $\mathcal{F}_2$ occur in approximately the order given. The only difficulties occur near $\bar{\mu}_2 + 2\bar{\mu}_7 = \frac{1}{3}, \frac{2}{3}, 1$, where two crossings occur close together. Thus unless $(\bar{\mu}, \epsilon)$ is between the two surfaces associated with one of these values of $\bar{\mu}_2 + 2\bar{\mu}_7$, we could draw the analog of Figure 7. We leave this to the interested reader.

Remark 2.4 If u_- is treated as a parameter, then everything depends smoothly on u_- also. More precisely, the nine transition surfaces are given by smooth functions of the form $v_- = v_-(u_-; b, \bar{\mu}, \epsilon)$; the various intersection points are smooth functions of $(u_-; \bar{\mu}, \epsilon)$; and

the eighteen functions of Theorem 2.8 depend smoothly on $(u_-; \bar{\mu}, \epsilon)$. The various functions $\rho(\bar{\mu})$ become $\rho(u_-; \bar{\mu})$.

Finally we discuss the bifurcation diagrams for $U \in \mathcal{U}$, $-\frac{2}{3}u_- + \delta \leq s \leq -2u_- + \delta$. For $\dot{U} = G(U, s, 0)$, as s increases past $-2u_-$, the saddle and node on the u -axis approach and cross in a transcritical bifurcation; see Figure 2. The two equilibria remain connected along the u -axis for all s . Since there is an equilibrium at (u_-, v_-) for all M , the same bifurcation diagram is valid on $\mathcal{U} \times [-\frac{2}{3}u_- + \delta, -2u_- + \delta]$ for all small M . For some s in this interval, one or two of the saddles near the node in Figure 2 may leave the region $\mathcal{U}$. We shall ignore this phenomenon, since these saddles are not connected to (u_-, v_-) , hence are not important for shocks.

3 Admissible Shocks

In this section, we interpret the results of the last section in a form that leads to a description of admissible shock wave solutions of system (1.1), with F given by (1.3). Admissible shocks correspond to heteroclinic orbits of (2.1) from $U_- = (u_-, v_-)$ to some U_+ . Portions of some of the transition surfaces identified in Section 2 are not relevant to such orbits, so we shall discard them.

Let us write $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where, in $\alpha\beta\mu$ -coordinates (2.4), $\mathcal{B}_1 = \{(\alpha, \beta, \mu) : \alpha = 0, \beta \leq 0\}$, $\mathcal{B}_2 = \{(\alpha, \beta, \mu) : \alpha = 0, \beta > 0\}$. In (2.1) we set $\mu = \epsilon\bar{\mu}$ with $\bar{\mu}_5 = 1$ and we fix a small $\epsilon > 0$. The nine transition surfaces $\mathcal{B}$, etc., intersect the set $\{(v_-, b, \mu) : |v_-| \text{ and } |b| \text{ small, } \mu = \epsilon\bar{\mu}\}$ in nine curves which we shall label with the same letters $\mathcal{B}$, etc. The intersection points $P_1, \dots, P_6, Q$ were defined in Section 2. We consider only heteroclinic orbits in $\mathcal{U}$ and s only in the range $2u_- - \delta < s < -2u_- + \delta$ discussed in Section 2.

Proposition 3.1 *The curves $\mathcal{B}_2, \mathcal{H} \setminus \{P_1, P_2, P_3, P_6\}, \mathcal{E}_2 \setminus \{Q\}$, and $\mathcal{F}_1 \setminus \{Q\}$ are not relevant to admissible shocks.*

Proof Consider $\mathcal{E}_2 \setminus \{Q\}$. One component of $\mathcal{E}_2 \setminus \{Q\}$ in Figure 5 separates regions 2 and 3; the other separates regions 7 and 8. From Figure 6, we see that in passing from region 2 to 3, the sequence $I_l J_l$ in the bifurcation diagram is replaced by $I_l H_l$. In all three phase portraits I_l, J_l, H_l , there is no heteroclinic orbit from U_- to any U_+ . Therefore the transition is not relevant to admissible shocks. A similar argument applies to regions 7 and 8.

Consider $\mathcal{F}_1 \setminus \{Q\}$. One component of $\mathcal{F}_1 \setminus \{Q\}$ in Figure 5 separates regions 4 and 5; the other separates regions 8 and 9. From Figure 6, we see that in passing from region 4 to 5, the sequence $G_l H_l B$ in the bifurcation diagram is replaced by $G_l A B$. In either case there is exactly one s value greater than the transcritical bifurcation value for which there is a heteroclinic orbit from U_- to U_+ . Therefore the transition is not relevant to admissible shocks. A similar argument applies to regions 8 and 9.

The proof is completed by applying similar arguments to the remaining curves $\mathcal{B}_2, \mathcal{H} \setminus \{P_1, P_2, P_3, P_6\}$. $\square$

In Figure 10, we redraw Figure 7, retaining only those transition curves that are relevant to admissible shocks. As an aid to intuition in solving Riemann problems in Section 4, we have reversed the v_- and b axes in Figure 10. In Figure 11 we give bifurcation diagrams for representative (b, v_-) in the open regions of Figure 10, indicating pairs (U_-, s) and (U_+, s) that correspond to a heteroclinic orbit in $\mathcal{U}$. Heavier curves in Figure 11 locate points (U_+, s) for which the shock (1.5) is compressive, while pairs of dots mark triples (U_1, U_2, s^*) for which there is a special heteroclinic orbit from U_1 to U_2 with $s = s^*$ in (1.10). These bifurcation diagrams include the transcritical bifurcation near $s = -2u_-$, discussed at the end of Section 2. This bifurcation plays a part in the solution of the Riemann problems (1.1),(1.2) with U_R near U_L .

Finally, we introduce some terminology to be used in Section 4. The points $(U_-, \lambda_k(U_-))$ in the bifurcation diagrams of Figure 11 are called *primary bifurcation points*. If a transcritical or pitchfork bifurcation occurs at a point (U, s) with $U \neq U_-$, we call this point a *secondary bifurcation point*. We define the *Hugoniot locus* $H(U_-)$ of a point U_- to be the set of U in $\mathcal{U}$ such that (U, s) satisfies the Rankine-Hugoniot condition (1.6) for some $s \in (2u_- - \delta, -2u_- + \delta)$.

4 Riemann Problems

In this section, we solve representative Riemann problems (1.1),(1.2) that show the role of undercompressive shocks. The function $F(U)$ is given by (1.3), with parameters b and $\epsilon \neq 0$ both near zero. The construction of solutions provided here uniquely defines a solution for U_L near $(-1, 0)$ and U_R in $\mathcal{U}$.

Recall from [16] the structure of solutions of Riemann problems (1.1),(1.2). The Riemann problem is solved by finding a piecewise smooth weak solution $U = U(x/t)$ of (1.1),(1.2). The solution consists of a sequence of shock waves and rarefaction waves, either adjacent to one another in the (x,t) -plane, or separating wedges in which U is constant. For each fixed U_L , the U_R plane is divided into U_R regions with the property that for each U_R in a region, the sequence of shock and rarefaction waves appearing in the solution of the Riemann problem is the same, with the strengths of the waves varying as U_R varies within the region. Any curve separating U_R regions is called a U_R boundary. As U_L varies, the U_R regions distort. The pattern of U_R regions undergoes qualitative changes as U_L crosses what are called U_L boundaries. The strategy for solving the Riemann problem is to identify U_L boundaries, and to determine the patterns of U_R regions with their associated sequences of waves, for representative U_L not on a U_L boundary.

A *centered wave* from U_- to $U_+ \neq U_-$ is a centered solution of (1.1) of the form

$$U(x, t) = \begin{cases} U_- & \text{if } x < at \\ \tilde{U}(x/t) & \text{if } at < x < bt \\ U_+ & \text{if } x > bt, \end{cases} \quad (4.1)$$

for some $a \leq b$, in which $\tilde{U}(x/t) \in \mathcal{U}$ is not constant for x/t in any subinterval of $[a, b]$. The numbers a and b will be referred to as the left and right speeds, respectively.

A *centered rarefaction wave* is a centered wave (4.1) that is continuous for $t > 0$, and in which $\tilde{U}$ is differentiable. $U = \tilde{U}(\xi)$ is necessarily of the following form, for $k = 1$ or $k = 2$:

$$\begin{aligned} U'(\xi) &= r_k(U(\xi)) \\ \xi &= \lambda_k(U(\xi)), \end{aligned} \quad (4.2)$$

where $r_k(U)$ is the eigenvector of $dF(U)$ associated with the eigenvalue $\lambda_k(U)$, and normalised by $d\lambda_k(U).r_k(U) = 1$. Curves in the U plane on which $d\lambda_k(U).r_k(U) = 0$ are called *inflection loci*. If $k = 1$ in (4.2), then the rarefaction wave is *slow*, whereas for $k = 2$, the

rarefaction wave is *fast*. Equation (4.2), with initial condition

$$U(\lambda_k(U_-)) = U_-,$$

and with $\xi > \lambda_k(U_-)$ increasing, defines two curves with endpoint U_- , called fast and slow *rarefaction curves*. Note that in a rarefaction wave, U is constant on *characteristics* $x/t = \lambda_k(U)$.

Centered waves consist of adjacent shocks and centered rarefaction waves. A centered wave is *admissible* if each shock in the wave is admissible. A centered wave is *slow* or *fast* depending on whether the individual shocks and rarefactions are slow or fast. In addition to centered waves consisting of individual shocks or rarefactions, we shall also encounter slow rarefaction-shocks and fast shock-rarefactions. In these combination waves, the speed of the shock is that of the characteristic forming one edge of the rarefaction.

Consider the collection of all slow admissible centered waves, with U_L on the left of the wave in the (x, t) -plane. Each such wave defines a value U_- of U on the right of the wave. The *attached slow wave curve* $W_1(U_L)$ is defined to be the connected component containing U_L of this set of U_- values that lie in the neighborhood $\mathcal{U}$. $W_1(U_L)$ is constructed from rarefaction curves, shock curves and rarefaction-shock curves as for strictly hyperbolic equations []. All Lax shocks are admissible in this construction because U_- is close to U_L . Note that $W_1(U_L)$ is a C^1 curve that has a vertical intersection with the u -axis when $\epsilon = b = v_L = 0$. Therefore, $W_1(U_L)$ is roughly vertical, and we parameterize $W_1(U_L)$ by v_- . That is, we shall consider u_- as a function of v_- for $U_- = (u_-, v_-)$ on $W_1(U_L)$.

There is also a *detached slow wave curve* $W_1^*(U_L)$ in our problem. This wave curve consists of all $U_1 \in \mathcal{U} \setminus W_1(U_L)$ for which there is an admissible slow centered wave with U_L on the left of the wave and U_1 on the right.

The *undercompressive wave curve* Σ consists of points U_+ for which there is an undercompressive shock from U_- to U_+ for some $U_- \in W_1(U_L)$.

We remark that there are no undercompressive shocks with $U_- \in W_1^*(U_L)$ on the left of the shock. This assertion is based on a calculation [19] showing that for $v_- = b = \epsilon = 0, u_- > 0$, there are no undercompressive shocks with $U_- = (u_-, 0)$ on the left. Since for $b = \epsilon = 0$, and $v_L = 0, u_L < 0$, $W_1^*(U_L)$ consists of a portion of the positive u axis, the assertion follows by continuity.

The *fast wave curve* $W_2(U_-)$ through a point U_- consists of points U_+ in $\mathcal{U}$ for which

there is an admissible fast centered wave from U_- to U_+ . The fast wave curve may or may not be connected. Toward the end of the section, we refer to the detached portion of the fast wave curve, and label it $W_2^*(U_-)$.

Let $\bar{\mu}$ be fixed, with $\bar{\mu}_5 = 1, \bar{\mu}_2 + 2\bar{\mu}_7 < 0$. Consider $\epsilon > 0$ fixed, near zero, and let $b = b_o$ be fixed, also near zero. Fixing these two parameters specifies the system of equations (1.1), (1.3). Now let $U_L = (u_L, v_L)$ with $u_L < 0$, and v_L near zero.

We identify $W_1(U_L)$ with the corresponding vertical line $b = b_o$ in Figure 10. To do so, we need to reinterpret Figure 10 as follows. The curves in Figure 10 for fixed small ϵ are given by functions of the form

$$v_- = f(b, u_-; \epsilon), \quad (4.3)$$

with u_- fixed. Allowing $u_- < 0$ to vary in (4.3) defines a corresponding transition surface in (u_-, v_-, b) -space. For fixed $\epsilon \neq 0$, transition surfaces intersect along curves $\mathcal{C}$ of the form

$$\mathcal{C} : v_- = g(u_-; \epsilon), \quad b = h(u_-; \epsilon). \quad (4.4)$$

The functions g and h and hence the curves $\mathcal{C}$ extend smoothly to $\epsilon = 0$. (See Theorem 2.4 and Remark 2.4.) For each U_L , $W_1(U_L)$ is a curve in (u_-, v_-) -space of the form

$$\mathcal{S} : u_- = \psi(v_-; b, \epsilon, U_L), \quad (4.5)$$

with parameters b, ϵ, U_L . Now consider $\mathcal{S}$, for fixed ϵ and U_L , as a surface in (u_-, v_-, b) -space.

We claim that there is a neighborhood N of $(-1, 0, 0)$ in (u_-, v_-, b) -space such that for ϵ, v_L, b near zero, and U_L near $(-1, 0)$, each curve $\mathcal{C}$ is transverse to the surface $\mathcal{S}$ in N . First note that by continuity, it is sufficient to prove the claim for $\epsilon = 0, U_L = (-1, 0)$. Then to first order in (b, v_-) ,

$$\psi = -1.$$

Therefore, the normal to $\mathcal{S}$ at $(u_-, v_-, b) = (-1, 0, 0)$ is $(1, 0, 0)$. Now $\mathcal{C}$ also passes through $(-1, 0, 0)$, by Theorem 2.4, and the tangent vector to $\mathcal{C}$ there is $(1, g'(-1; 0), h'(-1; 0))$. Thus for $\epsilon = 0, U_L = (-1, 0)$, the normal vector to $\mathcal{S}$ and the tangent vector to $\mathcal{C}$ at $(u_-, v_-, b) = (-1, 0, 0)$ are not orthogonal. By continuity, $\mathcal{S}$ and $\mathcal{C}$ are transversal in some neighborhood of $(u_-, v_-, b) = (-1, 0, 0)$, and the claim is proved.

From the claim it follows that the surface $\mathcal{S}$ also intersects each transition surface transversally in a curve. We can now interpret Figure 10 as the projection of these curves onto the (b, v_-) -plane. By transversality, the dependence of the curves on u_- does not affect the qualitative structure of the curves in Figure 10, nor their intersections.

Each curve $W_1(U_L)$ with $b = \text{constant}$ in $\mathcal{S}$ projects as a vertical line in Figure 10. With this identification of $W_1(U_L)$ with a vertical line in the (b, v_-) -plane, we may speak of $W_1(U_L)$ intersecting the transition curves of Figure 10. Each such intersection may correspond to a significant point in the U_R plane, at which one or more boundaries of U_R regions meet. The precise arrangement of U_R regions in a neighborhood of these special U_R points may depend on the sequence of waves joining U_L to U_- , which in turn depends on b and the location of U_L , or more precisely, on the location of (b, v_L) with respect to the transition curves in Figure 10. This local analysis reveals new U_R boundaries associated with undercompressive shocks.

Using the identification of $W_1(U_L)$ with a vertical line in the (b, v_-) -plane, let us describe the undercompressive shock curve Σ more precisely. From Figure 11 we see that there is a saddle-to-saddle connection $U_- \rightarrow U_+$ for some value of s and some U_+ if and only if (b, v_-) lies between the transition curves $\mathcal{E}$ and $\mathcal{E}_1$. Correspondingly, for each U_- in the portion of $W_1(U_L)$ that projects into Figure 11 as a vertical line between $\mathcal{E}$ and $\mathcal{E}_1$, there is a $U_+ = U_+(U_-)$ such that $U_- \rightarrow U_+$ is a saddle-to-saddle connection for some $s = s_+$. For $U_- \in W_1(U_L)$, let the slow centered wave joining U_L to U_- have right speed s_- . The curve Σ is then defined by

$$\Sigma = \{U = U_+(U_-) : U_- \in W_1(U_L), \text{ between } \mathcal{E} \text{ and } \mathcal{E}_1, \quad s_+ > s_-\}.$$

We proceed by considering each of the transition curves $\mathcal{B}, \mathcal{P}, \mathcal{E}, \mathcal{F}, \mathcal{E}_1, \mathcal{F}_2$ of Figure 10 in turn, establishing how they affect the local structure of the U_R -plane. We then show in one example how to combine the local pieces of the U_R -plane to provide a more global picture of the solution of the Riemann problem.

