**ON A LINEAR, PARTLY HYPERBOLIC MODEL
OF VISCOELASTIC FLOW PAST A PLATE**

By

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The paper presents solutions, for a class of shear relaxation functions, of a linear problem formulated by Joseph (1985) to elucidate the steady, supercritical flow of a viscoelastic fluid past a semi-infinite flat plate. The velocity $(U,0)$ at infinity is parallel to the plate, and ‘supercritical flow’ means that U is greater than the propagation speed C of shear waves. As a result, the vorticity satisfies a hyperbolic equation and is confined to the region downstream of a shock wave from the leading edge of the plate. The disturbance velocity field extends upstream of the shock and is continuous across it. In contrast to the case of a Newtonian fluid, the solutions are unique under the condition that the functions representing the vorticity on the two sides of the plate belong to a certain Banach space.

1. Introduction

1.1. Description and defence of the problem

This paper implements, with some slight mathematical changes, a proposal of Joseph (1985) for description of the viscoelastic flow past a semi-infinite flat plate. The flow is two-dimensional and independent of time; the plate P is along the positive x -axis of the (x,y) -plane, as shown in Figure 1; there is a constant velocity $(U,0)$ at infinity, outside a neighbourhood of the plate that may be called a boundary layer; and the total velocity $(U + u, v)$ is zero on the plate. Because the viscoelastic Mach number M (defined below) is greater than 1, the vorticity $\zeta := v_x - u_y$ is confined to the region $\mathcal{D}$ downstream of a linearized shock wave S of vorticity. The disturbance velocity (u, v) extends to the upstream region $\mathcal{U}$, and is continuous across the shock.

The linearization may be regarded as a (far-reaching) extension of the Oseen approximation for a Newtonian fluid; like that approximation for the case of flow past an unbounded object, it cannot be justified by arguments involving a small parameter. Rather, its justification must be pragmatic and on grounds of expediency: a linearized

theory reveals some features of a situation that otherwise might remain totally obscure. In Joseph's view (expressed during a conversation), it is no more wicked to ignore non-linear effects in the present context than it is to ignore, in the Navier-Stokes equations, those properties of a real fluid that the Newtonian model lacks.

One can also argue (and Joseph would do so more emphatically) that as long as experiments remain indecisive, and non-linear theories are implemented mainly by numerical calculations unsupported by analysis, it is worth while to record an explicit linear theory. The mathematical aspect of such a theory contains no mystery, and the results are unlikely to be *uniformly* wrong as a description of real flows.

In the present case of flow past the plate P , a further case can be made for the linearization: comparison with the solution of Lewis and Carrier (1949), for the linearized flow past P of a Newtonian fluid, may be not without interest. That solution, of the Oseen equation (1.2b) below, represents a symmetrical flow (u is an even function of y , while v is odd) and fails to be unique, as was shown by Olmstead and Hector (1966), and more explicitly by Olmstead (1975). The additional solutions are no more singular than the symmetrical one, and cannot be ruled out (it would seem) by any natural condition other than insistence on symmetry for its own sake; see Appendix D. There is a uniqueness theorem for Oseen flow past a *bounded* obstacle, so that the difficulty is due to the infinite length of the plate and the relatively large disturbance velocity near infinity.

For viscoelastic flow with $M > 1$, we follow Lewis and Carrier in constructing symmetrical solutions, but now for a class of shear relaxation functions G (discussed in detail below); each function G characterizes a particular (hypothetical) liquid. When G allows a certain regularity of the solution, the disturbance velocity near infinity is precisely that of the solution of Lewis and Carrier to the lowest order. (The function G provides an effective Newtonian viscosity at large distances from the origin.) However, the present solutions are unique under an additional requirement: that the functions representing the vorticity ζ on the upper and lower sides of the plate belong to the Banach space A_μ introduced in § 5.1. This condition is amply satisfied

by our solutions for the more pleasant functions G , and seems appropriate on physical grounds.

In the Newtonian case, the condition is also satisfied amply by all the known solutions, but fails to imply uniqueness.

1.2. The vorticity equation and shear kernels

For steady, two-dimensional flow in the (x,y) -plane, with velocity $(U+u,v)$ and with linearization about a constant velocity $(U,0)$, the vorticity equation proposed by Joseph may be written

$$\zeta_x(x,y) = \Delta \int_{-\infty}^{\infty} K(x-x') \zeta(x',y) dx', \quad (1.1)$$

where

$$(\cdot)_x = \frac{\partial}{\partial x} (\cdot), \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \zeta = v_x - u_y,$$

and K is a given function characterizing the material. For a Newtonian fluid with kinematic viscosity ν , the appropriate function K is

$$K_0(x) = \frac{\nu}{U} \delta(x) \quad (-\infty < x < \infty), \quad (1.2a)$$

where δ is the Dirac generalized function; there results the Oseen vorticity equation

$$\nu \Delta \zeta - U \zeta_x = 0. \quad (1.2b)$$

We now consider fluids, of constant density ρ , that are characterized not merely by one or two further material constants but by a *shear relaxation function* $G : \mathbf{R} \rightarrow [0, \infty)$. The argument, say τ , of G is a reversed time such that (a) $\tau = 0$ refers to ‘now’, the moment of observation; (b) $\tau \rightarrow \infty$ refers to ‘long ago’, and $G(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$ because the memory of the material fades; (c) $\tau < 0$ refers to the future, and $G(\tau) = 0$ for $\tau < 0$. The number $G(0) =: \rho C^2$ defines the propagation speed C of shear waves; a linearized flow is *supercritical* when the *viscoelastic Mach number* $M := U/C$ exceeds 1.

Following Joseph (1985, pp. 278-282), we assume that $M > 1$ and define a *shear kernel* K by

$$K(x) := \frac{1}{\rho U^2} G\left(\frac{x}{U}\right) \quad (-\infty < x < \infty), \quad (1.3)$$

so that

$$K(x) = 0 \quad \text{for } x < 0, \quad (1.4a)$$

$$K(0) = \frac{1}{M^2} \in (0, 1), \quad (1.4b)$$

$$K(x) > 0 \quad \text{for } 0 < x < \infty. \quad (1.4c)$$

These three conditions are basic and physical; we add three that are partly for mathematical convenience, but that seem consistent with most shear relaxation functions proposed in the literature, and certainly with the simpler ones.

$$K \text{ is convex on } [0, \infty), \quad (1.4d)$$

$$K \in L_1(0, \infty) \quad [\text{that is, } K \text{ is Lebesgue integrable on } (0, \infty)], \quad (1.4e)$$

there are constants $C_0 > 0$ and $\alpha \in (0, 1]$ such that

$$K(0) - K(x) \leq C_0 x^\alpha \quad \text{for } 0 \leq x \leq 1. \quad (1.4f)$$

Note that (1.4d) and (1.4e) imply that $K(x) \geq 0$ for $x \geq 0$; the role of (1.4c) is now merely to rule out compact support of K . Regarding the qualification $x \leq 1$ in (1.4f), see item (vi) of §2.1.

It is worth while to record, for later use, certain implications of (1.4d), (1.4e) and (1.4c). The proof, which merely adds the L_1 property to standard results for a function convex on an interval, is omitted.

LEMMA 1.1. (a) K is strictly decreasing on $[0, \infty)$.

(b) The left-hand derivative $K'_-(x)$ and right-hand derivative $K'_+(x)$ exist for all $x > 0$, and

$$\frac{1}{\delta} \{K(x) - K(x-\delta)\} \leq K'_-(x) \leq K'_+(x) \leq \frac{1}{h} \{K(x+h) - K(x)\} < 0$$

whenever $x \geq \delta > 0$ and $h > 0$.

(c) K is absolutely continuous on $[0, \infty)$. [Hence the derivative $K'(x)$ exists almost everywhere in $(0, \infty)$, and

$$K(x) - K(0) = \int_0^x K'(t) dt \quad \text{for all } x \geq 0.]$$

(d) $K' \in L_1(0, \infty)$.

(e) As $x \rightarrow \infty$, $K(x) = o(x^{-1})$ and $K'_-(x) = o(x^{-2})$.

The prototype of a shear kernel satisfying (1.4) is that of the linearized *Maxwell model* (Joseph 1985, p.279), defined by

$$K_1(x) = M^{-2} \exp(-x/UT) \quad \text{for } x \geq 0, \quad (1.5)$$

where T is a relaxation time of the elastic stress. (Some results for this case have been announced previously (Fraenkel 1988); there is little duplication between that paper and the present one.) However, there is a genuine case for kernels having less smoothness, or a slower decay at infinity, than K_1 . In particular, kernels having the behaviour, for a constant exponent $\alpha \in (0, 1)$,

$$K_R(x) = M^{-2} - cx^\alpha + o(x^\alpha) \quad \text{as } x \rightarrow 0+$$

(as x tends to zero from above) are of interest; here c is a positive constant. Note that $K'_R(x) \rightarrow -\infty$ as $x \rightarrow 0+$ (in the sense that the right-hand derivative, a fortiori the left-hand one, tends to minus infinity). We call such kernels *Renardy kernels* because Renardy (1982) discovered the following phenomenon in the context of the first Stokes problem (also called the Rayleigh problem), in which an infinite plane boundary is set into motion at time zero and has a constant velocity parallel to itself thereafter. (The function called a by Renardy (1982) is our $-G'$.) Whereas the shear relaxation function G_1 of the Maxwell model leads to a simple discontinuity (finite jump) of velocity across a linearized shock wave propagating into the fluid, shear functions G_R , essentially like those giving K_R under the transformation (1.3), lead to an infinitely differentiable velocity field.

Similar but less simple results will be found for the present problem of flow past the plate P : for the linearized Maxwell model the vorticity tends to infinity as the shock is approached from the downstream region $\mathfrak{A}$, but Renardy kernels lead to velocity fields in $C^\infty(\mathbb{R}^2 \setminus \bar{P})$. We shall illustrate certain remarks by reference to the Maxwell kernel K_1 , to the particular Renardy kernels defined by

$$K_2(x) = M^{-2} \{ \exp(-x/UT) - a_\alpha (x/UT)^\alpha \exp(-2x/UT) \}, \quad x \geq 0, \quad (1.6)$$

where $\alpha \in (0,1)$ and the positive constant a_α is sufficiently small for convexity of K_2 on $[0, \infty)$, and to the *long-tailed kernels* defined by

$$K_3(x) = M^{-2} (1+x/UT)^{-1-l}, \quad x \geq 0, \quad (1.7)$$

where the constant $l \in (0,1)$. A later paper (Fraenkel 1990a) with a more physical bias will present detailed results for the kernels K_1 to K_3 , with $\alpha = \frac{1}{2}$ in (1.6).

1.3. Hyperbolicity

It is not obvious that the vorticity equation (1.1) is now something close to a hyperbolic partial differential equation. To demonstrate this, we use the notation

$$(A * f)(x, y) := \int_{-\infty}^{\infty} A(x-x') f(x', y) dx' = \int_{-\infty}^{\infty} A(t) f(x-t, y) dt \quad (1.8)$$

for convolutions; initially, A will be either K or its derivative K' . Here we mean the classical derivative $K'(x)$ for $x > 0$, with no delta function at the origin. Provided that either K or ζ is sufficiently smooth, manipulation (§ 2.4) transforms (1.1) into

$$\left\{ (M^2 - 1) \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right\} (K * \zeta) - M^2 \frac{\partial}{\partial x} (K' * \zeta) = 0, \quad (1.9)$$

which differs from a hyperbolic partial differential equation only in the occurrence of K' in the last term.

For the Maxwell model we have $K'_1 = -K_1/UT$ and (1.9) reduces to the telegraph equation

$$\left\{ (M^2 - 1) \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{M^2}{UT} \frac{\partial}{\partial x} \right\} (K_1 * \zeta) = 0. \quad (1.10)$$

The vorticity equation will be coupled with the elliptic equation $-\Delta\psi = \zeta$, where ψ is the disturbance stream function, such that $(u,v) = (\psi_y, -\psi_x)$, and the boundary conditions specify values of u and v . Thus the problem is (in Joseph's phrase) of *composite type*: an essentially hyperbolic equation is coupled with an elliptic one.

1.4. Declarations

Missing proofs. For the sake of acceptable length, the proofs of many claims in the paper (both in and outside theorems and lemmas) are omitted or are incomplete. I shall be happy to send a copy of such details to any skeptical or interested reader who writes to me and specifies the material in question.

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2. Formulation of the plate problem

2.1. Notation

(i) Let Ω be an open (usually unbounded) set in the Euclidean plane $\mathbf{R}^2$. As is conventional, $C(\Omega)$ denotes the set of functions $\Omega \rightarrow \mathbf{C}$ (complex-valued functions) that are continuous in Ω ; the subset $C^m(\Omega)$ consists of functions having all partial derivatives of order m or less continuous in Ω ; we write $C^\infty(\Omega) := \bigcap_{m=1}^\infty C^m(\Omega)$; and the suffix 0 of $C_0^\infty(\Omega)$ refers to functions having compact support in Ω .

We shall often write $z = (x, y)$ for points of $\mathbf{R}^2$, and combine this with the use of double indices such as $\mu = (\mu_1, \mu_2)$, where μ_1 and μ_2 are non-negative integers; thus

$$z^\mu := x^{\mu_1} y^{\mu_2}, \quad \partial^\nu := \left(\frac{\partial}{\partial x}\right)^{\nu_1} \left(\frac{\partial}{\partial y}\right)^{\nu_2} =: \partial_1^{\nu_1} \partial_2^{\nu_2}, \quad |\mu| := \mu_1 + \mu_2.$$

As is also usual, $\mathcal{S}(\mathbf{R}^2)$ denotes the space of rapidly decreasing test functions on $\mathbf{R}^2$; that is, $\varphi \in \mathcal{S}(\mathbf{R}^2)$ if it is in $C^\infty(\mathbf{R}^2)$ and if, for each pair μ, ν of double indices, the semi-norm

$$\|\varphi\|_{\mu, \nu} := \sup_{z \in \mathbf{R}^2} |z^\mu \partial^\nu \varphi(z)| < \infty.$$

With Ω as before, we define

$$\mathcal{S}(\Omega) := \{\varphi \in \mathcal{S}(\mathbf{R}^2) \mid \text{supp } \varphi \subset \Omega\},$$

where $\text{supp } \varphi$ denotes the support of φ .

(In this § 2, all functions could equally well be real-valued, but it is conventional that functions in $\mathcal{S}(\mathbf{R}^2)$ be complex-valued, and we shall have to consider complex-valued functions when we come to use the Fourier transform.)

(ii) Our main complex variable (from § 3 onwards) will be $s = \sigma + i\tau$. For sets in the real or complex planes, expressions

$$\{(x, y) \in \mathbf{R}^2 \mid x < 0\}, \quad \{s = \sigma + i\tau \in \mathbf{C} \mid \tau \geq 0\}, \dots$$

will often be abbreviated to $\{x < 0\}$, $\{\tau \geq 0\}$, ...; the context will prevent ambiguity.

(iii) The statement $f(s) = O(s^{-\alpha})$ as $s \rightarrow \infty$ with $\tau \geq 0$ is to mean that there are constants A and B such that $|f(s)| \leq A|s|^{-\alpha}$ whenever $|s| \geq B$ and $\tau \geq 0$.

(iv) Let

$$b := (M^2 - 1)^{\frac{1}{2}}, \quad \beta := \tan^{-1} b \in (0, \pi/2), \quad (2.1)$$

so that $\pi/2 - \beta$ is the *viscoelastic Mach angle*. The little identity $1 - ib = Me^{-i\beta}$ is used repeatedly in § 3.3 and Appendix B. Subsets of $\mathbf{R}^2$ already encountered in Figure 1 are defined precisely by

$$P := \{(x, y) \mid x > 0, y = 0\},$$

$$S := \{(x, y) \mid x = b|y|, -\infty < y < \infty\},$$

$$\mathcal{U} := \{(x, y) \mid x < b|y|, -\infty < y < \infty\},$$

$$\mathcal{X} := \{(x, y) \mid x > b|y|, y \neq 0\}.$$

(v) The norm of the shear kernel K will always be the L_1 norm; thus

$$\|K\| := \int_0^\infty K(x) dx.$$

(vi) In the right-hand members of (1.3) and (1.5) to (1.7), the variable x has the physical dimension of length. When we write $x \leq 1$ or $1 + x$, for example, it is to be understood that a transformation to a non-dimensional variable, such as $x_* = x/UT$, has been made a priori, and that the additional label has then been omitted.

2.2. The governing equations

Let the total velocity field be $(U+u, v)$, where $(U, 0)$ is the constant velocity at infinity in $\mathcal{U}$, and (u, v) is the disturbance velocity field. We demand that u and v be continuous in the whole plane $\mathbf{R}^2$ because the shock S is to be one of vorticity, not of velocity, and because the total velocity is to be zero on both sides of the plate. The vorticity ζ vanishes in $\mathcal{U}$ because it propagates from the plate with speed $C < U$; it induces an irrotational velocity field everywhere outside the plate, as well as corresponding to a rotational velocity field in $\mathcal{X}$.

Recalling the notation introduced after (1.1) and in (1.8), we seek (u, v) such that

$$\left. \begin{aligned} u, v &\rightarrow 0 \text{ at infinity in } \mathcal{U}, \\ u(x, y), v(x, y) &\text{ are } o(x^{\frac{1}{2}}) \text{ at infinity in } \mathcal{X}, \text{ uniformly over } y, \end{aligned} \right\} \quad (2.2a)$$

$$u_x + v_y = 0 \quad \text{in } \mathcal{U} \text{ and in } \mathcal{X}, \quad (2.2b)$$

$$\zeta := v_x - u_y = 0 \quad \text{in } \mathcal{U}, \quad (2.2c)$$

$$\zeta_x = \Delta(K * \zeta) \quad \text{in } \mathcal{X}, \quad (2.2d)$$

$$(u, v) = (-U, 0) \quad \text{on } \bar{P}, \quad (2.2e)$$

where $\bar{P}$ denotes the closure of P . We expect the disturbance velocity to be bounded (to be $O(1)$ at infinity in $\mathcal{X}$), and to tend to zero at infinity not merely in $\mathcal{U}$ but outside some boundary layer. However, we shall encounter a difficulty in bounding u when K decays slowly at infinity, for example, if

$$K(x) \sim \frac{\text{const.}}{x(\log x)^2} \quad \text{as } x \rightarrow \infty;$$

the nature of the boundary layer is not known a priori; and (2.2a) will be sufficient for uniqueness.

2.3. Weak solutions

The conditions (1.4) on K , despite their pleasant implications in Lemma 1.1, are not enough to lead to a solution of (2.2) for which all the derivatives appearing there exist in a classical sense. In the first instance we consider weak solutions.

DEFINITION 2.1. A velocity field $(u, v) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ will be called a *weak solution of the plate problem* (2.2) if it

$$(a) \text{ is continuous } [u, v \in C(\mathbf{R}^2)] \text{ and satisfies (2.2a);} \quad (2.3a)$$

$$(b) \text{ is weakly solenoidal in } \mathbf{R}^2:$$

$$\iint_{\mathbf{R}^2} (u\varphi_x + v\varphi_y) \, dx \, dy = 0 \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}^2); \quad (2.3b)$$

(c) is weakly irrotational in $\mathcal{U}$:

$$\iint_{\mathcal{U}} (u\varphi_y - v\varphi_x) dx dy = 0 \quad \text{for all } \varphi \in \mathcal{S}(\mathcal{U}); \quad (2.3c)$$

(d) satisfies the vorticity equation weakly:

$$\iint_{\mathbb{R}^2} \left(u \frac{\partial}{\partial y} - v \frac{\partial}{\partial x} \right) \{ \varphi_x + \Delta(K_r * \varphi) \} dx dy = 0 \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^2 \setminus \bar{P}), \quad (2.3d)$$

where the reflection K_r of K is defined by $K_r(t) = K(-t)$ for all $t \in \mathbb{R}$;

(e) satisfies pointwise the boundary condition (2.2e). (2.3e)

The function $K_r * \varphi$ in (2.3d) is (in general) not rapidly decreasing as $x \rightarrow -\infty$, but this causes no difficulty because (2.3a) implies that $u, v \in L_\infty(\mathcal{U})$ and because one can prove the following inequality (one of many that would serve). For every $\varphi \in \mathcal{S}(\mathbb{R}^2)$, double index ν and $x \leq 0$,

$$|\partial^\nu(K_r * \varphi)(x, y)| \leq A_{\varphi, \nu} \{ K_r(x/2)(a^2 + y^2)^{-3/2} + \|K\|(a^2 + x^2 + y^2)^{-2} \}, \quad (2.4)$$

where $A_{\varphi, \nu}$ is a linear combination of semi-norms $\|\varphi\|_{\mu, \nu}$ with $|\mu| \leq 4$, and a is a positive constant. Since $K_r * \varphi$ is rapidly decreasing in $\{x > 0\}$, it follows that $\partial^\nu(K_r * \varphi) \in L_1(\mathcal{U})$.

Condition (2.3d) is more than a weak form of (2.2d), because test functions in $\mathcal{S}(\mathbb{R}^2 \setminus \bar{P})$, rather than in $\mathcal{S}(\mathbb{R}^2)$, are used. For a solution that is smooth in $\mathbb{R}^2 \setminus \bar{P}$, condition (2.3d) implies integrability up to the shock of certain second derivatives of the velocity; this can be seen from the proof of Theorem 2.4.

For some purposes it is useful to define a *disturbance stream function* ψ by

$$\psi(z_0) := \int_0^{z_0} (u dy - v dx) \quad \text{for all } z_0 \in \mathbb{R}^2, \quad (2.5)$$

where the path of integration consists of finitely many straight-line segments, each parallel to a co-ordinate axis. The condition (2.3b) of weak solenoidality is sufficient to make $\psi(z_0)$ independent of the choice of path, for fixed z_0 ; it follows that $(\psi_y, -\psi_x) = (u, v)$, so that $\psi \in C^1(\mathbb{R}^2)$.

The apparatus of tempered distributions (Friedlander 1982, Hörmander 1983, Schwartz 1950) will also be useful. Elements of $\mathcal{S}'(\mathbf{R}^2)$ (that is, continuous linear functionals on $\mathcal{S}(\mathbf{R}^2)$) will be denoted by symbols like $\langle g, \cdot \rangle$, and we call g the *generalized function* corresponding to the *tempered distribution* $\langle g, \cdot \rangle$. In particular, if $f \in L_{1,\text{loc}}(\mathbf{R}^2)$ and has polynomial growth at infinity (that is, $f(z) = O(|z|^N)$ for some N as $|z| \rightarrow \infty$), then the rule

$$\langle f, \varphi \rangle := \iint_{\mathbf{R}^2} f \varphi \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}^2) \quad (2.6)$$

defines $\langle f, \cdot \rangle \in \mathcal{S}'(\mathbf{R}^2)$, and $\langle \partial^\nu f, \cdot \rangle \in \mathcal{S}'(\mathbf{R}^2)$ is defined by

$$\langle \partial^\nu f, \varphi \rangle := (-1)^{|\nu|} \iint_{\mathbf{R}^2} f \partial^\nu \varphi \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}^2) \quad (2.7)$$

and for all double indices ν . (In (2.6), (2.7) and similar expressions elsewhere, the integration element $dx dy$ is implied.)

For a weak solution, the tempered distributions $\langle u, \cdot \rangle$, $\langle v, \cdot \rangle$ and $\langle \psi, \cdot \rangle$ are all of the form (2.6), and the vorticity distribution

$$\langle \zeta, \cdot \rangle := \langle \partial_1 v - \partial_2 u, \cdot \rangle = -\langle \Delta \psi, \cdot \rangle \in \mathcal{S}'(\mathbf{R}^2) \quad (2.8)$$

is then described more explicitly by (2.7); moreover, (2.3c) states that

$$\text{supp } \zeta := \text{supp } \langle \zeta, \cdot \rangle \subset \bar{\mathcal{A}}. \quad (2.9)$$

(The support of a distribution is defined in all the texts cited above.) *The weak form (2.3d) of the vorticity equation is equivalent to the statement*

$$\langle \partial_1 \zeta - \Delta(K * \zeta), \varphi \rangle = 0 \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}^2 \setminus \bar{\mathcal{P}}); \quad (2.10)$$

this follows from the definition of $\langle K * \zeta, \cdot \rangle$ in Appendix C, from Lemma C.4(a) there, and from a calculation.

We can now derive a useful formula for the rotational part of the velocity field in terms of the vorticity. One would expect that, under various sets of conditions, the equation

$$-\Delta \psi_x = \zeta_x = \Delta(K * \zeta)$$

should imply that

$$\psi(x,y) = - \int_{-\infty}^x (K*\zeta)(t,y) dt + \psi_0(x,y),$$

where ψ_0 is a harmonic function ($\Delta\psi_0 = 0$). The following theorem establishes such a result for weak solutions of the plate problem; it involves the *Heaviside function* H , defined by

$$H(t) := \begin{cases} 0 & \text{if } t < 0, \\ \text{any convenient value in } [0,1] & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases} \quad (2.11)$$

THEOREM 2.2. *Given $\langle \zeta, \cdot \rangle \in \mathcal{S}'(\mathbb{R}^2)$ satisfying (2.9) and (2.10), define*

$$\langle \psi_1, \varphi \rangle := - \langle H*(K*\zeta), \varphi \rangle \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^2). \quad (2.12)$$

Then $\langle \psi_1, \cdot \rangle \in \mathcal{S}'(\mathbb{R}^2)$ and

$$- \langle \Delta\psi_1, \cdot \rangle = \langle \zeta, \cdot \rangle \quad \text{in } \mathcal{S}'(\mathbb{R}^2 \setminus \bar{P}). \quad (2.13)$$

It follows that $\psi_0 := \psi - \psi_1$ is a genuine function harmonic ($\Delta\psi_0 = 0$) in $\mathbb{R}^2 \setminus \bar{P}$. Accordingly, ψ_0 is real-analytic in $\mathbb{R}^2 \setminus \bar{P}$, and, if $(\partial_2\psi, -\partial_1\psi)$ is a weak solution, then $\psi_1 \in C^1(\mathbb{R}^2 \setminus \bar{P})$.

