**DISAPPEARANCE OF PHASE
IN THE STEFAN PROBLEM:
ONE SPACE DIMENSION**

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I. Introduction

We consider a classical two phase Stefan problem with a bounded piece of ice in a very large bath of water. At some finite time T , the ice phase disappears and it is natural to ask if the ice approaches an asymptotic shape as $t \nearrow T$. Faced with this question most well-trained applied mathematicians will immediately seek a self-similar solution, and will soon discover that no such solution exists.

Extensive numerical studies of two-phase Stefan problems carried out by Professor M. Fremond at the Laboratoire Central des Ponts et Chaussées in Paris provide some very strong evidence for the existence and properties of an asymptotic shape [Fre]. In Fremond's calculations it is very clear that as $t \nearrow T$, the temperature in the disappearing (ice) phase falls rapidly and uniformly to zero. Thus, close to the extinction time T , we are essentially dealing with a one-phase problem. Consequently, the asymptotic shape of the disappearing phase in the two-phase problem should be the same as the asymptotic phase of the disappearing phase in the one-phase problem (if it exists).

For one space dimension, it is well known that the asymptotic shape of the disappearing phase in the one-phase Stefan problem is a space-time wedge, with its tip at $t = T$. In this paper, we prove that the same is true for the two-phase problem, thus verifying that, at least in one space dimension, Fremond's calculation represents the general situation. Our proof relies on various known facts about the Stefan problem in one-space demension and some new comparison results which show that the heat flux across the boundary of the ice phase goes to zero and that the temperature in the ice phase goes exponentially to zero as $t \nearrow T$.

In subsequent papers in this series, we will deal with the problem in multi-dimensional space. Here things are much more difficult, since very little is known about the asymptotic shape of the disappearing phase for even the one-phase problem.

We begin with the formulation of the one-phase Stefan problem in one space dimension:

(P) Given functions $\varphi_1(x), \varphi_2(x), \varphi_3(x), f_1(t)$ and $f_2(t)$ find functions $u_1(x, t), u_2(x, t), u_3(x, t), s_1(t)$ and $s_2(t)$ such that

$$\frac{\partial u_1}{\partial t} = c_1^2 \frac{\partial^2 u_1}{\partial x^2} \quad \text{in } D_1, \quad (1.1)$$

$$\frac{\partial u_2}{\partial t} = c_2^2 \frac{\partial^2 u_2}{\partial x^2} \quad \text{in } D_2, \quad (1.2)$$

$$\frac{\partial u_3}{\partial t} = c_1^2 \frac{\partial^2 u_3}{\partial x^2} \quad \text{in } D_3, \quad (1.3)$$

where

$$\begin{aligned}
D_1 &= \{(x, t) , t_1 < t < 0 , s_1(t) < x < \ell\} , \\
D_2 &= \{(x, t) , t_1 < t < 0 , s_2(t) < x < s_1(t)\} , \\
D_3 &= \{(x, t) , t_1 < t < 0 , -\ell < x < s_2(t)\} , \\
u_i &\in C^{1,2}(\overline{D_i}) , u_1 > 0 , u_2 < 0 , u_3 > 0 , \\
u_i(x, t_1) &= \varphi_i(x) , \varphi_1 > 0 , \varphi_3 > 0 , \varphi_2 < 0 , \\
u_1(\ell, t) &= f_1(t) > 0 , u_3(-\ell, t) = f_2(t) > 0 ,
\end{aligned}$$

$$\begin{aligned}
u_1(s_1(t), t) &= u_2(s_1(t), t) = 0 , \\
u_2(s_2(t), t) &= u_3(s_2(t), t) = 0 ,
\end{aligned}$$

and

$$\begin{aligned}
\frac{ds_1}{dt} &= \left[-\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} \right]_{x=s_1(t)} , \\
\frac{ds_2}{dt} &= \left[-\frac{\partial u_3}{\partial x} + \frac{\partial u_2}{\partial x} \right]_{x=s_2(t)} .
\end{aligned}$$

We define a weak solution of (P) in the following way: Let $D = \{x, t : x \in (-\ell, \ell) , t \in (t_1, 0)\}$ and let $\varphi(x) = \varphi_i(x)$ ($i = 1, 2, 3$) if x belongs to the interval $(s_1(0), \ell)$, $(s_2(0), s_1(0))$, $(-\ell, s_2(0))$ respectively. We define the enthalpy function $a(u)$ by

$$a(u) = \begin{cases} \frac{1}{c_1^2} u & \text{if } u < 0 \\ \frac{1}{c_2^2} u + 1 & \text{if } u > 0 \end{cases}$$

and $a(u)$ at zero temperature can take any value in $[0, 1]$. Let u be a continuous function in $\overline{D}$ which satisfies $u(x, 0) = \varphi(x)$, $u(-\ell, t) = f_2(t)$, $u(\ell, t) = f_1(t)$. We say that such $u(x, t)$ is a weak solution of (P) if it satisfies the equation

$$\frac{\partial a(u)}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

in the sense of distributions.