The transition curve labelled $\mathcal{B}$ in Figure 10 corresponds to the occurrence of a secondary bifurcation point $(\tilde{B}, \tilde{s})$. The corresponding special value of U_R is labelled $\tilde{B}$ in Figures 15-17. As explained in [16], there is no U_R boundary (in the sense defined here) through $\tilde{B}$.

The transition curve $\mathcal{P}$ of Figure 10 corresponds to a primary pitchfork bifurcation at

$(U_-, \lambda_1(U_-))$ in the bifurcation diagram. $\mathcal{P}$ is therefore an inflection locus (see [16]). The associated special value of U_R is marked $U'_\mathcal{P}$ in Figure 16; it is defined by a saddle-to-saddle connection $U_\mathcal{P} \rightarrow U'_\mathcal{P}$ at some value of s . The fast wave curve through $U'_\mathcal{P}$ is a U_R boundary because the slow wave in the solution of the Riemann problem changes from a rarefaction wave to a rarefaction-shock as U_R crosses this curve. Although this is a new U_R boundary, it involves a phenomenon (namely pitchfork bifurcation) that was explained previously [16].

The remaining four transition curves $\mathcal{E}, \mathcal{F}, \mathcal{E}_1$ and $\mathcal{F}_2$ of Figure 10 give rise to new U_L and U_R boundaries. Each of these boundaries involves undercompressive shocks. Transition curves $\mathcal{E}, \mathcal{F}$ also affect fast shocks, while $\mathcal{E}_1$ and $\mathcal{F}_2$ also affect slow shocks. (This can be deduced by observing which parts of the bifurcation diagrams change as (b, v_-) crosses a transition curve in Figure 10.)

4.1 Transition curves for fast waves

In Figure 12 we show the role of the transition curves $\mathcal{E}$ and $\mathcal{F}$ in solving Riemann problems. In this Figure we show local coordinate systems of fast wave curves. We also label U_R regions by the sequence of waves appearing in the solution of the Riemann problem for U_R in that region. The letter Σ represents an undercompressive shock, while the letters $R, S, (SR)$ indicate the nature of a wave (here, the fast wave) as rarefaction, shock or shock-rarefaction, respectively. Wave curves that are also U_R boundaries are drawn thicker in the Figure. We use the letter W to represent the slow wave, which may be a shock, rarefaction, or rarefaction-shock.

In Figure 12, there are two new types of U_R boundary. The undercompressive shock curve Σ (defined above) locates values of U_+ for which $U_- \rightarrow U_+$ is an undercompressive shock, as U_- moves along $W_1(U_L)$. The other new type of boundary is labeled S_σ in Figure 12.

To define the curve S_σ , let (b, v_-) lie between transition curves $\mathcal{E}$ and $\mathcal{F}$ in Figure 10. We see from the corresponding bifurcation diagrams in Figure 11 that the curve of fast compressive shocks terminates at a point (U_1, s^*) because of the presence of a saddle-to-saddle connection. The corresponding phase portrait for (1.10), with $s = s^*$, is shown in Figure 13(c). In this Figure, we have labelled the equilibria as U_-, U_+, U_1 , where $U_- \rightarrow U_+$ is the saddle-to-saddle connection, and $U_1 = U_1(U_-)$ is the stable node. In

particular, $U_+ \in \Sigma$. The curve S_σ is defined by:

$$S_\sigma = \{U = U_1(U_-) : U_- \in W_1(U_L) \text{ between } \mathcal{E} \text{ and } \mathcal{F}\}.$$

Note that the curve of fast Lax shocks extends beyond U_1 (to the right of S_σ in Figure 12), but the corresponding fast shocks are not admissible.

Now U_1 lies in the Hugoniot locus $H(U_-)$ of U_- , and marks the end of a fast compressive shock curve, as explained above. However, U_1 also lies in the Hugoniot locus $H(U_+)$ of U_+ , and marks the end of a fast compressive shock curve $S_2(U_+)$ through U_+ . The curve $S_2(U_+)$ terminates at U_1 because the shock speed s decreases along $S_2(U_+)$ as U moves away from U_+ , and becomes s^* precisely at $U = U_1$. Therefore, although there are compressive shocks from U_+ to points $U \in H(U_+)$ beyond U_1 (to the left of S_σ in Figure 12), these shocks cannot be used to solve the Riemann problem because their speeds are smaller than the speed s^* of the undercompressive shock $U_- \rightarrow U_+$. Note that this observation preserves the property of uniqueness in our construction of solutions of Riemann problems.

At this point, it is convenient to assume that Σ is transverse to the fast wave curves through points of Σ :

Assumption 4.1 For each $U \in \Sigma$, Σ is transverse to the right eigenvector $r_2(U)$ of $DF(U)$.

The status of Assumption 4.1 is discussed at the end of this section.

In Figure 12(a) we represent the transition in the solution of the Riemann problem when U_- crosses the curve $\mathcal{E}$, while Figures 12(b),(c) show the solution as U_- crosses the curve $\mathcal{F}$. In drawing these figures, we rely upon Assumption 4.1 to ensure that the fast wave curves $W_2(U), U \in \Sigma$ do not intersect.

4.1.1 Transition curve $\mathcal{E}$

As $W_1(U_L)$ crosses $\mathcal{E}$, at $U_\mathcal{E}$, there is a point $U'_\mathcal{E}$ and a heteroclinic orbit $U_\mathcal{E} \rightarrow U'_\mathcal{E}$ for $s = \lambda_2(U_\mathcal{E})$. For U_R below $mU'_\mathcal{E}n$ in Figure 12(a) the Riemann problem solution has $U_- = (u_-, v_-) \in W_1(U_L)$ corresponding to (b, v_-) below $\mathcal{E}$ in Figure 10. Using the corresponding bifurcation diagrams from Figure 11, and using the definition of S_σ given above, we make the following observations about the structure of the Riemann problem solution. For U_R to the left of S_σ , the Riemann problem solution has a slow wave W and a fast shock, as shown in Figure 13(a). However, for U_R on the right of S_σ , the Riemann

problem solution has a slow wave, an undercompressive shock and a fast shock. As U_R approaches S_σ from the right, the fast shock speed decreases and approaches that of the undercompressive shock. In the limit, i.e., for U_R exactly on S_σ , there is a slow wave and a fast shock.

As (b, v_-) approaches the transition curve $\mathcal{E}$ of Figure 10 from below, the U_R boundaries S_σ and Σ approach and meet at $U'_\mathcal{E}$ in Figure 12(a). For U_- above $\mathcal{E}$ (i.e., (b, v_-) above $\mathcal{E}$ in Figure 10), there are no undercompressive shocks with U_- on the left of the shock. Moreover, the fast shock curve now includes all fast Lax shocks, so that the fast shock curve terminates at a limit point, on the dotted line labeled **L** in Figure 12(a) and emanating from $U'_\mathcal{E}$. Note that $U'_\mathcal{E}n$ is a U_R boundary, but that $mU'_\mathcal{E}$ is not a U_R boundary.

4.1.2 Transition curve $\mathcal{F}$

The transition curve $\mathcal{F}$ has to be considered in two parts, one part being the boundary between regions 9 and 12 in Figure 10, and the other part being the rest. Solutions of the Riemann problem involving $\mathcal{F}$ are shown in Figures 12(b,c).

As $W_1(U_L)$ crosses $\mathcal{F}$, at $U_\mathcal{F}$, there are points $(U'_\mathcal{F}, s), (Q, s)$ in the bifurcation diagram for (1.8) such that $s = \lambda_2(Q)$ and there is a heteroclinic orbit $U_\mathcal{F} \longrightarrow U'_\mathcal{F}$ at speed s . The points $U'_\mathcal{F}$ and Q are shown in Figures 12(b,c).

We first explain Figure 12(b), in which $W_1(U_L)$ crosses $\mathcal{F}$ between regions 9 and 11, or 8 and 9, or 3 and 4. The admissible portion of the fast shock curve for U_- shrinks to Q as U_- approaches $\mathcal{F}$ from above. Since there are undercompressive shocks for U_- on each side of $\mathcal{F}$, the undercompressive shock curve Σ persists. The role of Σ is to provide starting points for fast shock curves that replace the shrinking fast shock curve segments associated with U_- on $W_1(U_L)$. To put it another way, the solution of the Riemann problem involves the same sequence of waves for U_R above Qn in Figure 12(b) as for U_R below Qn , so that Qn is not a U_R boundary. Note however, that mQ is clearly a U_R boundary.

In Figure 12(c), we show the solution of the Riemann problem when $W_1(U_L)$ crosses $\mathcal{F}$ from region 9 to region 12. Here, the curve of limit points through Q is not monotonic because the shock curve to the left of S_σ does not shrink to Q as U_- approaches $\mathcal{F}$.

The curves $S_\sigma \cup U'_\mathcal{E}n$ in Figure 12(a) and $S_\sigma \cup mQ$ in Figures 12(b,c) have the following additional significance as U_R boundaries. For U_R on the left of these curves, the solution

of the Riemann problem consists of a slow wave and a fast shock-rarefaction wave. As U_R crosses the curve, an undercompressive shock wave splits off from the fast wave, as in Figure 13, so that the solution consists of three waves.

The transition curves $\mathcal{E}, \mathcal{F}$ are also U_L boundaries. To see this, first note that the fast wave curve $W_2(U_L)$ is a U_R boundary because (except for $U_L \in \mathcal{P}$) when U_R is on one side of $W_2(U_L)$, the slow wave is a shock, and when U_R is on the other side of $W_2(U_L)$, the slow wave is a rarefaction. As U_L passes through $\mathcal{E}$ or $\mathcal{F}$, the curve $W_2(U_L)$ passes through the corresponding point $U'_\mathcal{E}$ or $U'_\mathcal{F}$, thus rearranging the pattern of U_R regions.

4.2 Transition curves for slow waves

We described the transition curves $\mathcal{E}, \mathcal{F}$ above primarily in terms of their influence on U_R regions. By contrast, we focus on the role of the transition curves $\mathcal{E}_1, \mathcal{F}_2$ as U_L boundaries. Solutions of Riemann problems involving these boundaries are illustrated in Figure 14.

In Figure 14, we modify the conventions of Figure 12 as follows. We draw fewer fast wave curves, and we show the U_R regions only near the detached wave curve labelled CD in the Figure. Here, W stands for a fast centered wave, while the letters $R, S, (RS)$ indicate the nature of the slow wave.

Before explaining Figure 14, we discuss the relevant portions of the bifurcation diagrams of Figure 11. In these bifurcation diagrams, we shall refer to the primary bifurcation branch through $(U_-, \lambda_1(U_-))$ as ab and the detached branch as cd . In each bifurcation diagram, there are four dots indicating the presence of certain special heteroclinic orbits, as discussed in Section 2. At one value of s , there is a pair of dots, one dot on the s axis at a point $D = (U_-, s)$, with the second dot at a point $D' = (U'_-, s)$ on cd . Correspondingly, there is a heteroclinic orbit $U_- \rightarrow U'_-$ at speed s . The second pair of dots is at a different value s^* of s , and signifies a *remote* heteroclinic orbit $U_1 \rightarrow U_+$ from $U_1 \neq U_-$ near U_- (i.e., $Q = (U_1, s^*) \in ab$) to U_+ , with $Q' = (U_+, s^*) \in cd$. These features are labeled in region 9 of Figure 11. Note that U'_- and U_+ are saddle points, while U_- is a saddle for $s > \lambda_1(U_-)$, and an unstable node for $s < \lambda_1(U_-)$. U_1 is a saddle point for $s < \lambda_1(U_-)$.

We now turn to an explanation of the wave curves represented in Figure 14. In each figure, we show a portion AB of the attached slow wave curve $W_1(U_L)$ and a detached wave curve labelled CD . The detached wave curve CD consists of the detached slow wave curve $W_1^*(U_L)$ and part of the undercompressive shock curve Σ . Note that the curve

CD in Figure 14 is roughly horizontal so that it has transversal intersection with the fast wave curves, which are roughly vertical. Consequently, fast wave curves originating at distinct points of CD do not intersect. It follows by continuous dependence that we have existence and uniqueness of solutions of the Riemann problem obtained by the wave curve construction.

For each U_L , the slow wave curve $W_1(U_L)$ cuts the three transition curves $\mathcal{F}_2, \mathcal{P}, \mathcal{E}_1$. These intersections give rise to transitions in the detached wave curve CD . In Figure 14, the intersection of $W_1(U_L)$ with each of $\mathcal{F}_2, \mathcal{P}, \mathcal{E}_1$ is labeled with the corresponding letter $U_{\mathcal{F}_2}, U_{\mathcal{P}}, U_{\mathcal{E}_1}$. For definiteness, we suppose $W_1(U_L)$ lies to the left of $\mathcal{F}_2 \cap \mathcal{E}_1$, with sections labeled 4,5,10 on $W_1(U_L)$ corresponding to regions in Figure 10. (For $W_1(U_L)$ lying to the right of the intersection, the sequences of events along wave curves are different only in detail.) As remarked above, the location of U_L with respect to $\mathcal{F}_2, \mathcal{P}, \mathcal{E}_1$ is important. All the new features of this dependence are captured by taking U_L successively in regions 4,5,10 of Figure 10.

For U on $W_1(U_L)$, we use the notation U' to indicate that $U \longrightarrow U'$ is a heteroclinic orbit. We use the notation $S_1^* = S_1^*(U_L)$ to denote the detached portion of the slow shock curve. (I.e., S_1^* is a portion of the Hugoniot locus $H(U_L)$ not lying on $W_1(U_L)$ and consisting of admissible slow Lax shocks.)

4.2.1 U_L in region 4, above $\mathcal{E}_1$.

Here, all slow Lax shocks in $H(U_L)$ are admissible. It follows that the rarefaction-shock construction may be used with the end points of the detached shock curves $S_1^*(U_-)$ as U_- moves along the portion of the rarefaction curve $U_L U_{\mathcal{E}_1}$ lying in region 4. This explains the section labeled $U'_{\mathcal{E}_1} D$ of the detached wave curve CD .

At $U = U_{\mathcal{E}_1}$, the rarefaction shock involves a shock wave for which there is a separatrix connection. For each U_- in the section $AU_{\mathcal{E}_1}$ of $W_1(U_L)$, there is a corresponding U'_- in the section $CU'_{\mathcal{E}_1}$ such that $U_- \longrightarrow U'_-$ is an undercompressive shock with speed greater than the right speed of the wave from U_L to U_- .

For U_- in sections 5,10 however, the corresponding shocks $U_- \longrightarrow U'_-$ are too slow. For example, consider U_- in section 5a of $W_1(U_L)$. Then there is a slow rarefaction wave from U_L to U_- , with fastest speed $\lambda_1(U_-)$. The only admissible shocks on segment cd of the bifurcation diagram for U_- have speeds strictly less than $\lambda_1(U_-)$. Therefore these

shocks cannot be used in the construction of Riemann problem solutions for this value of U_L . In particular, the heteroclinic orbit $U_- \rightarrow U'_-$ has speed less than $\lambda_1(U_-)$. The only shock in the bifurcation diagram that can be used here corresponds to a rarefaction-shock in which the shock has speed $\lambda_1(U_-)$ and connects U_- to a point U_1 on the attached rarefaction-shock curve, shown as a railroad track in Figure 14. However, the shock speed $\lambda_1(U_-)$ is strictly greater than $\lambda_1(U_1)$, so there can be no faster slow shock with U_1 on the left. Moreover, since U_1 lies in region 5b or region 10 of Figure 10, the admissible shock $U_1 \rightarrow U'_1$ has speed less than $\lambda_1(U_1)$, so is too slow to be used. Exactly the same argument applies for U_- in section 5b of $W_1(U_L)$ except that the point U_- may be reached by a shock or a rarefaction-shock.

For U_- in region 10 however, there are admissible shocks from U_- to U_1 on cd with speeds $s \leq \lambda_1(U_-)$. But U_L is joined to U_- by a shock or rarefaction-shock whose right speed is strictly larger than $\lambda_1(U_-)$. From Figure 11, region10, we see that the only admissible shocks with U_- on the left and speed larger than $\lambda_1(U_-)$ are fast shocks. We conclude that Figure 14(a) is a complete local picture of the slow wave curves and nearby undercompressive shocks, for U_L in region 4. The solution of the Riemann problem for U_R near CD is completed by filling in the fast wave curves $W_2(U)$, with $U \in CD$.