Proof. By Lemma C.3, first $\langle K*\zeta, \cdot \rangle$ and then $\langle \psi_1, \cdot \rangle$ are in $\mathcal{S}'(\mathbb{R}^2)$ and have supports in $[0, \infty) \times \mathbb{R}$. Lemma C.4(b) states that, if $\langle w, \cdot \rangle \in \mathcal{S}'(\mathbb{R}^2)$ and $\text{supp } w \subset [0, \infty) \times \mathbb{R}$, then

$$\langle \partial_1(H*w), \cdot \rangle = \langle w, \cdot \rangle = \langle H*\partial_1 w, \cdot \rangle. \quad (2.14)$$

Now the definition (2.12) and the left-hand identity in (2.14) imply that

$$- \langle \partial_1\psi_1, \cdot \rangle = \langle K*\zeta, \cdot \rangle \quad \text{in } \mathcal{S}'(\mathbb{R}^2), \quad (2.15)$$

whence

$$\begin{aligned} - \langle \Delta\psi_1, \cdot \rangle &= \langle \partial_1(K*\zeta) + H*\partial_2^2(K*\zeta), \cdot \rangle \quad \text{in } \mathcal{S}'(\mathbb{R}^2) \\ &= \langle \partial_1(K*\zeta) + H*(\partial_1\zeta - \partial_1^2(K*\zeta)), \cdot \rangle \quad \text{in } \mathcal{S}'(\mathbb{R}^2 \setminus \bar{P}), \end{aligned} \quad (2.16)$$

by (2.10). Applying the right-hand identity in (2.14) to the convolution with H in

(2.16), we obtain (2.13).

It now follows (by the definitions $\zeta = -\Delta\psi$ and $\psi_0 = \psi - \psi_1$ of the generalized functions ζ and ψ_0) that

$$\langle \Delta\psi_0, \cdot \rangle = \langle 0, \cdot \rangle \text{ in } \mathcal{S}'(\mathbf{R}^2 \setminus \bar{P}),$$

and we apply elliptic regularity theory (Friedlander 1982, p. 109 ; Hörmander 1983, p. 271) to conclude that $\psi_0 \in C^\infty(\mathbf{R}^2 \setminus \bar{P})$. It is then classical that ψ_0 is real-analytic on $\mathbf{R}^2 \setminus \bar{P}$. The C^1 property of ψ_1 is implied by $\psi \in C^1(\mathbf{R}^2)$ and by the smoothness of ψ_0 .

□

2.4. Pointwise solutions

To discuss the pointwise solutions that occur for some shear kernels K , we define *semi-characteristic co-ordinates* ξ, η by

$$\xi := x - b|y|, \quad \eta := by.$$

(This terminology will be justified after equation (2.19).) With the notation $\omega(\xi, \eta) := \zeta(x, y)$, so that $\omega(\xi, \eta) = 0$ for $\xi < 0$, the vorticity equation (2.2d) becomes

$$\left. \begin{aligned} \omega_\xi &= \left(M^2 \frac{\partial^2}{\partial \xi^2} \mp 2b^2 \frac{\partial^2}{\partial \xi \partial \eta} + b^2 \frac{\partial^2}{\partial \eta^2} \right) (K * \omega) \\ &\text{in } \mathfrak{A}_c := \{(\xi, \eta) \mid \xi > 0, \eta \neq 0\}. \end{aligned} \right\} \quad (2.17)$$

Here and elsewhere the upper sign of $\mp$ or $\pm$ refers to $\{\eta > 0\}$ and the lower to $\{\eta < 0\}$; the suffix c denotes the characteristic-variable image of a set or a function.

The identity (in which a.e. means almost everywhere)

$$(K * \omega)_{\xi\xi}(\xi, \eta) = K(0) \omega_\xi(\xi, \eta) + (K' * \omega)_\xi(\xi, \eta) \quad \text{a.e. in } \mathbf{R}^2 \quad (2.18)$$

can be justified for two different situations; we state sufficient conditions that can be weakened in various ways.

(i) Assume that $K \in C^2[0, \infty)$, that $\omega(\cdot, \eta) \in L_{1, \text{loc}}(\mathbf{R})$ and that $\omega_\xi(\xi, \eta)$ exists for $\xi \neq 0$ (and $\eta \neq 0$). We wish to treat the case in which ω_ξ is not integrable on line

segments $(0, \delta) \times \{\eta\}$, $\delta > 0$; this is the situation for the Maxwell kernel. The definition

$$(K * \omega)(\xi, \eta) = \int_0^\xi K(\xi - t) \omega(t, \eta) dt$$

yields

$$(K * \omega)_{\xi\xi}(\xi, \eta) = K(0)\omega_\xi(\xi, \eta) + K'(0+)\omega(\xi, \eta) + \int_0^\xi K''(\xi - t) \omega(t, \eta) dt$$

for $\xi \neq 0$, and this is (2.18).

(ii) Assume that K is merely as in (1.4) and that $\omega(\cdot, \eta) \in C^2(\mathbf{R})$ for $\eta \neq 0$. This is essentially the situation for Renardy kernels. The definition

$$(K * \omega)(\xi, \eta) = \int_0^\xi K(\theta) \omega(\xi - \theta, \eta) d\theta$$

yields

$$\begin{aligned} (K * \omega)_{\xi\xi}(\xi, \eta) &= \int_0^\xi K(\theta) \omega_{\xi\xi}(\xi - \theta, \eta) d\theta \\ &= K(0) \omega_\xi(\xi, \eta) + \int_0^\xi K'(\theta) \omega_\xi(\xi - \theta, \eta) d\theta \end{aligned}$$

upon integration by parts and appeal to Lemma 1.1; again we have (2.18).

Substituting (2.18) into (2.17), and recalling that $K(0) = M^{-2}$, we observe that the term ω_ξ , which is not locally integrable when K is the Maxwell kernel, is cancelled, leaving

$$M^2(K' * \omega)_\xi \mp 2b^2(K * \omega)_{\xi\eta} + b^2(K * \omega)_{\eta\eta} = 0 \quad \text{in } \mathfrak{A}_c. \quad (2.19)$$

Transforming back to x and y , one obtains (1.9), which shows that lines $\{x \pm by = \text{const.}\}$ are characteristic lines in the Maxwell case, and may be so called in the general case.

We make (2.19), rather than (2.2d), a basis of our definition of pointwise solution. Note that $K * \omega$, which is now to be a genuine function continuous outside $\bar{P}$, is given a simple meaning by (2.15): $K * \omega$ is the vertical component, say v_1 , of the rotational part of the velocity field. The significance of $K' * \omega$ is less simple: it equals

$v_{1\xi} - M^{-2}\omega$ and is the less evil part of $v_{1\xi}$, or of $-M^{-2}\omega$, when ω_ξ is not locally integrable.

DEFINITION 2.3. A velocity field $(u,v) : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ will be called a *pointwise solution of the plate problem* (2.2) if

$$u, v \in C(\mathbf{R}^2) \cap C^1(\mathcal{U}) \cap C^1(\mathcal{X}), \quad (2.20a)$$

$$(K' * \omega)_\xi, (K * \omega)_{\xi\eta}, (K * \omega)_{\eta\eta} \in C(\mathcal{X}_c), \quad (2.20b)$$

$$K' * \omega, (K * \omega)_\eta, K * \omega \in C(\mathbf{R}^2 \setminus \bar{P}); \quad (2.20c)$$

if (2.2a, b, c, e) hold; and if (2.19) is satisfied.

Condition (2.20c) is, in effect, the requirement that $K' * \omega$, $v_{1\eta}$ and v_1 be continuous at the shock S . (In the case of v_1 , this is implied by Theorem 2.2, but we have yet to relate pointwise solutions and weak ones, so that Theorem 2.2 is not yet applicable to pointwise solutions.)

THEOREM 2.4. Let (u,v) have the continuity properties (2.20), and let $K' * \omega$, $(K * \omega)_\eta$ and $K * \omega$ have only polynomial growth at infinity. Then (u,v) is a weak solution of the plate problem if and only if it is a pointwise solution.

Proof. To prove equivalence of (2.2b) and (2.3b), we note that (2.2a) and (2.20a) are sufficient for the identity

$$\iint_{\mathcal{U} \cup \mathcal{X}} (u_x + v_y) \varphi = - \iint_{\mathbf{R}^2} (u \varphi_x + v \varphi_y) \quad \text{whenever } \varphi \in \mathcal{S}(\mathbf{R}^2).$$

This shows that (2.2b) implies (2.3b); for the converse, one applies the 'fundamental lemma of the calculus of variations'. The equivalence of (2.2c) and (2.3c) is proved similarly, with integration only over $\mathcal{U}$.

Equivalence of (2.19) and (2.3d), under the hypotheses of the theorem, requires a more elaborate identity. Let $u_c(\xi, \eta) := u(x, y)$, and similarly for v and φ ; denote by

Δ_c the operator on the right-hand side of (2.17), and recall the convention stated after that equation. We claim that, for all $\varphi \in \mathcal{S}(\mathbb{R}^2 \setminus \bar{P})$,

$$\begin{aligned} \iint_{\mathfrak{A}_c} \{M^2(K' * \omega)_\xi \mp 2b^2(K * \omega)_{\xi\eta} + b^2(K * \omega)_{\eta\eta}\} \varphi_c d\xi d\eta \\ = \iint_{\mathbb{R}^2} \{bu_c \frac{\partial}{\partial \eta} - (\pm bu_c + v_c) \frac{\partial}{\partial \xi}\} \{\varphi_{c,\xi} + \Delta_c(K_r * \varphi_c)\} d\xi d\eta \\ = b \iint_{\mathbb{R}^2} (u \frac{\partial}{\partial y} - v \frac{\partial}{\partial x}) \{\varphi_x + \Delta(K_r * \varphi)\} dx dy. \end{aligned} \quad (2.21)$$

To prove the identity, we note that $K' * \omega$ and $K * \omega$ vanish outside $\overline{\mathfrak{A}_c}$, and extend the integration over $\mathfrak{A}_c$ to one over $\mathbb{R}^2$, considering separately $\mathbb{R}_+^2 := \{\eta > 0\}$ and $\mathbb{R}_-^2 := \{\eta < 0\}$. In the following calculation we begin with an integration by parts that is justified by (2.20b) and (2.20c), and by the condition of polynomial growth at infinity; we use abbreviations (such as φ for φ_c) that are self-explanatory.

$$\begin{aligned} \iint_{\mathbb{R}_+^2} (K' * \omega)_\xi \varphi d\xi d\eta &= - \iint (K' * \omega) \varphi_\xi \\ &= \iint \omega \{(K_r)' * \varphi_\xi\} \\ &= \iint \omega \{K_r(0) \varphi_\xi + K_r * \varphi_{\xi\xi}\} \\ &= \iint \{(bu + v)_\xi - bu_\eta\} \{M^{-2} \varphi_\xi + (K_r * \varphi)_{\xi\xi}\} \\ &= \int_\Gamma bu \{M^{-2} \varphi_\xi + (K_r * \varphi)_{\xi\xi}\} d\xi \\ &\quad + \iint_{\mathbb{R}_-^2} \{bu \frac{\partial}{\partial \eta} - (bu + v) \frac{\partial}{\partial \xi}\} \{M^{-2} \varphi_\xi + (K_r * \varphi)_{\xi\xi}\} d\xi d\eta, \end{aligned}$$

where $\Gamma := (-\infty, 0) \times \{0\}$. Regarding the last integration by parts: since u and v satisfy (2.2a) and (2.20a), and $\varphi \in \mathcal{S}(\mathbb{R}^2 \setminus \bar{P})$, the only possible objection concerns the term involving $K_r * \varphi$ as $\xi \rightarrow -\infty$; but there (2.4) is sufficient. Integration of $(K' * \omega)_\xi \varphi$ over $\mathbb{R}_-^2$, and corresponding (but simpler) calculations for $(K * \omega)_{\xi\eta}$ and $(K * \omega)_{\eta\eta}$, prove the identity; the integrals over Γ cancel each other.

Under the hypotheses of the theorem, it now follows from the identity (2.21) that (2.19) implies (2.3d), and also, by way of the ‘fundamental lemma’, that (2.3d) implies (2.19). However, for this latter implication, fewer hypotheses are sufficient, because it

is enough to have the identity (2.21) for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, so that (2.20c) and the condition of polynomial growth at infinity are not needed. $\square$

3. The construction of solutions

3.1. Initial steps

Our notation for the Fourier transform of appropriate functions f is

$$\tilde{f}(s, y) := \int_{-\infty}^{\infty} e^{isx} f(x, y) dx, \quad s = \sigma + i\tau \in \mathbb{C}.$$

Where more precise terminology is desirable, $\tilde{f}$ will be called (a) a *partial* transform (because it is with respect to only the first co-ordinate, in contrast to the Fourier transform used by Friedlander 1982, Hörmander 1983 and Schwartz 1950), (b) a partial *Fourier* transform when s is restricted to the real axis, (c) a partial *Fourier-Laplace* transform when $\text{supp } f \subset [0, \infty) \times \mathbb{R}$ and $\tau > 0$. Occasionally we shall distinguish case (b) by the notation

$$f^{\dagger}(\sigma, y) := \int_{-\infty}^{\infty} e^{i\sigma x} f(x, y) dx, \quad \sigma \in \mathbb{R}.$$

The partial Fourier and Fourier-Laplace transforms of tempered distributions are defined in Appendix C; equations (2.8) and (2.10) transform to

$$\langle (\partial_2^2 - p_{2,0})\psi^{\dagger} + \zeta^{\dagger}, \cdot \rangle = 0 \quad \text{in } S'(\mathbb{R}^2), \quad (3.1)$$

where $p_{2,0}(\sigma, y) := \sigma^2$ and $\langle 0, \cdot \rangle$ is abbreviated to 0, and

$$\langle \tilde{K}(s) (\partial_2^2 - s^2) \tilde{\zeta}(s, \cdot) + is \tilde{\zeta}(s, \cdot), \cdot \rangle = 0 \quad \text{in } S'(\mathbb{R} \setminus \{0\}) \quad \text{for } \tau > 0. \quad (3.2)$$

The functional $\langle \zeta^{\dagger}, \cdot \rangle$ can, in principle, be determined from $\langle \tilde{\zeta}(s, \cdot), \cdot \rangle$ with $\tau > 0$, but the formula in question (Lemma C.7(c)) is not helpful at this stage.

We consider (3.2) first. Since $\tilde{K}$ has no zero in $\{\tau \geq 0\}$, by Lemma A.2, the equation may be written

$$\langle (\partial_2^2 - k(s)^2) \tilde{\zeta}(s, \cdot), \cdot \rangle = 0 \quad \text{in } S'(\mathbb{R} \setminus \{0\}) \quad \text{for } \tau > 0, \quad (3.3)$$

where

$$k(s)^2 := s^2 - is/\tilde{K}(s). \quad (3.4)$$

By Theorem A.6, the square root of $s^2 - is/\tilde{K}(s)$ has a branch, denoted by $k(s)$

henceforth, that is holomorphic in $\{\tau > 0\}$ and satisfies $\operatorname{Re} k(s) < 0$ there.

THEOREM 3.1. *Let $\langle \zeta, \cdot \rangle \in \mathcal{S}'(\mathbb{R}^2)$ and satisfy (2.9) and (2.10). Then the generalized function $\tilde{\zeta}$ is a genuine function on $\{\tau > 0\} \times (\mathbb{R} \setminus \{0\})$ with values*

$$\tilde{\zeta}(s, y) = \begin{cases} \tilde{\zeta}(s, 0+) e^{k(s)y}, & y > 0, \\ \tilde{\zeta}(s, 0-) e^{-k(s)y}, & y < 0; \end{cases} \quad (3.5)$$

the complex numbers $\tilde{\zeta}(s, 0+)$ are given by

$$\begin{aligned} \tilde{\zeta}(s, 0+) &:= \lim_{y \rightarrow 0+} \tilde{\zeta}(s, y) \\ &= \lim_{n \rightarrow \infty} \langle \zeta, e_{is} \otimes a_n \rangle, \quad \tau > 0, \end{aligned} \quad (3.6)$$

where

$$(e_{is} \otimes a_n)(x, y) = e^{isx} a_n(y) \quad \text{for } x \geq 0$$

and a_n is an 'averaging kernel': $\operatorname{supp} a_n \subset [1/n, 3/n]$, say, $a_n \in C_0^\infty(0, \infty)$, $\int_0^\infty a_n(y) dy = 1$ and $a_n(y) \geq 0$.

There is a corresponding formula for $\tilde{\zeta}(s, 0-)$.

Proof. It suffices to prove the result for $y > 0$. Let s be fixed in $\{\tau > 0\}$. If we regard (3.3) as a conventional ordinary differential equation for $\tilde{\zeta}(s, \cdot)$ on $\mathbb{R}_+ := \{y \in \mathbb{R} | y > 0\}$, then

$$\tilde{\zeta}(s, y) = A(s) e^{k(s)y} + B(s) e^{-k(s)y} \quad (\tau > 0, y > 0), \quad (3.7)$$

for some complex numbers $A(s)$ and $B(s)$. Let $\mathcal{D}'(\mathbb{R}_+)$ denote the space of distributions acting on $C_0^\infty(\mathbb{R}_+)$; it is known (Hörmander 1983, p.58; Schwartz 1950, vol.I, p.128) that, if we consider solutions $\langle \tilde{\zeta}(s, \cdot), \cdot \rangle \in \mathcal{D}'(\mathbb{R}_+)$ of (3.3), then $\tilde{\zeta}(s, \cdot) \in C^2(\mathbb{R}_+)$, so that $\tilde{\zeta}(s, \cdot)$ is again as in (3.7). But the condition $\langle \tilde{\zeta}(s, \cdot), \cdot \rangle \in \mathcal{S}'(\mathbb{R}_+)$ requires that $B(s) = 0$, because $\operatorname{Re} k(s) < 0$. Now letting $y \rightarrow 0+$, we obtain $A(s) = \tilde{\zeta}(s, 0+)$ and hence (3.5).

It remains to prove (3.6). From (3.5) and Definition C.6 we have

$$\tilde{\zeta}(s, 0+) \int_0^\infty e^{k(s)y} \gamma(y) dy = \langle \zeta, e_{is} \otimes \gamma \rangle$$

whenever $\tau > 0$ and $\gamma \in \mathcal{S}(\mathbf{R}_+)$. Choose $\gamma = a_n$ and let $n \rightarrow \infty$; the left-hand member tends to $\tilde{\zeta}(s, 0+)$, and this proves (3.6). $\square$.

COROLLARY 3.2. *Let $\langle \zeta, \cdot \rangle$ be as in Theorem 3.1, let ψ_1 be the stream function introduced in Theorem 2.2, and let $u_1 := \psi_{1y}$, $v_1 := -\psi_{1x}$. Then the generalized functions $\tilde{\psi}_1$, $\tilde{u}_1$ and $\tilde{v}_1$ are also genuine functions on $\{\tau > 0\} \times (\mathbf{R} \setminus \{0\})$, with values*

$$\tilde{\psi}_1(s, y) = -\frac{i}{s} \tilde{K}(s) \tilde{\zeta}(s, 0+) e^{k(s)y}, \quad y > 0, \quad (3.8a)$$

$$\tilde{u}_1(s, y) = -\frac{i}{s} \tilde{K}(s) \tilde{\zeta}(s, 0+) k(s) e^{k(s)y}, \quad y > 0, \quad (3.8b)$$

$$\tilde{v}_1(s, y) = \tilde{K}(s) \tilde{\zeta}(s, 0+) e^{k(s)y}, \quad y > 0; \quad (3.8c)$$

there are corresponding formulae for $y < 0$.

Proof. These formulae are implied by (2.12), (2.14), (3.5) and the rule in Lemma C.7(b) for the Fourier-Laplace transform of distributional convolutions. $\square$

These results will be used in full for the proof of uniqueness. Our construction will use the formulae (3.8), but will be rigorous only in the description of ‘known’ functions like k and the all-important E and F introduced in §3.3; it will be merely formal or speculative in our treatment of $\tilde{\zeta}$ and $\tilde{\psi}$. Guesses about $\tilde{\zeta}$ and $\tilde{\psi}$ seem necessary because (3.1) and (3.2) do not yet give sufficient information; such guesses are allowable because we shall verify in §4 that the construction yields a weak solution of the plate problem. Our principal assumptions for the construction (but not for the uniqueness proof) are that

(a) *the velocity field is symmetrical: u is an even function of y , while v , ζ and ψ are odd;*

(b) the function $\tilde{\zeta}(\cdot, 0+)$, described by (3.6), has an extension that is continuous on $\{\tau \geq 0, s \neq 0\}$. (It then follows from Lemma C.7(a) that $\tilde{\zeta}(\cdot, 0+)$ is holomorphic in $\{\tau > 0\}$.)

Cuts in the complex plane $\mathbb{C}$ are required for the definition of branches of various set-valued functions. Let s_+ and s_- denote, respectively, points s in

$$\mathcal{G}_+ := \mathbb{C} \setminus \{i\tau \mid \tau \leq 0\} \quad \text{and} \quad \mathcal{G}_- := \mathbb{C} \setminus \{i\tau \mid \tau \geq 0\}, \quad (3.9)$$

with the convention that

$$-\frac{\pi}{2} < \arg s_+ < \frac{3\pi}{2} \quad \text{and} \quad -\frac{3\pi}{2} < \arg s_- < \frac{\pi}{2}. \quad (3.10)$$

To write the irrotational part of $\tilde{\psi}$ (the ‘complementary function’ of (3.1)), we define

$$q(s) := s_+^{\frac{1}{2}} s_-^{\frac{1}{2}} \quad (s \in \mathbb{C}, \sigma \neq 0). \quad (3.11)$$

Note that $q(s) = s$ if $\sigma > 0$ and $q(s) = -s$ if $\sigma < 0$.

3.2. Outline of the construction

There are three stages to the construction, as follows.

(i) We extend (3.1) formally to $\{\tau \geq 0, \sigma \neq 0\}$; a formal solution, having limiting values $\tilde{\psi}(s, 0\pm) = 0$ and consistent with (3.8a), is given by

$$\begin{aligned} \tilde{\psi}(s, y) &= \frac{i}{s} \tilde{K}(s) \tilde{\zeta}(s, 0+) \{e^{-q(s)|y|} - e^{k(s)|y|}\} \operatorname{sgn} y \\ & \quad (\sigma \neq 0, \tau \geq 0, y \neq 0), \end{aligned} \quad (3.12)$$

where $\operatorname{sgn} y := \pm 1$ for $y > 0$ or $y < 0$, respectively. A conceivable term $\exp(q(s)|y|)$ has been rejected in order that $\psi^\dagger \in \mathcal{S}'(\mathbb{R}^2)$. For manipulation of (3.12), we recall that

$$q(s)^2 - k(s)^2 = s^2 - k(s)^2 = is / \tilde{K}(s). \quad (3.13)$$

(ii) Determination of $\tilde{\zeta}(s, 0+) = -\tilde{\psi}_{yy}(s, 0+)$, or equivalently of $\tilde{\psi}_y(s, 0)$, cannot be immediate: $\tilde{\psi}_{yy}(s, 0+)$ is unknown because we do not know $\psi_{yy}(x, 0+)$ for $x > 0$,

and $\tilde{\psi}_y(s,0)$ is unknown because we do not know $\psi_y(x,0)$ for $x < 0$. The *Wiener-Hopf technique* (Noble 1958; Paley and Wiener 1934, Chapter IV; Titchmarsh 1948, p.339) was devised for precisely this type of situation, and proceeds in the present case as follows. Let

$$\tilde{\xi}_+(s) := \tilde{\xi}(s,0+),$$

$$u_-(x) := \psi_y(x,0) H(-x) = \psi_y(x,0) + UH(x),$$

where we have used the boundary condition $u = -U$ on $\bar{P}$. The suffix $+$ now labels functions of x that vanish for $x < 0$, and, correspondingly, functions of s that are holomorphic in $\{\tau > 0\}$; similarly, the suffix $-$ labels functions of x that vanish for $x > 0$ and functions of s that are holomorphic in $\{\tau < 0\}$. It follows from (3.12) and (3.13), upon continuing the Fourier-Laplace transform of H analytically to the real axis punctured at the origin, that

$$\tilde{\xi}_+(s) = \{q(s) - k(s)\} \{\tilde{u}_-(s) - \frac{iU}{s}\} \quad \text{for } s \in \mathbf{R} \setminus \{0\}. \quad (3.14)$$

The Wiener-Hopf recipe for this equation requires an identity on the real axis

$$q(s) - k(s) = G_+(s) G_-(s) \quad \text{for } s \in \mathbf{R}, \quad (3.15)$$

in which G_+ and G_- are known functions such that $1/G_+$ is holomorphic in $\{\tau > 0\}$ and continuous on $\{\tau \geq 0\}$, while G_- is holomorphic in $\{\tau < 0\}$ and continuous on $\{\tau \leq 0\}$. (Of course, such decompositions are not unique: if G_+, G_- is a pair satisfying these conditions, and f is an entire function, then $G_+/f, G_-f$ is another such pair. However, at the third stage growth conditions at infinity and properties of the other functions in (3.14) narrow the choice.)

We shall need details of a decomposition (3.15) in order to write and verify the solutions, and in order to prove uniqueness.