The unique weak solution of this problem exists and is a continuous function [K]. It is proved in [Fri] that under our assumptions the set $u = 0$ consists of two continuous curves $s_1(t), s_2(t)$, for $t < T$, with $s_1(T) = s_2(T)$. The solution is positive everywhere for $t > T$. Thus $t = T$ is the time of the disappearance of the solid phase. To simplify the writing we assume, without loss of

generality, that $T = 0$ and the point of disappearance is $s_1(T) = s_2(T) = 0$. It is shown in [FPK] that $s_i(t) \in C^\infty(t_1, 0)$. Moreover, the derivatives $\frac{\partial u_i}{\partial t}$, $\frac{\partial u_i}{\partial x}$, $\frac{\partial^2 u_i}{\partial x^2}$ exist and are continuous for $t < 0$.

The main result of the paper is the

Theorem. *Let $\{u_i(x, t), s_i(t)\}$ be the classical solution of problem (P). There exists constants $C \in \mathbb{R}^+$, $\alpha \in \mathbb{R}^+$, $\tilde{t} \in \mathbb{R}^-$, $\lambda_1 \in \mathbb{R}^-$ and $\lambda_2 \in \mathbb{R}^+$ such that*

$$|u_2(x, t)| \leq C e^{\alpha/t} \quad \text{for } t \in (\tilde{t}, 0), \quad s_2(t) \leq x \leq s_1(t). \quad (1.4)$$

Moreover,

$$\lim_{t \nearrow 0} \frac{\partial u_2}{\partial x} \Big|_{x=s_i(t)} = 0, \quad i = 1, 2, \quad (1.5)$$

and

$$\lim_{t \nearrow 0} \frac{ds_i}{dt} = \lambda_i, \quad i = 1, 2, \quad -\infty < \lambda_1 < 0, \quad 0 < \lambda_2 < \infty. \quad (1.6)$$

II. Proof of the Theorem

The proof of the theorem is contained in the following three lemmas. In Lemma 1 we prove the boundedness of the $\left| \frac{ds_i}{dt} \right|$ and the existence of the limits of $\frac{\partial u_1}{\partial x}(s_1(t), t)$ and $\frac{\partial u_3}{\partial x}(s_2(t), t)$ as $t \nearrow 0$. This lemma was first proved in [CDH] by means of comparison methods and some regularity results of Gevrey [G]. We give an independent proof which also uses comparison methods but avoids the necessity of referring to [G] and so is selfcontained. In Lemma 2 we prove (1.5) which says that the heat flux from the ice phase vanishes as $t \nearrow 0$. Lemmas 1 and 2 imply the existence of the limits of the ds_i/dt as $t \nearrow 0$ and so prove (1.6). Finally, we prove (1.4) in Lemma 3. The proof uses a comparison argument which exploits the existence of the limits (1.6).

Lemma 1. *There exists a constant K such that*

$$\left| \frac{ds_i}{dt} \right| \leq K. \quad (i = 1, 2)$$

Moreover

$$\lim_{t \nearrow 0} \frac{\partial u_1}{\partial x}(s_1(t), t) \quad \text{and} \quad \lim_{t \nearrow 0} \frac{\partial u_3}{\partial x}(s_2(t), t)$$

exist.

Proof:

Fig. 1

Since $u(x, 0) > 0$ for $x > 0$, by continuity we can choose $x_0 > 0$ and $t_0 < 0$ such that $u(\pm x_0, t) > 0$ for $t \in [t_0, 0]$. Let $R \equiv \{(x, t) : |x| \leq x_0, t_0 \leq t \leq 0\}$ and note that

$$-x_0 < s_2(t) \leq s_1(t) < x_0 \quad \text{for} \quad t \in [t_0, 0],$$

i.e., the free boundaries $x = s_i(t)$ intersect the boundary of R only at the origin and on the open lower boundary (cf. Fig. 1). Since $u(x, t) \rightarrow 0$ as $(x, t) \rightarrow (0, 0_-)$ we can always choose x_0 and t_0 so small that

$$|u(x, t)| \leq \frac{1}{4} \min\{c_1^2, c_2^2\} \quad \text{in} \quad R. \quad (2.1)$$

For $\tau \in (t_0, 0)$ define

$$\sigma(\tau) \equiv \sup \left\{ \left| \frac{ds_i}{dt} \right| : t \in [t_0, \tau], i = 1, 2 \right\} \quad (2.2)$$

We shall derive a bound for $\sigma(\tau)$ which depends only on the values of u on the parabolic boundary

$$\partial_p R \equiv \{(x, t) : x = \pm x_0, t_0 \leq t \leq 0\} \cup \{(x, t_0) : |x| \leq x_0\}$$

of R and which is independent of τ .