4.2.2 U_L in region 5.

For U_L in regions 5 or 10, the end point U^* of S_1^* has corresponding speed $s^* < \lambda_1(U_L)$. Therefore, the curve S_1^* of detached admissible shocks shown in Figures 14(b,c,d) omits some Lax shocks. The point Q of Figure 11, region 5 is projected in Figure 14(b) as U_Q . Then $U_Q \rightarrow U'_Q$ is a saddle-to-saddle connection, with U'_Q lying at the end of S_1^* . Thus $U_L \rightarrow U'_Q$ is an admissible slow shock that is the superposition of a slow shock $U_L \rightarrow U_Q$ and an undercompressive shock $U_Q \rightarrow U'_Q$ at the same speed. For intermediate states U above U_Q on $W_1(U_L)$, as shown, there is also a saddle-to-saddle connection $U \rightarrow U'$, with the locus of U' (as U moves along $S_1(U_L)$) forming a curve shown as $U'_Q C$ in Figure 14(b).

For U_- between $U_{\mathcal{E}_1}$ and U_Q , there is an undercompressive shock $U_- \rightarrow U'_-$, but its speed s_- is smaller than the speed of the slow shock $U_L \rightarrow U_-$, even though it is larger than the characteristic speed $\lambda_1(U_-)$. For example, for $U_- = U_{\mathcal{E}_1}$, the shock $U_- \rightarrow U'_-$ has speed $\lambda_1(U_-)$. But $\lambda_1(U_-) < s$ by the Lax condition on slow shocks, where s is the

speed of the slow shock joining U_L to U_- . Therefore, these shocks cannot be used in solving the Riemann problem for this U_L .

As for Figure 14(a), there are no undercompressive shocks associated with U in section $U_{\mathcal{E}_1}B$, so the picture of wave curves in Figure 14(b) is complete.

The explanation of Figure 14(c), in which U_L lies between $\mathcal{P}$ and $\mathcal{F}_2$, is similar to that of Figure 14(b). The point $U_{\mathcal{E}_1}$ may or may not lie within the railroad track. Comparing Figure 14(b),(c), we see that the transition curve $\mathcal{P}$ has no effect on the wave curve CD or the U_R regions near CD .

4.2.3 U_L in region 10, below $\mathcal{F}_2$.

Consider the rarefaction curve $U_L U_{\mathcal{F}_2}$, which lies in region 10. From Figure 11, region 10, we see that there is a detached rarefaction-shock curve connected to the end of S_1^* . That is, there is a rarefaction wave from U_L to a point U_- on $U_L U_{\mathcal{F}_2}$ followed by a shock at speed $\lambda_1(U_-)$ from U_- to a point U_1 on $W_1^*(U_L)$. At $U_- = U_{\mathcal{F}_2}$, this rarefaction-shock construction breaks down due to the presence of a remote saddle-to-saddle connection, $\tilde{U}_{\mathcal{F}_2} \rightarrow \tilde{U}'_{\mathcal{F}_2}$ at speed $s = \lambda_1(U_{\mathcal{F}_2})$, for some point $\tilde{U}_{\mathcal{F}_2}$ on the attached rarefaction-shock curve (the railroad track portion of $W_1(U_L)$). We claim that

$\tilde{U}_{\mathcal{F}_2}$ must lie above $U_{\mathcal{E}_1}$ as shown in Figure 14(d).

To prove the claim, let $U_- \in \mathcal{F}_2 \cap W_1(U_L)$ be the point labeled $U_{\mathcal{F}_2}$ in Figure 14(d). Let $\bar{s} = \lambda_1(U_-)$ denote the speed of the shock in the rarefaction-shock construction, and let $\bar{U} = \tilde{U}_{\mathcal{F}_2}$ denote the value of U on the right of the wave. Then

$$\lambda_1(\bar{U}) < \bar{s} = \lambda_1(U_-) < \lambda_2(\bar{U}). \quad (4.6)$$

Also, if $s' = \lambda_1(U_-)$ is the speed of the undercompressive shock $\bar{U} \rightarrow \bar{U}' = \tilde{U}'_{\mathcal{F}_2}$, then

$$\lambda_1(\bar{U}') < s' = \lambda_1(U_-) < \lambda_2(\bar{U}'), \quad (4.7)$$

since this corresponds to the detached heteroclinic orbit for U_- on $\mathcal{F}_2$. From (4.6),(4.7), we deduce that

$$\lambda_1(\bar{U}) < s'. \quad (4.8)$$

It follows that $\bar{U}$ lies in region 4, the only region in which the pair of dots labelled D, D' in Figure 11 lies to the right of the plane $s = \lambda_1(\bar{U})$. The proof of the claim is complete.

The importance of the location of $\tilde{U}_{\mathcal{F}_2}$ relative to $U_{\mathcal{E}_1}$ is that it allows the construction of the undercompressive shock curve Σ attached to $\tilde{U}'_{\mathcal{F}_2}$.

The structure of Figure 14(d) is now readily comprehended. The section $C\tilde{U}'_{\mathcal{F}_2}$ corresponds to undercompressive shocks $U \rightarrow U'$ with U as shown in the section $A\tilde{U}_{\mathcal{F}_2}$ of $W_1(U_L)$.

4.3 Solution of a sample Riemann problem

The solution of Riemann problems involves fixing b , and constructing wave curves for each U_L , as above, by varying an intermediate state U_- on $W_1(U_L)$. As U_- moves along $W_1(U_L)$, the corresponding point (b, v_-) moves along a vertical path in Figure 10, intersecting a sequence of U_L boundaries. It is clear that there are a large number of different cases to consider, depending on the location of (U_L, b) . To illustrate the solution of Riemann problems, we choose a representative (U_L, b) in sector 2 of Figure 10, such that (b, v_-) intersects the sequence 1,2,7,8,9,10 of sectors in Figure 10. This sequence is chosen so that transitions in slow waves occur both before and after transitions involving fast waves. Despite this mixing, there is no interaction between the phenomena that we have discussed separately above.

In Figures 15-17 we represent the solution of the Riemann problem for this example. In Figure 15, points are labeled on the attached slow wave curve $W_1(U_L)$ to indicate where (b, v_-) crosses a transition curve of Figure 10. These points have counterparts on the detached slow wave curve $W_1^*(U_L)$, on the undercompressive shock curve Σ and on the curve S_σ , as explained below. The solution of the Riemann problem may be read off in its entirety from Figure 15, by understanding this correspondence, and by using the structure of the wave curves shown in Figure 15.

Figure 16 shows the division of the U_R plane into U_R regions. Each region is labeled according to the sequence of waves occurring in the solution of the Riemann problem when U_R lies in that region. The first letters S, R , or (RS) indicate the nature of the slow wave. Similarly, the final letters S, R , or (SR) indicate the nature of the fast wave. The letter Σ indicates the use of an undercompressive shock. In Figure 17, we show the broad structure of the coordinate system of wave curves used to construct solutions of the Riemann problem.

First we explain the correspondence of points in Figure 15. The point $\tilde{B}$ indicates the

secondary bifurcation point associated with $U_- = U_B$ on $W_1(U_L)$. For U_- above U_B on $W_1(U_L)$, the fast shock curve $S_2(U_-)$ is connected, whereas for U_- below U_B , and above $\mathcal{F}$, there is a detached portion $S_2^*(U_-)$ of the fast shock curve. From Subsection 4.1, we see that the end points of $S_2^*(U_-)$ form two curves as U_- varies along $W_1(U_L)$ below U_B , and above $U_{\mathcal{F}}$. The locus of one end point of $S_2^*(U_-)$ defines the curve labelled $\tilde{B}U'_{\mathcal{E}}U''_{\mathcal{F}}$ in Figure 15, while the locus of the other end forms part of the curve of limit points labelled L' . Correspondingly, for each $U_- \in W_1(U_L)$ between U_B , and $U_{\mathcal{F}}$, the fast wave curve has an attached portion containing U_- and a detached portion $W_2^*(U_-)$. In particular, $W_2^*(U_L)$ is a U_R boundary. As U_R crosses $W_2^*(U_L)$, $W_2^*(U_L)$, the slow wave changes from a shock to a rarefaction. The solution of the Riemann problem for U_R on $W_2^*(U_L)$ consists of a fast wave only. Note that this is true also of the attached fast wave curve $W_2(U_L)$ through U_L .

The points $U'_{\mathcal{E}}, U'_{\mathcal{F}}, U''_{\mathcal{F}}$ and the curves joining them, correspond to a combination of the constructions of Figure 12.

The point I' simply labels the intersection of the locus of inflection points with the undercompressive shock curve Σ . The corresponding point I shown on $W_1(U_L)$ is defined by the property that $I \rightarrow I'$ is an undercompressive shock. Note that, although the bifurcation from $U = I', s = \lambda_2(I')$ is supercritical, the connection $I \rightarrow I'$ joins two hyperbolic saddle points.

The shock $U_{\mathcal{F}_2} \rightarrow \tilde{U}_{\mathcal{F}_2}$ is special because it marks the transition between the undercompressive waves, and the detached rarefaction-shock curve, at $\tilde{U}'_{\mathcal{F}_2}$. This transition is the same as that of Figure 14(d).

In Figure 17, we point out the broad structure of the pattern of wave curves in Figure 15. This broad structure is common to all the possible Riemann problems obtained by varying b and U_L in the neighborhood considered in this paper. In the area labeled 1, the solution is obtained using the classical construction of Lax [11], as generalized by Liu [12] to allow for loss of genuine nonlinearity. For U_R in area 2, the solution involves a classical slow wave joining U_L to a point U_m on $W_1(U_L)$, and a fast wave joining U_m to U_R . The fast wave curve $W_2^*(U_m)$ involved in this construction passes through U_R , in area 2, but does not include any point of $W_1(U_L)$. In area 3, the solution involves a slow wave to a point U_m on $W_1(U_L)$, an undercompressive shock from U_m to U'_m on Σ , and a fast wave from U'_m to U_R . In area 4, the solution involves a slow wave joining U_L to a point U_m on the detached slow wave curve labeled $W_1^*(U_L)$, followed by a fast wave from U_m to U_R . Note

that if the slow wave is a rarefaction-shock, then the intermediate states in the rarefaction wave lie on $W_1(U_L)$.

As $\epsilon \rightarrow 0$, area 2 in Figure 17 shrinks to a curve, and the undercompressive shock curve Σ becomes a straight line. We recover the solution of the Riemann problem described in [20] for quadratic nonlinearities. The new features of the construction presented here, with $\epsilon \neq 0$, are the construction of the curve Σ , and the appearance of detached fast shock curves.

We end this section with a discussion of Assumption 4.1. It is not hard to verify that Σ is transverse to $r_2(U)$ for each U outside a small neighborhood of $U = (\frac{1}{3}, 0)$. In other words, we can verify Assumption 4.1 except in some neighborhood of the point U'_ϵ . If Assumption 4.1 should turn out to be false, then solutions of the Riemann problem can still be constructed as described in this section, except that fast wave curves through some points of Σ overlap in the U -plane. This implies that the solution is not unique for some initial data (1.2). We conjecture however, that Assumption 4.1 is true, in which case the solution obtained through our construction is unique.

5 Bifurcation Analysis

In this section we prove the results of Section 2 about the family of bifurcation problems $\dot{U} = G(U, s, M)$ given by (2.1). We study separately the intervals $2u_- - \delta \leq s \leq 2u_- + \delta$, $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$, and $-\frac{2}{3}u_- - \delta \leq s \leq -\frac{2}{3}u_- + \delta$. We study these intervals in the order: second, third, first. In a final subsection we prove those results of Section 2 that use the analysis of more than one interval.

We recall that $U \in \mathcal{U}$, and we restrict to $|M| < \delta$.

We first note that for $2u_- - \delta \leq s \leq -\frac{2}{3}u_- + \delta$, the differential equation $\dot{U} = G(U, s, M)$ has an equilibrium at (u_-, v_-) with a unique invariant curve tangent to the eigendirection for the larger eigenvalue. This curve meets the line $u = -\frac{s}{2}$ in a point $\tilde{U}_1(s, M)$. See Figure 18. Also, for $2u_- - \delta \leq s < -\frac{2}{3}u_-$ and M near 0 (depending on s) there is a hyperbolic saddle $\bar{U}(s, M)$ near $(-u_- - s, 0)$. Its stable manifold meets the line $u = -\frac{s}{2}$ in a point $\tilde{U}_2(s, M)$. We define a function $d(s, M)$ that measures the separation of $\tilde{U}_1(s, M)$ and $\tilde{U}_2(s, M)$:

$$d(s, M) = G\left(-\frac{s}{2}, 0, s, 0\right) \wedge (\tilde{U}_1(s, M) - \tilde{U}_2(s, M)), \quad (5.1)$$

where $\wedge$ means determinant. We have $d(s, 0) \equiv 0$. The function d is C^k , where $k \rightarrow \infty$ as $\delta \rightarrow 0$ [21].

5.1 Melnikov integrals

In this subsection we evaluate the partial derivatives of d at $(s, 0)$ for $2u_- < s < -\frac{2}{3}u_-$.

For $2u_- - \delta \leq s \leq -\frac{2}{3}u_- + \delta$, let $\gamma_s(t) = (u_s(t), 0)$ be the solution of $\dot{U} = G(U, s, 0)$ that passes through $(-\frac{s}{2}, 0)$ at $t = 0$. Then

$$u_s(t) = \left[e^{(2u_- + s)t} + 1 \right]^{-1} \left[e^{(2u_- + s)t} u_- - (u_- + s) \right].$$

The solution $\gamma_s(t)$ runs along the u -axis from $(u_-, 0)$ to $(-u_- - s, 0)$. For $2u_- \leq s \leq -\frac{2}{3}u_-$, the image of $\gamma_s(t)$ is a separatrix connection.

Let

$$J_s(t) = \exp \left[- \int_0^t \operatorname{div} G(\gamma_s(r), s, 0) dr \right] = e^{2st}.$$

Let p denote any of $b, v_-, \mu_1, \dots, \mu_8$. If both $(u_-, 0)$ and $(-u_- - s, 0)$ are saddles (i.e., if

$2u_- < s < -\frac{2}{3}u_-$), then $\frac{\partial d}{\partial p}(s, 0)$ is given by the Melnikov integral:

$$\frac{\partial d}{\partial p}(s, 0) = \int_{-\infty}^{\infty} J_s(t) G(\gamma_s(t), s, 0) \wedge \frac{\partial G}{\partial p}(\gamma_s(t), s, 0) dt = \int_{-\infty}^{\infty} e^{2st} \dot{u}_s(t) \frac{\partial G_2}{\partial p}((u_s(t), 0), s, 0) dt.$$

See [18]. For $i \neq 5$, $\frac{\partial G_2}{\partial \mu_i}((u, 0), s, 0) = 0$, so $\frac{\partial d}{\partial \mu_i}(s, 0) = 0$.

The other integrals are of the form

$$\int_{-\infty}^{\infty} e^{2st} \dot{u}_s P(u_s) dt, \quad (5.2)$$

where P is a polynomial. We make the substitution

$$u = u_s(t), \quad du = \dot{u}_s dt = (u_- - u_s)(u_s + u_- + s) dt$$

This substitution and the initial condition $u_s(0) = -\frac{s}{2}$ imply that

$$t = \frac{1}{2u_- + s} \ln \left[\frac{u + u_- + s}{u_- - u} \right].$$

Therefore (5.2) becomes

$$\int_{u_-}^{-u_- - s} \left[\frac{u + u_- + s}{u_- - u} \right]^{2s/(2u_- + s)} P(u) du. \quad (5.3)$$

If $s = 0$, the integral is elementary. If $2u_- < s < -\frac{2}{3}u_-$, $s \neq 0$, we substitute

$$z = \frac{u + u_- + s}{u_- - u}, \quad u = u_- - \frac{2u_- + s}{z + 1}, \quad du = \frac{2u_- + s}{(z + 1)^2} dz.$$

Then (5.3) becomes

$$\int_{\infty}^0 z^{2s/(2u_- + s)} Q(z) dz,$$

where

$$Q(z) = \frac{2u_- + s}{(z + 1)^2} P\left(u_- - \frac{2u_- + s}{z + 1}\right).$$

Thus Q is a polynomial in $\frac{1}{z+1}$. This type of integral is evaluated by residues ([14], p. 304); the only pole of Q is at $z = -1$.