(iii) At the third stage, one restores the damping constant $\varepsilon > 0$ used in Appendix B, at the same time replacing $H(x)$ by $\exp(-\varepsilon x) H(x)$, and casts the perturbed version of (3.14) into the form

$$\frac{\tilde{\zeta}_+(s, \varepsilon)}{(s+i\varepsilon) G_+(s, \varepsilon)} + U\varphi_+(s, \varepsilon) = \frac{G_-(s, \varepsilon) \tilde{u}_-(s, \varepsilon)}{s+i\varepsilon} + U\varphi_-(s, \varepsilon), \quad -\varepsilon < \tau < \varepsilon, \quad (3.16)$$

the left-hand member being holomorphic in $\{\tau > -\varepsilon\}$ and the right-hand member in $\{\tau < \varepsilon\}$. (The function $G_-(\cdot, \varepsilon)$ has a zero at $-i\varepsilon$; see (B.14) in Appendix B.) One then determines both $\tilde{\zeta}_+(\cdot, \varepsilon)$ and $\tilde{u}_-(\cdot, \varepsilon)$ by hypotheses on their orders of magnitude at infinity, and by recognizing that both sides of (3.16) are the restriction to $\{-\varepsilon < \tau < \varepsilon\}$ of the same entire function. In the present application of the method, this entire function must be the zero function by virtue of estimates at infinity. Finally, one sets $\varepsilon = 0$.

We shall need only the final result of this procedure.

3.3. Decomposition of $q - k$: the functions E and F

We begin with definitions and descriptions of the functions involved in the decomposition. For q we refer to (3.11), for k to Theorem A.6. Perturbed versions of the following functions g , E and F are discussed in Appendix B; where a result proved there for $\varepsilon > 0$ is easily seen to remain true for $\varepsilon = 0$, it is merely stated here. The critical aspects of passing to $\varepsilon = 0$ will be taken up in Lemma 3.3 and Theorem 3.4.

$$g(s) := \frac{e^{i\beta}}{M} \left(\frac{s_-}{s_+} \right)^{\beta/\pi} \left\{ 1 - \frac{k(s)}{q(s)} \right\}, \quad \sigma \neq 0, \quad \tau \geq 0. \quad (3.17)$$

There is a branch $\log g$ (of the set-valued function $\text{Log } g$) that is holomorphic in $\{\sigma \neq 0, \tau > 0\}$, is continuous, has limiting values as $\sigma \rightarrow 0+$ with $\tau > 0$ and as $\sigma \rightarrow 0-$ with $\tau > 0$ that define continuous functions on $(0, \infty)$, and is such that

$$\log g(s) = O(s^{-\alpha}) \quad \text{as } s \rightarrow \infty \quad \text{with } \sigma \neq 0, \tau \geq 0. \quad (3.18)$$

Here α is the exponent in (1.4f).

The real constants $\kappa > 0$ and $\lambda > 0$ in the following definitions of E and F are chosen at pleasure; $\arg(s+i\kappa)$ and $\arg(s-i\lambda)$ are restricted, respectively, to the same intervals as $\arg s_+$ and $\arg s_-$ are in (3.10). The path of integration for E is the real axis, that for F the two sides (taken in opposite directions) of the cut along the positive imaginary axis.

$$\begin{aligned} E(s, \kappa) &:= \left(\frac{s_+}{s+i\kappa} \right)^{\beta/\pi} \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log g(w) \frac{dw}{w-s} \right\}, \quad \tau > 0, \\ &= \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log \left(g(w) \left(\frac{w_+}{w+i\kappa} \right)^{\beta/\pi} \right) \frac{dw}{w-s} \right\}; \end{aligned} \quad (3.19)$$

$E(\cdot, \kappa)$ is holomorphic and has no zero; as $s \rightarrow \infty$ with $\tau > 0$,

$$E(s, \kappa) = \begin{cases} 1 + O(s^{-\alpha}) & \text{if } \alpha \in (0, 1), \\ 1 + O(s^{-1} \log s) & \text{if } \alpha = 1. \end{cases} \quad (3.20)$$

With the notation $w =: \omega + it$, with $\mathcal{G}_-$ as in (3.9), and with H still denoting the Heaviside function,

$$\begin{aligned} F(s, \lambda) &:= \left(\frac{s_-}{s-i\lambda} \right)^{\frac{1}{2}-\beta/\pi} \exp \left\{ -\frac{1}{2\pi i} \int_0^{\infty} [\log g(w)]_{\omega \rightarrow 0^-}^{\omega \rightarrow 0^+} \frac{dw}{w-s} \right\}, \quad s \in \mathcal{G}_-, \\ &= \exp \left\{ \frac{1}{\pi} \int_0^{\infty} \left(\chi(t) - \beta - \left(\frac{1}{2}\pi - \beta \right) H(\lambda - t) \right) \frac{dt}{t+is} \right\}, \end{aligned} \quad (3.21)$$

where

$$\chi(t) := \tan^{-1} \left(\frac{1}{t\tilde{K}(it)} - 1 \right)^{\frac{1}{2}} \in (\beta, \frac{1}{2}\pi]. \quad (3.22)$$

(The lower bound for $\chi(t)$ comes from

$$t\tilde{K}(it) = t \int_0^{\infty} e^{-tx} K(x) dx < tK(0) \int_0^{\infty} e^{-tx} dx = M^{-2};$$

that $\chi(t) - \beta = O(t^{-\alpha})$ as $t \rightarrow \infty$ is implied by (3.18).) The function $F(\cdot, \lambda)$ is holomorphic and has no zero; as $s \rightarrow \infty$ in $\mathcal{G}_-$,

$$F(s, \lambda) = \begin{cases} 1 + O(s^{-\alpha}) & \text{if } \alpha \in (0, 1), \\ 1 + O(s^{-1} \log s) & \text{if } \alpha = 1. \end{cases} \quad (3.23)$$

LEMMA 3.3. Define a number $F(0, \lambda) > 0$ by setting $s = 0$ in (3.21); the integral converges. Then $F(s, \lambda) - F(0, \lambda) = O(s^{\frac{1}{2}})$ as $s = s_- \rightarrow 0$, so that $F(\cdot, \lambda)$ is defined and continuous on $\mathcal{G}_- \cup \{0\}$.

Proof. If the integral in (3.21) is written as one over $(0, \lambda)$ plus one over (λ, ∞) , then the latter is holomorphic in $\{|s| \leq \frac{1}{2}\lambda\}$; hence it suffices to prove the following.

(a) Given $f : [0, \lambda] \rightarrow \mathbf{R}$ such that, for some fixed exponent $\mu \in (0, 1)$,

$$|f(t)| \leq \text{const. } t^\mu \quad \text{whenever } 0 \leq t \leq \lambda,$$

$$|f(t_1) - f(t_2)| \leq \text{const. } t_1^{\mu-1}(t_2 - t_1) \quad \text{whenever } 0 < t_1 < t_2 \leq \lambda,$$

define

$$h(s) := \int_0^\lambda \frac{f(t)}{t+is} dt, \quad s \in \mathcal{G}_- \cup \{0\}.$$

Then

$$|h(s) - h(0)| \leq \text{const. } |s|^\mu \quad \text{for } |s| \leq \frac{1}{2}\lambda \text{ and } s \in \mathcal{G}_-.$$

(b) $0 \leq \pi/2 - \chi(t) \leq \text{const. } t^{\frac{1}{2}}$ whenever $t \geq 0$,

$$0 < \chi(t_1) - \chi(t_2) \leq \text{const. } t_1^{-\frac{1}{2}}(t_2 - t_1) \quad \text{whenever } 0 < t_1 < t_2 \leq \lambda.$$

Here (a) is an exercise of classical type for Cauchy integrals; the solution is omitted. For (b), we begin with an integration by parts:

$$i\tilde{K}(it) = K(0) + \int_0^\infty e^{-tx} K'(x) dx \quad (t \geq 0),$$

thus using Lemma 1.1. Let $0 \leq t_1 < t_2$. Since $K'(x) < 0$ wherever it exists,

$$\begin{aligned} 0 < t_2 \tilde{K}(it_2) - t_1 \tilde{K}(it_1) &= \int_0^\infty (e^{-t_2 x} - e^{-t_1 x}) K'(x) dx \\ &< \int_0^\infty (t_1 x - t_2 x) K'(x) dx \\ &= \|K\|(t_2 - t_1), \end{aligned}$$

upon integrating by parts again and using Lemma 1.1(e). (This Lipschitz continuity results from the factor t ; the function $\tilde{K}$ may be not even Hölder continuous at the origin.) The estimates (b) now follow from the definition (3.22) of χ . $\square$

In what follows,

$$W(s) := is + 1/\tilde{K}(s), \quad \tau \geq 0; \quad (3.24)$$

this function and the branch $\arg W$ determining $W(s)^{\frac{1}{2}}$ are described in Lemma A.5.

THEOREM 3.4. (a) *The decomposition equation*

$$s_{\pm}^{\frac{1}{2}} - e^{i3\pi/4} W(s)^{\frac{1}{2}} = M e^{-i\beta} (s+i\kappa)^{\beta/\pi} E(s, \kappa) (s-i\lambda)^{\frac{1}{2}-\beta/\pi} F(s, \lambda), \quad (3.25)$$

in which the left-hand member is $s_{\pm}^{-\frac{1}{2}}\{q(s) - k(s)\}$, holds for $\sigma \neq 0$ and $\tau > 0$.

(b) *The function $E(\cdot, \kappa)$ has an extension (denoted by the same symbol) that is continuous on $\{\tau \geq 0\}$, has no zero there and satisfies (3.20) as $s \rightarrow \infty$ with $\tau \geq 0$. [In addition, $E(\cdot, \kappa)$ is holomorphic in $\{\tau > 0\}$.]*

(c) *With $E(\cdot, \kappa)$ thus extended, (3.25) holds also for all $s \in \mathbb{R}$.*

Proof. (a) Consider the functions $q(\cdot, \varepsilon)$, $k(\cdot, \varepsilon)$, $E(\cdot, \kappa, \varepsilon)$ and $F(\cdot, \lambda, \varepsilon)$ defined by (B.1), (B.2), (B.11) and (B.12), respectively. For any fixed s with $\sigma \neq 0$ and $\tau > 0$, let $\varepsilon \rightarrow 0$; then $q(s, \varepsilon) \rightarrow q(s), \dots, F(s, \lambda, \varepsilon) \rightarrow F(s, \lambda)$, and (3.25) follows from (B.13).

(b), (c) We now have

$$E(s, \kappa) = \frac{e^{i\beta} \{s_{\pm}^{\frac{1}{2}} - e^{i3\pi/4} W(s)^{\frac{1}{2}}\}}{M(s+i\kappa)^{\beta/\pi} (s-i\lambda)^{\frac{1}{2}-\beta/\pi} F(s, \lambda)} \quad (3.26)$$

for $\sigma \neq 0$ and $\tau > 0$. Denote the right-hand member of this equation by $R(s)$. Then R is continuous on $\{\sigma \neq 0, \tau \geq 0\} \cup \{0\}$ because $W^{\frac{1}{2}}$ is continuous on $\{\tau \geq 0\}$ and $1/F(\cdot, \lambda)$ is continuous on $\mathcal{G}_{-} \cup \{0\}$. But $E(\cdot, \kappa)$ is holomorphic in $\{\tau > 0\}$; hence the singularities of R on $\{\sigma = 0, \tau > 0\}$ are removable. (One can test (3.26) by a calculation: the limiting values of R , as $\sigma \rightarrow 0+$ and as $\sigma \rightarrow 0-$, are equal for fixed $\tau \geq 0$.)

Accordingly, $E(\cdot, \kappa) = R$ in $\{\tau > 0\}$, and R is continuous on $\{\tau \geq 0\}$. Extending $E(\cdot, \kappa)$ to the real axis by $E(\sigma, \kappa) := R(\sigma)$, we extend (3.25) at the same time. That

$E(\cdot, \kappa)$ has no zero on the real axis, and now satisfies (3.20) on $\{\tau \geq 0\}$, is also shown by (3.26). $\square$

On formulae for E . In the context of general proofs, (3.26) and (3.21) seem to give a more useful representation of E than does (3.19). However, if $K(x)$ decays exponentially as $x \rightarrow \infty$, then its transform $\tilde{K}$ is holomorphic in $\{\tau > -c\}$ for some $c > 0$, and the path of integration in (3.19) can be deformed to be at least partly in the lower half-plane, so that a variant of (3.19) may give a convenient representation of E .

For the Maxwell kernel, if one chooses a suitable non-dimensional co-ordinate x and chooses $\kappa = 1$, both $W^{\frac{1}{2}}$ and $E(\cdot, 1)$ have analytic continuations onto $\mathbb{C} \setminus \{it | t \leq -1\}$, and $E(\cdot, 1)$ can be written as an integral along the cut $\{it | t \leq -1\}$. In many contexts, this representation of E (Fraenkel 1988) is more useful than (3.26) and (3.21) are (for that particular shear kernel).

4. Weak solutions : formulae and verification

In order to record the velocity field resulting from step (iii) of §3.2, we introduce a function Φ related to the tentative solution (3.12) by

$$\Phi(s) = is^{\frac{1}{2}} \tilde{K}(s) \tilde{\zeta}(s, 0+),$$

and defined explicitly by

$$\Phi(s) := C_1 \tilde{K}(s) (s+i\kappa)^{\beta/\pi} E(s, \kappa), \quad \tau \geq 0, \quad (4.1a)$$

where

$$C_1 := \frac{U\kappa^{-\beta/\pi} e^{-i\pi/4-i\beta/2}}{\|K\|^{\frac{1}{2}} E(0, \kappa)}. \quad (4.1b)$$

[The elaborate constant C_1 has a simple consequence. Since $\tilde{K}(0) = \|K\|$, we have

$$\Phi(0) = U\|K\|^{\frac{1}{2}} e^{-i\pi/4}. \quad (4.1c)$$

This leads to the complex velocity in Lemma 4.1(b), which is precisely the far velocity field, outside the boundary layer adjacent to the plate, for symmetrical, linearized flow past P of a Newtonian fluid with kinematic viscosity $U\|K\|$. Observe that in (1.2a) we also had $v = U \int K_0$. Thus the complicated C_1 leads to the simplest possible value of the effective Newtonian viscosity, at large distances from the origin, that was mentioned in §1.1.] Lemma A.2 and Theorem 3.4(b) show that Φ is *holomorphic* in $\{\tau > 0\}$ and *continuous on* $\{\tau \geq 0\}$, also that

$$\Phi(s) = O(s^{-1+\beta/\pi}) \quad \text{as } s \rightarrow \infty \quad \text{with } \tau \geq 0. \quad (4.1d)$$

The *irrotational velocity field* is

$$\begin{bmatrix} u_0(x, y) \\ v_0(x, y) \end{bmatrix} := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixs-|y|q(s)} \begin{bmatrix} -q(s) \\ is \operatorname{sgn} y \end{bmatrix} \frac{\Phi(s)}{s_+^{3/2}} ds, \quad (4.2)$$

where the path of integration is the real axis. For the complex velocity $u_0 - iv_0$, we use the notation

$$z = x + iy = re^{i\theta}, \quad P = \{z \in \mathbb{C} \mid x > 0, y = 0\};$$

it follows from (4.2) (since $q(s) = \pm s$ for $\sigma > 0$ or $\sigma < 0$, respectively) that

$$(u_0 - iv_0)(z) = \frac{1}{\pi} \int_{-\infty}^0 e^{-izs} \frac{\Phi(s)}{s_+^{\frac{1}{2}}} ds, \quad y \geq 0, \quad (4.3)$$

where the abbreviation $y \geq 0$ means that $y > 0$ or that limiting values as $y \rightarrow 0+$ are taken.

The *rotational velocity field* is

$$\begin{bmatrix} u_1(x, y) \\ v_1(x, y) \end{bmatrix} := -\frac{1}{2\pi} \int_{ic-\infty}^{ic+\infty} e^{-ixs+|y|k(s)} \begin{bmatrix} k(s) \\ is \operatorname{sgn} y \end{bmatrix} \frac{\Phi(s)}{s_+^{3/2}} ds, \quad c > 0, \quad (4.4)$$

where the path of integration $\{\tau = c > 0\}$ has been specified for the sake of definiteness, and because the integrand for u_1 behaves like s^{-1} as $s \rightarrow 0$ with $\tau \geq 0$. However, other paths of integration may and will be used. The path for u_1 may be deformed to one departing from the real axis only to pass above the origin, say to

$$\{|\sigma| > \delta, \tau = 0\} \cup \{|s| = \delta, 0 \leq \arg s \leq \pi\} \quad \text{for any } \delta > 0, \quad (4.5)$$

and the path for v_1 may be the real axis.

The following estimates of the integrands in (4.3) and (4.4) will be used repeatedly; they come from Theorem A.6 and (4.1d).

$$|e^{-izs}| \leq 1 \quad (y \geq 0, s = \sigma \leq 0), \quad (4.6a)$$

$$|e^{-ixs+|y|k(s)}| \leq e^{(x-b|y|)\tau} \quad (\tau \geq 0), \quad (4.6b)$$

$$\Phi(s)/s_+^{\frac{1}{2}} = O(s^{-\frac{1}{2}}) \quad \text{as } s \rightarrow 0 \quad \text{with } \tau \geq 0, \quad (4.7a)$$

$$k(s)\Phi(s)/s_+^{3/2} = O(s^{-1}) \quad \text{as } s \rightarrow 0 \quad \text{with } \tau \geq 0, \quad (4.7b)$$

$$\Phi(s)/s_+^{\frac{1}{2}}, k(s)\Phi(s)/s_+^{3/2} \quad \text{are } O(s^{-3/2+\beta/\pi}) \quad \text{as } s \rightarrow \infty \quad \text{with } \tau \geq 0. \quad (4.8)$$

Note that *the non-exponential parts of the integrands in (4.2), (4.3) and (4.4), and the integrands as a whole, are absolutely integrable (are in L_1) on the paths of integration that have been specified.*

We proceed to show that the disturbance velocity field $(u_0 + u_1, v_0 + v_1)$ is a weak solution, in the sense of Definition 2.1. The results for (u_0, v_0) are somewhat stronger

than that definition requires.

LEMMA 4.1. (a) $u_0, u_1, v_0 + v_1 \in C(\mathbf{R}^2)$ and $v_0, v_1 \in C(\mathbf{R}^2 \setminus P)$.

(b) For any $\gamma > 0$, as $r \rightarrow \infty$ with $\gamma \leq \theta \leq 2\pi - \gamma$ (in particular, as $z \rightarrow \infty$ in $\mathcal{U}$),

$$(u_0 - iv_0)(z) = -iU(\|K\|/\pi)^{\frac{1}{2}} z^{-\frac{1}{2}} + o(r^{-\frac{1}{2}}). \quad (4.9)$$

(c) $u_1 = v_1 = 0$ on $\overline{\mathcal{U}}$.

(d) In $\mathcal{A}$, u_0, v_0 and v_1 are bounded, and

$$|u_1(x, y)| \leq \text{const.} \log \left(2 + \frac{x - b|y|}{a} \right),$$

where a is a positive constant.

Proof. (a) We first consider $u_0, \dots, v_1$ on the closed half-plane $Z := \mathbf{R} \times [0, \infty)$; in this context, $\text{sgn } y = 1$ for $y \geq 0$. Since the exponentials are continuous functions of x and y , and the integrals converge uniformly on each compact subset of Z , we have $u_0, \dots, v_1 \in C(Z)$. Since u_0 and u_1 are even functions of y , their continuity on Z implies continuity on $\mathbf{R}^2$.

For v_0 and v_1 , which are odd functions of y , it now suffices to prove that

$$(v_0 + v_1)(x, 0+) = 0 \text{ for all } x, \quad v_0(x, 0+) = v_1(x, 0+) = 0 \text{ for } x \leq 0. \quad (4.10)$$

The first of these is shown, in view of continuity on Z , by

$$v_0(x, 0+) = \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{-ixs} \Phi(s) s_+^{-\frac{1}{2}} ds = -v_1(x, 0+).$$

The second of (4.10) is shown by contour integration: one uses a large semi-circle in $\{\tau > 0\}$ and the estimates (4.6b), with $y = 0$, and (4.8).

(b) We need prove (4.9) only for $\theta \in [\gamma, \pi]$; we use (4.3). Let $s =: \rho e^{i\varphi}$. For $\theta \in [\gamma, \frac{3}{2}\pi]$ we choose $\varphi = \pi$ ($0 < \rho < \infty$), thus using the path of integration in (4.3); for $\theta \in (\frac{3}{2}\pi, \pi]$ we deform the path to one on which $\varphi = \frac{3}{2}\pi$ ($0 < \rho < \infty$); then

$$|e^{-izs}| = \exp\{r\rho \sin(\theta + \varphi)\} \leq \exp(-cr\rho), \quad c > 0.$$

The explicit part of the result is due to replacing $\Phi(s)$ by $\Phi(0)$. The o -term is due to the continuity of Φ at 0 (of which, however, we lack detailed knowledge) and the following inequality. If $f(\rho) = o(\rho^{-\frac{1}{2}})$ as $\rho \rightarrow 0+$, and $f \in L_1(0,1) \cap L_\infty(1,\infty)$, then (for $c > 0$)

$$\int_0^\infty e^{-cr\rho} f(\rho) d\rho = o(r^{-\frac{1}{2}}) \quad \text{as } r \rightarrow \infty.$$

(c) Again the result follows from contour integration, the use of a large semi-circle in $\{\tau > 0\}$, and the estimates (4.6b) and (4.8).

(d) Integrating along the real axis $\{\tau = 0\}$ to calculate v_1 , we observe that bounds for u_0, v_0 and v_1 are immediate from (4.6) and our knowledge of Φ . Now consider u_1 . We use the path of integration (4.5), choose a constant $a > 0$, set $\delta = 1/a$ for $\xi = x - b|y| \leq a$, and set $\delta = 1/\xi$ for $\xi > a$. Then $|u_1(x,y)|$ is bounded by a constant for $\xi \leq a$. For $\xi > a$, the straight part $\{|\sigma| > 1/\xi, \tau = 0\}$ of the path contributes a term of order $\log \xi$ for large ξ , and the contribution of the semi-circle $\{|s| = 1/\xi\}$ is bounded, in view of (4.6b) and (4.7b). $\square$

The logarithmic estimate in Lemma 4.1(d) is unsatisfactory but sufficient for our needs. For the particular kernels K_1 to K_3 in §1.2, one readily proves that u_1 is bounded. In the general case, the difficulty is the lack of detailed information about $\exp\{|y|k(s)\}$ near $s = 0$.

LEMMA 4.2. (a) $u_0 - iv_0$ is holomorphic in $\mathbb{C} \setminus \bar{P}$. [Consequently, (u_0, v_0) is solenoidal and irrotational pointwise in $\mathbb{R}^2 \setminus \bar{P}$, and satisfies the vorticity equation trivially.]

(b) $(u_0 + u_1, v_0 + v_1)$ is weakly solenoidal in $\mathbb{R}^2$. [Hence (u_1, v_1) is weakly solenoidal in $\mathbb{R}^2 \setminus \bar{P}$.]

(c) (u_1, v_1) satisfies the vorticity equation weakly, in the sense of (2.3d).

Proof. (a) We apply Morera's theorem (Rudin 1970, p.209; Titchmarsh 1932, p.82). Since $u_0 - iv_0$ is continuous in $\mathbb{C} \setminus \bar{P}$, it is holomorphic there if

$$\int_{\partial T} (u_0 - iv_0)(z) dz = 0$$

whenever ∂T is the boundary of a closed triangle $T \subset \mathbb{C} \setminus \bar{P}$. Let $T_+ := T \cap \{y \geq 0\}$ and $T_- := T \cap \{y \leq 0\}$. It is sufficient to prove that the integral of $u_0 - iv_0$ over the boundary ∂T_+ of T_+ vanishes whenever T_+ has non-empty interior. The same will be true for ∂T_- by the parity of $u_0 - iv_0$, and, when T_+ and T_- both have non-empty interiors, the integral over ∂T is the sum of those over ∂T_+ and ∂T_- , provided that all three boundaries have the same orientation. Now the integrand in (4.3) belongs to $L_1(\partial T_+ \times (-\infty, 0))$; by Fubini's theorem,

$$\int_{\partial T_+} (u_0 - iv_0)(z) dz = \frac{1}{\pi} \int_{-\infty}^0 \frac{\Phi(s)}{s_+^{\frac{1}{2}}} ds \int_{\partial T_+} e^{-isz} dz = 0.$$

(b) To prove weak solenoidality in $\mathbb{R}^2$, it is enough to use test functions $\varphi \in C_0^\infty(\mathbb{R}^2)$, because this set is dense in $\mathcal{S}(\mathbb{R}^2)$ (Friedlander 1982, p.94; Hörmander 1983, p.163; Schwartz 1950, vol. II, p.93). Integrating first over $\mathbb{R}_+^2 := \{y > 0\}$, we may apply Fubini's theorem again to obtain

$$\begin{aligned} & 2\pi \iint_{\mathbb{R}^2} \{(u_0 + u_1)\varphi_x + (v_0 + v_1)\varphi_y\} dx dy \\ &= \int_{-\infty}^{\infty} \frac{\Phi(s)}{s_+^{3/2}} I_{0,\varphi}(s) ds + \int_{ic-\infty}^{ic+\infty} \frac{\Phi(s)}{s_+^{3/2}} I_{1,\varphi}(s) ds, \end{aligned} \quad (4.12)$$

where $c > 0$ and

$$I_{0,\varphi}(s) := \iint_{\mathbb{R}_+^2} e^{-isx - q(s)y} \{-q(s)\varphi_x + is\varphi_y\} dx dy \quad (\tau = 0),$$

$$I_{1,\varphi}(s) := -\iint_{\mathbb{R}_+^2} e^{-isx + k(s)y} \{k(s)\varphi_x + is\varphi_y\} dx dy \quad (\tau = c).$$

Integration by parts gives

$$I_{0,\varphi}(s) = -is \int_{-\infty}^{\infty} e^{-isx} \varphi(x, 0) dx,$$

$$I_{1,\varphi}(s) = is \int_{-\infty}^{\infty} e^{-isx} \varphi(x,0) dx.$$

Since $I_{1,\varphi}(s) = O(s)$ as $s \rightarrow 0$ with $\tau \geq 0$, we may now let $c \rightarrow 0$ in (4.12); the integral there over $\mathbf{R}_+^2$ is seen to vanish, and the argument for $\mathbf{R}_-^2$ is similar.