Let $R_1 \subset R$ be given by

$$R_1 \equiv \{(x, t) : s_1(t) \leq x \leq x_0, t_0 \leq t \leq 0\}.$$

In R_1 , we compare u with

$$w(x, t) \equiv \frac{c_1^2}{3} \left\{ 1 - e^{-\frac{1}{c_1^2}(\sigma + \gamma)(x - s_1(t))} \right\} \quad (2.3)$$

where $\gamma > 0$ will be specified later. By direct computation and (2.2)

$$e^{(\sigma + \gamma)(x - s_1)/c_1^2} (w_t - c_1^2 w_{xx}) = \frac{\sigma + \gamma}{3} \{\sigma + \gamma - s_1'(t)\} > 0 \quad \text{in} \quad R_{1\tau}, \quad (2.4)$$

where $R_{1\tau} = R_1 \cap \{t : t \leq \tau\}$. Therefore, w is a supersolution of the heat equation (1.1) in $R_{1\tau}$.

It is clear that

$$w(s_1(t), t) = u_1(s_1(t), t) = 0. \quad (2.5)$$

Moreover,

$$w(x_0, t) \geq \frac{c_1^2}{3} \{1 - e^{-(\sigma+\gamma)\delta_1/c_1^2}\},$$

where $\delta_1 = \min_{[t_0, 0]} \{x_0 - s_1(t)\}$. Thus we can choose γ so large that

$$w(x_0, t) > \frac{c_1^2}{4} \geq u_1(x_0, t) \quad \text{on} \quad [t_0, 0]. \quad (2.6)$$

Let

$$M = \max \left\{ \left| \frac{\partial u_1}{\partial x}(x, t_0) \right| : s_1(t_0) \leq x \leq x_0 \right\},$$

then

$$u(x, t_0) \leq M \{x - s_1(t_0)\} \quad \text{on} \quad [s_1(t_0), x_0].$$

Since

$$\frac{\partial w_1}{\partial x}(s_1(t_0), t_0) = \frac{\sigma + \gamma}{3}$$

we can choose $\gamma = \gamma_0$ so that

$$w_x(s_1(t_0), t_0) = \frac{\sigma + \gamma_0}{3} > M > u_{1x}(x, t_0).$$

This implies the existence of a $\rho_0 = \rho(\gamma_0)$ such that

$$w(x, t_0) \geq u_1(x, t_0) \quad \text{on} \quad [s_1(t_0), s_1(t_0) + \rho_0]. \quad (2.7)$$

However, since w is an increasing function of γ it follows that (2.7) holds for all $\gamma \geq \gamma_0$. On the other hand, $w(x, t_0) \rightarrow \frac{c_1^2}{3}$ as $\gamma \rightarrow \infty$ uniformly on $[s_1(t_0) + \rho_0, x_0]$. Therefore, in view of (2.1), if γ is sufficiently large,

$$w(x, t_0) \geq u_1(x, t_0) \quad \text{on} \quad [s_1(t_0), x_0]. \quad (2.8)$$

Choose γ_1 so large that both (2.6) and (2.8) hold. Note that γ_1 depends only on the values of u_1 on $\partial_p R$ and that γ_1 is independent of τ . In view of (2.4), (2.5), (2.6) and (2.8) it follows by comparison that if $\gamma \geq \gamma_1$ then

$$u_1(x, t) \leq w(x, t) \quad \text{in} \quad R_1. \quad (2.9)$$

Moreover, since $u_1 = w_1 = 0$ on $x = s_1(t)$ we have for $\gamma \geq \gamma_1$

$$0 \leq \frac{\partial u_1}{\partial x}(s_1(t), t) \leq \frac{1}{3}(\sigma + \gamma) \quad \text{on} \quad [t_0, \tau] . \quad (2.10)$$