The results are as follows. Let

$$\zeta(s) = \begin{cases} \frac{\pi s(s - 2u_-)}{\sin \frac{2u_- + 3s}{2u_- + s} \pi}, & 2u_- < s < -\frac{2}{3}u_-, \quad s \neq 0; \\ 2u_-^2, & s = 0. \end{cases}$$

Then

$$\begin{aligned}\frac{\partial d}{\partial v_-}(s, 0) &= 2\zeta(s), \\ \frac{\partial d}{\partial b}(s, 0) &= \frac{2}{3}u_- \zeta(s), \\ \frac{\partial d}{\partial \mu_5}(s, 0) &= \frac{1}{3}u_-(s + 3u_-)\zeta(s).\end{aligned}\tag{5.4}$$

5.2 Bifurcation of the separatrix connection for $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$.

For s in this interval and $|M| < \delta$, $d(s, M) = 0$ if and only if the saddles (u_-, v_-) and $\overline{U}(s, M)$ are joined by a separatrix connection near the u -axis. For fixed M , condition (II) of Section 2 is satisfied on the interval $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$ provided $d(s, M) = 0$ implies $\frac{\partial d}{\partial s}(s, M) \neq 0$.

We recall the curve (2.2),

$$v_- = \phi(b) = -\frac{1}{3}u_-b + \dots.$$

For $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$, $d(s, v_-, b, 0) = 0$ if and only if $v_- = \phi(b)$.

Fix $\bar{\mu}$ with $\bar{\mu}_5 = 1$. Let

$$\hat{d}(s, v_-, b, \epsilon) = d(s, v_-, b, \epsilon \bar{\mu}).$$

Theorem 5.1 *There are smooth functions $\xi(b, \epsilon)$, $\eta(b, \epsilon)$, with*

$$\begin{aligned}\xi(b, \epsilon) &= \epsilon \left(-\frac{1}{6}u_-(5u_- + \delta) + \mathcal{O}(|(b, \epsilon)|) \right), \\ \eta(b, \epsilon) &= \epsilon \left(-\frac{1}{6}u_-(\frac{7}{3}u_- - \delta) + \mathcal{O}(|(b, \epsilon)|) \right),\end{aligned}$$

such that for (v_-, b, ϵ) small, the equation

$$\hat{d}(s, v_-, b, \epsilon) = 0\tag{5.5}$$

has a solution in $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$ if and only if

$$\xi(b, \epsilon) \leq v_- - \phi(b) \leq \eta(b, \epsilon).\tag{5.6}$$

If (v_-, b, ϵ) is sufficiently small and satisfies (5.6), and $\epsilon \neq 0$, then (5.5) has a unique solution in $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$. At that solution $\frac{\partial \hat{d}}{\partial s} \neq 0$. The solution occurs at

$s = 2u_- + \delta$ (resp. $s = -\frac{2}{3}u_- - \delta$) if and only if $v_- - \phi(b) = \xi(b, \epsilon)$ (resp. $v_- - \phi(b) = \eta(b, \epsilon)$). Moreover, ξ and η are smooth functions of $(b, \epsilon, \bar{\mu})$.

See Figure 19. This theorem implies that for $\bar{\mu}_5 = 1$ and (v_-, b, ϵ) small, the bifurcation problem $\dot{U} = G(U, s, v_-, b, \epsilon \bar{\mu})$ satisfies conditions (I)–(III) of Section 2 on the interval $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$. The strip (5.6) has width $\mathcal{O}(\epsilon)$.

Proof. Since

$$\hat{d}(s, \phi(b), b, 0) \equiv 0,$$

we write

$$\hat{d}(s, v_-, b, \epsilon) = \frac{\partial \hat{d}}{\partial v_-}(s, \phi(b), b, 0)(v_- - \phi(b)) + \frac{\partial \hat{d}}{\partial \epsilon}(s, \phi(b), b, 0)\epsilon + \mathcal{O}(|(v_- - \phi(b), \epsilon)|^2).$$

Since $\frac{\partial \hat{d}}{\partial v_-}(s, 0, 0, 0) \neq 0$ by (5.4), by the Implicit Function Theorem we have that for $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$ and (v_-, b, ϵ) small, $\hat{d} = 0$ if and only if

$$v_- - \phi(b) = \gamma(s, b, \epsilon) = \epsilon[\gamma_1(s, b) + \mathcal{O}(\epsilon)]. \quad (5.7)$$

For fixed (v_-, b, ϵ) , there exists s , $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$, such that $\hat{d}(s, v_-, b, \epsilon) = 0$ if and only if

$$\min\{\gamma(s, b, \epsilon) : 2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta\} \leq v_- - \phi(b) \leq \max\{\gamma(s, b, \epsilon) : 2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta\}.$$

The extrema of $\gamma(s, b, \epsilon)$ on $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$ are also the extrema of $\gamma_1(s, b) + \mathcal{O}(\epsilon)$ on this interval. Using (5.4), we easily compute:

$$\gamma_1(s, b) = -\frac{\partial \hat{d}}{\partial \epsilon}(s, \phi(b), b, 0) / \frac{\partial \hat{d}}{\partial v_-}(s, \phi(b), b, 0) = -\frac{1}{6}u_-(s + 3u_-). \quad (5.8)$$

Therefore $\gamma_1(s, b) + \mathcal{O}(\epsilon)$ has positive derivative on $2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta$. We conclude:

$$\begin{aligned} \min\{\gamma(s, b, \epsilon) : 2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta\} &= \gamma(2u_- + \delta, b, \epsilon) \stackrel{\text{def}}{=} \xi(b, \epsilon), \\ \max\{\gamma(s, b, \epsilon) : 2u_- + \delta \leq s \leq -\frac{2}{3}u_- - \delta\} &= \gamma(-\frac{2}{3}u_- - \delta, b, \epsilon) \stackrel{\text{def}}{=} \eta(b, \epsilon). \end{aligned}$$

The expressions for ξ and η in Theorem 5.1 follow from (5.7) and (5.8). Also,

$$\begin{aligned} & \frac{\partial \hat{d}}{\partial s}(s, \phi(b) + \gamma(s, b, \epsilon), b, \epsilon) = \\ & \epsilon \left[\frac{\partial^2 \hat{d}}{\partial v_- \partial s}(s, \phi(b), b, 0) \gamma_1(s, b) + \frac{\partial^2 \hat{d}}{\partial \epsilon \partial s}(s, \phi(b), b, 0) + \mathcal{O}(\epsilon) \right] = \\ & \epsilon \left[\frac{1}{3} u_- \zeta(s) + \mathcal{O}(\epsilon) \right] \neq 0 \quad \text{for } \epsilon \neq 0. \quad \square \end{aligned}$$

5.3 Bifurcation near $s = -\frac{2}{3}u_-$.

For $(s, M) = (-\frac{2}{3}u_-, 0)$, the equilibrium at $(-\frac{1}{3}u_-, 0)$ has a zero eigenvalue with eigenvector $(0, 1)$. We let $r = s + \frac{2}{3}u_-$ and consider

$$\dot{U} = G(U, -\frac{2}{3}u_- + r, M). \quad (5.9)$$

To study equilibria of (5.9) near $(U, r, M) = ((-\frac{1}{3}u_-, 0), 0, 0)$, we first solve the equation $G_1(U, -\frac{2}{3}u_- + r, M) = 0$ near that point by the Implicit Function Theorem, obtaining

$$u = -\frac{1}{3}u_- + \psi(v, r, M),$$

where ψ is C^∞ and $\psi(0) = 0$. We then define

$$k(v, r, M) = G_2((-\frac{1}{3}u_- + \psi(v, r, M), v), -\frac{2}{3}u_- + r, M).$$

This is of course the Liapunov-Schmidt method of studying equilibria. It gives the same information as center manifold reduction [17], but has the advantage that the bifurcation function k is C^∞ . This is important for us since we shall use Golubitsky-Schaeffer bifurcation theory [6].

Let us preview the remainder of this subsection. We first show that $k(v, r, 0)$ undergoes a pitchfork bifurcation at $(v, r) = (0, 0)$. Using Golubitsky-Schaeffer theory, we put the unfolding $k(v, r, M)$ into normal form. In the normal form variables, it is easy to obtain a C^∞ parameterization of the equilibria of (5.9) near $(U, r, M) = ((-\frac{1}{3}u_-, 0), 0, 0)$ by a vector of parameters P . We write this parameterization as $(U(P), r(P), M(P))$. The differential equation

$$\dot{U} = G(U, -\frac{2}{3}u_- + r(P), M(P))$$

has equilibria at (u_-, v_-) and at $U(P)$. The former is a saddle. The latter has a unique invariant curve $W(P)$ tangent to the eigendirection for the smaller eigenvalue. We define a function $d(P)$ that measures the separation between the unstable manifold of (u_-, v_-) and $W(P)$, and we compute its partial derivatives at $P = 0$. Using the normal form for the function k and the function d , we then study how conditions (I)–(III) of Section 2 can fail near $s = -\frac{2}{3}u_-$. We conclude that failure can occur only on one of the transition surfaces $\mathcal{B}$, $\mathcal{H}$, $\mathcal{E}$ and $\mathcal{F}$ of Section 2, and we study how these surfaces fit together.

We begin by writing k as a series in v and r with coefficients series in the coordinates of M . We retain in the vr series the terms $1, v, r, v^2, vr, v^3, vr^2, v^3r$, and v^5 . We compute the coefficients to order one for the first four terms and to order zero for the last two. The result is

$$\begin{aligned} k(v, r, M) = & \left(-\frac{8}{3}u_-v_- - \frac{8}{9}u_-^2b - \frac{28}{27}u_-^3\mu_5 + \cdots\right) + \left(\frac{14}{9}u_-^2\mu_1 + \frac{1}{9}u_-^2\mu_6 + \cdots\right)v \\ & + \left(v_- + \frac{2}{3}u_-b - \frac{1}{3}u_-^2\mu_5 + \cdots\right)r + \left(\frac{3}{2}b - \frac{1}{6}u_- \mu_2 - \frac{1}{4}u_- \mu_5 - \frac{1}{3}u_- \mu_7 + \cdots\right)v^2 + (-3 + \cdots)vr \\ & + \left(-\frac{3}{2u_-} + \cdots\right)v^3 + (0 + \cdots)vr^2 + \left(\frac{9}{8u_-^2} + \cdots\right)v^3r + \left(\frac{27}{32u_-^3} + \cdots\right)v^5 + \cdots. \end{aligned} \quad (5.10)$$

Since

$$k(v, r, 0) = -\frac{3}{2u_-}v^3 - 3vr + \cdots, \quad (5.11)$$

we have a pitchfork bifurcation at $(v, r) = (0, 0)$ when $M = 0$.

Let

$$f(x, \lambda, \alpha, \beta) = x^3 - \lambda x + \alpha + \beta x^2,$$

the universal unfolding of the pitchfork; the coefficients of x^3 and λx are chosen to agree in sign with those of v^3 and vr in (5.11). By [6] there are smooth functions $X(v, r, M)$, $\Lambda(r, M)$, $A(M)$, $B(M)$, $S(v, r, M)$, with $X(0) = \Lambda(0) = A(0) = B(0) = 0$, $S > 0$, $\frac{\partial X}{\partial v} > 0$, $\frac{\partial \Lambda}{\partial r} > 0$, such that

$$k = S \cdot f(X, \Lambda, A, B). \quad (5.12)$$

Proposition 5.1 *There exist smooth functions X , Λ , A , B and S as above such that to first order,*

$$X(v, r, M) = \frac{1}{\sqrt{-2u_-}}v + \frac{\sqrt{-2u_-}}{9}b - \frac{1}{3\sqrt{-2u_-}}v_- - \frac{u_- \sqrt{-2u_-}}{18}\mu_5 + \cdots,$$

$$\begin{aligned}
\Lambda(r, M) &= r - \frac{14}{27}u_-^2\mu_1 - \frac{u_-^2}{27}\mu_6 + \dots, \\
A(M) &= \frac{4u_- \sqrt{-2u_-}}{27}b + \frac{4\sqrt{-2u_-}}{9}v_- + \frac{14u_-^2 \sqrt{-2u_-}}{81}\mu_5 + \dots, \\
B(M) &= \frac{\sqrt{-2u_-}}{12}b + \frac{3}{2\sqrt{-2u_-}}v_- - \frac{u_- \sqrt{-2u_-}}{18}\mu_2 - \frac{u_- \sqrt{-2u_-}}{72}\mu_5 - \frac{u_- \sqrt{-2u_-}}{9}\mu_7 + \dots.
\end{aligned}$$

Proof. Let m denote any coordinate of M . Write

$$\begin{aligned}
X(v, r, 0) &= a_1v + a_2r + a_3v^2 + \dots, \\
S(v, r, 0) &= s_o + s_1v + s_2r + s_3v^2 + \dots, \\
\frac{\partial k}{\partial m}(v, r, 0) &= b_o + b_1v + b_2r + b_3v^2 + \dots.
\end{aligned} \tag{5.13}$$

Differentiate both sides of (5.13) with respect to m , set $M = 0$, write both sides as series in v, r , and retain only the terms $1, v, r, v^2$. We obtain the equations

$$\begin{aligned}
b_o &= s_o \frac{\partial A}{\partial m}(0), \\
b_1 &= -s_o a_1 \frac{\partial \Lambda}{\partial m}(0) + s_1 \frac{\partial A}{\partial m}(0), \\
b_2 &= -s_o \left(\frac{\partial X}{\partial m}(0) + a_2 \frac{\partial \Lambda}{\partial m}(0) \right) + s_2 \frac{\partial A}{\partial m}(0), \\
b_3 &= s_o \left(3a_1^2 \frac{\partial X}{\partial m}(0) - a_3 \frac{\partial \Lambda}{\partial m}(0) + a_1^2 \frac{\partial B}{\partial m}(0) \right) - s_1 a_1 \frac{\partial \Lambda}{\partial m}(0) + s_3 \frac{\partial A}{\partial m}(0).
\end{aligned} \tag{5.14}$$

The numbers a_i and s_i can be determined as follows. Since $\dot{U} = G(U, -\frac{2}{3}u_- + r, 0)$ is symmetric about the u -axis, we find that $k(v, r, 0) = -k(-v, r, 0)$. Therefore

$$k(v, r, 0) = v(v^2 g_1(v) - r g_2(v, r)),$$

where $g_1(0) = -\frac{3}{2u_-} > 0$, $g_2(0, 0) = 3$, g_1 and g_2 are even functions of v . We write

$$k(v, r, 0) = g_2(v, r)v(v^2 \frac{g_1(v)}{g_2(v, r)} - r),$$

which suggests

$$\begin{aligned}
X(v, r, 0) &= v g_1^{\frac{1}{2}} g_2^{-\frac{1}{2}}, \\
\Lambda(r, 0) &= r, \\
S(v, r, 0) &= g_1^{-\frac{1}{2}} g_2^{\frac{3}{2}}.
\end{aligned} \tag{5.15}$$

It follows from the Golubitsky-Schaeffer theory that any choice of $X(v, r, 0)$, $\Lambda(r, 0)$, $S(v, r, 0)$ that works for $M = 0$ can be used, so we shall use (5.15). We see immediately that

$$a_1 = \frac{1}{\sqrt{-2u_-}}, \quad a_2 = 0, \quad s_o = 3\sqrt{-2u_-}.$$

Moreover, since g_1 and g_2 are even functions of v , we have $g_1'(0) = \frac{\partial g_2}{\partial v}(0, 0) = 0$. Then short computations show that $a_3 = s_1 = 0$. The computation of s_2 and s_3 from (5.15) requires that we know $\frac{\partial g_2}{\partial r}(0, 0)$, $g_1''(0)$, and $\frac{\partial^2 g_2}{\partial v^2}(0, 0)$; these are determined by the vr^2 , v^5 , and v^3r terms of (5.10). The results are $s_2 = 0$, $s_3 = -\frac{27}{32u_-}\sqrt{-2u_-}$.

Now we can solve the system (5.14) for each m . Using the results together with a_1 , a_2 , and (5.15), we have X , Λ , A and B to first order. $\square$

Let

$$\begin{aligned} \nu &\in \mathbb{R}^8, \\ N &= (\alpha, \beta, \nu) \in \mathbb{R}^{10}. \end{aligned}$$

Define a diffeomorphism T of neighborhoods of the origin in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{10}$ by $T(v, r, M) = (x, \lambda, N)$, where

$$\begin{aligned} x &= X(v, r, M), \\ \lambda &= \Lambda(r, M), \\ \alpha &= A(M), \\ \beta &= B(M), \\ \nu &= \mu. \end{aligned} \tag{5.16}$$

T^{-1} has the form

$$\begin{aligned} v &= V(x, \lambda, N) \\ r &= R(\lambda, N) \\ v_- &= V_-(N) \\ b &= C(N) \\ \mu &= \nu. \end{aligned} \tag{5.17}$$

Note that there is a diffeomorphism Φ from a neighborhood of the origin in M -space to one in N -space given by the last three lines of (5.16).