(c) The method of proof is basically the same as in (b). $\square$

LEMMA 4.3. *The boundary condition on $\bar{P}$ is satisfied:*

$$(u_0 + u_1, v_0 + v_1)(x,0) = (-U,0) \quad \text{for } x \geq 0. \quad (4.13)$$

Proof. We know already, from (4.10) and the fact that $v_0 + v_1$ is odd in y , that $v_0 + v_1$ vanishes on $\bar{P}$. So far, only qualitative properties of Φ have been used; the proof that $u_0 + u_1 = -U$ on $\bar{P}$ requires the full force of Theorem 3.4.

Substituting into (4.1) the formula (3.26) for E , and also using the definitions (A.9) of k and (3.24) of W , we ultimately obtain

$$\frac{k(s)}{s_+^{\frac{1}{2}}} \Phi(s) = iU \left(\frac{-i\lambda}{s-i\lambda} \right)^{\frac{1}{2}-\beta/\pi} \frac{F(0,\lambda)}{F(s,\lambda)} \{1 - e^{-i3\pi/4} s_+^{\frac{1}{2}} \tilde{K}(s) W(s)^{\frac{1}{2}} + is \tilde{K}(s)\}, \quad \tau \geq 0, \quad (4.14)$$

$$\frac{q(s)+k(s)}{s_+^{\frac{1}{2}}} \Phi(s) = iU \left(\frac{-i\lambda}{s-i\lambda} \right)^{\frac{1}{2}-\beta/\pi} \frac{F(0,\lambda)}{F(s,\lambda)}, \quad s \in \mathcal{G}_- \cup \{0\}. \quad (4.15)$$

In (4.14), the singularities on $\{\sigma=0, \tau>0\}$ of the right-hand member are removable (because it equals the left-hand member elsewhere on $\{\tau \geq 0\}$). In (4.15), the right-hand member continues the left-hand one analytically into all of $\mathcal{G}_-$.

From (4.14) and Lemma 3.3 we see that

$$\frac{k(s)}{s_+^{3/2}} \Phi(s) = \frac{iU}{s} \{1 + O(s^{\frac{1}{2}})\} \quad \text{as } s \rightarrow 0 \text{ with } \tau \geq 0.$$

Hence, in the equation

$$UH(x) + u_1(x,0) = \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{ic-N}^{ic+N} e^{-ixs} \left\{ \frac{iU}{s} - \frac{k(s)}{s_+^{3/2}} \Phi(s) \right\} ds, \quad c > 0,$$

we may let $c \rightarrow 0$. Adding $u_0(x,0)$ and then using (4.15), we obtain

$$\begin{aligned} UH(x) + u_0(x,0) + u_1(x,0) &= \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-N}^N e^{-ixs} \left\{ \frac{iU}{s} - \frac{q(s)+k(s)}{s_+^{3/2}} \Phi(s) \right\} ds \\ &= \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-N}^N e^{-ixs} \frac{iU}{s} \left\{ 1 - \left(\frac{-i\lambda}{s-i\lambda} \right)^{\frac{1}{2}-\beta/\pi} \frac{F(0,\lambda)}{F(s,\lambda)} \right\} ds \\ &= 0 \quad \text{for } x > 0 \end{aligned}$$

by contour integration and Jordan's lemma; we use the semi-circle $\{|s| = N, \tau < 0\}$.

The result (4.13) for $x \geq 0$ follows by the continuity of $u_0 + u_1$ on $\mathbf{R}^2$. $\square$

Lemmas 4.1 to 4.3 establish

THEOREM 4.4. *The velocity field $(u_0 + u_1, v_0 + v_1)$ defined by (4.2) and (4.4) is a weak solution of the plate problem.*

Note that, if (u_1, v_1) has the continuity properties (2.20), then $(u_0 + u_1, v_0 + v_1)$ is also a pointwise solution. [That (u_0, v_0) satisfies (2.20) is clear from Lemma 4.2(a). In Theorem 2.4, the condition of polynomial growth on $K' * \omega, \dots$, was needed only to pass from pointwise solutions to weak ones; see the final remark of the proof.]

5. Uniqueness under the condition $\zeta(\cdot, 0\pm) \in A_\mu$

5.1. Introduction of the Banach space A_μ

For the shear kernels K_1 to K_3 introduced in §1.2, one has pointwise solutions and can compute without great difficulty. It turns out that, for these functions K and for the solutions established in §4, the plate-vorticity function $\zeta(\cdot, 0+)$ exists on $(0, \infty)$ both as the inverse transform of the function $\tilde{\zeta}(\cdot, 0+)$ in Theorem 3.1 and as $\lim_{y \rightarrow 0+} \zeta(\cdot, y)$, the latter being derived from (4.4). In these cases, $\zeta(\cdot, 0+) \in C^\infty(0, \infty)$ and

$$\zeta(x, 0+) \sim -c_1 x^{-1/2-\beta/\pi} \quad \text{as } x \rightarrow 0+, \quad (5.1a)$$

$$\zeta(x, 0+) = -c_2 x^{-1/2} - c_3 x^{-1/2-l} + O(x^{-1-\delta}) \quad \text{as } x \rightarrow \infty, \quad (5.1b)$$

where c_1 and c_2 are positive constants, $c_3 = 0$ when $K = K_1$ or K_2 , $c_3 > 0$ when $K = K_3$, and $\delta > 0$.

It is clear that such functions $\zeta(\cdot, 0+)$ are in $L_{1,\text{loc}}(\mathbf{R})$, but fail for each $p \in [1, \infty]$ to be in $L_p(0, \infty)$; nor are they in any of the more familiar Banach spaces. It seems necessary for a proof of uniqueness to have some control over the functions $\zeta(\cdot, 0\pm)$ and their Fourier-Laplace transforms; therefore we define a Banach space A_μ that contains, with some room to spare, functions having the behaviour in (5.1).

Notation. (a) Given a function $h : [0, \infty) \rightarrow \mathbf{R}$ and a number $x \geq 0$, we write $V_x^\infty h$ for the *total variation of h over $[x, \infty)$* ; that is,

$$V_x^\infty h := \sup \sum_{j=1}^n |h(t_j) - h(t_{j-1})|, \quad x = t_0 \leq t_1 \leq \dots \leq t_n,$$

the supremum being over all such finite sets $\{t_0, \dots, t_n\}$.

(b) The *sum $X + Y$ of two Banach spaces X and Y* over the same field consists of the set

$$Z := \{u+v \mid u \in X \text{ and } v \in Y\},$$

and of the norm defined by

$$\|z|Z\| := \inf \{ \|x|X\| + \|y|Y\| \mid x + y = z \}, \quad (5.2)$$

the infimum being over all decompositions $x + y$ of $z \in Z$ such that $x \in X$ and $y \in Y$.

It is known (Triebel 1978, p.18) that $\|\cdot|Z\|$ is indeed a norm, and that $Z = X + Y$ is complete (is a Banach space).

(c) Norm symbols $\|\cdot|X\|$ will often be abbreviated to $\|\cdot\|$ when the space X is implied by the context.

DEFINITION 5.1. Choose and fix a number $\mu \in (0, \frac{1}{2})$. The real Banach space B_μ consists of the norm defined by

$$\|h|B_\mu\| := \sup_{x \geq 0} (1+x)^\mu V_x^\infty h, \quad (5.3)$$

and of the set

$$\{h : [0, \infty) \rightarrow \mathbf{R} \mid h(x) \rightarrow 0 \text{ as } x \rightarrow \infty, \quad \|h|B_\mu\| < \infty\}.$$

DEFINITION 5.2. Let $L_1 := L_1(0, \infty)$ and let it contain only real-valued functions. The real Banach space A_μ is the sum $L_1 + B_\mu$ [normed as in (5.2), with $X = L_1$ and $Y = B_\mu$].

Remarks. 1. That (5.3) does indeed define a norm, and that B_μ is complete, follows from standard results for functions of bounded variation (Dunford and Schwartz 1958, pp.241 and 337).

2. The simplest function in B_μ is perhaps that with values $h(x) = (1+x)^{-\mu}$ and norm $\|h\| = 1$. (We have $V_x^\infty h = h(x)$ whenever h is non-increasing on $[x, \infty)$ and tends to zero at infinity.) Functions in B_μ that decay more rapidly than this can oscillate a little; for example, the function with values $(1+x)^{-1/2} \sin \{(1+x)^\gamma\}$ is in B_μ for $\gamma \leq \frac{1}{2} - \mu$.

3. Functions as in (5.1) certainly belong to A_μ . For example, let

$$h(x) := -c_2(1+x)^{-1/2} - c_3(1+x)^{-1/2-l} \quad (x \geq 0),$$

or

$$h(x) := -\{c_2x^{-1/2} + c_3x^{-1/2-l}\} H(x-1);$$

then $h \in B_\mu$. Defining $g := \zeta(\cdot, 0+) - h$, we have $g \in L_1$ and hence $\zeta(\cdot, 0+) \in A_\mu$.

4. The main properties of functions in B_μ are derived elsewhere (Fraenkel 1990b). We quote a result that is almost immediate (given the Jordan decomposition theorem for functions of bounded variation) and shows why the Fourier-Laplace transform of functions in B_μ can be estimated. The second statement combines such estimates with standard ones for the Fourier-Laplace transform of functions in L_1 .

(a) If $h \in B_\mu$, then there is a decomposition $h = h_1 - h_2$ such that h_1 and h_2 are non-increasing, and

$$0 \leq h_j(x) \leq \|h\| (1+x)^{-\mu} \quad \text{for } j=1,2 \text{ and all } x \geq 0. \quad (5.4)$$

(b) If $f \in A_\mu$, then its Fourier-Laplace transform $\tilde{f}$ is continuous on the set $\{\tau \geq 0, s \neq 0\}$ and holomorphic in $\{\tau > 0\}$; moreover,

$$|\tilde{f}(s)| \leq \begin{cases} (4\sqrt{2} + \frac{1}{1-\mu}) \|f\| |s|^{-1+\mu} & \text{for } 0 < |s| < 1, \quad \tau \geq 0, \\ 4\sqrt{2} \|f\| & \text{for } |s| \geq 1, \quad \tau \geq 0. \end{cases} \quad (5.5a)$$

$$(5.5b)$$

A little more than this is known. If $f = g + h$, with $g \in L_1$ and $h \in B_\mu$, then $\tilde{g}(s) \rightarrow 0$ as $s \rightarrow \infty$ with $\tau \geq 0$, but (in general) no rate can be stated, and $\tilde{h}(s) = O(s^{-1})$ as $s \rightarrow \infty$ with $\tau \geq 0$.

5.2. The uniqueness theorem

The condition stated loosely as $\zeta(\cdot, 0\pm) \in A_\mu$, which we shall add to the definition of weak solution to prove uniqueness, has the following precise form.

(A). *The functions $\tilde{\zeta}(\cdot, 0+)$ and $\tilde{\zeta}(\cdot, 0-)$ in Theorem 3.1 are the Fourier-Laplace transforms of functions in A_μ .*

A related condition, which avoids mention of the Fourier-Laplace transform, is stated in terms of ω and η . Recall from §2.4 that $\omega(\xi, \eta) = \zeta(\xi + |\eta|, \eta/b)$; it is natural to consider the vorticity on characteristic lines $\{\xi = \text{const.}\}$.

(B). *There exist a number $\eta_1 > 0$ and a function $\zeta_0 \in A_\mu$ such that $\omega(\cdot, \eta) \in A_\mu$ for $0 < \eta \leq \eta_1$, and*

$$\|\zeta_0 - \omega(\cdot, \eta)\|_{A_\mu} \rightarrow 0 \quad \text{as } \eta \rightarrow 0+;$$

similarly for $\eta < 0$.

It is not difficult to prove that (B) implies (A). One would expect that (A) need not imply (B), but I have not resolved this question.

For the solutions in §4, condition (B) holds not only when K is one of K_1 to K_3 , but also for certain shear kernels (§6, Remarks 4, 8 and 9) such that the weak solution is not a pointwise one.

From the physical point of view, a solution failing, for every $\mu \in (0, \frac{1}{2})$, to satisfy condition (A) would seem outrageous; essentially, such a plate-vorticity function either would be not locally integrable, or would have improbable behaviour far downstream. (On the other hand, it will be seen in §6 that rather exotic wave propagation can occur in $\mathfrak{S}$.)

We are now in a position to state the uniqueness theorem.

THEOREM 5.3. *For a given shear kernel K as in (1.4), and a given velocity U at infinity, there is at most one velocity field that satisfies both (2.3) and condition (A).*

The proof requires many steps; before setting to work seriously, we dispose of some easy preliminaries.

Let $\psi = \psi_I - \psi_{II}$ be the difference between the stream functions of two weak solutions that satisfy condition (A). Then $(\psi_y, -\psi_x)$ is a weak solution of the *homogeneous* plate problem, which means that $(\psi_y, -\psi_x)$ satisfies (2.3) with the change that U is replaced by 0 in the boundary condition on $\bar{P}$. Define the odd and even parts of $\psi(x, \cdot)$ by

$$\psi^o(x, y) := \frac{1}{2} \{ \psi(x, y) - \psi(x, -y) \}, \quad (5.6a)$$

$$\psi^e(x, y) := \frac{1}{2} \{ \psi(x, y) + \psi(x, -y) \}, \quad (5.6b)$$

and let

$$u^o := \partial_2 \psi^o, \quad v_1^o := -\partial_1 \psi^o, \dots, \quad \langle \zeta^e, \cdot \rangle := -\langle \Delta \psi^e, \cdot \rangle. \quad (5.6c)$$

Then both (u^o, v^o) and (u^e, v^e) are weak solutions of the homogeneous plate problem, by invariance of that problem under the transformation $y \mapsto -y$ and by linearity; also, both (u^o, v^o) and (u^e, v^e) satisfy condition (A), again by linearity.

In our construction of solutions, equation (3.14) related $\tilde{\zeta}_+(s)$ to $\tilde{u}_-(s)$ on the real axis punctured at the origin; the next step of the uniqueness proof is to derive the counterparts of that equation for (u^o, v^o) and (u^e, v^e) . At the same time we add precise information about the analogues u_-^o and v_-^e of the previous u_- .

Notation. (a) For definiteness in what follows, we now specify the value $H(0) = 0$ of the Heaviside function. This is not essential.

(b) Given $f : (-\infty, 0] \rightarrow \mathbf{R}$, we say that $f \in A_\mu(-\infty, 0]$ if its reflection $f_r \in A_\mu$. [Here $f_r(x) := f(-x)$.] The space $B_\mu(-\infty, 0]$ is defined similarly.

PROPOSITION 5.4. (a) Let $\tilde{\zeta}_+ := \tilde{\zeta}^o(\cdot, 0+)$ [this is now the Fourier-Laplace transform of some function $\zeta_+ \in A_\mu$], and let $u_-(x) := u^o(x, 0)H(-x)$. Then $u_- \in A_\mu(-\infty, 0]$ and

$$\tilde{\zeta}_+(s) = \{q(s) - k(s)\} \tilde{u}_-(s) \quad \text{for } s \in \mathbb{R} \setminus \{0\}. \quad (5.7)$$

(b) Let $\tilde{\zeta}_+ := \tilde{\zeta}^e(\cdot, 0+)$ [again this is the transform of some $\zeta_+ \in A_\mu$], and let $v_-(x) := v^e(x, 0)H(-x)$. Then $v_- \in A_\mu(-\infty, 0]$ and

$$\tilde{\zeta}_+(s) = -i(s_+/s_-)^{\frac{1}{2}} \{q(s) - k(s)\} \tilde{v}_-(s) \quad \text{for } s \in \mathbb{R} \setminus \{0\}. \quad (5.8)$$

Unlike the (largely formal) derivation of (3.14), the proof of the proposition is not short. Preliminary estimates are made in §5.3 and the proposition is proved in §5.4.

Proof of Theorem 5.3. (a) Equation (5.7) and Theorem 3.4 imply that

$$\frac{s_+^{1/2} \tilde{\zeta}_+(s)}{(s+i\kappa)^{\beta/\pi} E(s, \kappa)} = Me^{-i\beta} (s-i\lambda)^{1/2-\beta/\pi} F(s, \lambda) s \tilde{u}_-(s) \quad \text{for } s \in \mathbb{R} \setminus \{0\}.$$

Denote the left-hand and right-hand members of this equation by $L(s)$ and $R(s)$, respectively. Since $u_- \in A_\mu(-\infty, 0]$, it follows from (5.5a) that $R(s) = O(s^\mu)$ as $s \rightarrow 0$ with $\tau \leq 0$; therefore $L(s) = O(s^\mu)$ as $s \rightarrow 0$ with $\tau = 0$. Again by (5.5a), $L(s) = O(s^{-1/2+\mu})$ as $s \rightarrow 0$ with $\tau > 0$. Then the Phragmén-Lindelöf theorem for holomorphic functions (Titchmarsh 1932, p.176) shows that $L(s) = O(s^\mu)$ as $s \rightarrow 0$ with $\tau \geq 0$; indeed, for this conclusion a certain exponentially large order could replace $O(s^{-1/2+\mu})$. We set $L(0) = 0$ and $R(0) = 0$; then L is continuous on $\{\tau \geq 0\}$, R is continuous on $\{\tau \leq 0\}$, and they are equal on the real axis.

In addition, L is holomorphic in $\{\tau > 0\}$ and R is holomorphic in $\{\tau < 0\}$. Hence we define w by $w(s) := L(s)$ for $\tau \geq 0$, and by $w(s) := R(s)$ for $\tau < 0$. It follows from Morera's theorem (rather as in the proof of Lemma 4.2(a)) that w is an entire function. By (5.5b),

$$w(s) = O(s^{1/2-\beta/\pi}) \quad \text{as } s \rightarrow \infty \quad \text{with } \tau \geq 0, \quad (5.9a)$$

$$w(s) = O(s^{3/2-\beta/\pi}) \quad \text{as } s \rightarrow \infty \quad \text{with } \tau \leq 0. \quad (5.9b)$$

Using $O(s^{3/2-\beta/\pi})$ for all large s , we conclude from a well known extension of Liouville's theorem that $w(s) = a_0 + a_1 s$ for some complex constants a_0 and a_1 . But

then $a_1 = 0$, otherwise (5.9a) is contradicted. Thus $R(s) = a_0$ and

$$\tilde{u}_-(s) = \frac{a_0 e^{i\beta}}{M(s-i\lambda)^{1/2-\beta/\pi} F(s,\lambda)} \frac{1}{s} \quad \text{for } s \neq 0, \tau \leq 0;$$

indeed, for $s \in \mathcal{G}_-$. If $a_0 \neq 0$, then $\tilde{u}_-(s)$ is not $O(s^{-1+\mu})$ as $s \rightarrow 0$ with $\tau \leq 0$, which contradicts the result $u_- \in A_\mu(-\infty, 0]$.

We now have $a_0 = 0$, $w = 0$ and hence $\tilde{\zeta}_+ = 0$; by Corollary 3.2, the rotational part (u_1, v_1) of the vector field (u^o, v^o) vanishes. The irrotational part (u_0, v_0) , introduced in Theorem 2.2, is now harmonic in $\mathbf{R}^2 \setminus \bar{P}$, continuous on $\mathbf{R}^2$, zero on $\bar{P}$, and $o(r^{1/2})$ near infinity. Either applying the Phragmén-Lindelöf theorem for harmonic functions (Protter and Weinberger 1967, pp.94-96) to u_0 and v_0 , or applying the Phragmén-Lindelöf theorem for holomorphic functions to $u_0 - iv_0$, we conclude that (u_0, v_0) vanishes.

(b) Equation (5.8) and Theorem 3.4 imply that

$$\frac{s \tilde{\zeta}_+(s)}{(s+i\kappa)^{\beta/\pi} E(s,\kappa)} = -i M e^{-i\beta} (s-i\lambda)^{1/2-\beta/\pi} F(s,\lambda) s_-^{3/2} \tilde{v}_-(s) \quad \text{for } s \in \mathbf{R} \setminus \{0\}.$$

We proceed essentially as in (a). The left-hand member is now like $s_+^{1/2}$ times the previous one; the right-hand member is like $s_-^{1/2}$ times the previous one. Each is seen a priori to tend to zero as $s \rightarrow 0$ in the relevant half-plane. The orders in (5.9) are increased by a factor $s^{1/2}$, but this leads again to an entire function with values $a_0 + a_1 s$, then to $a_1 = 0$, and finally to $a_0 = 0$. $\square$

5.3. The component v_1 of the rotational field when $\zeta(\cdot, 0\pm) \in A_\mu$

Let (u_0, v_0) be the irrotational part (introduced in Theorem 2.2) of (u^o, v^o) or of (u^e, v^e) . In §5.4, we shall analyze this irrotational field by concentrating on v_0 ; in order to have knowledge of v_0 near $\bar{P}$ and near infinity, we need knowledge of v_1 there. We recall from Theorem 2.2 and (2.15) that $v_1 = K * \zeta$ and is continuous in $\mathbf{R}^2 \setminus \bar{P}$; we shall see that, under condition (A), v_1 has a certain continuity as $y \rightarrow 0+$ or

$y \rightarrow 0-$ with $x \geq 0$. The difference between the original plate problem and its homogeneous form is irrelevant in this §5.3.

Given $\tilde{\zeta}_0 := \tilde{\zeta}(\cdot, 0+)$ as in Theorem 3.1, and given condition (A), we define

$$V(x, y) := \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-N}^N e^{-ix\sigma + yk(\sigma)} \tilde{K}(\sigma) \tilde{\zeta}_0(\sigma) d\sigma, \\ x \in \mathbf{R}, \quad y \geq 0, \quad \zeta_0 \in A_\mu. \quad (5.10)$$

The essential difference between the v_1 in (4.4) and the present V is that in (4.4) we had $\tilde{K}(\sigma) \tilde{\zeta}_0(\sigma) = O(|\sigma|^{-3/2+\beta/\pi})$ as $\sigma \rightarrow \pm\infty$, so that $\tilde{K} \tilde{\zeta}_0$ was in $L_1(\mathbf{R})$, whereas now it is known only that $\tilde{K}(\sigma) \tilde{\zeta}_0(\sigma) = o(|\sigma|^{-1})$ as $\sigma \rightarrow \pm\infty$.

LEMMA 5.5. (a) For $x < by$, $V(x, y) = 0$.

(b) The function $V(\cdot, y) \in L_{p'}(\mathbf{R})$ for

$$\frac{1}{\mu} < p' < \infty, \text{ equivalently, for } 1 < p < \frac{1}{1-\mu} \text{ (where } \frac{1}{p} + \frac{1}{p'} = 1), \quad (5.11)$$

and $\|V(\cdot, y)\|_{L_{p'}(\mathbf{R})}$ is bounded independently of y .

(c) The Fourier-Laplace transform of V is

$$\tilde{V}(s, y) = e^{k(s)y} \tilde{K}(s) \tilde{\zeta}_0(s), \quad \tau \geq 0, \quad s \neq 0; \quad (5.12)$$

hence V is the function v_1 , in $\{y > 0\}$, of a weak solution with $\tilde{\zeta}(\cdot, 0+) = \tilde{\zeta}_0$.

Proof. (a) This results from contour integration and Jordan's lemma.

(b), (c) Let

$$f(\sigma, y) := e^{yk(\sigma)} \tilde{K}(\sigma) \tilde{\zeta}_0(\sigma), \quad \sigma \in \mathbf{R} \setminus \{0\}, \quad y \geq 0.$$

Using inequalities in Theorem A.6, Lemma A.2 and (5.5), we find that $f(\cdot, y) \in L_p(\mathbf{R})$, with norm bounded independently of y , for the values of p in (5.11). The result (b), and (5.12) for $\tau = 0$, follow from the L_p theory ($1 < p < 2$) of the Fourier transform (Butzer and Nessel 1971, pp. 208-217; Titchmarsh 1948, chapter IV). One obtains (5.12) for $\tau > 0$ by using the transform of the convolution having values

$$\int_{-\infty}^{\infty} f(\rho, y) \frac{\tau}{(\sigma - \rho)^2 + \tau^2} d\rho \quad (\tau > 0).$$

Corollary 3.2 and the injectivity of the operator $(\tilde{\cdot})$ imply the final remark in (c). $\square$

LEMMA 5.6. *Let $V_0 := V(\cdot, 0)$. Then $V_0 = K * \zeta_0$. Moreover,*

- (a) $V_0 \in C(\mathbf{R})$ [which implies that $V_0(0) = 0$, by Lemma 5.5(a)];
- (b) $V_0 \in \{L_1(\mathbf{R}) \cap L_\infty(\mathbf{R})\} + B_\mu$ [which implies that $V_0 \in L_{p'}(\mathbf{R})$ for $1/\mu < p' \leq \infty$]. Here functions in B_μ are defined to be zero in $(-\infty, 0)$.

Proof. Since $\zeta_0 \in A_\mu$, we have

$$K * \zeta_0 = K * \zeta_1 + K * \zeta_2 \quad \text{with } \zeta_1 \in L_1 \text{ and } \zeta_2 \in B_\mu. \quad (5.13)$$

The theory of the Fourier-Laplace transform acting on L_1 and on B_μ then ensures that $(K * \zeta_0)^\sim = \tilde{K} \tilde{\zeta}_0$ in $\{\tau \geq 0, s \neq 0\}$. Using (5.12) (and the injectivity of the transform), we conclude that $K * \zeta_0 = V_0$.

(a) That $K * \zeta_0 \in C(\mathbf{R})$ is proved as follows: $K * \zeta_1 \in C(\mathbf{R})$ because $K \in L_\infty$ and $\zeta_1 \in L_1$, while $K * \zeta_2 \in C(\mathbf{R})$ because $K \in L_p$ and $\zeta_2 \in L_{p'}$ whenever $1/\mu < p' \leq \infty$. (If necessary, see Butzer and Nessel 1971, p.4.)

(b) Young's inequality for convolutions shows that $K * \zeta_1 \in L_1 \cap L_\infty$, since $K \in L_1 \cap L_\infty$, and $K * \zeta_2 \in B_\mu$ because $K * (\cdot)$ is a bounded linear operator from B_μ into itself (Fraenkel 1990 b). $\square$

We are estimating $V = v_1$ because the definition of weak solution concerns $v_0 + v_1$. Since v_0 is a harmonic function, it has the property that $v_0(x_0, y_0)$ equals the mean value of v_0 over any ball (or disk) in $\mathbf{R}^2 \setminus \bar{P}$ with centre (x_0, y_0) . It will be sufficient to estimate such means of V .

We define a *mean-value operator* $\mathcal{M}$ by

$$(\mathcal{M}f)(z_0, \rho) := \frac{1}{\pi \rho^2} \iint_{\mathcal{B}(z_0, \rho)} f(x, y) dx dy, \quad (5.14)$$

where $z = (x, y)$ and the ball $\mathcal{B}(z_0, \rho) := \{z \mid |z - z_0| < \rho\}$ is in $\mathbf{R}_2 \setminus \bar{\mathcal{P}}$; here $\rho \in (0, \infty)$.

LEMMA 5.7. *There are constants $C_2(p)$ and $C_3(p)$, depending only on K, μ and p , as follows.*

(a) *Whenever $\mathcal{B}(z_0, \rho) \subset \{y > 0\}$,*

$$|(\mathcal{M}V)(z_0, \rho)| \leq C_2(p) \|\zeta_0\| \rho^{-1/p'}, \quad \frac{1}{\mu} < p' < \infty. \quad (5.15)$$

(b) *For $x_0 \in \mathbf{R}$ and $0 < y_0 \leq 1$,*

$$|(\mathcal{M}V)(z_0, y_0) - V(x_0, 0)| \leq \delta(z_0, V) + C_3(p) \|\zeta_0\| y_0^c, \quad (5.16)$$

where

$$0 \leq \delta(z_0, V) \leq \max \{|V(t, 0) - V(x_0, 0)| \mid |t - x_0| \leq (M+b)y_0\},$$

$$c := \begin{cases} \frac{\alpha(p-1)}{p(1-\alpha)} & \text{if } \alpha < 1 \text{ and } 1 < p < \min\{\frac{1}{\alpha}, \frac{1}{1-\mu}\}, \\ \frac{1}{p} & \text{if } \alpha = 1 \text{ and } 1 < p < \frac{1}{1-\mu}. \end{cases}$$

Here $\|\zeta_0\| = \|\zeta_0\|_{A_\mu}$ and α is the exponent in (1.4f).

Proof. We sketch the derivation of (5.16) for the case $\alpha < 1$; the other two inequalities are found more easily. In order to estimate effectively, we transform to the co-ordinates $\xi = x - by$ and $\eta = by$ used in §2.4. Let

$$h(s) := \frac{1}{b} \{k(s) - ibs\}, \quad \tau \geq 0, \quad (5.17)$$

so that

$$\operatorname{Re} h(s) \leq 0, \quad h(s) = O(s^{1-\alpha}) \text{ as } s \rightarrow \infty \text{ with } \tau \geq 0, \quad (5.18)$$

by Theorem A.6. Then

$$V(x, y) - V(x - by, 0) = V_c(\xi, \eta) - V_c(\xi, 0) = \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-N}^N e^{-i\xi\sigma} g(\sigma, \eta) d\sigma,$$

where

$$g(\sigma, \eta) := \{e^{\eta h(\sigma)} - 1\} \tilde{K}(\sigma) \tilde{\zeta}_0(\sigma).$$

As was noted in the proof of Lemma 5.5, a bound for $g(\cdot, \eta)$ in $L_p(\mathbf{R})$ yields a bound for $V_c(\cdot, \eta) - V_c(\cdot, 0)$ in $L_{p'}(\mathbf{R})$. Now, by (5.18),

$$|e^{\eta h(\sigma)} - 1| \leq \begin{cases} \text{const. } \eta(1+\sigma)^{1-\alpha} & \text{for } 0 \leq \sigma \leq \eta^{-1/(1-\alpha)}, \\ 2 & \text{for } \sigma > \eta^{-1/(1-\alpha)}, \end{cases}$$

and similarly for $\sigma < 0$. Using Lemma A.2 to bound $\tilde{K}(\sigma)$, and (5.5) to bound $\tilde{\zeta}_0(\sigma)$, one obtains (for the values of p stated in the lemma)

$$\|V_c(\cdot, \eta) - V_c(\cdot, 0)\|_{L_p(\mathbf{R})} \leq \text{const. } \|\zeta_0\| \eta^a, \quad a := \frac{p-1}{p(1-\alpha)}.$$

Integrating $|V_c(\xi, \eta) - V_c(\xi, 0)|$ over the smallest parallelogram in the (x, y) -plane that contains $\mathcal{B}(z_0, y_0)$ and has sides $\{\xi = \text{const.}\}$ and $\{\eta = \text{const.}\}$, we apply Hölder's inequality to the integral with respect to ξ ; this gives the last term in (5.16). The term $\delta(z_0, V)$ is the mean value of $|V(x-by, 0) - V(x_0, 0)|$ over $\mathcal{B}(z_0, y_0)$, where $V(\cdot, 0) \in C(\mathbf{R})$ by Lemma 5.6 (a). $\square$

COROLLARY 5.8. (a) *We can so choose p and $\rho = \rho(y_0)$ in Lemma 5.7 that*

$$|(\mathcal{M}V)(z_0, \rho)| \leq \text{const. } \|\zeta_0\| \quad \text{whenever } y_0 > 0;$$

the constant depends only on K and μ .

$$(b) (\mathcal{M}V)(z_0, y_0) \rightarrow V(x_0, 0) \quad \text{as } y_0 \rightarrow 0+, \quad \text{for each } x_0 \in \mathbf{R}.$$

Proof. (a) Choose any p permitted in both parts of Lemma 5.7. Since $V(\cdot, 0) \in L_\infty(\mathbf{R})$ by Lemma 5.6(b), the inequality (5.16) gives

$$|(\mathcal{M}V)(z_0, y_0)| \leq 3 \|V(\cdot, 0)\|_{L_\infty(\mathbf{R})} + C_3(p) \|\zeta_0\| \leq \text{const. } \|\zeta_0\|, \quad 0 < y_0 \leq 1.$$

We use this inequality for $y_0 \leq 1$, and (5.15) with $\rho = 1$ for $y_0 > 1$.

(b) This is immediate from (5.16): the continuity of $V(\cdot, 0)$ ensures that $\delta(z_0, V) \rightarrow 0$. $\square$

5.4. The proof of Proposition 5.4

The proof takes the form of three more lemmas. For $u_0 - iv_0$, we use the notation introduced after (4.2).

LEMMA 5.9. *Let $\zeta^o(\cdot, 0+)$ and $\zeta^e(\cdot, 0+)$ be given, these functions being as in Theorem 3.1 and condition (A).*

(a) *In accord with Theorem 2.2, define $\psi_0 := \psi^o + H * (K * \zeta^o)$ and $(u_0, v_0) := (\partial_2 \psi_0, -\partial_1 \psi_0)$. There is exactly one such irrotational field; it is determined by*

$$(u_0 - iv_0)(z) = \frac{1}{\pi} \int_0^\infty \frac{(K * \zeta_0)(t)}{t - z} dt, \quad z \in \mathbb{C} \setminus \bar{P}, \quad (5.19)$$

where $\zeta_0 := \zeta^o(\cdot, 0+)$.

(b) *Now define $\psi_0 := \psi^e + H * (K * \zeta^e)$ and $(u_0, v_0) := (\partial_2 \psi_0, -\partial_1 \psi_0)$.*

Again there is exactly one such irrotational field; it is determined by

$$(u_0 - iv_0)(z) = \frac{1}{\pi} \int_0^\infty \frac{(K * \zeta_0)(t)}{t - z} \frac{z^{\frac{1}{2}}}{t^{\frac{1}{2}}} dt, \quad z \in \mathbb{C} \setminus \bar{P}, \quad (5.20)$$

where now $\zeta_0 := \zeta^e(\cdot, 0+)$ and $\theta \in (0, 2\pi)$.

Proof. The proof will be given for case (b). That for case (a) is similar; the formula (5.19) may be more familiar than (5.20); and for (u^o, v^o) the condition that u, v are $o(x^{\frac{1}{2}})$ at infinity in $\mathcal{A}$ could be weakened to $o(x)$.

Suppose then that $\zeta_0 = \zeta^e(\cdot, 0+)$ is given, and write $g := K * \zeta_0$. By Lemma 5.6,

$$g \in C[0, \infty) \cap L_{p'}(0, \infty) \quad \text{for} \quad \frac{1}{\mu} < p' \leq \infty, \quad g(0) = 0. \quad (5.21)$$

We claim that v_0 satisfies the following conditions.

$$\Delta v_0 = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \bar{P}; \quad (5.22a)$$

$$v_0(x, y) = o(r^{\frac{1}{2}}) \quad \text{at infinity in} \quad \mathbb{R}^2 \setminus \bar{P}; \quad (5.22b)$$

$$\text{as } y \rightarrow 0+ \text{ or } y \rightarrow 0- \text{ with } x \geq 0, \quad v_0(x,y) \rightarrow -g(x). \quad (5.22c)$$

Here (5.22a) is implied by Theorem 2.2. To prove the rest of (5.22), we note that $v_0 = v^e - v_1$, where $v_1 = K * \zeta^e$ and equals the V of Lemma 5.5 and Corollary 5.8 in $\{y>0\}$; both v_0 and v_1 are even functions of y by definition.

Now v_0 has the mean-value property noted before (5.14). Let $z = (x,y)$ and let $\rho = \rho(y)$ be as in Corollary 5.8(a); then

$$v_0(z) = (\mathcal{M}v_0)(z,\rho) = (\mathcal{M}v^e)(z,\rho) - (\mathcal{M}v_1)(z,\rho) = o(r^{\frac{1}{2}}) \text{ at infinity in } \mathbf{R}^2 \setminus \bar{P},$$

because $v^e \in C(\mathbf{R}^2)$ and satisfies (2.2a), while $(\mathcal{M}v_1)(z,\rho)$ is bounded. Also, as $y \rightarrow 0+$ or $y \rightarrow 0-$ with $x \geq 0$,

$$v_0(z) = (\mathcal{M}v_0)(z,|y|) = (\mathcal{M}v^e)(z,|y|) - (\mathcal{M}v_1)(z,|y|) \rightarrow -g(x)$$

because $v^e \in C(\mathbf{R}^2)$ and $v^e = 0$ on $\bar{P}$, while $(\mathcal{M}v_1)(z,|y|) \rightarrow g(x)$ by Corollary 5.8(b). There can be no discontinuity of v_0 at the origin because, in addition to (5.22c) and $g(0) = 0$, we have $v^e \in C(\mathbf{R}^2)$ and $v_1(x,y) = 0$ for $x < b|y|$.

Let v_d be the difference of two functions v_0 satisfying (5.22). Then v_d is harmonic in $\mathbf{R}^2 \setminus \bar{P}$, is $o(r^{\frac{1}{2}})$ at infinity there, is zero on $\bar{P}$ and continuous on $\mathbf{R}^2$. By the Phragmén-Lindelöf theorem for harmonic functions (Protter and Weinberger 1967, pp.94-96), $v_d = 0$.

The formula (5.20) results from the conformal transformation

$$X + iY = (x+iy)^{\frac{1}{2}} \quad [\arg(x+iy) = \theta \in (0, 2\pi)],$$

which maps $\mathbf{R}^2 \setminus \bar{P}$ onto $\{Y > 0\}$. Let $v_*(X,Y) := v_0(x,y)$ and $g_*(X) := g(x) = g(X^2)$ for all $X \in \mathbf{R}$. One checks without difficulty that, in view of (5.21),

$$g_* \in C(\mathbf{R}) \cap L_{p'}(\mathbf{R}) \quad \text{for } \frac{1}{\mu} < p' \leq \infty,$$

(that $g_* \in C(\mathbf{R})$ is immediate); then the Poisson integral for a half-plane determines v_* in terms of g_* . (See Butzer and Nessel 1971, p.126 or Titchmarsh 1948, pp. 133-139. Titchmarsh's proofs contain results beyond those in the statements of his theorems.)

Transforming back to the (x,y) -plane, we obtain the imaginary part of (5.20).

The definition of u_0 shows it to be a harmonic conjugate of $-v_0$; the Cauchy-Riemann equations and v_0 then determine u_0 up to an additive constant, and that constant is fixed here by the condition that $u_0 \rightarrow 0$ at infinity in $\mathcal{U}$. $\square$

LEMMA 5.10. (a) Let $u_-(x) := u^o(x,0) H(-x)$; then $u_- \in A_\mu(-\infty, 0]$.

(b) Let $v_-(x) := v^e(x,0) H(-x)$; then $v_- \in \{L_1 \cap L_\infty\} + B_\mu(-\infty, 0]$, where $L_p = L_p(-\infty, 0)$.

Proof. Again we prove the result (b), because (a) is slightly easier. Since $v_1 = 0$ in $\mathcal{U}$ (both by Theorem 2.2 and by Lemma 5.5), we have $v_-(x) = v_0(x,0)$ for $x < 0$. Let $w := (v_-)_r$ denote the reflection of v_- ; then (5.20) implies that

$$w(r) = -\frac{1}{\pi} \int_0^\infty \frac{g(t)}{r+t} \frac{r^{\frac{1}{2}}}{t^{\frac{1}{2}}} dt, \quad r > 0, \quad (5.23)$$

where

$$g := K * \zeta^e(\cdot, 0+) \in \{L_1 \cap L_\infty\} + B_\mu,$$

by Lemma 5.6; here $L_p = L_p(0, \infty)$ and $B_\mu \subset L_p$ for $1/\mu < p \leq \infty$. Note first that

$$|w(r)| \leq \|g\|_{L_\infty} \frac{1}{\pi} \int_0^\infty \frac{1}{r+t} \frac{r^{\frac{1}{2}}}{t^{\frac{1}{2}}} dt = \|g\|_{L_\infty}.$$

Define $w_a(r) := w(r) H(1-r)$ and $w_b := w - w_a$; then $w_a \in L_1 \cap L_\infty$ (because $w \in L_\infty$ and $\text{supp } w_a \subset [0, 1]$), and it suffices to prove that $w_b \in B_\mu$. Define

$$A(\rho, t) := \frac{1}{\pi} \frac{\rho^{\frac{1}{2}}}{\rho+t}, \quad \rho \geq 1, \quad t \geq 0.$$

Since $w_b \in C^1[1, \infty)$ and Fubini's theorem applies in what follows, we have for $r \geq 1$

$$\begin{aligned} V_r^\infty w_b &= \int_r^\infty |w'(\rho)| d\rho \quad (r \geq 1) \\ &\leq \int_r^\infty d\rho \int_0^\infty |A_\rho(\rho, t)| |g(t)| t^{-\frac{1}{2}} dt \\ &= \int_0^\infty |g(t)| t^{-\frac{1}{2}} dt \int_r^\infty |A_\rho(\rho, t)| d\rho \\ &= \frac{1}{\pi} \int_0^r \frac{|g(t)|}{r+t} \frac{r^{\frac{1}{2}}}{t^{\frac{1}{2}}} dt + \frac{1}{\pi} \int_r^\infty \frac{r+t-(rt)^{\frac{1}{2}}}{r+t} |g(t)| \frac{dt}{t}. \end{aligned} \quad (5.24)$$

Here $g = g_1 + g_2$ with $g_1 \in L_1 \cap L_\infty$ and $g_2 \in B_\mu$; also, it is convenient to write the integral over $(0, r)$ as one over $(0, 1)$ plus one over $(1, r)$; we display only the more critical bounds.

$$\begin{aligned} \int_1^r \frac{|g_2(t)|}{r+t} \frac{r^{\frac{1}{2}}}{t^{\frac{1}{2}}} dt &\leq \|g_2\|_{B_\mu} \int_1^r \frac{1}{(1+t)^\mu (r+t)} \frac{r^{\frac{1}{2}}}{t^{\frac{1}{2}}} dt \\ &\leq \|g_2\|_{B_\mu} r^{-\mu} \int_0^1 \frac{\theta^{-\frac{1}{2}-\mu}}{1+\theta} d\theta \quad (t=r\theta), \\ \int_r^\infty \frac{r+t-(rt)^{\frac{1}{2}}}{r+t} |g_2(t)| \frac{dt}{t} &\leq \int_r^\infty |g_2(t)| \frac{dt}{t} \\ &\leq \|g_2\|_{B_\mu} \int_r^\infty (1+t)^{-\mu} \frac{dt}{t} \\ &\leq \|g_2\|_{B_\mu} 2(1+r)^{-\mu}/\mu, \end{aligned}$$

since $t \geq \frac{1}{2}(1+t)$ for $t \geq r \geq 1$. Thus $V_r^\infty w_b \leq \text{const. } (1+r)^{-\mu}$ for $r \geq 1$, and $V_r^\infty w_b = 2V_1^\infty w_b$ for $r \in [0, 1)$. The integrand in (5.24) dominates the modulus of that in (5.23); therefore $w_b(r) \rightarrow 0$ as $r \rightarrow \infty$, and $w_b \in B_\mu$. $\square$

The formula (5.20) has now served its main purpose; for the final stage of the proof we use the Hilbert transform to pass from v_0 to u_0 , on the whole x -axis, for the irrotational parts of both (u^o, v^o) and (u^e, v^e) .

The *Hilbert transform* T may be defined by

$$(Tf)(x) := \lim_{y \rightarrow 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t-x}{(t-x)^2 + y^2} f(t) dt, \quad x \in \mathbf{R}, \quad (5.25)$$

and $T : L_p(\mathbf{R}) \rightarrow L_p(\mathbf{R})$ is a bounded linear operator if $1 < p < \infty$ (Butzer and Nessel 1971, pp. 312-313; Titchmarsh 1948, pp. 132-138).

Denote by $B_\mu(\mathbf{R})$ the space of functions $f : \mathbf{R} \rightarrow \mathbf{R}$ the restrictions of which to $(-\infty, 0]$ and to $[0, \infty)$ are in $B_\mu(-\infty, 0]$ and B_μ , respectively. Then (by the L_p property of the transform) $T : B_\mu(\mathbf{R}) \rightarrow L_p(\mathbf{R})$ is a bounded linear operator if $1/\mu < p < \infty$. Note that this p was called p' in Lemma 5.5(b).

For our purposes, the *signum rule* for the Fourier transform of Tf is

$$(Tf)^\sim(\sigma) = -i \operatorname{sgn} \sigma \tilde{f}(\sigma) \quad \text{a.e. on } \mathbf{R} \quad \text{if } f \in L_p(\mathbf{R}), \quad 1 < p \leq 2, \quad (5.26a)$$

$$\text{on } \mathbf{R} \setminus \{0\} \quad \text{if } f \in B_\mu(\mathbf{R}). \quad (5.26b)$$

For the first case, see Butzer and Nessel (1971), p.324; for the second, see the paper on B_μ (Fraenkel 1990b). [In §3.2, the signum rule was used in the following way. Let $h_1 + ih_2$ be holomorphic in $\{y > 0\}$; we said, in effect, that the Cauchy-Riemann equation $\partial_1 h_1 = \partial_2 h_2$ and the formula, used in (3.12),

$$\tilde{h}_2(\sigma, y) = \tilde{h}_2(\sigma, 0+) e^{-q(\sigma)y} \quad (y > 0, q(\sigma) = \sigma \operatorname{sgn} \sigma)$$

combine to $-i\sigma \tilde{h}_1(\sigma, 0+) = -\sigma \operatorname{sgn} \sigma \tilde{h}_2(\sigma, 0+).$]

We can now complete the proof of Proposition 3.4.

LEMMA 5.11. *Equations (5.7) and (5.8) are valid.*

Proof. (a) Consider the weak solution (u^σ, v^σ) of the homogeneous problem. Equations (5.19), (5.25) and (5.26) imply that

$$\tilde{u}_0(\sigma, 0\pm) = -i \operatorname{sgn} \sigma \tilde{K}(\sigma) \tilde{\zeta}_0(\sigma), \quad \sigma \in \mathbf{R} \setminus \{0\},$$

because $K * \zeta_0 \in \{L_1 \cap L_\infty\} + B_\mu$ by Lemma 5.6 (and, by interpolation, $L_1 \cap L_\infty \subset L_p$ whenever $1 < p \leq 2$). The equation holds on $\mathbf{R} \setminus \{0\}$ because $\tilde{K} \tilde{\zeta}_0$ is continuous there.

In addition, $u_0 + u_1 = 0$ on $\bar{P}$, and $u_0(x, 0) = u_-(x)$ for $x < 0$ (because $u_1 = 0$ in $\mathcal{U}$ by Theorem 2.2). Under condition (A), the results in Corollary 3.2 extend to $\{\tau \geq 0, s \neq 0\}$ because the functions in (3.8) are continuous on this set. Accordingly,

$$\tilde{u}_0(\sigma, 0\pm) = \frac{i}{\sigma} \tilde{K}(\sigma) \tilde{\zeta}_0(\sigma) k(\sigma) + \tilde{u}_-(\sigma), \quad \sigma \in \mathbf{R} \setminus \{0\}.$$

Equating these two expressions for $\tilde{u}_0(\sigma, 0\pm)$ and manipulating a little, we obtain (5.7).

(b) Now consider the weak solution (u^e, v^e) . In view of the properties of its irrotational part (u_0, v_0) , established in Lemmas 5.9 and 5.10, we can replace (5.20) by

$$(u_0 - iv_0)(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(t)}{t-z} dt, \quad y > 0, \quad (5.27a)$$

where

$$g := \begin{cases} K * \zeta_0 & \text{on } [0, \infty), \\ -v_- & \text{on } (-\infty, 0), \end{cases} \quad (5.27b)$$

so that $g \in \{L_1(\mathbf{R}) \cap L_\infty(\mathbf{R})\} + B_\mu(\mathbf{R})$. Equations (5.27), (5.25) and (5.26) imply that

$$\tilde{u}_0(\sigma, 0+) = -i \operatorname{sgn} \sigma \{ \tilde{K}(\sigma) \tilde{\zeta}_0(\sigma) - \tilde{v}_-(\sigma) \}, \quad \sigma \in \mathbf{R} \setminus \{0\}.$$

In addition, $u_0 + u_1 = 0$ on $\bar{P}$, $u_0(x, 0) = 0$ for $x < 0$ by (5.20), and $u_1 = 0$ in $\mathcal{U}$; by the extended form of Corollary 3.2,

$$\tilde{u}_0(\sigma, 0+) = \frac{i}{\sigma} \tilde{K}(\sigma) \tilde{\zeta}_0(\sigma) k(\sigma), \quad \sigma \in \mathbf{R} \setminus \{0\}.$$

Equating these two expressions for $\tilde{u}_0(\sigma, 0+)$ and manipulating a little, we obtain (5.8). $\square$

6. Examples of regularity and irregularity

Throughout this section we consider the solutions in §4; the emphasis is on examples because a general regularity theory is beyond my powers. We shall see that properties of solutions, beyond those established in §4, are sensitive to properties of K beyond those specified in (1.4). For example, making the Maxwell kernel K_1 less smooth by adding a Renardy term makes the solution much smoother away from $\bar{P}$. On the other hand, if we add to K_1 a simple discontinuity (finite jump) of K' at some point $c > 0$, then the solution becomes decidedly less smooth in $\mathcal{X}$. Therefore a general regularity theory (specifying conditions on K that make the weak solution in §4 a pointwise one, and possibly smoother) would have to be long, elaborate and probably difficult. The following conditions and examples, while far from exhaustive, at least indicate some important properties of solutions.

Notation. (a) Non-dimensional co-ordinates are used in this §6, with the possible exception of item (vii) (where they are not necessary, but do no harm). With T denoting a relaxation time of the elastic stress, as in (1.5) to (1.7), we define

$$x_* = x/UT, \quad y_* = y/UT, \quad K_*(x_*) = K(UTx_*),$$

and then omit the subscript $*$. Thus the Maxwell kernel becomes $K_1(x) = M^{-2} e^{-x}$ for $x \geq 0$.

(b) It will be convenient to use the abbreviation

$$p := e^{-i\pi/2} s_+ \quad (-\pi < \arg p < \pi), \quad (6.1)$$

and to let J be a generic symbol (possibly having a different meaning each time that it appears) for any function of the form

$$J(s) = 1 + a_1 p^{-\alpha_1} + \dots + a_n p^{-\alpha_n}, \quad 0 < \alpha_1 < \dots < \alpha_n,$$

where all the coefficients a_j and exponents α_j are real, and in (6.4a), for example, $1 + \alpha + \alpha_n < 3$.

(c) As in §2.4 and in condition (B) of §5.2, the form $\omega(\xi, \eta)$ of the vorticity will be considered; correspondingly, we have in (4.4)

$$e^{-ixs+|y|k(s)} = e^{-i\xi s+|\eta|h(s)}, \quad (6.2)$$

where h is as in (5.17) and (5.18).