Corollary 5.1 *In (5.17) we have, to first order,*

$$\begin{aligned} V(x, \lambda, N) &= \sqrt{-2u_-}x + \frac{21}{16\sqrt{-2u_-}}\alpha - \frac{\sqrt{-2u_-}}{6}\beta + \frac{u_-^2}{54}\nu_2 - \frac{u_-^2}{3}\nu_5 + \frac{u_-^2}{27}\nu_7 + \cdots, \\ R(\lambda, N) &= \lambda + \frac{14}{27}u_-^2\nu_1 + \frac{u_-^2}{27}\nu_6 + \cdots, \\ V_-(N) &= -\frac{9\sqrt{-2u_-}}{32u_-}\alpha + \frac{\sqrt{-2u_-}}{2}\beta - \frac{u_-^2}{18}\nu_2 - \frac{u_-^2}{9}\nu_5 - \frac{u_-^2}{9}\nu_7 + \cdots, \\ C(N) &= \frac{81}{16\sqrt{-2u_-}}\alpha - \frac{3\sqrt{-2u_-}}{2u_-}\beta + \frac{u_-}{6}\nu_2 - \frac{5u_-}{6}\nu_5 + \frac{u_-}{3}\nu_7 + \cdots. \end{aligned}$$

Next we use the normal form variables to parameterize the equilibria of (5.9) near $((-\frac{1}{3}u_-, 0), 0, 0)$. Let

$$g(x, \lambda, N) = f(x, \lambda, \alpha, \beta) = x^3 - \lambda x + \alpha + \beta x^2.$$

By (5.12), $T(k^{-1}(0)) = g^{-1}(0) = \{(x, \lambda, N) : \alpha = -x^3 + \lambda x - \beta x^2\}$. Now $g^{-1}(0)$ is smoothly parameterized by

$$P = (x, \lambda, \beta, \nu).$$

Applying T^{-1} we obtain a smooth parameterization of $k^{-1}(0)$. The equations are (5.17) with $\alpha = -x^3 + \lambda x - \beta x^2$. We obtain a smooth parameterization of the equilibria of (5.9) near $((-\frac{1}{3}u_-, 0), 0, 0)$ by adding the equation $u = -\frac{1}{3}u_- + \psi(v, r, M)$, where (v, r, M) is given by (5.17) with $\alpha = -x^3 + \lambda x - \beta x^2$. We write this parameterization as $(U(P), r(P), M(P))$.

The differential equation

$$\dot{U} = G(U, -\frac{2}{3}u_- + r(P), M(P))$$

has equilibria at (u_-, v_-) and at $U(P)$. The former is a saddle. The latter has a unique invariant curve $W(P)$ tangent to the eigendirection for the smaller eigenvalue. The unstable manifold of (u_-, v_-) (resp. the curve $W(P)$) meets the line $u = \frac{1}{3}u_-$ in a point $\tilde{U}_1(P)$ (resp. $\tilde{U}_2(P)$). We measure the separation by

$$d(P) = G((\frac{1}{3}u_-, 0), -\frac{2}{3}u_-, 0) \wedge (\tilde{U}_1(P) - \tilde{U}_2(P)).$$

Since $U(P)$ is smooth, so is the function d .

We have:

Proposition 5.2 $d(0, \lambda, \beta, 0) = 0$.

Proof. Let Γ be the curve in M -space defined by $v_- = \phi(b)$, $\mu = 0$. Let $L(b)$ be the line near the u -axis that is invariant under the flow of $\dot{U} = G(U, s, (\phi(b), b, 0))$ for all s . $\Phi(\Gamma)$ is a curve in the plane $\nu = 0$. For each point $(\phi(b), b, 0) \in \Gamma$ there is an s near $2u_-$ such that a transcritical or pitchfork bifurcation occurs at an equilibrium on $L(b)$ near $(-\frac{1}{3}u_-, 0)$. But for the function g , $g = \frac{\partial g}{\partial x} = \frac{\partial g}{\partial \lambda} = 0$ if and only if $x = \lambda = \alpha = 0$. Therefore $\Phi(\Gamma)$ is an interval on the β -axis.

Consider in P -space the point $P = (x, \lambda, \beta, \nu) = (0, 0, \beta, 0)$. At the corresponding point in $x\lambda N$ -space, $(x, \lambda, \alpha, \beta, \nu) = (0, 0, 0, \beta, 0)$, we have $g = \frac{\partial g}{\partial x} = \frac{\partial g}{\partial \lambda} = 0$. Therefore $(U(P), r(P), M(P))$ is a point at which a transcritical or pitchfork bifurcation occurs. Therefore $M(P) = (v_-(P), b(P), 0)$ is a point of Γ , $U(P) \in L(b(P))$, and $r(P)$ is the value of r at which the bifurcation occurs. Therefore $d(0, 0, \beta, 0) = 0$.

Of course, $M(0, \lambda, \beta, 0) = M(0, 0, \beta, 0)$ independent of λ . Also, $r(0, \lambda, 0, 0) = R(\lambda, 0) = \lambda$. Therefore $U(0, \lambda, 0, 0) = (-\frac{1}{3}u_- - \lambda, 0)$, since no other possible choice of U would be smooth. Then in order that U be smooth, we must have $U(0, \lambda, \beta, 0) \in L(b(0, \lambda, \beta, 0))$ for all λ, β . Therefore $d(0, \lambda, \beta, 0) \equiv 0$. $\square$

Theorem 5.2 *The only nonzero partial derivatives of d at the origin are:*

$$\begin{aligned}\frac{\partial d}{\partial x}(0) &= -\frac{16}{9}u_-^2\sqrt{-2u_-}, \\ \frac{\partial d}{\partial \nu_5}(0) &= \frac{64}{243}u_-^4.\end{aligned}$$

Proof. It is convenient to define the differential equation

$$\dot{U} = K(U, P) = (K_1(U, P), K_2(U, P)) = G\left(U, -\frac{2}{3}u_- + r(P), M(P)\right),$$

which has the smooth families of equilibria $U_1(P) = (u_-, v_-)$ and $U(P)$. We recall that

$$\dot{U} = K(U, 0) = G\left(U, -\frac{2}{3}u_-, 0\right)$$

has the solution

$$\gamma_{-\frac{2}{3}u_-}(t) = (u_{-\frac{2}{3}u_-}(t), 0)$$

that passes through $(\frac{1}{3}u_-, 0)$ at $t = 0$. Its image is the separatrix connection along the u -axis from $(u_-, 0)$ to $(-\frac{1}{3}u_-, 0)$.

We have

$$d(P) = K \left(\left(\frac{1}{3}u_-, 0 \right), 0 \right) \wedge \left(\tilde{U}_1(P) - \tilde{U}_2(P) \right).$$

Let p stand for any coordinate of P . Since for $P = 0$, the equilibrium at $t = \infty$ on the heteroclinic orbit has a zero eigenvalue, a boundary term must be added to Melnikov's integral for $\frac{\partial d}{\partial p}(0)$ [18]. We have

$$\begin{aligned} \frac{\partial d}{\partial p}(0) &= \lim_{t \rightarrow \infty} \left[-J_{-\frac{2}{3}u_-}(t) K(\gamma_{-\frac{2}{3}u_-}(t), 0) \right] \wedge \frac{\partial U}{\partial p}(0) \\ &\quad + \int_{-\infty}^{\infty} -J_{-\frac{2}{3}u_-}(t) K(\gamma_{-\frac{2}{3}u_-}(t), 0) \wedge \frac{\partial K}{\partial p}(\gamma_{-\frac{2}{3}u_-}(t), 0) dt \\ &= -\frac{16}{9}u_-^2 \frac{\partial V}{\partial p}(0) + \int_{-\infty}^{\infty} e^{-\frac{4}{3}u_-t} \dot{u}_{-\frac{2}{3}u_-}(t) \frac{\partial K_2}{\partial p}((u_{-\frac{2}{3}u_-}(t), 0), 0) dt \end{aligned}$$

From (5.17), Corollary 5.1, the definitions of $r(P)$ and $M(P)$, and (2.1) we compute the following. Partial derivatives of K_2 are evaluated at $((u, 0), 0)$, of G_2 at $((u, 0), -\frac{2}{3}u_-, 0)$.

$$\begin{aligned} \frac{\partial K_2}{\partial x} &= 0 \quad \text{because} \quad \alpha = -x^3 + \lambda x - \beta x^2, \\ \frac{\partial K_2}{\partial \nu_1} &= \frac{14}{27}u_-^2 \frac{\partial G_2}{\partial r} + \frac{\partial G_2}{\partial \mu_1} = 0, \\ \frac{\partial K_2}{\partial \nu_2} &= -\frac{u_-^2}{18} \frac{\partial G_2}{\partial v_-} + \frac{u_-}{6} \frac{\partial G_2}{\partial b} + \frac{\partial G_2}{\partial \mu_2}, \\ \frac{\partial K_2}{\partial \nu_5} &= -\frac{1}{9}u_-^2 \frac{\partial G_2}{\partial v_-} - \frac{5}{6}u_- \frac{\partial G_2}{\partial b} + \frac{\partial G_2}{\partial \mu_5}, \\ \frac{\partial K_2}{\partial \nu_6} &= \frac{1}{27}u_-^2 \frac{\partial G_2}{\partial r} + \frac{\partial G_2}{\partial \mu_6} = 0, \\ \frac{\partial K_2}{\partial \nu_7} &= -\frac{1}{9}u_-^2 \frac{\partial G_2}{\partial v_-} + \frac{1}{3}u_- \frac{\partial G_2}{\partial b} + \frac{\partial G_2}{\partial \mu_7}. \end{aligned}$$

For $i = 3, 4, 8$, $\frac{\partial K_2}{\partial \nu_i} = \frac{\partial G_2}{\partial \mu_i} = 0$.

To compute the integrals in $\frac{\partial d}{\partial \nu_2}(0)$ and $\frac{\partial d}{\partial \nu_7}(0)$, we note that $\frac{\partial G_2}{\partial \mu_2}((u, 0), -\frac{2}{3}u_-, 0) = \frac{\partial G_2}{\partial \mu_7}((u, 0), -\frac{2}{3}u_-, 0) = 0$, so that $\frac{\partial K_2}{\partial \nu_7}((u, 0), 0) = 2\frac{\partial K_2}{\partial \nu_2}((u, 0), 0)$. Thus only one of these integrals need be computed. The integral in $\frac{\partial d}{\partial \nu_5}(0)$ is essentially different. Computing the integrals, which in the form (5.3) are elementary, and using the first order formula for V in Corollary 5.1, yields

$$\frac{\partial d}{\partial x}(0) = -\frac{16}{9}u_-^2 \sqrt{-2u_-} + 0 = -\frac{16}{9}u_-^2 \sqrt{-2u_-},$$

$$\begin{aligned}
\frac{\partial d}{\partial \nu_2}(0) &= -\frac{16}{9}u_-^2 \cdot \frac{u_-^2}{54} + \frac{8}{243}u_-^4 = 0, \\
\frac{\partial d}{\partial \nu_5}(0) &= -\frac{16}{9}u_-^2 \cdot -\frac{1}{3}u_-^2 - \frac{80}{243}u_-^4 = \frac{64}{243}u_-^4, \\
\frac{\partial d}{\partial \nu_7}(0) &= -\frac{16}{9}u_-^2 \cdot \frac{u_-^2}{27} + \frac{16}{243}u_-^4 = 0.
\end{aligned}$$

Of course, $\frac{\partial d}{\partial \lambda}(0) = \frac{\partial d}{\partial \beta}(0) = 0$ by Proposition 5.1. The other partial derivatives of d are trivially zero. $\square$

We define the following sets:

$$\begin{aligned}
\tilde{\mathcal{B}} &= \{(x, \lambda, \alpha, \beta, \nu) : g = \frac{\partial g}{\partial x} = \frac{\partial g}{\partial \lambda} = 0\}, \\
\tilde{\mathcal{H}} &= \{(x, \lambda, \alpha, \beta, \nu) : g = \frac{\partial g}{\partial x} = \frac{\partial^2 g}{\partial x^2} = 0\}, \\
\tilde{\mathcal{E}} &= \{(x, \lambda, \alpha, \beta, \nu) : g = \frac{\partial g}{\partial x} = 0 \text{ and } d(x, \lambda, \beta, \nu) = 0\}, \\
\tilde{\mathcal{F}} &= \text{the closure of } \{(x, \lambda, \alpha, \beta, \epsilon) : g = 0, d = 0, \text{ and there exists } y \neq x \\
&\quad \text{such that } g = \frac{\partial g}{\partial x} = 0 \text{ at } (y, \lambda, \alpha, \beta, \nu)\},
\end{aligned}$$

If we project these sets to $\alpha\beta\nu$ -space and apply Φ^{-1} , we obtain the sets $\mathcal{B}$, $\mathcal{H}$, $\mathcal{E}$ and $\mathcal{F}$ defined in Section 2.

Theorem 5.3 *There is a continuous positive function $\rho(\bar{\mu})$, defined for $\{\bar{\mu} : \bar{\mu}_5 = 1\}$, such that if $\sup\{|v_-|, |b|, |\epsilon|\} < \rho(\bar{\mu})$, then the bifurcation problem $\dot{U} = G(U, s, v_-, b, \epsilon\bar{\mu})$ satisfies conditions (I)–(III) of Section 2 on the interval $-\frac{2}{3}u_- - \delta \leq s \leq -\frac{2}{3}u_- + \delta$ unless $(b, v_-, \epsilon\bar{\mu}) \in \mathcal{B} \cup \mathcal{H} \cup \mathcal{E} \cup \mathcal{F}$.*

Proof. The question of whether $\dot{U} = G(U, s, v_-, b, \mu)$ satisfies (I)–(III) on $-\frac{2}{3}u_- - \delta \leq s \leq -\frac{2}{3}u_- + \delta$ can be answered by studying the functions $g(x, \lambda, \alpha, \beta, \nu)$ and $d(x, \lambda, \beta, \nu)$, with x small, $-\delta \leq \lambda \leq \delta$, and $(\alpha, \beta, \nu) = \Phi(v_-, b, \mu)$.

Condition (I) holds on $-\frac{2}{3}u_- - \delta \leq s \leq -\frac{2}{3}u_- + \delta$ provided $g = \frac{\partial g}{\partial x} = 0$ implies $\frac{\partial g}{\partial \lambda} \neq 0$ and $\frac{\partial^2 g}{\partial x^2} \neq 0$ for x small and $-\delta \leq \lambda \leq \delta$. Therefore (I) holds unless there exists (x, λ) such that $(x, \lambda, \alpha, \beta, \nu) \in \tilde{\mathcal{B}} \cup \tilde{\mathcal{H}}$.

Next, we discuss condition (III), the simultaneous occurrence of bifurcations. In the unfolding of the pitchfork bifurcation, it is never the case that two equilibrium bifurcations

occur at the same value of λ . Moreover, since $\frac{\partial d}{\partial x} \neq 0$, we may solve $d = 0$ for x as a function of (λ, β, ν) . Therefore it is never the case that two heteroclinic bifurcations occur at the same value of λ . Thus (III) fails only when an equilibrium and a heteroclinic bifurcation occur simultaneously, i.e., when there exists (x, λ) such that $(x, \lambda, \alpha, \beta, \nu) = \tilde{\mathcal{F}}$.

A sufficient condition for (II) to hold is that whenever $g(x, \lambda, \alpha, \beta, \nu) = 0$ and $d(x, \lambda, \beta, \nu) = 0$, then $\frac{\partial g}{\partial x}(x, \lambda, \alpha, \beta, \nu) \neq 0$ and the connection breaks in a nondegenerate manner when (α, β, ν) is held fixed and λ varies. Thus (II) can fail if there exists (x, λ) such that $(x, \lambda, \alpha, \beta, \nu) \in \tilde{\mathcal{E}}$ or if the connection fails to break in a nondegenerate manner. We shall rule out the second possibility when $\nu_5 \neq 0$.