(d) Since $\omega(\xi, \cdot)$ is an odd function, we shall restrict attention to ω on $\mathbf{R} \times [0, \infty)$, defining $\text{sgn } \eta = 1$ for $\eta \geq 0$. (This avoids repeated mention of limiting values as $\eta \rightarrow 0+$.)

(i) *Existence of the vorticity as a genuine function.* Any condition allowing differentiation of the integrals defining u_1 and v_1 in (4.4) cannot be simple; see Remarks 2 and 3 below. In order to relate the condition adopted here to concrete examples, we perturb the kernels K_1 to K_3 in §1.2 by adding to their graphs what we shall call a *kink*:

$$K(x) = K_j(x) + [K']f(x-c), \quad j=1,2 \text{ or } 3, \quad (6.3a)$$

where

$$[K'] := K'(c+) - K'(c-) \geq 0, \quad c > 0, \quad f(t) = te^{-2t}H(t) \text{ (say)}, \quad (6.3b)$$

and where $[K']$ is so small that K remains convex on $[0, \infty)$. Then the Fourier-Laplace transform of K satisfies the following condition.

(I). As $s \rightarrow \infty$ with $\tau \geq 0$,

$$\tilde{K}(s) = M^{-2}p^{-1} - \gamma_\alpha p^{-1-\alpha}J(s) + [K']e^{ics}p^{-2} + O(s^{-3}), \quad (6.4a)$$

where

$$\gamma_\alpha > 0, \quad \alpha \in (0, 1], \quad [K'] \geq 0, \quad c > 0. \quad (6.4b)$$

Here α is the exponent in (1.4f) and $\gamma_\alpha = \alpha! \lim_{x \rightarrow 0+} \frac{K(0)-K(x)}{x^\alpha}$.

THEOREM 6.1. *Let condition (I) hold, and, as in §4, let*

$$\tilde{\zeta}(s, 0+) = -iC_1 s_+^{-\frac{1}{2}} (s+i\kappa)^{\beta/\pi} E(s, \kappa). \quad (6.5)$$

Then the vorticity of the weak solution in §4 is given by

$$\omega(\xi, \eta) = \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-N}^N e^{-i\xi s + \eta h(s)} \tilde{\zeta}(s, 0+) ds, \quad \eta \geq 0, \quad (6.6)$$

for $\xi \neq 0$ if $[K'] = 0$, and for $\xi \notin \{0, c, 2c, 3c, \dots\}$ if $[K'] > 0$. Let G denote the union of open intervals to which ξ has been restricted; then ω is continuous on $G \times [0, \infty)$.

Remarks. 1. Condition (I) obviously holds for shear kernels K other than those in (6.3); a sufficient condition on K itself is stated in Appendix D, item (ii).

2. It follows from condition (I) by direct calculation that, as $s \rightarrow \infty$ with $\tau \geq 0$,

$$h(s) = -\Gamma_\alpha p^{1-\alpha} J(s) + h_1 e^{ics} J(s) + O(s^{-1}), \quad (6.7a)$$

where

$$\Gamma_\alpha := \frac{1}{2} \frac{M^4}{b^2} \gamma_\alpha > 0, \quad h_1 := \frac{1}{2} \frac{M^4}{b^2} [K'] \geq 0. \quad (6.7b)$$

The proof of Theorem 6.1, summarized in Remark 3, requires exact specification of the part of $h(s)$ that does not tend to zero as $s \rightarrow \infty$; it is for this reason that a rather long asymptotic approximation to $\tilde{K}(s)$ is needed in condition (I). Since $O(s^{-1})$ is the natural error term in (6.7a) when $\alpha = 1$, we have chosen to keep that error term for all cases. The proof also requires exact knowledge of that part of $\tilde{\zeta}(s, 0+)$ which is not absolutely integrable at infinity, but for this a shorter approximation to $\tilde{K}(s)$ suffices.

3. In (6.6) we have $\tilde{\zeta}(s, 0+) \sim -iC_1 s_+^{-\frac{1}{2} + \beta/\pi}$ as $s \rightarrow \infty$; therefore only conditional convergence is possible, in the sense of an improper Riemann integral, unless $\alpha < 1$ and η is bounded away from zero. The proof of Theorem 6.1 amounts to a sequence of estimates establishing uniform convergence of the integral in (6.6) on any compact subset of $G \times [0, \infty)$. I found it necessary to consider separately the cases $[K'] = 0$ and $[K'] > 0$, and, when proving some of the results below, to consider separately the cases $\alpha = 1$ and $\alpha \in (0, 1)$.

Even when $\alpha < 1$, so that $\text{Re } p^{1-\alpha} > 0$ on $\{\tau \geq 0, s \neq 0\}$ and

$$\exp(-\eta \Gamma_\alpha p^{1-\alpha}) \rightarrow 0 \text{ exponentially as } \sigma \rightarrow \pm \infty \text{ if } \eta > 0, \quad (6.8)$$

uniform convergence on each compact subset of $G \times [0, \infty)$ cannot be proved if we

know only that $h(s) = -\Gamma_\alpha p^{1-\alpha} + O(1)$.

(ii) *The singularity at the shock S in the simplest cases.* Let

$$\alpha = 1, \quad J(s) = 1 \text{ and } [K'] = 0 \text{ in condition (I).} \quad (6.9)$$

For example, the Maxwell kernel K_1 and the long-tailed kernel K_3 satisfy these conditions. Then

$$h(s) = h_0 + h_*(s), \quad h_0 := \frac{1}{2} \frac{M^4}{b^2} K'(0+) < 0, \quad (6.10)$$

$$\left. \begin{array}{l} h_*(s) = O(s^{-1}) \text{ as } s \rightarrow \infty \text{ with } \tau \geq 0, \\ h_* \text{ is holomorphic in } \{\tau > 0\} \text{ and continuous on } \{\tau \geq 0\}. \end{array} \right\} \quad (6.11)$$

It is not difficult to deduce the following result from Theorem 6.1; in order to bound $\omega_2(\xi, \eta)$ for $\xi \rightarrow 0+$, one can use $\{\tau = \text{const.} = 1/\xi\}$ as a path of integration.

COROLLARY 6.2. *Given condition (I) and (6.9), we have $\omega = \omega_0 + \omega_1$, where*

$$\omega_0(\xi, \eta) := - \frac{|C_1|}{(-\frac{1}{2} - \beta/\pi)!} \xi^{-\frac{1}{2} - \beta/\pi} e^{h_0 \eta} H(\xi), \quad \eta \geq 0, \quad (6.12)$$

and ω_1 is continuous on $\mathbf{R} \times [0, \infty)$, bounded on $\mathbf{R} \times [0, \eta_1]$ for any $\eta_1 \in (0, \infty)$, and $O(\xi^{\frac{1}{2} - \beta/\pi} \log \xi)$ as $\xi \rightarrow 0+$ with $\eta \in [0, \eta_1]$.

(iii) *The Renardy effect.* Now let $\alpha \in (0, 1)$. Then in (6.7a) the term $-\Gamma_\alpha p^{1-\alpha}$ is the dominant part of $h(s)$ as $\sigma \rightarrow \pm\infty$; for $\eta > 0$, the integral in Theorem 6.1 converges exponentially at infinity, as was noted in (6.8). The same is true for the integrals defining u_1 and v_1 in (4.4), and for the integral representing any derivative of u_1 or v_1 . Recalling that u_1 and v_1 vanish on $\overline{\mathcal{U}}$, and that u_0 and v_0 are real-analytic in $\mathbf{R}^2 \setminus \overline{P}$, we have

(c) As in §2.4 and in condition (B) of §5.2, the form $\omega(\xi, \eta)$ of the vorticity will be considered; correspondingly, we have in (4.4)

$$e^{-ixs+|y|k(s)} = e^{-i\xi s+|\eta|h(s)}, \quad (6.2)$$

where h is as in (5.17) and (5.18).

(d) Since $\omega(\xi, \cdot)$ is an odd function, we shall restrict attention to ω on $\mathbf{R} \times [0, \infty)$, defining $\text{sgn } \eta = 1$ for $\eta \geq 0$. (This avoids repeated mention of limiting values as $\eta \rightarrow 0+$.)

(i) *Existence of the vorticity as a genuine function.* Any condition allowing differentiation of the integrals defining u_1 and v_1 in (4.4) cannot be simple; see Remarks 2 and 3 below. In order to relate the condition adopted here to concrete examples, we perturb the kernels K_1 to K_3 in §1.2 by adding to their graphs what we shall call a *kink*:

$$K(x) = K_j(x) + [K']f(x-c), \quad j=1,2 \text{ or } 3, \quad (6.3a)$$

where

$$[K'] := K'(c+) - K'(c-) \geq 0, \quad c > 0, \quad f(t) = te^{-2t}H(t) \text{ (say)}, \quad (6.3b)$$

and where $[K']$ is so small that K remains convex on $[0, \infty)$. Then the Fourier-Laplace transform of K satisfies the following condition.

(I). As $s \rightarrow \infty$ with $\tau \geq 0$,

$$\tilde{K}(s) = M^{-2}p^{-1} - \gamma_\alpha p^{-1-\alpha}J(s) + [K']e^{ics}p^{-2} + O(s^{-3}), \quad (6.4a)$$

where

$$\gamma_\alpha > 0, \quad \alpha \in (0, 1], \quad [K'] \geq 0, \quad c > 0. \quad (6.4b)$$

Here α is the exponent in (1.4f) and $\gamma_\alpha = \alpha! \lim_{x \rightarrow 0+} \frac{K(0)-K(x)}{x^\alpha}$.

THEOREM 6.1. *Let condition (I) hold, and, as in §4, let*

$$\tilde{\zeta}(s, 0+) = -iC_1 s_+^{-1} (s+i\kappa)^{\beta/\pi} E(s, \kappa). \quad (6.5)$$

Then the vorticity of the weak solution in §4 is given by

COROLLARY 6.3. Under condition (I), or a weaker form of it with error $o(s^{-1-\alpha})$,

$$u, v \in C^\infty(\mathbb{R}^2 \setminus \bar{P}) \text{ whenever } \alpha \in (0, 1).$$

(iv) *The effect of a kink.* The next theorem describes the simplest example of this effect. The proof, which really forms a part of the proof of Theorem 6.1, amounts to justifying term-by-term integration when the series

$$\exp(\eta h_1 e^{ics}) = \sum_{n=0}^{\infty} e^{incs} \frac{(h_1 \eta)^n}{n!}$$

is used in the integral for ω in Theorem 6.1. Fortunately, given ξ , one can dispose of the sum of the terms with $n > \xi/c$ by integration along the semi-circle $\{|s| = N, \tau > 0\}$ and appeal to Jordan's lemma.

THEOREM 6.4. Let $\alpha = 1$, $J(s) = 1$ and $[K'] > 0$ in condition (I), so that

$$h(s) = h_0 + h_1 e^{ics} + h_*(s),$$

where

$$h_0 = \frac{1}{2} \frac{M^4}{b^2} K'(0+) < 0, \quad h_1 = \frac{1}{2} \frac{M^4}{b^2} [K'] > 0,$$

and h_* is as in (6.11). Define

$$Z(\xi, \eta) := \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-N}^N e^{-i\xi s + \eta h_0 + \eta h_*(s)} \tilde{\zeta}(s, 0+) ds, \quad \eta \geq 0. \quad (6.13)$$

Then $Z = \omega_0 + Z_1$, where ω_0 is as in Corollary 6.2 and Z_1 has the properties of ω_1 there; moreover, for $\xi \notin \{0, c, 2c, 3c, \dots\}$ and $\eta \geq 0$,

$$\omega(\xi, \eta) = Z(\xi, \eta) + \sum_{n=1}^{\infty} Z(\xi - nc, \eta) \frac{(h_1 \eta)^n}{n!}, \quad (6.14)$$

where the terms with $n > \xi/c$ are zero because $Z(t, \eta) = 0$ for $t < 0$.

Remarks. 4. The theorem gives our first example of a shear kernel for which the weak solution cannot be a pointwise one. Since

$$Z(t, \eta) \sim \omega_0(t, \eta) \rightarrow -\infty \text{ like } -t^{-\frac{1}{2}-\beta/\pi} \text{ as } t \rightarrow 0+, \quad (6.15)$$

it follows from (6.14) that $\omega(\xi, \eta) \rightarrow -\infty$ whenever $\xi \rightarrow nc+$ with $\eta > 0$ and $n \in \{0, 1, 2, \dots\}$. Thus u and v are not in $C^1(\mathcal{L})$.

There are (more complicated) versions of the theorem for the general form of condition (I) and for more than one kink.

5. For the first Stokes problem, mentioned in §1.2, Joseph and Renardy (1990) have found a result analogous to (6.14) in that a kink of the shear relaxation function induces a singularity on a sequence of equally spaced characteristic lines. In their case, the singularity is a finite jump of velocity (there being only one component of velocity).

(v) *Some shear kernels for which condition (A) holds.* Condition (I) has enabled us to discuss the singularities (or smoothness) of the vorticity. To verify condition (A) in §5.2, one must also estimate the plate-vorticity function $\omega(\xi, 0+)$ for $\xi \rightarrow \infty$, and this requires additional information about $\tilde{K}$. On the other hand, for $s \rightarrow \infty$ a shorter asymptotic approximation to $\tilde{K}(s)$ serves, because $h(s)$ need not be considered. Here is a sufficient condition.

(II). (a) *The transform $\tilde{K}$ has an analytic continuation that is holomorphic on*

$$D_1 := \{s | \tau \geq -\delta_1\} \setminus \{i\tau | \tau \leq 0\} \text{ for some } \delta_1 > 0;$$

(b) *for $s \in D_1$ and $|s| \leq \delta_1$,*

$$\tilde{K}(s) = \|K\| \{1 - a_l p^l + O(s)\}, \quad a_l \geq 0, \quad l \in (0, 1), \quad (6.16)$$

where a_l and l are real constants;

(c) *as $s \rightarrow \infty$ in D_1 ,*

$$\tilde{K}(s) = M^{-2} p^{-1} - \gamma_\alpha p^{-1-\alpha} J(s) + O(s^{-2}), \quad (6.17)$$

where $\gamma_\alpha \geq 0$ and $\alpha \in (0, 1)$.

Note that the hypotheses about γ_α and α differ from those in condition (I): if $\alpha = 1$ in condition (I), and if that condition extends to $\{\tau \geq -\delta_1\}$, then the term $-\gamma_1 p^{-2} J(s)$ in condition (I) is now absorbed in the error term of (6.17).

THEOREM 6.5. *If condition (II) holds, then for the solution in §4 the function $\tilde{\zeta}(\cdot, 0+)$ in Theorem 3.1 and in (6.5) is the Fourier-Laplace transform of a function in A_μ .*

Remarks. 6. Condition (II) does not imply that the functions $\tilde{K}$ and W [recall that $W(s) := is + 1/\tilde{K}(s)$] have no zero in D_1 , but it does imply existence of a number $\delta_0 \in (0, \delta_1]$ such that $\tilde{K}$ and W have no zero in the corresponding set D_0 ; then k , h and $E(\cdot, \kappa)$ are holomorphic on D_0 . To estimate $\omega(\xi, 0+)$ for $\xi \rightarrow \infty$ by means of (6.6), one can deform the path of integration to one along the boundary ∂D_0 , both sides of the cut $\{i\tau | -\delta_0 \leq \tau \leq 0\}$ being taken into account.

7. I am unable to give a useful, general condition on K itself that implies condition (II), but here are some examples.

(a) The functions K in (6.3) certainly satisfy condition (II); their transforms $\tilde{K}$ are given in Appendix D, item (iii).

(b) Let K be regularly piecewise affine (Definition A.3) with values

$$K(n) = M^{-2} a^n \text{ for } n = 0, 1, 2, \dots, \quad a = \text{const.} \in (0, 1), \quad (6.18)$$

the graph of K being a straight line on each interval $[n, n+1]$. Then $\tilde{K}$ has an analytic continuation given by

$$\tilde{K}(s) = M^{-2} \left\{ \frac{i}{s} + \frac{(1-a)(1-e^{is})}{1-ae^{is}} \frac{1}{s^2} \right\}; \quad (6.19)$$

the poles at $\pm 2n\pi - i \log(1/a)$, $n \in \{0, 1, 2, \dots\}$, are the only singularities. (Near the origin, the two terms in the formula combine to a holomorphic function.) Accordingly, condition (II) is satisfied for any $\delta_1 < \log(1/a)$, with $a_l = 0$ in part (b) and with $\gamma_\alpha = 0$ in part (c).

Rather as in Remark 4, $\omega(\xi, \eta) \rightarrow -\infty$ whenever $\xi \rightarrow n+$ with $\eta > 0$, but that is hardly surprising for this kernel, since a kink is associated with each such characteristic line.

(c) Consider the perturbation of K_3 defined by

$$K(x) = M^{-2} \{ (1+x)^{-1-l} + a(1+x)^{-3-l} \sin \gamma x \}, \quad x \geq 0, \quad (6.20)$$

where $l \in (0, 1)$ and the coefficient a is sufficiently small for convexity on $[0, \infty)$. Convexity requires that the oscillatory term be $O(x^{-3-l})$ as $x \rightarrow \infty$. Condition (II) is not satisfied in this case, because $\tilde{K}$ has branch points at $s = \pm \gamma$, so that cuts $\{\pm \gamma + i\tau \mid \tau \leq 0\}$ must be introduced. However, the oscillatory term in (6.20) is harmless: the additional cuts contribute to $\omega(\xi, 0+)$ a term that is in $L_1(0, \infty)$ by a wide margin.

(vi) *Some shear kernels for which condition (B) holds.* Condition (B) in §5.2 involves the vorticity away from the plate, hence its verification requires something like a combination of conditions (I) and (II), the former for knowledge of singularities of $\omega(\xi, \eta)$ with $\eta \geq 0$ and the latter for behaviour as $\xi \rightarrow \infty$. Here is such a condition.

(III). *The transform $\tilde{K}$ satisfies condition (II) with part (c) strengthened to the following. As $s \rightarrow \infty$ in D_1 ,*

$$\tilde{K}(s) = M^{-2} p^{-1} - \gamma_\alpha p^{-1-\alpha} J(s) + \left\{ \sum_{m=1}^{\infty} [K']_m e^{ic_m s} \right\} p^{-2} + O(s^{-3}), \quad (6.21a)$$

where

$$\gamma_\alpha > 0, \quad \alpha \in (0, 1], \quad [K']_m := K'(c_m+) - K'(c_m-) \geq 0, \quad (6.21b)$$

$$c_m \geq \text{const.} > 0 \text{ for all } m, \quad \sum_{m=1}^{\infty} [K']_m e^{c_m \delta_1} < \infty, \quad (6.21c)$$

and δ_1 is the constant in the definition of D_1 .

There is a natural extension of Theorem 6.4 when (6.21) holds; for the moment we do not need the rest of condition (III), and it suffices for Theorem 6.6 that (6.21) hold in $\{\tau \geq 0\}$ with δ_1 replaced by 0.

Let $\rho = (\rho_1, \dots, \rho_r)$ be multi-indices with any number r of entries; as always, $|\rho| := \sum \rho_m$. In what follows at most n entries ρ_m differ from zero because $|\rho| = n$.

THEOREM 6.6. *If (6.21) holds with $\alpha = 1$ and $J(s) = 1$, then for $\eta \geq 0$*

$$\omega(\xi, \eta) = Z(\xi, \eta) + \lim_{r \rightarrow \infty} \sum_{n=1}^{\infty} \eta^n \sum_{|\rho|=n} \frac{h_1^{\rho_1} \dots h_r^{\rho_r}}{\rho_1! \dots \rho_r!} Z(\xi - \langle \rho, c \rangle, \eta),$$

where

$$h_m := \frac{1}{2} \frac{M^4}{b^2} [K']_m, \quad \langle \rho, c \rangle := \sum_{m=1}^r \rho_m c_m,$$

the function Z is as in Theorem 6.4, and

$$\xi \notin \{\langle \rho, c \rangle \mid \rho \text{ is a multi-index}\}.$$

THEOREM 6.7. *If condition (III) holds, then the vorticity of the solution in §4 satisfies condition (B), for any $\eta_1 \in (0, \infty)$.*

Remarks. 8. We have observed that the kernels in (6.3) satisfy conditions (I) and (II); inspection of their transforms (Appendix D, item (iii)) shows that they also satisfy (III). The regularly piecewise affine kernel described by (6.18) also satisfies condition (III), for any $\delta_1 < \log(1/a)$; to pass from (6.19) to (6.21) one simply expands the factor $(1 - ae^{is})^{-1}$ in powers of ae^{is} . Accordingly, the solutions corresponding to these kernels have vorticity $\omega(\cdot, \eta)$ in A_μ for each $\eta \geq 0$.

9. Here is a more interesting example of a kernel for which the weak solution is not a pointwise one. Let $c_1, c_2, c_3, \dots$ be an enumeration of the rational numbers in $[\frac{1}{2}, 1]$, and define

$$K(x) = M^{-2} \left\{ e^{-x} + \frac{1}{27} \sum_{m=1}^{\infty} 2^{-m} f(x - c_m) \right\}, \quad x \geq 0, \quad (6.22)$$

where again $f(t) = t e^{-2t} H(t)$. It can be shown that the coefficient $1/27$ is sufficiently small for convexity of K on $[0, \infty)$. Inspection of the transform

$$\tilde{K}(s) = M^{-2} \left\{ \frac{1}{1-is} + \frac{1}{27} \frac{1}{(2-is)^2} \sum_{m=1}^{\infty} 2^{-m} e^{ic_m s} \right\} \quad (6.23)$$

shows that condition (III) is satisfied (with $a_l = 0$ and $\alpha = 1$) for any $\delta_1 < 1$. Consequently, the vorticity $\omega(\cdot, \eta)$ is again in A_μ for each $\eta \geq 0$, and therefore has the good properties noted in §§5.1 and 5.2.

On the other hand, the set of excluded values of ξ in Theorem 6.6 consists of zero and of all rational numbers in $[\frac{1}{2}, \infty)$; let A denote this set. We observe from Theorem 6.6 and (6.15) that the vorticity is infinite on all characteristic lines $\{\xi = a, \eta > 0\}$ such that $a \in A$, in the sense that $\omega(\xi, \eta) \rightarrow -\infty$ whenever $\xi \rightarrow a+$ with $\eta > 0$ and $a \in A$.

(vii) *The velocity field near infinity.* Here we consider *shear kernels for which condition (III) holds and which have no kink* ($[K']_m = 0$ for all m). We shall substantiate the claim in §1.1 that, to the lowest order, the disturbance velocity near infinity is that of the symmetrical solution for a Newtonian fluid with kinematic viscosity $U\|K\|$.

Let $k_0(s) := e^{i3\pi/4} (s_+/\|K\|)^{\frac{1}{2}}$; this is a first approximation to $k(s)$ as $s \rightarrow 0$ in D_1 . For the irrotational field, polar co-ordinates r, θ are again convenient. Formal approximations for $r \rightarrow \infty$ are calculated as follows from (4.2) and (4.4).

$$\begin{aligned} \begin{bmatrix} u_{(0)}(x, y) \\ v_{(0)}(x, y) \end{bmatrix} &:= \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-N}^N e^{-ixs - |y|q(s)} \begin{bmatrix} -q(s) \\ is \operatorname{sgn} y \end{bmatrix} \frac{\Phi(0)}{s_+^{3/2}} ds \\ &= U (\|K\|/\pi)^{\frac{1}{2}} r^{-\frac{1}{2}} \begin{bmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix}, \quad 0 < \theta < 2\pi, \end{aligned}$$

and

$$\begin{bmatrix} u_{(1)}(x, y) \\ v_{(1)}(x, y) \end{bmatrix} := -\lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{ic-N}^{ic+N} e^{-ixs + yk_0(s)} \begin{bmatrix} k_0(s) \\ is \end{bmatrix} \frac{\Phi(0)}{s_+^{3/2}} ds \quad (c > 0, y \geq 0)$$

$$= -U \left[\frac{\operatorname{erfc} Y}{(\|K\|/\pi)^{\frac{1}{2}} x^{-\frac{1}{2}} e^{-Y^2}} \right],$$

where

$$Y := \frac{1}{2} \|K\|^{-\frac{1}{2}} x^{-\frac{1}{2}} y \geq 0,$$

and where erfc denotes the complementary error function (the definition of which is recalled in (D.3)). Note that $(u_{(0)} + u_{(1)}, v_{(0)} + v_{(1)})$ satisfies the boundary condition on $\bar{P}$ exactly.

Reference to equations (D.5) and (D.6) shows that $(u_{(0)}, v_{(0)})$ is exactly the irrotational velocity of the Newtonian case (with kinematic viscosity $\nu = U\|K\|$), and that $(u_{(1)}, v_{(1)})$ is the boundary-layer approximation (essentially for $x \rightarrow \infty$ with Y fixed) to the rotational velocity of that case.

Condition (II) is sufficient for bounding the error of $(u_{(0)}, v_{(0)})$ more effectively than was done in Lemma 4.1. One finds that, for $r \geq \|K\|$ (say),

$$|(u_0 - u_{(0)}, v_0 - v_{(0)})(x, y)| \leq \text{const.} (a_l r^{-\frac{1}{2}-l} + r^{-3/2}), \quad (6.24)$$

where a_l and l are as in (6.16). The estimate is uniform in $\mathbf{R}^2 \setminus \bar{P}$.

To find bounds for the errors of $u_{(1)}$ and $v_{(1)}$ that are uniform in $\mathcal{X}$, and as analogous as possible to the bounds (D.7) for the Newtonian case, seems to require a long sequence of estimates. The behaviour of $k(s)$ away from the origin must be taken into account, and one must find a path of integration on which both terms in $\exp\{-ixs + yk(s)\}$ are favourable. It turns out that *there is a constant $\delta_2 > 0$ (depending on K and M but independent of x and y) such that*

$$|(u_1 - u_{(1)})(x, y)| \leq \text{const.} (a_l x^{-l} + x^{-1}) Y \exp(-\delta_2 Y^2), \quad (6.25a)$$

$$|(v_1 - v_{(1)})(x, y)| \leq \text{const.} (a_l x^{-\frac{1}{2}-l} + x^{-3/2}) \exp(-\delta_2 Y^2), \quad (6.25b)$$

whenever $x \geq \|K\|$ (say) and $0 \leq by \leq x$. Again a_l and l are as in (6.16).

(viii) *Regularity of the solutions for the kernels K_1 to K_3 .* Consider the functions u_{1c} and v_{1c} on $\mathbf{R} \times [0, \infty)$; here $u_{1c}(\xi, \eta) = u_1(x, y)$, similarly for v_1 , and $\operatorname{sgn} \eta = 1$ for $\eta \geq 0$. These functions are defined by (4.4) with the exponential transformed as in

(6.2). We shall show that, *when K is one of K_1 to K_3 , the functions u_{1c} and v_{1c} are infinitely differentiable (are in C^∞) on $(0, \infty) \times [0, \infty)$.*

For $\tilde{K}_1$ to $\tilde{K}_3$ [which are displayed in Appendix D, (iii)], the asymptotic formula for $s \rightarrow \infty$ stated in conditions (I) and (III) holds not merely in D_1 , but in $\mathbb{C} \setminus \{i\tau \mid \tau \leq 0\}$. Since this asymptotic form shows that $\tilde{K}$ and W have no zero outside some large ball, we can find paths of integration on which

(a) $\tau \rightarrow -\infty$ as $s \rightarrow \infty$, so that $|e^{-i\xi s}| = e^{\xi\tau} \rightarrow 0$ exponentially if $\xi > 0$,

(b) the functions k, h and Φ are defined and holomorphic, with the asymptotic behaviour noted previously.

The exponential decay at infinity of $e^{-i\xi s}$ allows repeated differentiation of the integrals defining u_{1c} and v_{1c} , *provided that $\xi > 0$* and that the factor $\exp\{\eta h(s)\}$ is not adverse. This latter is the case: for K_1 and K_3 , we have h as in (6.10), and any path with the properties (a) and (b) will serve. For K_2 , we can find asymptotic directions for the path such that, if $\eta > 0$, both $\exp(-i\xi s)$ and $\exp\{\eta h(s)\}$ tend to zero exponentially.

For K_1 to K_3 , the weak solution is a pointwise one. It remains only to check condition (2.20c). For K_2 , Corollary 6.3 shows that this condition is satisfied. For K_1 and K_3 , one need only verify that $\partial v_{1c}/\partial \eta$ is continuous at the shock (as v_1 is), because K' is no worse than K for these two kernels.

Appendix A. Properties of $\tilde{K}$ and k

We begin by quoting a known result (Titchmarsh 1948, pp. 16 and 169) and adding the details of its exceptional case.

LEMMA A.1. *Let $f:(0,\infty)\rightarrow\mathbf{R}$ be non-increasing, integrable on $(0,1)$, and such that $f(x)\rightarrow 0$ as $x\rightarrow\infty$. Define*

$$\begin{aligned}\tilde{f}_c(\sigma) &:= \lim_{N\rightarrow\infty} \int_0^N \cos\sigma x f(x) dx, \quad \sigma \in \mathbf{R} \setminus \{0\}, \\ \tilde{f}_s(\sigma) &:= \lim_{N\rightarrow\infty} \int_0^N \sin\sigma x f(x) dx, \quad \sigma \in \mathbf{R};\end{aligned}$$

these limits exist in $\mathbf{R}$. Then

- (a) $\tilde{f}_s(\sigma) \geq 0$ for all $\sigma \geq 0$;
- (b) $\tilde{f}_s(\sigma_0) = 0$ for some $\sigma_0 > 0$ if and only if $f(x)$ is constant in each interval $(2n\pi/\sigma_0, (2n+2)\pi/\sigma_0)$, $n=0,1,2,\dots$, and in that case $\tilde{f}_c(\sigma_0)=0$.

The exceptional case, (b) of the lemma, arises in this paper when K is regularly piecewise affine (Definition A.3), and is tiresome at this stage. However, it would be artificial and equally tiresome to lengthen the list of conditions on K in order to exclude this case.

LEMMA A.2. *The Fourier-Laplace transform $\tilde{K}$ of the function K in (1.4) has the following properties.*

- (a) $\tilde{K}$ is holomorphic in $\{\tau > 0\}$ and continuous on $\{\tau \geq 0\}$.
- (b) $\tilde{K}$ has no zero in $\{\tau \geq 0\}$.
- (c) $\tilde{K}(s) = \frac{i}{M^2 s} \{1 + O(s^{-\alpha})\}$ as $s \rightarrow \infty$ with $\tau \geq 0$.

Proof. (a) This follows from the fact that $K \in L_1(0, \infty)$; see, for example, Widder (1946), pp. 57 and 50.

(b) For $\sigma=0$,

$$\tilde{K}(i\tau) = \int_0^\infty e^{-\tau x} K(x) dx > 0.$$

For $\sigma>0$,

$$\text{Im } \tilde{K}(s) = \int_0^\infty \sin \sigma x e^{-\tau x} K(x) dx > 0,$$

by Lemma A.1. For $\sigma<0$, $\text{Im } \tilde{K}(s)<0$ because it is an odd function of σ .

(c) Lemma 1.1 allows us to integrate by parts the integral defining $\tilde{K}(s)$; thus

$$\tilde{K}(s) = \frac{iK(0)}{s} - \frac{1}{is} \int_0^\infty e^{isx} K'(x) dx \quad (s \neq 0), \quad (\text{A.1})$$

where $K(0)=M^{-2}$. We shall derive the inequalities

$$\left| \int_0^\infty e^{isx} K'(x) dx \right| \leq \begin{cases} \text{const. } \tau^{-\alpha}, & \tau > 0, \\ \text{const. } (|\sigma|^{-1} \tau^{1-\alpha} + |\sigma|^{-\alpha}), & \sigma \neq 0, \tau \geq 0. \end{cases} \quad (\text{A.2a})$$

Use of (A.2a) for $|\sigma| \leq \tau$, and of (A.2b) for $|\sigma| > \tau$, gives

$$\left| \int_0^\infty e^{isx} K'(x) dx \right| \leq \text{const. } |s|^{-\alpha} \quad (|s| \geq 1, \text{ say}); \quad (\text{A.3})$$

then (A.1) and (A.3) imply the result (c) of the lemma.

Condition (1.4f) and Lemma 1.1 imply existence of a constant such that

$$0 < -K'_-(x) \leq \text{const. } x^{\alpha-1} \quad \text{for } x > 0. \quad (\text{A.4})$$

(For $0 < x \leq 1$, we choose $\delta=x$ in Lemma 1.1 (b).) Combining this with $|e^{isx}| = e^{-\tau x}$, we obtain (A.2a).

Now let $g(x) := e^{-\tau x} K'(x)$, so that $g(x)=0$ for $x<0$, and let $\sigma>0$. Then

$$\begin{aligned} \int_0^\infty e^{isx} K'(x) dx &= \frac{1}{2} \left\{ \int_{-\infty}^\infty e^{i\sigma x} g(x) dx + \int_{-\infty}^\infty e^{i\sigma t + i\pi} g(t+\pi/\sigma) dt \right\} \\ &= \frac{1}{2} \int_{-\infty}^\infty e^{i\sigma x} \{g(x) - g(x+\pi/\sigma)\} dx \\ &= \frac{1}{2} \int_{-\infty}^\infty e^{i\sigma x} \left[\{e^{-\tau x} - e^{-\tau(x+\pi/\sigma)}\} K'(x) \right. \\ &\quad \left. + e^{-\tau(x+\pi/\sigma)} \{K'(x) - K'(x+\pi/\sigma)\} \right] dx, \end{aligned}$$

and

$$\begin{aligned} \left| \int_{-\infty}^{\infty} e^{i\sigma x} \{ e^{-\tau x} - e^{-\tau(x+\pi/\sigma)} \} K'(x) dx \right| &\leq - \int_0^{\infty} e^{-\tau x} \frac{\tau\pi}{\sigma} K'(x) dx \\ &\leq \text{const. } \sigma^{-1} \tau^{1-\alpha}, \end{aligned}$$

in view of (A.4); also,

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} e^{i\sigma x - \tau(x+\pi/\sigma)} \{ K'(x) - K'(x+\pi/\sigma) \} dx \right| \\ & \leq - \int_{-\pi/\sigma}^0 K'(x+\pi/\sigma) dx + \int_0^{\infty} \{ K'(x+\pi/\sigma) - K'(x) \} dx \\ & = 2 \{ K(0) - K(\pi/\sigma) \} \\ & \leq \text{const. } \sigma^{-\alpha}. \end{aligned}$$

This proves (A.2b) for $\sigma > 0$, and there is no significant change for $\sigma < 0$. $\square$

DEFINITION A.3. We shall say that K is *regularly piecewise affine* (r.p.a.) if it satisfies (1.4) and there is a number $\lambda > 0$ such that $K''(x) = 0$ whenever $x \notin \{0, \lambda, 2\lambda, 3\lambda, \dots\}$. The largest such λ will be called the *interval length* of K , and will be denoted by l .

LEMMA A.4. For $\sigma > 0$ and $\tau \geq 0$, $\text{Re } \tilde{K}(s) > 0$ and $\text{Re } s\tilde{K}(s) > 0$ unless K is r.p.a. and $s \in \{2\pi/l, 4\pi/l, 6\pi/l, \dots\}$; in that case, $\tilde{K}(s) = i/M^2\sigma$.

Proof. Let $\tilde{K}_1 := \text{Re } \tilde{K}$ and $\tilde{K}_2 := \text{Im } \tilde{K}$. Integration by parts of the integral defining $\tilde{K}_1$ gives

$$\begin{aligned} \tilde{K}_1(s) &= \frac{1}{\sigma} \int_0^{\infty} \sin \sigma x f_1(x, \tau) dx, \quad f_1(x, \tau) := e^{-\tau x} \{ \tau K(x) - K'(x) \}, \\ \sigma \tilde{K}_1(s) - \tau \tilde{K}_2(s) &= \int_0^{\infty} \sin \sigma x f_2(x, \tau) dx, \quad f_2(x, \tau) := -e^{-\tau x} K'(x). \end{aligned}$$

If $\tau > 0$, then $f_1(\cdot, \tau)$ and $f_2(\cdot, \tau)$ are strictly decreasing functions, and the result follows from Lemma A.1. If $\tau = 0$, the exceptional case occurs if and only if $K'(x)$ is constant in each of the intervals in Lemma A.1 (b), and this is what we have claimed (in different words) in the statement of the lemma. That $\tilde{K}_2(s) = 1/M^2\sigma$ in the exceptional

case follows from integration by parts of the integral defining $\tilde{K}_2$. $\square$

As a stepping-stone to the function k introduced in (3.4), we define

$$W(s) := is + 1/\tilde{K}(s), \quad \tau \geq 0. \quad (\text{A.5})$$

LEMMA A.5. (a) W is holomorphic in $\{\tau > 0\}$ and continuous on $\{\tau \geq 0\}$.

(b) W has no zero in $\{\tau \geq 0\}$.

(c) We can choose a (continuous) branch $\arg W$ of the set-valued function $\text{Arg } W$ such that (for $\tau \geq 0$)

$$-\frac{1}{2}\pi \leq \arg W(s) \leq \frac{1}{2}\pi, \quad (\text{A.6})$$

with equality on the left if and only if K is r.p.a. and $s \in \{2\pi/l, 4\pi/l, \dots\}$, equality on the right if and only if K is r.p.a. and $s \in \{-2\pi/l, -4\pi/l, \dots\}$.

Proof. (a) This is immediate from (a) and (b) of Lemma A.2.

(b) For $\sigma=0$, we have

$$0 < \tilde{K}(i\tau) = \int_0^\infty e^{-\tau x} K(x) dx \leq \min \{ \|K\|, 1/M^2 \tau \},$$

upon using first $e^{-\tau x} \leq 1$ and then $K(x) \leq K(0)$; of course, the minimum is taken at fixed τ . Hence

$$W(i\tau) \geq \max \{ -\tau + 1/\|K\|, b^2 \tau \} \quad (b^2 := M^2 - 1). \quad (\text{A.7})$$

Now let $\sigma > 0$ (and $\tau \geq 0$). Since $\tilde{K}$ has no zero, a zero of W corresponds to $s\tilde{K}(s)=i$. By Lemma A.4, the real part of this equation is satisfied only if K is r.p.a. and $s \in \{2\pi/l, 4\pi/l, \dots\}$; then $\text{Im } s\tilde{K}(s) = M^{-2}$, so that the imaginary part is not satisfied. The same reasoning applies for $\sigma < 0$ because $\sigma\tilde{K}_1$ and $\tilde{K}_2$ (where $\tilde{K} =: \tilde{K}_1 + i\tilde{K}_2$) are odd functions of σ .

(c) We note that, by (a) and (b), any branch of $\text{Arg } W$ ($= \text{Im Log } W$) is a harmonic function in $\{\tau > 0\}$ and continuous on $\{\tau \geq 0\}$. Since $W(i\tau)$ is real and positive,

by (A.7), we choose the branch $\arg W$ satisfying $\arg W(i\tau) = 0$. Lemma A.2 (c) shows that, as $s \rightarrow \infty$ with $\tau \geq 0$,

$$W(s) = -ib^2 s \{ 1 + O(s^{-\alpha}) \}, \quad \arg W(s) \sim \arg s_+ - \frac{1}{2}\pi, \quad (\text{A.8})$$

where s_+ is as in (3.10). Thus the harmonic function $\arg W$ is bounded at infinity; it is sufficient to prove (A.6) for $\tau=0$, by the Phragmén-Lindelöf theorem for harmonic functions (Protter and Weinberger 1967, pp. 94-96). Remark (i) on p.96 of that text implies strict inequality in (A.6) for $\tau>0$, because $\arg W$ is not a constant.

We prove (A.6), and the remark following it, for $\tau=0$ by observing that $\arg W(0) = 0$, that $\operatorname{Re} W(\sigma) > 0$ wherever $\operatorname{Re} \tilde{K}(\sigma) > 0$ (see Lemma A.4), and that at the exceptional points, when K is r.p.a., $W(\sigma) = -ib^2\sigma$. $\square$

We come, at long last, to

$$k(s) := e^{i3\pi/4} s_+^{\frac{1}{2}} W(s)^{\frac{1}{2}}, \quad \tau \geq 0, \quad (\text{A.9})$$

where s_+ is as in (3.10) and $\arg W(s)$ is as in (A.6). (Of course, $\arg f(s)^{\frac{1}{2}} = \frac{1}{2} \arg f(s)$ whenever $\arg f(s)$ is defined.)

THEOREM A.6. (a) k is holomorphic in $\{\tau>0\}$ and continuous on $\{\tau\geq 0\}$.

(b) $k(s) = 0$ only at $s=0$.

(c) $k(s) = ibs \{ 1 + O(s^{-\alpha}) \}$ as $s \rightarrow \infty$ with $\tau \geq 0$.

(d) $\operatorname{Re} k(s) \leq -b\tau$, with equality if and only if either $s=0$ or K is r.p.a. and $s \in \{\pm 2\pi/l, \pm 4\pi/l, \pm 6\pi/l, \dots\}$.

Proof. (a),(b),(c) These are immediate from (a) and (b) of Lemma A.5 and from (A.8).

(d) Define $f(\sigma, \tau) := \operatorname{Re} k(s) + b\tau$ ($\tau \geq 0$). Then f is a harmonic function in $\{\tau>0\}$, continuous on $\{\tau\geq 0\}$ and $O(s^{1-\alpha})$ at infinity. Again we apply the Phragmén-Lindelöf theorem for harmonic functions: if $f(\sigma, 0) \leq 0$ for all $\sigma \in \mathbf{R}$, then

$f(\sigma, \tau) \leq 0$ whenever $\tau \geq 0$, with strict inequality for $\tau > 0$.

Consider $f(\sigma, 0)$. We have $f(0, 0) = 0$, and, by (A.6) and (A.9),

$$\begin{aligned} \frac{1}{2}\pi &\leq \arg k(\sigma) \leq \pi && \text{for } \sigma > 0, \\ \pi &\leq \arg k(\sigma) \leq \frac{3}{2}\pi && \text{for } \sigma < 0, \end{aligned}$$

with $\arg k(\sigma) = \frac{1}{2}\pi$ if and only if K is r.p.a. and $s \in \{2\pi/l, 4\pi/l, \dots\}$, and with $\arg k(\sigma) = 3\pi/2$ if and only if K is r.p.a. and $s \in \{-2\pi/l, -4\pi/l, \dots\}$. This proves that $\operatorname{Re} k(\sigma) \leq 0$, with equality as in the statement of the theorem. $\square$

Appendix B. Decomposition of $q(.,\varepsilon)-k(.,\varepsilon)$.

Here we use a traditional device: we decompose a perturbed function $q(.,\varepsilon)-k(.,\varepsilon)$ that is handled more easily than is $q-k$. Let $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$, and define

$$q(s, \varepsilon) := (s+i\varepsilon)^{\frac{1}{2}} (s-i\varepsilon)^{\frac{1}{2}}, \quad s \in \mathbb{C} \setminus \{i\tau \mid \tau \leq -\varepsilon\} \setminus \{i\tau \mid \tau \geq \varepsilon\}, \quad (\text{B.1})$$

where $\arg(s+i\varepsilon)$ and $\arg(s-i\varepsilon)$ are restricted, respectively, to the same intervals as $\arg s_+$ and $\arg s_-$ are in (3.10). Also define

$$k(s, \varepsilon) := e^{i3\pi/4} (s+i\varepsilon)^{\frac{1}{2}} \{W(s+i\varepsilon) + 2\varepsilon\}^{\frac{1}{2}}, \quad \tau \geq -\varepsilon, \quad (\text{B.2})$$

where W is as in (A.5) and Lemma A.5, and, corresponding to our choice there, $\arg \{W(s+i\varepsilon) + 2\varepsilon\}$ is the branch that is zero when $s=i\tau$, $\tau \geq -\varepsilon$. The function $W(.,\varepsilon) + 2\varepsilon$ has no zero in $\{\tau \geq -\varepsilon\}$ because $\operatorname{Re} W(s+i\varepsilon) \geq 0$ there, by (A.6). Hence $k(.,\varepsilon)$ is holomorphic in $\{\tau > -\varepsilon\}$ and continuous on $\{\tau \geq -\varepsilon\}$.

A variant of $q(.,\varepsilon) - k(.,\varepsilon)$, which is chosen because it leads to (B.5), is defined by

$$g(s, \varepsilon) := \frac{e^{i\beta}}{M} \left[\frac{s-i\varepsilon}{s+i\varepsilon} \right]^{\beta/\pi} \left\{ 1 - \frac{k(s, \varepsilon)}{q(s, \varepsilon)} \right\} \quad (\text{B.3})$$

for points s in

$$D_\varepsilon := \{s \mid \tau \geq -\varepsilon\} \setminus \{i\tau \mid \tau \geq \varepsilon\} \setminus \{-i\varepsilon\}. \quad (\text{B.4})$$

We denote the interior of D_ε by $\overset{\circ}{D}_\varepsilon$ and the boundary of D_ε by ∂D_ε .

LEMMA B.1. *There is a branch $\log g(.,\varepsilon)$ of $\operatorname{Log} g(.,\varepsilon)$ that is holomorphic in $\overset{\circ}{D}_\varepsilon$, continuous on D_ε , has limiting values as $\sigma \rightarrow 0+$ with $\tau > \varepsilon$ and as $\sigma \rightarrow 0-$ with $\tau > \varepsilon$ that define continuous functions on (ε, ∞) , and is such that*

$$\log g(s, \varepsilon) = O(s^{-\alpha}) \quad \text{as } s \rightarrow \infty \text{ in } D_\varepsilon. \quad (\text{B.5})$$

Proof. We have noted that $k(.,\varepsilon)$ is holomorphic in $\{\tau > -\varepsilon\}$ and continuous on $\{\tau \geq -\varepsilon\}$; the other ingredients of $g(.,\varepsilon)$ are elementary functions the behaviour of which is seen by inspection. This leaves two things to be proved, as follows.

(a) We show that $g(.,\varepsilon)$ has no zero. Suppose that $1-k(.,\varepsilon)/q(.,\varepsilon)$ has a zero at some point $s \in D_\varepsilon$, or as a limiting value when $\sigma \rightarrow 0^\pm$ with $\tau > \varepsilon$. Then

$$1 = \frac{k(s,\varepsilon)^2}{q(s,\varepsilon)^2} = - \frac{i}{s-i\varepsilon} \{ W(s+i\varepsilon) + 2\varepsilon \} = 1 - \frac{i}{(s-i\varepsilon) \tilde{K}(s+i\varepsilon)} .$$

This implies that $1/\tilde{K}(s+i\varepsilon) = 0$, which is a contradiction for $\tau \geq -\varepsilon$, by Lemma A.2 (a).

(b) It remains to choose a branch $\arg g(.,\varepsilon)$ such that (B.5) holds. By (2.1) and Theorem A.6 (c),

$$1 - \frac{k(s,\varepsilon)}{q(s,\varepsilon)} = \begin{cases} Me^{-i\beta} \{1+O(s^{-\alpha})\} & \text{as } s \rightarrow \infty \text{ with } \sigma > 0, \tau \geq -\varepsilon, \\ Me^{i\beta} \{1+O(s^{-\alpha})\} & \text{as } s \rightarrow \infty \text{ with } \sigma < 0, \tau \geq -\varepsilon. \end{cases}$$

We choose $\arg \{1-k(.,\varepsilon)/q(.,\varepsilon)\}$ to be the branch (continuous on D_ε) that has limiting values in $(-\frac{1}{2}\pi, 0)$ as $\sigma \rightarrow 0^+$ with $\tau > \varepsilon$. Then it has limiting values in $(0, \frac{1}{2}\pi)$ as $\sigma \rightarrow 0^-$ with $\tau > \varepsilon$, tends to $-\beta$ as $s \rightarrow \infty$ with $\sigma > 0$, and tends to β as $s \rightarrow \infty$ with $\sigma < 0$. The result (B.5) now follows from the definition of g . $\square$

The function g can be decomposed by a standard procedure for functions holomorphic in a strip $\{-\varepsilon < \tau < \varepsilon\}$ and having suitable decay at infinity there (Noble 1958, p.15 ; Paley and Wiener 1934, p.51 ; Titchmarsh 1948, p. 339). Using this procedure, but also exploiting the additional properties of $\log g$, we adopt $w = \omega + it$ as a complex variable of integration and define

$$g_+(s,\varepsilon) := \exp \left\{ \frac{1}{2\pi i} \int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty} \log g(w,\varepsilon) \frac{dw}{w-s} \right\}, \quad \tau > -\varepsilon, \quad (\text{B.6})$$

where the integral is along the line $\{w | t = -\varepsilon\}$; and

$$g_-(s, \varepsilon) := \exp \left\{ -\frac{1}{2\pi i} \int_{i\varepsilon}^{i\infty} [\log g(w, \varepsilon)]_{\omega \rightarrow 0^-}^{\omega \rightarrow 0^+} \frac{dw}{w-s} \right\},$$

$$s \in \mathbb{C} \setminus \{i\tau \mid \tau \geq \varepsilon\}, \quad (\text{B.7})$$

where, in effect, the integral is in opposite directions along the two sides of the cut $\{w \mid \omega=0, \tau \geq \varepsilon\}$. Obviously the singularities of $\log g(w, \varepsilon)$ at $w = \mp i\varepsilon$ are integrable, and Lemma B.1 implies (absolute) integrability elsewhere.

$$\text{LEMMA B.2. (a) } g_+(s, \varepsilon) g_-(s, \varepsilon) = g(s, \varepsilon) \quad \text{for } s \in \overset{\circ}{D}_\varepsilon. \quad (\text{B.8})$$

(b) $g_+(\cdot, \varepsilon)$ is holomorphic in $\{\tau > -\varepsilon\}$ and has no zero there; as $s \rightarrow \infty$ with $\tau > -\varepsilon$,

$$g_+(s, \varepsilon) = \begin{cases} 1 + O(s^{-\alpha}) & \text{if } \alpha \in (0, 1), \\ 1 + O(s^{-1} \log s) & \text{if } \alpha = 1. \end{cases} \quad (\text{B.9})$$

(c) $g_-(\cdot, \varepsilon)$ is holomorphic in $\mathbb{C} \setminus \{i\tau \mid \tau \geq \varepsilon\}$ and has no zero there; as $s \rightarrow \infty$ in $\mathbb{C} \setminus \{i\tau \mid \tau \geq \varepsilon\}$,

$$g_-(s, \varepsilon) = \begin{cases} 1 + O(s^{-\alpha}) & \text{if } \alpha \in (0, 1), \\ 1 + O(s^{-1} \log s) & \text{if } \alpha = 1. \end{cases} \quad (\text{B.10})$$

Proof. (a) This identity follows from the residue theorem, from (B.5) and from the following fact. If we define $g_+(s, \varepsilon, \delta)$ and $g_-(s, \varepsilon, \delta)$ by using paths of integration in $\overset{\circ}{D}_\varepsilon$ on which $\text{dist}(w, \partial D_\varepsilon) = \delta$, where $\delta \in (0, \varepsilon)$ and $\delta < \text{dist}(s, \partial D_\varepsilon)$, then $g_+(s, \varepsilon) = g_+(s, \varepsilon, \delta)$ and $g_-(s, \varepsilon) = g_-(s, \varepsilon, \delta)$.