Fix $\bar{\nu}$ with $\bar{\nu}_5 = 1$ and let

$$\begin{aligned}\hat{g}(x, \lambda, \alpha, \beta, \epsilon) &= g(x, \lambda, \alpha, \beta, \epsilon\bar{\nu}) = x^3 - \lambda x + \alpha + \beta x^2, \\ \hat{d}(x, \lambda, \beta, \epsilon) &= d(x, \lambda, \beta, \epsilon\bar{\nu}) = -\frac{16}{9}u_-^2\sqrt{-2u_-}x + \frac{64}{243}u_-^4\epsilon + \dots\end{aligned}\tag{5.18}$$

By Proposition 5.1 we also have

$$\hat{d}(0, \lambda, \beta, 0) = 0.\tag{5.19}$$

Suppose $\hat{g}(x_o, \lambda_o, \alpha_o, \beta_o, \epsilon_o) = 0$, $\hat{d}(x_o, \lambda_o, \beta_o, \epsilon_o) = 0$, $\frac{\partial \hat{g}}{\partial x}(x_o, \lambda_o, \alpha_o, \beta_o, \epsilon_o) \neq 0$, and $\epsilon_o \neq 0$. Near $(x_o, \lambda_o, \alpha_o, \beta_o, \epsilon_o)$ we can solve the equation $\hat{g} = 0$ for x by the Implicit Function Theorem:

$$x = x(\lambda, \alpha, \beta, \epsilon) \quad \text{with} \quad x(\lambda_o, \alpha_o, \beta_o, \epsilon_o) = x_o.$$

Nondegenerate breaking of the heteroclinic connection means that

$$\frac{\partial}{\partial \lambda} \hat{d}(x(\lambda, \alpha, \beta, \epsilon), \lambda, \beta, \epsilon) \neq 0 \quad \text{at} \quad (\lambda, \alpha, \beta, \epsilon) = (\lambda_o, \alpha_o, \beta_o, \epsilon_o).\tag{5.20}$$

Since $\frac{\partial x}{\partial \lambda} = -\frac{\partial \hat{g}}{\partial \lambda} / \frac{\partial \hat{g}}{\partial x}$, (5.20) is equivalent to

$$\frac{\partial \hat{d}}{\partial \lambda} \frac{\partial \hat{g}}{\partial x} - \frac{\partial \hat{d}}{\partial x} \frac{\partial \hat{g}}{\partial \lambda} \neq 0 \quad \text{at} \quad (x_o, \lambda_o, \alpha_o, \beta_o, \epsilon_o).\tag{5.21}$$

However, for $\epsilon \neq 0$, there are no small simultaneous solutions of $\hat{g} = 0$, $\hat{d} = 0$, and $\frac{\partial \hat{d}}{\partial \lambda} \frac{\partial \hat{g}}{\partial x} - \frac{\partial \hat{d}}{\partial x} \frac{\partial \hat{g}}{\partial \lambda} = 0$. In fact, by Theorem 2.2 and the definition of $\hat{g}$, these three functions have linearly independent derivatives at the origin, so their simultaneous solutions form a two-dimensional manifold. But from (5.19), this manifold is $\{(x, \lambda, \alpha, \beta, \epsilon) : x = 0, \alpha = 0, \epsilon = 0\}$. $\square$

Proof of Theorem 2.2 We define sets $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{M}}$ by using the definitions of $\mathcal{E}$ and $\mathcal{F}$ respectively but omitting the condition $d = 0$.

Fix $\bar{\nu}$ with $\bar{\nu}_5 = 1$ and define $\hat{g}$ and $\hat{d}$ as in the proof of Theorem 5.3. By abuse of notation, we shall write $(x, \lambda, \alpha, \beta, \epsilon) \in \tilde{\mathcal{E}}$ for $(x, \lambda, \alpha, \beta, \epsilon\bar{\nu}) \in \tilde{\mathcal{E}}$, and $(\alpha, \beta, \epsilon) \in \mathcal{E}$ for $\Phi^{-1}(\alpha, \beta, \epsilon\bar{\nu}) \in \mathcal{E}$, etc.

Of course $\hat{g}(x, \lambda, \alpha, \beta, \epsilon) = 0$ if and only if

$$\alpha = -x^3 + \lambda x - \beta x^2. \quad (5.22)$$

From (5.18) and (5.19), and the Implicit Function Theorem, we have that $\hat{d}(x, \lambda, \beta, \epsilon) = 0$ if and only if

$$x = c\epsilon(1 + h(\lambda, \beta, \epsilon)) \quad (5.23)$$

where h is smooth, $h(0, 0, 0) = 0$, and

$$c = -\frac{2}{27}u_- \sqrt{-2u_-} > 0.$$

In the remainder of the proof, the Implicit Function Theorem is repeatedly used in this way, and always guarantees that the functions produced are smooth. To find $\mathcal{E}$, we note that $(x, \lambda, \alpha, \beta, \epsilon) \in \tilde{\mathcal{L}}$ if and only if (5.22) holds and in addition

$$\lambda = 3x^2 + 2\beta x. \quad (5.24)$$

Equations (5.23) and (5.24) define a two-dimensional surface in $x\lambda\beta\epsilon$ -space:

$$x = c\epsilon(1 + i(\beta, \epsilon)), \quad (5.25a)$$

$$\lambda = \mathcal{O}(\|(\beta, \epsilon)\|^2), \quad (5.25b)$$

where $i(0, 0) = 0$. Adding equation (5.22) gives $\tilde{\mathcal{E}}$. Projecting to $\alpha\beta\epsilon$ -space by substituting first (5.24) and then (5.25a) into (5.22) gives: $(\alpha, \beta, \epsilon) \in \mathcal{E}$ if and only if

$$\alpha = c^2\epsilon^2(1 + i(\beta, \epsilon))^2(2c\epsilon(1 + i(\beta, \epsilon)) + \beta). \quad (5.26)$$

To find $\mathcal{F}$, we note that $(x, \lambda, \alpha, \beta, \epsilon) \in \tilde{\mathcal{M}}$ if and only if (5.22) holds and in addition

$$\lambda = \frac{3}{4}(x + \frac{1}{3}\beta)^2 - \frac{1}{9}\beta^2. \quad (5.27)$$

Equations (5.23) and (5.27) define a two-dimensional surface in $x\lambda\beta\epsilon$ -space

$$x = c\epsilon(1 + j(\beta, \epsilon)), \quad (5.28a)$$

$$\lambda = \mathcal{O}(\|(\beta, \epsilon)\|^2), \quad (5.28b)$$

where $j(0, 0) = 0$. Adding equation (5.22) gives $\tilde{\mathcal{F}}$. Projecting to $\alpha\beta\epsilon$ -space by substituting first (5.27) and then (5.28a) into (5.22) gives $(\alpha, \beta, \epsilon) \in \mathcal{F}$ if and only if

$$\alpha = -\frac{1}{4}c\epsilon(1 + j(\beta, \epsilon))(c\epsilon(1 + j(\beta, \epsilon)) + \beta)^2. \quad (5.29)$$

Since $(\alpha, \beta, \epsilon) \in \mathcal{B}$ if and only if $\alpha = 0$, to determine $\mathcal{B} \cap \mathcal{E}$ for $\epsilon \neq 0$ we set $\alpha = 0$ in (5.26) and divide by ϵ^2 . Then for $\epsilon \neq 0$, $(\alpha, \beta, \epsilon) \in \mathcal{B} \cap \mathcal{E}$ if and only if $\alpha = 0$ and

$$\beta = -2c\epsilon + \mathcal{O}(\epsilon^2).$$

The intersection is transverse since this is a simple root of (5.26).

To determine $\mathcal{B} \cap \mathcal{F}$ for $\epsilon \neq 0$ we set $\alpha = 0$ in (5.29) and divide by ϵ . Then for $\epsilon \neq 0$, $(\alpha, \beta, \epsilon) \in \mathcal{B} \cap \mathcal{F}$ if and only if $\alpha = 0$ and

$$\beta = -c\epsilon + \mathcal{O}(\epsilon^2).$$

$\mathcal{B}$ and $\mathcal{E}$ have a quadratic tangency at this point since this is a double root of (5.29).

To find $\mathcal{H} \cap \mathcal{E}$ and $\mathcal{H} \cap \mathcal{F}$, we first find $\tilde{\mathcal{H}} \cap \tilde{\mathcal{E}} = \tilde{\mathcal{H}} \cap \tilde{\mathcal{F}}$. We have $(x, \lambda, \alpha, \beta, \epsilon) \in \tilde{\mathcal{H}}$ if and only if (5.22) holds and

$$\begin{aligned} x &= -\frac{\beta}{3}, \\ \lambda &= -\frac{\beta^2}{3}. \end{aligned} \quad (5.30)$$

Equations (5.23) and (5.30) define a curve in $x\lambda\beta\epsilon$ -space.

$$\begin{aligned} x &= c\epsilon + \mathcal{O}(\epsilon^2), \\ \lambda &= \mathcal{O}(\epsilon^2), \\ \beta &= -3c\epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \quad (5.31)$$

Adding equation (5.22) gives $\tilde{\mathcal{H}} \cap \tilde{\mathcal{E}} = \tilde{\mathcal{H}} \cap \tilde{\mathcal{F}}$. Projecting to $\alpha\beta\epsilon$ -space by substituting (5.31) into (5.22) gives

$$\begin{aligned} \alpha &= \mathcal{O}(\epsilon^3), \\ \beta &= -3c\epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \quad (5.32)$$

These points are contained in both $\mathcal{H} \cap \mathcal{E}$ and $\mathcal{H} \cap \mathcal{F}$. Along $\tilde{\mathcal{H}}$, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{M}}$ are tangent, and for each fixed ϵ , projection of the common two-dimensional tangent space to $\alpha\beta$ -space has rank one. It follows that $\mathcal{H}$, $\mathcal{E}$, and $\mathcal{F}$ are tangent along (5.32).

An equation for $\mathcal{H} \cap \mathcal{E}$ comes from equating the equation for $\mathcal{H}$, $\alpha = \beta^3/27$, and (5.26). Setting $\hat{\epsilon} = \epsilon(1 + i(\beta, \epsilon))$ in (5.26), we obtain

$$\beta^3 - 27c^2\beta\hat{\epsilon}^2 - 54c^3\hat{\epsilon}^3 = 0,$$

or

$$(\beta + 3c\hat{\epsilon})^2(\beta - 6c\hat{\epsilon}) = 0. \quad (5.33)$$

The double root represents the tangential intersection of $\mathcal{H}$, $\mathcal{E}$, and $\mathcal{F}$ identified above. The single root $\beta = 6c\hat{\epsilon} + \mathcal{O}(\epsilon^2)$ is, for $\epsilon \neq 0$, a transverse intersection of $\mathcal{H}$ and $\mathcal{E}$.

Similarly, for $\mathcal{H} \cap \mathcal{F}$, we obtain, after setting $\tilde{\epsilon} = \epsilon(1 + j(\beta, \epsilon))$ in (5.29),

$$4\beta^3 + 27c\beta^2\tilde{\epsilon} + 54c^2\beta\tilde{\epsilon}^2 + 27c^3\tilde{\epsilon}^3 = 0,$$

or

$$(\beta + 3c\tilde{\epsilon})^2(4\beta + 3c\tilde{\epsilon}) = 0. \quad (5.34)$$

Again, the double root is the known tangential intersection of $\mathcal{H}$, $\mathcal{E}$, and $\mathcal{F}$, and the single root $\beta = -\frac{3}{4}c\tilde{\epsilon} + \mathcal{O}(\epsilon^2)$ is, for $\epsilon \neq 0$, a transverse intersection of $\mathcal{H}$ and $\mathcal{F}$.

To determine $\mathcal{E} \cap \mathcal{F}$, we equate (5.26) and (5.29), divide by $c^2\epsilon^2$, and obtain

$$(\beta + 3c\epsilon)^2 + \mathcal{O}(\|(\beta, \epsilon)\|^3) = 0. \quad (5.35)$$

Since (5.35) has a known double root $\beta = -3c\epsilon + \mathcal{O}(\epsilon^2)$, (5.36) is divisible by its square. Performing the division shows that there are no other small solutions of (5.35). Moreover, equations (5.33), (5.34), and (5.35) show that the pairwise intersections of $\mathcal{H}$, $\mathcal{E}$, and $\mathcal{F}$ along (5.32) are, for fixed $\epsilon \neq 0$, nondegenerate quadratic. $\square$

Proof of Remark 2.1. We define

$$\tilde{\mathcal{A}}_{\pm} = \{(x, \lambda, \alpha, \beta, \nu) : g = 0, \quad d(x, \lambda, \beta, \nu) = 0, \quad \text{and} \quad \lambda = \pm\delta\}.$$

Projecting to $\alpha\beta\nu$ -space and applying Φ^{-1} gives $\mathcal{A}_{\pm}$. We shall abuse this notation as in the previous proof.

To find $\tilde{\mathcal{A}}_{\pm}$ for $\nu = \epsilon\bar{\nu}$, we note that (5.23) and the equation $\lambda = \pm\delta$ define a two-dimensional surface in $x\lambda\beta\epsilon$ -space.

$$\begin{aligned} x &= c\epsilon(1 + h(\pm\delta, \beta, \epsilon)), \\ \lambda &= \pm\delta. \end{aligned} \tag{5.36}$$

Adding (5.22) gives $\tilde{\mathcal{A}}_{\pm}$. Projecting to $\alpha\beta\epsilon$ -space by substituting (5.36) into (5.22) yields that $(\alpha, \beta, \epsilon) \in \tilde{\mathcal{A}}_{\pm}$ if and only if

$$\alpha = c\epsilon(\pm\delta + \mathcal{O}(\|(\delta, \beta, \epsilon)\|^2)). \tag{5.37}$$

For $\epsilon = 0$, $\mathcal{A}_{\pm}$ coincides with $\mathcal{B}$, $\mathcal{E}$, and $\mathcal{F}$. For small $\delta > 0$, (β, ϵ) in a small ball about 0 whose size depends on δ , and $\epsilon \neq 0$, the sets $\tilde{\mathcal{A}}_{\pm}$ do not meet $\mathcal{B}$, $\mathcal{E}$, or $\mathcal{F}$. To see this for $\mathcal{B}$, we set $\alpha = 0$ in (5.37), divide by $c\epsilon$, and note that for small nonzero δ the function $\pm\delta + \mathcal{O}(\|(\delta, \beta, \epsilon)\|^2)$ is nonzero for $(\beta, \epsilon) = (0, 0)$. The assertion then follows by continuity. To see the assertion for $\mathcal{E}$ or $\mathcal{F}$, we equate equations (5.37) and (5.26) or (5.29), divide by $c\epsilon$, and again obtain $\pm\delta + \mathcal{O}(\|(\delta, \beta, \epsilon)\|^2) = 0$.

To find $\tilde{\mathcal{A}}_{\pm} \cap \mathcal{H}$, we equate (5.37) and $\alpha = \beta^3/27$, obtaining

$$\beta^3 \mp 27c\epsilon(\delta + \mathcal{O}(\|(\delta, \beta, \epsilon)\|^2)) = 0.$$

By the Implicit Function Theorem, this equation can be solved for ϵ as a function of β :

$$\epsilon = \frac{\beta^3}{27c\delta} + \mathcal{O}(\beta^4).$$

Therefore $\beta \sim 3^3\sqrt{c\delta\epsilon}$. $\square$

Remark 5.1 $\mathcal{A}_{-}$ coincides with the set $v_{-} = \phi(b) + \eta(b, \epsilon)$ of Theorem 5.1.

5.4 Bifurcation near $s = 2u_{-}$.

For $(s, M) = (2u_{-}, 0)$, the equilibrium at $(u, v) = (u_{-}, 0)$ has a zero eigenvalue with eigenvector $(0, 1)$. Since $(u, v) = (u_{-}, v_{-})$ is an equilibrium of (2.1) for all (s, M) , we let $y = u - u_{-}$, $z = v - v_{-}$, $Y = (y, z)$, $r = s - 2u_{-}$, and consider

$$\dot{Y} = H(Y, r, M) = G(U_{-} + Y, 2u_{-} + r, M). \tag{5.38}$$

Then $H(0, r, M) \equiv 0$. To study equilibria of (5.38) near $(Y, r, M) = (0, 0, 0)$, we first solve the equation $H_1 = 0$ near the origin by the Implicit Function Theorem, obtaining

$$y = \psi(z, r, M)$$

with $\psi(0, r, M) \equiv 0$. We then define

$$k(z, r, M) = H_2(\psi(z, r, M), r, M),$$

where $k(0, r, M) \equiv 0$.

The remainder of this subsection is organized like Subsection 5.3. We first show that $k(z, r, 0)$ undergoes a pitchfork bifurcation at $(z, r) = (0, 0)$. The fact that $k(0, r, M) \equiv 0$ makes the normal form for the unfolding $k(z, r, M)$ simpler in this case. On the other hand, there are two smooth families of equilibria of (5.38) near $(Y, r, M) = (0, 0, 0)$, one of them $Y \equiv 0$. Thus we need two separation functions.