(b),(c) The holomorphic nature of the Cauchy integrals in (B.6) and (B.7), at points outside the paths of integration, is a standard result. However, (B.9) and (B.10) require comment because s is not bounded away from these paths. Consider the definition (B.7) of g_- . Because $k(\cdot, \varepsilon)$ is holomorphic in $\{\tau > -\varepsilon\}$, one can use Theorem A.6 (c) to bound not merely $[\log g(w, \varepsilon)]$ but also its derivative for large $t (= \text{Im } w)$. This yields for (B.10) an inequality depending on $|s|$ but not on

$\text{dist}(s, \{it \mid t \geq \varepsilon\})$; then (B.9) follows from (B.8) and the known behaviour of g and g_- . $\square$

In order to have a variant of g_+ that remains bounded as $s \rightarrow -i\varepsilon$, and similarly for g_- as $s \rightarrow i\varepsilon$, we choose at pleasure real constants $\kappa > \varepsilon_0$ and $\lambda > \varepsilon_0$, and define

$$E(s, \kappa, \varepsilon) := \left[\frac{s+i\varepsilon}{s+i\kappa} \right]^{\beta/\pi} g_+(s, \varepsilon), \quad \tau > -\varepsilon, \quad (\text{B.11})$$

$$F(s, \lambda, \varepsilon) := \left[\frac{s-i\varepsilon}{s-i\lambda} \right]^{\frac{1}{2}-\beta/\pi} g_-(s, \varepsilon), \quad s \in \mathbb{C} \setminus \{it \mid t \geq \varepsilon\}, \quad (\text{B.12})$$

where $\arg(s+i\kappa)$ and $\arg(s-i\lambda)$ are restricted, respectively, to the same intervals as $\arg s_+$ and $\arg s_-$ are in (3.10). It follows that E has the properties stated for g_+ in Lemma B.2 (b), and F has the properties stated for g_- in Lemma B.2 (c). Of course, $|F(s, \lambda, \varepsilon)| \rightarrow \infty$ as $\sigma \rightarrow 0^\pm$ with $\tau = \lambda$, in contrast to g_- , but this will cause no difficulty.

THEOREM B.3. *If $s \in D_\varepsilon^\circ$, then*

$$q(s, \varepsilon) - k(s, \varepsilon) = Me^{-i\beta} (s+i\varepsilon)^{\frac{1}{2}} (s+i\kappa)^{\beta/\pi} E(s, \kappa, \varepsilon) (s-i\lambda)^{\frac{1}{2}-\beta/\pi} F(s, \lambda, \varepsilon). \quad (\text{B.13})$$

Proof. Let $s \in D_\varepsilon^\circ$. The definition (B.3) of g implies that

$$q(s, \varepsilon) - k(s, \varepsilon) = Me^{-i\beta} (s+i\varepsilon)^{\frac{1}{2}+\beta/\pi} (s-i\varepsilon)^{\frac{1}{2}-\beta/\pi} g(s, \varepsilon);$$

we use the result $g = g_+ g_-$ of Lemma B.2 and then apply (B.11) and (B.12). $\square$

A possible choice for equation (3.16) is

$$G_+(s, \varepsilon) := Me^{-i\beta} (s+i\varepsilon)^{-\frac{1}{2}} (s+i\kappa)^{\beta/\pi} E(s, \kappa, \varepsilon), \quad \tau > -\varepsilon, \quad (\text{B.14a})$$

$$G_-(s, \varepsilon) := (s+i\varepsilon) (s-i\lambda)^{\frac{1}{2}-\beta/\pi} F(s, \lambda, \varepsilon), \quad \tau < \varepsilon. \quad (\text{B.14b})$$

Theorem 3.4 shows that, with $\varepsilon=0$, these functions also have the properties demanded after (3.15).

Appendix C. Convolutions and transforms of the distributions in the problem

Our use of tempered distributions is a little unorthodox in that (a) we convolve a function of one variable with a generalized function of two variables, (b) we use Fourier transforms that are with respect to only one of two co-ordinates. These adaptations are not difficult, but it seems necessary to set out the following definitions and results.

Notation. Throughout this appendix, $A = K$ or $A = H$ (where K and H are as in (1.4) and (2.11), respectively), so that $\langle A, \cdot \rangle \in \mathcal{S}'(\mathbf{R})$ is defined by

$$\langle A, \gamma \rangle := \int_0^\infty A(t) \gamma(t) dt \quad \text{for all } \gamma \in \mathcal{S}(\mathbf{R});$$

also, $\langle w, \cdot \rangle \in \mathcal{S}'(\mathbf{R}^2)$ with $\text{supp } w \subset [0, \infty) \times \mathbf{R}$.

The content of Lemma C.1 is wholly orthodox (Friedlander 1982, pp. 40-41 and 47; Schwartz 1950, vol. I, pp. 104 and 108).

LEMMA C.1. *For any $\chi \in \mathcal{S}(\mathbf{R}^3)$, let $\langle A, \cdot \rangle$ act on the first argument of χ , and $\langle w, \cdot \rangle$ act on the second and third arguments of χ . Then $\langle A, \chi \rangle \in \mathcal{S}(\mathbf{R}^2)$ and $\langle w, \chi \rangle \in \mathcal{S}(\mathbf{R})$; moreover, defining a tensor product by*

$$\langle A \otimes w, \chi \rangle := \langle A, \langle w, \chi \rangle \rangle \quad \text{for all } \chi \in \mathcal{S}(\mathbf{R}^3), \quad (\text{C.1})$$

one finds that $\langle A \otimes w, \cdot \rangle \in \mathcal{S}'(\mathbf{R}^3)$, $\text{supp } A \otimes w \subset [0, \infty) \times [0, \infty) \times \mathbf{R}$ and

$$\langle A \otimes w, \chi \rangle = \langle w, \langle A, \chi \rangle \rangle \quad \text{for all } \chi \in \mathcal{S}(\mathbf{R}^3). \quad (\text{C.2})$$

DEFINITION C.2. Given $\varphi \in \mathcal{S}(\mathbf{R}^2)$, define $\varphi_{(+)} \in C^\infty(\mathbf{R}^3)$ by $\varphi_{(+)}(t, x, y) := \varphi(t+x, y)$. Let $X := [0, \infty) \times [0, \infty) \times \mathbf{R}$, and let $m \in C^\infty(\mathbf{R}^3)$ be a mollifier such that

$$m(t, x, y) = \begin{cases} 1 & \text{if } (t, x, y) \in X, \\ 0 & \text{if } \text{dist}((t, x, y), X) \geq 1. \end{cases}$$

Now define the *convolution of A and w* by

$$\langle A * w, \varphi \rangle := \langle A \otimes w, m\varphi_{(+)} \rangle \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}^2). \quad (\text{C.3})$$

[Note that $m\varphi_{(+)} \in \mathcal{S}(\mathbf{R}^3)$ and that $m\varphi_{(+)} = \varphi_{(+)}$ on $\text{supp } A \otimes w$.]

LEMMA C.3. $\langle A * w, \cdot \rangle \in \mathcal{S}'(\mathbf{R}^2)$, $\text{supp } A * w \subset [0, \infty) \times \mathbf{R}$, and $\langle A * w, \cdot \rangle$ is independent of the choice of m .

Proof. To show that $\langle A * w, \cdot \rangle \in \mathcal{S}'(\mathbf{R}^2)$, we derive a semi-norm estimate. Since $\langle A \otimes w, \cdot \rangle \in \mathcal{S}'(\mathbf{R}^3)$, there are constants C_a and N such that

$$|\langle A \otimes w, \chi \rangle| \leq C_a \sum_{|\alpha|, |\beta| \leq N} \|\chi\|_{\alpha, \beta} \quad \text{for all } \chi \in \mathcal{S}(\mathbf{R}^3), \quad (\text{C.4})$$

where α and β are triple indices replacing the double indices in §2.1(i). We set $\chi = m\varphi_{(+)}$ and observe that, if $(t, x, y) \in \text{supp } m$, then $|t| \leq |t+x| + 1$ and $|x| \leq |t+x| + 1$. Hence, whenever $|\alpha|, |\beta| \leq N$,

$$\begin{aligned} \|m\varphi_{(+)}\|_{\alpha, \beta} &= \sup |t^{\alpha_1} x^{\alpha_2} y^{\alpha_3} \partial^{\beta} \{m(t, x, y) \varphi(t+x, y)\}| \\ &\leq C_b \sum_{|\mu| \leq |\alpha|, |\nu| \leq |\beta|} \|\varphi\|_{\mu, \nu} \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}^2), \end{aligned} \quad (\text{C.5})$$

where C_b is a constant depending only on N and m ; we have used not only our bounds for $|t|$ and $|x|$, but also the Leibniz rule for differentiation of a product and the fact that each derivative of m is uniformly bounded. Insertion of (C.5) into (C.4) gives the desired estimate of $\langle A * w, \cdot \rangle$.

Let $z_0 \in (-\infty, 0) \times \mathbf{R}$ and let Ω be an open set such that $z_0 \in \Omega \subset (-\infty, 0) \times \mathbf{R}$. If $\varphi \in C_0^\infty(\Omega)$, then the supports of $A \otimes w$ and $\varphi_{(+)}$ are disjoint; hence $\langle A * w, \varphi \rangle = 0$, which shows that $\text{supp } A * w \subset [0, \infty) \times \mathbf{R}$.

Finally, if m_1 and m_2 are admissible mollifiers, then $m_1 = m_2$ on the support of $A \otimes w$, so that $\langle A \otimes w, (m_1 - m_2)\varphi_{(+)} \rangle = 0$. $\square$

Part (a) of the next lemma is an essential step in proving equivalence of the two forms, (2.3d) and (2.10), of the vorticity equation; part (b) is used in the proof of Theorem 2.2.

LEMMA C.4. (a) With $A_r(t) := A(-t)$ for all $t \in \mathbf{R}$, we have

$$\langle A * w, \varphi \rangle = \langle w, A_r * \varphi \rangle \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}^2).$$

$$(b) \quad \langle \partial_1(H * w), \cdot \rangle = \langle w, \cdot \rangle = \langle H * \partial_1 w, \cdot \rangle \quad \text{in } \mathcal{S}'(\mathbf{R}^2).$$

Proof. (a) By (C.3) and (C.2),

$$\langle A * w, \varphi \rangle = \langle A \otimes w, m\varphi_{(+)} \rangle = \langle w, \langle A, m\varphi_{(+)} \rangle \rangle \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}^2).$$

If $(x, y) \in \text{supp } w$, which implies that $x \geq 0$ and that $m(t, x, y) = 1$ when $t \in \text{supp } A$, we have

$$\begin{aligned} \langle A, m\varphi_{(+)} \rangle(x, y) &= \int_0^\infty A(t) \varphi(t+x, y) dt = \int_{-\infty}^0 A_r(t') \varphi(x-t', y) dt' \\ &= (A_r * \varphi)(x, y). \end{aligned}$$

(b) We shall prove the right-hand identity; the proof of the left-hand one is similar and one step shorter. For any $\varphi \in \mathcal{S}(\mathbf{R}^2)$,

$$\begin{aligned} \langle H * \partial_1 w, \varphi \rangle &= \langle H \otimes \partial_1 w, m\varphi_{(+)} \rangle = \langle \partial_1 w, \langle H, m\varphi_{(+)} \rangle \rangle \\ &= -\langle w, \partial_1 \langle H, m\varphi_{(+)} \rangle \rangle. \end{aligned}$$

If $(x, y) \in \text{supp } w$, which implies that $m(t, x, y) = 1$ when $t \in \text{supp } H$, we have

$$\begin{aligned} (\partial_1 \langle H, m\varphi_{(+)} \rangle)(x, y) &= \frac{\partial}{\partial x} \int_0^\infty \varphi(t+x, y) dt \\ &= \int_0^\infty (\partial_1 \varphi)(t+x, y) dt = -\varphi(x, y). \quad \square \end{aligned}$$

DEFINITION C.5. The *partial Fourier transform* $\langle g^\dagger, \cdot \rangle \in \mathcal{S}'(\mathbf{R}^2)$ of any tempered distribution $\langle g, \cdot \rangle \in \mathcal{S}'(\mathbf{R}^2)$ is defined by

$$\langle g^\dagger, \varphi \rangle := \langle g, \varphi^\dagger \rangle \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}^2),$$

where

$$\varphi^\dagger(\sigma, y) := \int_{-\infty}^{\infty} e^{i\sigma x} \varphi(x, y) dx, \quad \sigma \in \mathbf{R}.$$

DEFINITION C.6. Let $e_{is}(x) := \mu(x) e^{isx}$ for all $x \in \mathbf{R}$; here $\mu \in C^\infty(\mathbf{R})$ is a mollifier such that $\mu(x) = 0$ for $x \leq -1$ and $\mu(x) = 1$ for $x \geq 0$. The *partial Fourier-Laplace transform* $\langle \tilde{w}(s, \cdot), \cdot \rangle \in \mathcal{S}'(\mathbf{R})$ of $\langle w, \cdot \rangle \in \mathcal{S}'(\mathbf{R}^2)$ [with $\text{supp } w \subset [0, \infty) \times \mathbf{R}$ as we assume throughout this appendix] is defined by

$$\langle \tilde{w}(s, \cdot), \gamma \rangle := \langle w, e_{is} \otimes \gamma \rangle \quad \text{whenever } \tau > 0 \text{ and } \gamma \in \mathcal{S}(\mathbf{R}); \quad (\text{C.6})$$

here

$$(e_{is} \otimes \gamma)(x, y) := e_{is}(x) \gamma(y).$$

It seems worth while to state the result (c) in the next lemma because $\zeta^\dagger$ and $\tilde{\zeta}$ both arise in the plate problem. The proof is omitted; item (c) is not used in the paper, because the corresponding result for functions in A_μ is simpler (see (5.5) and the remark preceding it).

LEMMA C.7. (a) For each fixed $\gamma \in \mathcal{S}(\mathbf{R})$, the function with values $\langle \tilde{w}(s, \cdot), \gamma \rangle$ is holomorphic in $\{\tau > 0\}$.

$$(b) \langle (A * w)^\sim(s, \cdot), \cdot \rangle = \langle \tilde{A}(s) \tilde{w}(s, \cdot), \cdot \rangle \quad \text{in } \mathcal{S}'(\mathbf{R}).$$

(c) The partial Fourier transform of $\langle w, \cdot \rangle$ is implied by its partial Fourier-Laplace transform and the formula

$$\langle w^\dagger, \varphi \rangle = \lim_{\tau \rightarrow 0+} \int_{-\infty}^{\infty} \langle \tilde{w}(\sigma + i\tau, \cdot), \varphi(\sigma, \cdot) \rangle d\sigma \quad \text{for all } \varphi \in \mathcal{S}(\mathbf{R}^2).$$

Proof. (a) The proof does not differ significantly from that for other Laplace transforms of distributions (Schwartz 1952, p.202; Zemanian 1965, p.225).

(b) By the definitions (C.6), (C.3) and (C.1), for any $\gamma \in \mathcal{S}(\mathbf{R})$,

$$\langle (A * w)^\sim(s, \cdot), \gamma \rangle = \langle A * w, e_{is} \otimes \gamma \rangle \quad (\tau > 0)$$

$$\begin{aligned}
 &= \langle A \otimes w, m(e_{is} \otimes \gamma)_{(+)} \rangle \\
 &= \langle A, \langle w, m(e_{is} \otimes \gamma)_{(+)} \rangle \rangle.
 \end{aligned}$$

Now, for $t \geq 0$,

$$\langle w, m(e_{is} \otimes \gamma)_{(+)} \rangle(t) = e^{ist} \langle w, e_{is} \otimes \gamma \rangle,$$

where $\langle w, e_{is} \otimes \gamma \rangle$ is independent of t . Hence

$$\begin{aligned}
 \langle A, \langle w, m(e_{is} \otimes \gamma)_{(+)} \rangle \rangle &= \left\{ \int_0^\infty A(t) e^{ist} dt \right\} \langle w, e_{is} \otimes \gamma \rangle \\
 &= \tilde{A}(s) \langle \tilde{w}(s, \cdot), \gamma \rangle. \quad \square
 \end{aligned}$$

Appendix D. Various details

(i) *The case of a Newtonian fluid.* Lewis and Carrier (1949) solved the Oseen problem of flow past the plate P by means of the Wiener-Hopf technique; later it was found that a formulation in terms of parabolic co-ordinates yields an elementary derivation (Kaplun 1954; Van Dyke 1975, p.165). Writing $x + iy = re^{i\theta}$ once more, we define parabolic co-ordinates by

$$\xi = \left(\frac{U}{\nu}\right)^{\frac{1}{2}} r^{\frac{1}{2}} \cos \frac{\theta}{2} = \left(\frac{U}{2\nu}\right)^{\frac{1}{2}} (r+x)^{\frac{1}{2}} \operatorname{sgn} y,$$

$$\eta = \left(\frac{U}{\nu}\right)^{\frac{1}{2}} r^{\frac{1}{2}} \sin \frac{\theta}{2} = \left(\frac{U}{2\nu}\right)^{\frac{1}{2}} (r-x)^{\frac{1}{2}}, \quad 0 < \theta < 2\pi,$$

where ν is the kinematic viscosity of the Newtonian fluid. (There will be no confusion with the semi-characteristic co-ordinates called ξ and η elsewhere in this paper.)

(a) The disturbance stream function $\psi = \psi_0 + \psi_1$ of the solution for symmetrical flow (due to Lewis and Carrier) is given by

$$\psi_0(x, y) = -\frac{2\nu}{\sqrt{\pi}} \xi, \quad (\text{D.1})$$

$$\psi_1(x, y) = 2\nu\xi \left\{ \frac{1}{\sqrt{\pi}} e^{-\eta^2} - \eta \operatorname{erfc} \eta \right\}, \quad (\text{D.2})$$

where ψ_0 is the stream function of the irrotational part of the velocity field, and

$$\operatorname{erfc} \eta = 1 - \operatorname{erf} \eta = \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} e^{-t^2} dt. \quad (\text{D.3})$$

(b) A more general solution (Olmstead 1975) is given by $\psi_0 + \psi_1 + c\psi_2$, where c is an arbitrary real constant and

$$\psi_2(x, y) = \nu \left(\frac{2}{\sqrt{\pi}} \eta - \operatorname{erf} \eta \right). \quad (\text{D.4})$$

The stream function ψ_2 represents a flow along the parabolic streamlines $\{r-x = \text{const.}\}$ such that *both the velocity and the vorticity vanish on $\bar{P}$* . Therefore no

condition on the plate-vorticity function $\zeta(\cdot, 0+)$ can rule out ψ_2 . Moreover, the formula

$$\Delta(\psi_1 + c\psi_2)(x, y) = \frac{U}{\sqrt{\pi}} \frac{\xi + c\eta}{r} e^{-\eta^2}$$

shows that for $c \neq 0$ the vorticity is no worse, as $r \rightarrow 0$ and as $r \rightarrow \infty$, than it is for $c = 0$. Examination of the velocity yields a similar conclusion.

(c) It remains to present formulae for the solution $\psi_0 + \psi_1$ that are comparable with those in §§4 and 6(vii). The velocity

$$(u_0, v_0)(x, y) = \left(\frac{Uv}{\pi} \right)^{\frac{1}{2}} r^{-\frac{1}{2}} \left(-\sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right) \quad (\text{D.5})$$

is the disturbance velocity of irrotational flow, with velocity $(U, 0)$ at infinity, past the parabola $\{\eta = 1/\sqrt{\pi}\}$. For $x \rightarrow \infty$, it represents the displacement of ‘outer’ streamlines due to the retarded fluid near the plate.

The rotational velocity field

$$u_1(x, y) = U \left\{ \frac{1}{\sqrt{\pi}} \frac{\eta}{\xi^2 + \eta^2} e^{-\eta^2} - \operatorname{erfc} \eta \right\},$$

$$v_1(x, y) = -\frac{U}{\sqrt{\pi}} \frac{\xi}{\xi^2 + \eta^2} e^{-\eta^2},$$

is traditionally replaced, for $x \rightarrow \infty$ with η fixed, by the *boundary-layer approximation*

$$u_{(1)}(x, y) := -U \operatorname{erfc} Y, \quad Y := \frac{1}{2} \left(\frac{U}{v} \right)^{\frac{1}{2}} x^{-\frac{1}{2}} y \geq 0, \quad (\text{D.6a})$$

$$v_{(1)}(x, y) := -\left(\frac{Uv}{\pi} \right)^{\frac{1}{2}} x^{-\frac{1}{2}} \exp(-Y^2). \quad (\text{D.6b})$$

It is sufficient to bound the error of this approximation in a sector

$$S_0 := \{(x, y) \mid 0 \leq \theta \leq \theta_0\}, \quad \theta_0 \in (0, \frac{1}{2}\pi),$$

because outside S_0 both (u_1, v_1) and $(u_{(1)}, v_{(1)})$ are exponentially small as $r \rightarrow \infty$ with $y \geq 0$, and are dominated by (u_0, v_0) . One finds that, if $(x, y) \in S_0$ and $Ux/v \geq 1$, then

$$|(u_1 - u_{(1)})(x, y)| \leq \text{const. } x^{-1} Y \exp(-c_1 Y^2), \quad (\text{D.7a})$$

$$|(v_1 - v_{(1)})(x, y)| \leq \text{const. } x^{-3/2} Y^2 \exp(-c_1 Y^2), \quad (\text{D.7b})$$

for any $c_1 < 2/(1+\sec \theta_0)$. Of course, the other constants depend on c_1 .

(ii) *Properties of K that imply condition (I).* Let $\Gamma^3[0, \infty)$ denote the linear space of functions $\varphi : [0, \infty) \rightarrow \mathbf{R}$ such that

$$(a) \varphi(x) \rightarrow 0 \text{ as } x \rightarrow \infty,$$

$$(b) \varphi'' \text{ is absolutely continuous on } [0, \infty) \text{ and } \varphi''' \in L_1(0, \infty).$$

These conditions imply that $\varphi''(x) \rightarrow 0$ and $\varphi'(x) \rightarrow 0$ as $x \rightarrow \infty$. Accordingly, integration by parts yields

$$\tilde{\varphi}(s) = -\frac{\varphi(0)}{is} + \frac{\varphi'(0+)}{(is)^2} - \frac{\varphi''(0+)}{(is)^3} - \int_0^\infty \frac{e^{isx}}{(is)^3} \varphi'''(x) dx \quad (D.8)$$

whenever $\varphi \in \Gamma^3[0, \infty)$, $\tau \geq 0$ and $s \neq 0$. The integral is $o(s^{-3})$ as $s \rightarrow \infty$, by the Riemann-Lebesgue lemma.

Now let K be a shear kernel of the form

$$K(x) = f_0(x) - x^\alpha f_1(x) + [K'] f_2(x-c), \quad x \geq 0, \quad (D.9)$$

where $\alpha \in (0, 1)$, $[K']$ and c are as in (6.3), and (in order that the last term be a kink term) $f_2(t) = 0$ for $t \leq 0$, $f_2'(0+) = 1$.

It is sufficient for condition (I) that f_0, f_2 and the function with values $(1+x)^\alpha f_1(x)$ all belong to $\Gamma^3[0, \infty)$. (The factor $(1+x)^\alpha$ is significant for f_1 only as regards magnitudes near infinity.)

The desired expansion of $\tilde{K}(s)$ for $s \rightarrow \infty$ is obtained as follows. For f_0 and f_2 we simply apply (D.8), noting that $f_2(x-c)$ transforms to $e^{ics} \tilde{f}_2(s)$. For the middle term, we write

$$x^\alpha f_1(x) = g(x) + h(x),$$

where

$$g(x) := x^\alpha e^{-x} \{f_1(0)(1+x) + f_1'(0+)x\}, \quad x \geq 0.$$

Then $\tilde{g}$ is known explicitly, $h \in \Gamma^3[0, \infty)$ and $h(0) = h'(0+) = h''(0+) = 0$.

(iii) *Some Fourier-Laplace transforms.* The functions in (6.3) are defined for $x \geq 0$ by

$$K_1(x) = M^{-2} e^{-x},$$

$$K_2(x) = M^{-2} e^{-x} - \gamma_\alpha \frac{x^\alpha}{\alpha!} e^{-2x}, \quad \gamma_\alpha > 0, \quad \alpha \in (0, 1),$$

$$K_3(x) = M^{-2} (1+x)^{-1-l}, \quad l \in (0, 1),$$

$$g(x) = (x-c) e^{-2(x-c)} H(x-c).$$

In terms of the abbreviation p introduced in (6.1), their Fourier-Laplace transforms have analytic continuations as follows onto $\mathbb{C} \setminus \{i\tau \mid \tau \leq 0\}$. (Indeed, all but $\tilde{K}_3$ have analytic continuations onto bigger subsets of $\mathbb{C}$.)

$$\tilde{K}_1(s) = M^{-2} (p+1)^{-1},$$

$$\tilde{K}_2(s) = M^{-2} (p+1)^{-1} - \gamma_\alpha (p+2)^{-1-\alpha},$$

$$\tilde{K}_3(s) = M^{-2} l^{-1} \{ 1 - (-l)! e^p p^l + e^p \sum_{n=0}^{\infty} \frac{(-1)^n p^{n+1}}{n!(n-l+1)} \},$$

$$\tilde{g}(s) = (p+2)^{-2} e^{-cp}.$$

The series in the formula for $\tilde{K}_3$ defines an entire function. Also, as $s \rightarrow \infty$ in $\mathbb{C} \setminus \{i\tau \mid \tau \leq 0\}$,

$$\tilde{K}_3(s) = M^{-2} \{ p^{-1} - (1+l)p^{-2} \} + O(s^{-3}).$$

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Figure 1. Notation