We write k as a series in z and r with coefficients series in the coordinates of M . Of course only terms of the form $z^i r^j$ with $i > 0$ occur. We retain in the zr series the terms z , z^2 , zr , and z^3 . We retain the coefficients to order one for the first two terms and to order zero for the last two. The result is

$$\dot{z} = k(z, r, M) = u_-^2 \mu_6 z + \left(\frac{3}{2u_-} v_- + \frac{3}{2} b + \frac{1}{2} u_- \mu_2 + \frac{3}{4} u_- \mu_5 + u_- \mu_7 \right) z^2 - zr + \frac{1}{2u_-} z^3 + \dots \quad (5.39)$$

Since

$$k(z, r, 0) = \frac{1}{2u_-} z^3 - zr + \dots, \quad (5.40)$$

we again have the pitchfork bifurcation at $(z, r) = (0, 0)$, but this time with known trivial solution $z = 0$ for all values of (s, M) . Let

$$f(x, r, \beta) = -x^3 - \lambda x + \beta x^2.$$

The coefficients of x^3 and λx are chosen to agree in sign with those of z^3 and zr in (5.40). By [6] there are smooth functions $X(z, r, M)$, $\Lambda(r, M)$, $B(M)$, $S(z, r, M)$, with $X(0, r, M) \equiv 0$, $\Lambda(0, 0) = 0$, $B(0) = 0$, $\frac{\partial X}{\partial r} > 0$, $\frac{\partial \Lambda}{\partial r} > 0$, such that

$$k = S \cdot f(X, \Lambda, B). \quad (5.41)$$

Proposition 5.3 *There exist smooth functions X, Λ, B, S as above such that to first order,*

$$\begin{aligned} X(z, r, M) &= \frac{1}{\sqrt{-2u_-}} z + \cdots, \\ \Lambda(r, M) &= r - u_-^2 \mu_6 + \cdots, \\ B(M) &= \sqrt{-2u_-} \left(\frac{3}{2u_-} v_- + \frac{3}{2} b + \frac{1}{2} u_- \mu_2 + \frac{3}{4} u_- \mu_5 + u_- \mu_7 \right) + \cdots. \end{aligned}$$

Proof. The computations are similar to those in the proof of Proposition 5.1 but easier. We write

$$\begin{aligned} X(z, 0, 0) &= a_1 z + \cdots, \\ S(z, 0, 0) &= s_o + s_1 z + \cdots, \\ \frac{\partial k}{\partial m}(z, 0, 0) &= b_1 z + b_3 z^2 + \cdots. \end{aligned}$$

We differentiate both sides of (5.41) with respect to m , set $(r, M) = (0, 0)$, and obtain

$$\begin{aligned} b_1 &= -s_o a_1 \frac{\partial \Lambda}{\partial m}(0), \\ b_3 &= -s_1 a_1 \frac{\partial \Lambda}{\partial m}(0) + s_o a_1^2 \frac{\partial B}{\partial m}(0). \end{aligned} \tag{5.42}$$

We determine a_1 , s_o , and s_1 as in the proof of Proposition 5.1:

$$a_1 = \frac{1}{\sqrt{-2u_-}}, \quad s_o = \sqrt{-2u_-}, \quad s_1 = 0.$$

Now we can solve the system (5.42) for each m . Using the results together with a_1 and (5.15) yields Proposition 5.3. $\square$

Let $\nu \in \mathbb{R}^8$, $N = (\alpha, \beta, \nu) \in \mathbb{R}^{10}$. Define a diffeomorphism T of neighborhoods of the origin in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{10}$ by $T(z, r, M) = (x, \lambda, N)$ where

$$\begin{aligned} x &= X(z, r, M) \\ \lambda &= \Lambda(r, M) \\ \alpha &= b \\ \beta &= B(M) \\ \nu &= \mu. \end{aligned} \tag{5.43}$$

Then T^{-1} has the form

$$\begin{aligned}
z &= Z(x, \lambda, N), \\
r &= R(\lambda, N), \\
v_- &= C(N), \\
b &= \alpha, \\
\mu &= \nu.
\end{aligned} \tag{5.44}$$

Corollary 5.2 *In (5.44) we have, to first order,*

$$\begin{aligned}
Z(x, \lambda, N) &= \sqrt{-2u_-}x + \cdots, \\
R(\lambda, N) &= \lambda + u_-^2\nu_6 + \cdots, \\
C(N) &= -u_- \alpha - \frac{1}{3}\sqrt{-2u_-}\beta - \frac{1}{3}u_-^2\nu_2 - \frac{1}{2}u_-^2\nu_5 - \frac{2}{3}u_-^2\mu_7 + \cdots.
\end{aligned}$$

Let

$$g(x, \lambda, N) = f(x, \lambda, \beta) = -x^3 - \lambda x + \beta x^2.$$

From (5.42), $T(k^{-1}(0)) = g^{-1}(0) = \{(x, \lambda, N) : -x^3 - \lambda x + \beta x^2 = 0\}$. Now $g^{-1}(0) = Z_1 \cup Z_2$, where

$$\begin{aligned}
Z_1 &= \{(x, \lambda, N) : x = 0\}, \\
Z_2 &= \{(x, \lambda, N) : \lambda = -x^2 + \beta x\}.
\end{aligned}$$

Therefore for s near $2u_-$, the differential equation $\dot{U} = G(U, s, M)$ has two families of equilibria near $((u_-, 0), 2u_-, 0)$:

$$\begin{aligned}
(1) \quad & ((u_-, v_-), s, M), \\
(2) \quad & (u_- + y(x, N), v_- + z(x, N), 2u_- + R_2(x, N), M(N)),
\end{aligned}$$

where $z(x, N) = Z(x, -x^2 + \beta x, N)$, $R_2(x, N) = R(-x^2 + \beta x, N)$, $M(N)$ is given by the last three lines of (5.44), and $y(x, N) = \psi(z(x, N), R_2(x, N), M(N))$. The differential equation $\dot{U} = G(U, s, M)$ also has the saddle $\bar{U}(s, M)$ near $(-3u_-, 0)$ defined at the start of Section 5.

We now define two separation functions, one for each family of equilibria. We recall from (5.1) the smooth function $d(s, M)$ that is zero precisely when, for the differential equation $\dot{U} = G(U, s, M)$, the unique invariant manifold of (u_-, v_-) tangent to the eigendirection for

the larger eigenvalue meets the stable manifold of $\bar{U}(s, M)$ in a trajectory near the u -axis. We set $d_1(\lambda, N) = d(2u_- + R(\lambda, N), M(N))$. To define the second separation function, let $s = 2u_- + R_2(x, N)$ and $M = M(N)$. The equilibrium $(u_- + y(x, N), v_- + z(x, N))$ of $\dot{U} = G(U, s, M)$ has a unique invariant manifold tangent to the eigendirection for the larger eigenvalue. It meets the line $u = -\frac{s}{2}$ in a point $\tilde{U}_1(x, N)$. With $\tilde{U}_2(s, M)$ defined at the start of Section 5, we let

$$d_2(x, N) = G((-\frac{s}{2}, 0), s, 0) \wedge [\tilde{U}_1(x, N) - \tilde{U}_2(s, M)],$$

where $s = 2u_- + R_2(x, N)$, $M = M(N)$. Then

$$d_1(0, \alpha, \beta, \nu) = d_2(0, \alpha, \beta, \nu). \quad (5.45)$$

Since $d_1(\lambda, 0) \equiv 0$, we have

$$d_1(\lambda, \alpha, \beta, \nu) = a(\lambda)\alpha + b(\lambda)\beta + \sum c_i(\lambda)\nu_i + \mathcal{O}(\|(\alpha, \beta, \nu)\|^2). \quad (5.46)$$

Then by (5.45),

$$d_2(x, \alpha, \beta, \nu) = ex + a(0)\alpha + b(0)\beta + \sum c_i(0)\nu_i + \mathcal{O}(\|(\alpha, \beta, \nu)\|^2).$$

Theorem 5.4

$$\begin{aligned} a(0) &= -\frac{32}{3}u_-^3, & a'(0) &= -8u_-^2, \\ b(0) &= -\frac{16}{3}u_-^2\sqrt{-2u_-}, & b'(0) &= -4u_- \sqrt{-2u_-}, \\ c_2(0) &= -\frac{16}{3}u_-^4, & c'_2(0) &= -4u_-^3, \\ c_5(0) &= \frac{16}{3}u_-^4, & c'_5(0) &= \frac{20}{3}u_-^3, \\ c_7(0) &= -\frac{32}{3}u_-^4, & c'_7(0) &= -8u_-^3, \\ c_i(0) &= c'_i(0) = 0 & \text{for } i &= 1, 3, 4, 6, 8, \\ e &= 16u_-^2\sqrt{-2u_-}. \end{aligned}$$

Proof. For $\lambda > 0$, the partial derivatives of $d_1(\lambda, N) = d(2u_- + R(\lambda, N), M(N))$ at $(\lambda, 0)$ may be computed easily from (5.4). Since d_1 is smooth, partial derivatives of d_1 at $(0, 0)$

may then be obtained by taking the limit as $\lambda \rightarrow 0$. We use (5.4), (5.44), and Corollary 5.2, and let

$$\zeta_1(\lambda) = \begin{cases} 8u_-^2, & \lambda = 0, \\ \zeta(2u_- + \lambda), & \lambda > 0. \end{cases}$$

Then for $\lambda > 0$,

$$\begin{aligned} \frac{\partial d_1}{\partial \alpha}(\lambda, 0) &= -u_- \frac{\partial d}{\partial v_-}(2u_- + \lambda, 0) + \frac{\partial d}{\partial b}(2u_- + \lambda, 0) = -\frac{4}{3}u_- \zeta_1(\lambda), \\ \frac{\partial d_1}{\partial \beta}(\lambda, 0) &= -\frac{1}{3}\sqrt{-2u_-} \frac{\partial d}{\partial v_-}(2u_- + \lambda, 0) = -\frac{2}{3}\sqrt{-2u_-} \zeta_1(\lambda), \\ \frac{\partial d_1}{\partial \nu_2}(\lambda, 0) &= -\frac{1}{3}u_-^2 \frac{\partial d}{\partial v_-}(2u_- + \lambda, 0) + \frac{\partial d}{\partial \mu_2}(2u_- + \lambda, 0) = -\frac{2}{3}u_-^2 \zeta_1(\lambda), \\ \frac{\partial d_1}{\partial \nu_5}(\lambda, 0) &= -\frac{1}{2}u_-^2 \frac{\partial d}{\partial v_-}(2u_- + \lambda, 0) + \frac{\partial d}{\partial \mu_5}(2u_- + \lambda, 0) = -\frac{1}{3}u_-(s + 3u_-)\zeta_1(\lambda), \\ \frac{\partial d_1}{\partial \nu_6}(\lambda, 0) &= u_-^2 \frac{\partial d}{\partial \lambda}(2u_- + \lambda, 0) + \frac{\partial d}{\partial \mu_6}(2u_- + \lambda, 0) = 0, \\ \frac{\partial d_1}{\partial \nu_7}(\lambda, 0) &= -\frac{2}{3}u_-^2 \frac{\partial d}{\partial v_-}(2u_- + \lambda, 0) + \frac{\partial d}{\partial \mu_7}(2u_- + \lambda, 0) = -\frac{4}{3}u_-^2 \zeta_1(\lambda). \end{aligned}$$

For $i = 1, 3, 4, 8$, $\frac{\partial d_1}{\partial \nu_i}(\lambda, 0) = \frac{\partial d}{\partial \mu_i}(2u_- + \lambda, 0) = 0$. For $\lambda = 0$, the same formulas hold by passage to the limit. The formulas of Theorem 5.4, except that for e , now follow easily.

To compute $\frac{\partial d_2}{\partial x}(0, 0)$, we note that for $(x, N) = (0, 0)$ we have

$$\frac{\partial}{\partial x}G(U, 2u_- + R_2(x, N), M(N)) = 0.$$

Therefore

$$e = \frac{\partial d_2}{\partial x}(0, 0) = \lim_{t \rightarrow \infty} J_{2u_-}(t)G(\gamma_{2u_-}(t), 0, 0) \wedge \frac{\partial U_1}{\partial x}(0, 0) = 16u_-^2 \frac{\partial z}{\partial x}(0) = 16u_-^2 \sqrt{-2u_-}. \quad \square$$

We define the following sets:

$$\begin{aligned} \tilde{\mathcal{P}} &= \{(x, \lambda, \alpha, \beta, \nu) : x = \lambda = \beta = 0\}, \\ \tilde{\mathcal{E}}_1 &= \{(x, \lambda, \alpha, \beta, \nu) : x = 0, \frac{\partial g}{\partial x} = 0, \text{ and } d_1(\lambda, \alpha, \beta, \nu) = 0\}, \\ \tilde{\mathcal{E}}_2 &= \{(x, \lambda, \alpha, \beta, \nu) : \lambda = -x^2 + \beta x, \frac{\partial g}{\partial x} = 0, \text{ and } d_2(x, \alpha, \beta, \nu) = 0\}, \\ \tilde{\mathcal{F}}_1 &= \{(x, \lambda, \alpha, \beta, \nu) : x = 0, d_1 = 0, \text{ and there exists } y \text{ such that} \\ &\quad \lambda = -y^2 + \beta y \text{ and } \frac{\partial g}{\partial x}(y, \lambda, \alpha, \beta, \nu) = 0\}, \\ \tilde{\mathcal{F}}_2 &= \{(x, \lambda, \alpha, \beta, \nu) : \lambda = -x^2 + \beta x, d_2 = 0, \text{ and } \frac{\partial g}{\partial x}(0, \lambda, \alpha, \beta, \nu) = 0\} \end{aligned}$$

If we project these sets to N -space and apply the mapping $M(N)$ given by the last three lines of (5.44), we obtain the sets $\mathcal{P}$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{F}_1$, $\mathcal{F}_2$ defined in Section 2.

Theorem 5.5 *There is a continuous positive function $\rho(\bar{\mu})$, defined for $\{\bar{\mu} : \bar{\mu}_5 = 1\}$, such that if $\sup\{|v_-|, |b|, |\epsilon|\} < \rho(\bar{\mu})$, then the bifurcation problem $\dot{U} = G(U, s, v_-, b, \epsilon\bar{\mu})$ satisfies conditions (I)–(III) of Section 2 on the interval $2u_- - \delta \leq s \leq 2u_- + \delta$ unless $(b, v_-, \epsilon\bar{\mu}) \in \mathcal{P} \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2$.*

Proof. The question of whether $\dot{U} = G(U, s, v_-, b, \mu)$ satisfies (I)–(III) on $2u_- - \delta \leq s \leq 2u_- + \delta$ can be answered by studying the functions $g(x, \lambda, \alpha, \beta, \nu)$, $d_1(\lambda, \alpha, \beta, \nu)$, and $d_2(\lambda, \alpha, \beta, \nu)$, with $-\delta \leq \lambda \leq \delta$ and $(\alpha, \beta, \nu) = \Phi(v_-, b, \mu)$.

For fixed (α, β, ν) , (I) holds provided

$$(1) \quad g(0, \lambda, \alpha, \beta, \nu) = 0 \text{ and } \frac{\partial g}{\partial x}(0, \lambda, \alpha, \beta, \nu) = 0 \text{ imply } \frac{\partial^2 g}{\partial x^2}(0, \lambda, \alpha, \beta, \nu) \neq 0 \text{ and } \frac{\partial^2 g}{\partial x \partial \lambda}(0, \lambda, \alpha, \beta, \nu) \neq 0;$$

$$(2) \quad \text{for } x \neq 0, g = 0 \text{ and } \frac{\partial g}{\partial x} = 0 \text{ imply } \frac{\partial g}{\partial \lambda} \neq 0 \text{ and } \frac{\partial^2 g}{\partial x^2} \neq 0.$$

Both (1) and (2) are true unless $\beta = 0$, in which case (1) is violated at $(x, \lambda) = (0, 0)$. Note that $(0, 0, \alpha, 0, \nu) \in \tilde{\mathcal{P}}$.

Next we consider condition (III), the simultaneous occurrence of bifurcations. We note that by (5.45) we may write

$$\begin{aligned} d_1(\lambda, \alpha, \beta, \nu) &= h(\alpha, \beta, \nu) + \lambda i(\lambda, \alpha, \beta, \nu), \\ d_2(x, \alpha, \beta, \nu) &= h(\alpha, \beta, \nu) + x j(x, \alpha, \beta, \nu). \end{aligned}$$

We define

$$\tilde{d}(x, \lambda, \alpha, \beta, \nu) = h(\alpha, \beta, \nu) + (x^2 + \lambda - \beta x)i(\lambda, \alpha, \beta, \nu) + x j(x, \alpha, \beta, \nu).$$

Then $g = \tilde{d} = 0$ if and only if either $x = 0$ and $d_1 = 0$, or $\lambda = -x^2 + \beta x$ and $d_2 = 0$. But $j(0) = e \neq 0$, so the equation $\tilde{d} = 0$ may be solved for x as a function of $(\lambda, \alpha, \beta, \nu)$. This implies that it is never the case that two heteroclinic orbits occur at the same value of λ . Since simultaneous equilibrium bifurcations never occur in the unfolding of the pitchfork, (III) fails only when there exists λ such that $(0, \lambda, \alpha, \beta, \nu) \in \tilde{\mathcal{F}}_1$, or there exists (x, λ) such that $(x, \lambda, \alpha, \beta, \nu) \in \tilde{\mathcal{F}}_2$.

We shall consider (II) only for $\nu_5 \neq 0$. Fix $\bar{\nu}$ with $\bar{\nu}_5 = 1$ and let $c(\lambda) = \sum_{i=1}^{\infty} \bar{\nu}_i c_i(\lambda)$.
Let

$$\begin{aligned}\hat{g}(x, \lambda, \alpha, \beta, \epsilon) &= g(x, \lambda, \alpha, \beta, \epsilon \bar{\nu}) = -x^3 - \lambda x + \beta x^2, \\ \hat{d}_1(\lambda, \alpha, \beta, \epsilon) &= d_1(\lambda, \alpha, \beta, \epsilon \bar{\nu}) = a(\lambda)\alpha + b(\lambda)\beta + c(\lambda)\epsilon + \mathcal{O}(\|(\alpha, \beta, \epsilon)\|^2), \\ \hat{d}_2(x, \alpha, \beta, \epsilon) &= d_2(x, \alpha, \beta, \epsilon \bar{\nu}) = ex + a(0)\alpha + b(0)\beta + c(0)\epsilon + \mathcal{O}(\|(x, \alpha, \beta, \epsilon)\|^2).\end{aligned}$$

In order that (II) hold for $(\alpha, \beta, \epsilon \bar{\nu})$, it is sufficient that the following two conditions hold:

- (1) if $\hat{d}_1(\lambda, \alpha, \beta, \epsilon) = 0$, then $\frac{\partial \hat{g}}{\partial x}(0, \lambda, \alpha, \beta, \epsilon) \neq 0$ and $\frac{\partial \hat{d}_1}{\partial \lambda}(\lambda, \alpha, \beta, \epsilon) \neq 0$;
- (2) if $-x^2 - \lambda + \beta x = 0$ and $\hat{d}_2(x, \alpha, \beta, \epsilon) = 0$, then $\frac{\partial \hat{g}}{\partial x}(x, \lambda, \alpha, \beta, \epsilon) \neq 0$ and $\frac{\partial \hat{d}_2}{\partial x}(x, \alpha, \beta, \epsilon) \neq 0$.

The derivation of the condition $\frac{\partial \hat{d}_2}{\partial x}(x, \alpha, \beta, \epsilon) \neq 0$ in (2) is similar to that of (5.22).

Regarding (1), the functions x , $\hat{d}_1$, and $\frac{\partial \hat{d}_1}{\partial \lambda}$ have linearly independent derivatives at the origin provided $\bar{\nu}_5 \neq 0$, so the only simultaneous solutions of $x = 0$, $\hat{d}_1 = 0$, $\frac{\partial \hat{d}_1}{\partial \lambda} = 0$ are the known two-parameter family for which $\epsilon = 0$. Regarding (2), $\frac{\partial \hat{d}_2}{\partial x}(0) \neq 0$ by Theorem 3, so there are no small simultaneous solutions of $-x^2 - \lambda + \beta x = 0$, $\hat{d}_2 = 0$ and $\frac{\partial \hat{d}_2}{\partial x} = 0$. Thus (II) can only fail if there exists λ such that $(0, \lambda, \alpha, \beta, \epsilon \bar{\nu}) \in \tilde{\mathcal{E}}_1$ or there exists (x, λ) such that $(x, \lambda, \alpha, \beta, \epsilon \bar{\nu}) \in \tilde{\mathcal{E}}_2$. $\square$

Proof of Theorem 2.3. Fix $\bar{\nu}$ with $\bar{\nu}_5 = 1$, and define $c(\lambda)$, $\hat{g}$, $\hat{d}_1$, and $\hat{d}_2$ as in the proof of Theorem 5.5. By abuse of notation, we shall write $(x, \lambda, \alpha, \beta, \epsilon) \in \tilde{\mathcal{E}}_2$ for $(x, \lambda, \alpha, \beta, \epsilon \bar{\nu}) \in \tilde{\mathcal{E}}_2$, and $(\alpha, \beta, \epsilon) \in \mathcal{E}_2$ for $M(\alpha, \beta, \epsilon \bar{\nu}) \in \tilde{\mathcal{E}}_2$, etc.

Let us consider $\{(x, \lambda, \alpha, \beta, \epsilon) \in \mathcal{P} : \hat{d}_1 = 0\} = \{(x, \lambda, \alpha, \beta, \epsilon) \in \mathcal{P} : \hat{d}_2 = 0\}$. This set lies in $\tilde{\mathcal{P}} \cap \tilde{\mathcal{E}}_1 \cap \tilde{\mathcal{E}}_2 \cap \tilde{\mathcal{F}}_1 \cap \tilde{\mathcal{F}}_2$. Its equations are

$$\begin{aligned}x &= 0, \\ \lambda &= 0, \\ \beta &= 0, \\ \hat{d}_1(0, \alpha, 0, \epsilon) &= a(0)\alpha + c(0)\epsilon + \cdots = 0.\end{aligned}$$

Projecting this curve to $\alpha\beta\epsilon$ -space, we have the curve

$$\begin{aligned} a(0)\alpha + c(0)\epsilon + \cdots &= 0, \\ \beta &= 0, \end{aligned} \tag{5.47}$$

which lies in $\mathcal{P} \cap \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{F}_1 \cap \mathcal{F}_2$. The equations in Theorem 2.3 for the point Q now follow from Theorem 5.4 and the Implicit Function Theorem.

We have

$$\begin{aligned} \tilde{\mathcal{E}}_1 &= \{(x, \lambda, \alpha, \beta, \epsilon) : x = 0, \lambda = 0, \hat{d}_1(0, \alpha, \beta, \epsilon) = 0\}, \\ \tilde{\mathcal{F}}_1 &= \{(x, \lambda, \alpha, \beta, \epsilon) : x = 0, \lambda = \beta^2/4, \hat{d}_1(\beta^2/4, \alpha, \beta, \epsilon) = 0\}. \end{aligned}$$

Projecting to $\alpha\beta\epsilon$ -space,

$$\begin{aligned} \mathcal{E}_1 &= \{(\alpha, \beta, \epsilon) : \hat{d}_1(0, \alpha, \beta, \epsilon) = 0\}, \\ \mathcal{F}_1 &= \{(\alpha, \beta, \epsilon) : \hat{d}_1(\beta^2/4, \alpha, \beta, \epsilon) = 0\}. \end{aligned}$$

The equation in Theorem 2.3 for $\mathcal{E}_1$ and $\mathcal{F}_1$ follows.

The function $\hat{d}_1(\lambda, \alpha, \beta, \epsilon) = 0$ may be solved for α near 0 by the Implicit Function Theorem. Since for $\epsilon = 0$ the set $\hat{d}_1 = 0$ is independent of λ , we obtain: $\hat{d}_1(\lambda, \alpha, \beta, \epsilon) = 0$ if and only if

$$\alpha = m(\beta) + \epsilon n(\lambda, \beta, \epsilon).$$

Therefore $(\alpha, \beta, \epsilon) \in \mathcal{E}_1$ if and only if

$$\alpha = \alpha(\beta, \epsilon) = m(\beta) + \epsilon n(0, \beta, \epsilon), \tag{5.48}$$

and $(\alpha, \beta, \epsilon) \in \mathcal{F}_1$ if and only if

$$\alpha = \hat{\alpha}(\beta, \epsilon) = m(\beta) + \epsilon n(\beta^2/4, \beta, \epsilon), \tag{5.49}$$

Hence

$$\alpha(0, \epsilon) = \hat{\alpha}(0, \epsilon) \quad \text{and} \quad \frac{\partial \alpha}{\partial \beta}(0, \epsilon) = \frac{\partial \hat{\alpha}}{\partial \beta}(0, \epsilon).$$

Thus the curves $\alpha(\cdot, \epsilon)$ and $\hat{\alpha}(\cdot, \epsilon)$ are equal for $\epsilon = 0$ and agree to first order at $\beta = 0$ for $\epsilon \neq 0$. For $\epsilon \neq 0$, their intersection at $\beta = 0$ is of course given by (5.47). Now (5.48) and (5.49) imply that

$$\frac{\partial^2 \alpha}{\partial \beta^2}(0, 0) - \frac{\partial^2 \hat{\alpha}}{\partial \beta^2}(0, \epsilon) = -\frac{1}{2} \epsilon \frac{\partial n}{\partial \lambda}(0, 0, \epsilon).$$

An elementary calculation shows that

$$\frac{\partial n}{\partial \lambda}(0, 0, 0) = (a(0))^{-2}(a'(0)c(0) - a(0)c'(0)) = \frac{1}{4} \quad (5.50)$$

independent of $\bar{\nu}$ when $\bar{\nu}_5 = 1$. Therefore for fixed $\epsilon > 0$, the intersection of the curves $\alpha(\cdot, \epsilon)$ and $\hat{\alpha}(\cdot, \epsilon)$ at $\beta = 0$ is nondegenerate quadratic and is as shown in Figure 5.

$\tilde{\mathcal{E}}_2$ is defined by the equations

$$\begin{aligned} x^2 - \lambda &= 0, \\ 2x - \beta &= 0, \\ \hat{d}_2(x, \alpha, \beta, \epsilon) &= ex + a(0)\alpha + b(0)\beta + c(0)\epsilon + \cdots = 0. \end{aligned}$$

Projecting to $\alpha\beta\epsilon$ -space, we obtain: $(\alpha, \beta, \epsilon) \in \mathcal{E}_2$ if and only if

$$a(0)\alpha + (b(0) + e/2)\beta + c(0)\epsilon + \cdots = 0.$$

The equation in Theorem 2.3 for $\mathcal{E}_2$ follows.

$\tilde{\mathcal{F}}_2$ is defined by the equations

$$\begin{aligned} \lambda &= 0, \\ x - \beta &= 0, \\ \hat{d}_2(x, \alpha, \beta, \epsilon) &= ex + a(0)\alpha + b(0)\beta + c(0)\epsilon + \cdots = 0. \end{aligned}$$

Projecting to $\alpha\beta\epsilon$ -space, we obtain $(\alpha, \beta, \epsilon) \in \mathcal{F}_2$ if and only if

$$a(0)\alpha + (b(0) + e)\beta + c(0)\epsilon + \cdots = 0.$$

The equation in Theorem 2.3 for $\mathcal{F}_2$ follows. $\square$

Proof of Remark 2.2. We define

$$\tilde{\mathcal{A}}_{1\pm} = \{(x, \lambda, \alpha, \beta, \nu) : x = 0, d_1 = 0, \lambda = \pm\delta\},$$

Projecting to N -space and applying $M(N)$ gives $\mathcal{A}_{1\pm}$. We shall abuse this notation as in the previous proof.

To find $\tilde{\mathcal{A}}_{1\pm}$ for $\nu = \epsilon\bar{\nu}$, we note that $(\alpha, \beta, \epsilon) \in \mathcal{A}_{1\pm}$ if and only if

$$\alpha = m(\beta) + \epsilon n(\pm\delta, \beta, \epsilon). \quad (5.51)$$

For $\epsilon = 0$ these sets coincide with $\mathcal{E}_1$ and $\mathcal{F}_1$. For small $\delta > 0$, (β, ϵ) in a small ball about 0 whose size depends on δ , and $\epsilon \neq 0$, the sets $\mathcal{A}_{1\pm}$ do not meet $\mathcal{E}_1$ or $\mathcal{F}_1$. To see this for $\mathcal{E}_1$, we equate (5.48) and (5.51) and divide by ϵ to obtain

$$m(0, \beta, \epsilon) = n(\pm\delta, \beta, \epsilon). \quad (5.52)$$

By (5.50), $\frac{\partial n}{\partial \lambda}(0) > 0$. Therefore for small $\delta > 0$, $n(\pm\delta, 0, 0) \neq 0$. Thus equation (5.52) is false for $(\beta, \epsilon) = (0, 0)$. The assertion then follows by continuity. The same argument works for $\mathcal{F}_1$. On the other hand, for each fixed ϵ the intersection of the curves defined by $\mathcal{A}_{1\pm}$ and $\mathcal{E}_2$ or $\mathcal{F}_2$ is transverse. $\square$

Remark 5.2 $\mathcal{A}_{1+}$ coincides with the set $v_- = \phi(b) + \xi(b, \epsilon)$ of Theorem 5.1.

An easy argument shows that the analogous sets $\mathcal{A}_{2\pm}$ are empty, so the points defined by $d_2 = 0$ do not leave the region $-\delta \leq \lambda \leq \delta$.

5.5 Remaining Proofs

Proof of Theorem 2.1. It was noted in Section 2 that (IV) is satisfied for all small M . The first sentence of Theorem 2.1 then follows from Theorems 5.1, 5.3, and 5.5. The second sentence follows from the descriptions of the transition surfaces given in the proofs of Theorems 2.2 and 2.3. $\square$

Proof of Theorem 2.4 The formulas for P_i and Q follow from the corresponding formulas in Theorems 2.2 and 2.3 and the changes of variables. To see that the curves $\mathcal{E}$, $\mathcal{B}$, and $\mathcal{F}$ lie above $\mathcal{E}_1$ and $\mathcal{F}_1$, one uses Remarks 5.1 and 5.2 together with Remarks 2.1 and 2.2. The transversal intersections of $\mathcal{P}$, $\mathcal{E}_2$ and $\mathcal{F}_2$ with $\mathcal{B}$, $\mathcal{H}$, $\mathcal{E}$, and $\mathcal{F}$, and the fact that $\mathcal{H}$ falls to meet $\mathcal{E}_1$ and $\mathcal{F}_1$, all follow from the situation at $\epsilon = 0$ (Figure 9). $\square$

Proof of Proposition 2.1. The main point is that in the region between $\mathcal{F}_1$ and $\mathcal{C}$, there is exactly one AB transition. The proof uses Theorem 5.1 and Remarks 5.1 and 5.2. $\square$

Proof of Theorem 2.5. We shall derive the condition that P_i meets $\mathcal{P}$, $1 \leq i \leq 6$. From Theorem 2.3, $\mathcal{P}$ has the formula

$$v_- + u_- b + u_-^2 \left[\frac{1}{2} + \frac{1}{3}(\bar{\mu}_2 + 2\bar{\mu}_7) \right] \epsilon + \cdots = 0.$$

Substituting the equations for P_i from Theorem 2.2, we obtain

$$(\bar{\mu}_2 + 2\bar{\mu}_7 - 1 - \frac{1}{3}a_i)\epsilon + \mathcal{O}(\epsilon^2) = 0.$$

The condition is then obtained by dividing by ϵ and using the Implicit Function Theorem. $\square$

Proof of Remark 2.4. This remark is clear from the role played by u_- throughout this section. $\square$

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SADDLE
OR
WEAK SADDLE

SADDLE-NODE

FIGURE 1.

15

$$2u_- - \delta \leq s < 2u_-$$

$$2u_- \leq s \leq -\frac{2}{3}u_-$$

$$-\frac{2}{3}u_- < s \leq -\frac{2}{3}u_- + \delta$$

FIGURE 2

FIGURE 3

FIGURE 4 (1)

FIGURE 5

FIGURE 6

FIGURE 6 (2)

FIGURE 7

FIGURE 8

FIGURE 9

Figure 10. The (b, v) plane.

Figure 11.

(a) U_- crosses E .

(b) U_- crosses F .

Figure 12.

Figure 13

Figure 14(a). U_L in sector 4

Figure 14(b). U_L in sector 5(a)

Figure 14(c). U_L is sector 5(6)

Figure 14(d). U_L is sector 10.

Key:

—	slow shock curve
←←←	slow rarefaction curve
	slow rarefaction-shock curve
***	undercompressive shock curve Σ

Figure 14. Slow wave curves and undercompressive shock curve.

- - - Limit points
- fast shock curve
- → → fast rarefaction curve.

Figure 16 Solution of Riemann problems. U_R regions.

Figure 17. Solution of Riemann problems. Wave curve pattern.

FIGURE 18

FIGURE 19

structure boundaries
slow wave curves
undercompressive shock curve
S_0
fast wave curves

Structure of solution of Riemann problem:

- 1 Lax-Liu structure.
- 2 Attached slow wave, detached fast wave.
- 3 Slow wave, undercompressive shock, fast wave.
- 4 Detached slow wave, fast wave.

Figure 17. Legend.

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