By similar arguments applied to u_2 in R_2 and u_3 in R_3 (cf. Fig. 1), we obtain the existence of constants γ_2 and γ_3 depending only on the values of u on $\partial_p R$ and independent of τ such that for $\gamma > \gamma_2$

$$\left| \frac{\partial u_2}{\partial x}(s_1(t), t) \right| \leq \frac{1}{3}(\sigma + \gamma_2) \quad \text{and} \quad \left| \frac{\partial u_2}{\partial x}(s_2(t), t) \right| \leq \frac{1}{3}(\sigma + \gamma_2) \quad \text{on} \quad [t_0, \tau] , \quad (2.11)$$

and for $\gamma > \gamma_3$

$$\left| \frac{\partial u_3}{\partial x}(s_2(t), t) \right| \leq \frac{1}{3}(\sigma + \gamma_3) \quad \text{on} \quad [t_0, \tau] . \quad (2.12)$$

Finally, we take $\gamma = \max\{\gamma_1, \gamma_2, \gamma_3\}$.

By the Stefan condition

$$\left| \frac{ds_1}{dt} \right| = \left| \frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x} \right|_{x=s_1} .$$

In view of (2.11), (2.10), and the definition of γ we have

$$\left| \frac{ds_1}{dt} \right| \leq \frac{2}{3}(\sigma + \gamma) \quad \text{on} \quad [t_0, \tau] .$$

It follows from the definition of σ that $\sigma \leq 2\gamma$, where γ is independent of τ . Therefore, we have

$$\left| \frac{ds_1}{dt} \right| \leq 2\gamma \quad \text{on} \quad [t_0, 0) \quad (2.13)$$

and, by a similar argument

$$\left| \frac{ds_2}{dt} \right| \leq 2\gamma \quad \text{on} \quad [t_0, 0) .$$

Next, we prove the second assertion of the lemma. It will suffice to prove the existence of $\lim_{t \nearrow 0} \frac{\partial u_1}{\partial x}(s_1(t), t)$ since the proof for u_2 is completely analogous. In view of (2.10) we have

$$0 \leq \frac{\partial u_1}{\partial x}(s_1(t), t) \leq \gamma \quad \text{on} \quad [t_0, 0) . \quad (2.14)$$

Moreover, $\frac{\partial u_1}{\partial x}$ is certainly bounded on the remainder of $\partial_p R_1$. Thus, by the maximum principle, $\frac{\partial u_1}{\partial x}$ is uniformly bounded in R_1 .

Since

$$u_t(s_1(t), t) = -\frac{\partial u_1}{\partial x}(s_1(t), t) \frac{ds_1}{dt}$$

it follows from (2.13) and (2.14) that

$$|u_t(s_1(t), t)| \leq 2\gamma^2 . \quad (2.15)$$

Thus, by (1.1), we have

$$\left| \frac{\partial^2 u_1}{\partial x^2}(s_1(t), t) \right| = \frac{1}{c_1^2} \left| \frac{\partial u_1}{\partial t}(s_1(t), t) \right| \leq \frac{2\gamma^2}{c_1^2} .$$

Again, since $\partial^2 u_1 / \partial x^2$ is bounded on the remainder of $\partial_p R_1$, we conclude, using the maximum principle, that $\frac{\partial^2 u_1}{\partial x^2}$ is uniformly bounded in R_1 . Note that, in particular, this implies that the family of functions $\left\{ \frac{\partial u_1}{\partial x}(\cdot, t) \right\}$ is equicontinuous with respect to x in R_1 .

Let $\delta > 0$ be so small that

$$s_1(t) + \delta < x_0 \quad \text{on} \quad [t_0, 0] .$$

Multiply both sides of the heat equation (1.1) by u_{xx} and integrate over the set

$$\{(x, t) : s_1(t) < x < s_1(t) + \delta, \ t_1 < t < t_2\} ,$$

where $t_0 < t_1 < t_2 < 0$. After integration-by-parts we obtain

$$\frac{1}{2} \int_{s_1(t)}^{s_1(t)+\delta} (u_x)^2 dx \Big|_{t_1}^{t_2} \leq \int_{t_1}^{t_2} u_t u_x dt \Big|_{s_1(t)}^{s_1(t)+\delta} . \quad (2.16)$$

By (2.10), (2.13) and (2.15) the right hand side of (2.16) is bounded by $C|t_2 - t_1|$, where C is a constant which is independent of t_1 and t_2 . Therefore

$$\left| \int_{s_1(t_2)}^{s_1(t_2)+\delta} u_x^2(x, t_2) dx - \int_{s_1(t_1)}^{s_1(t_1)+\delta} u_x^2(x, t_1) dx \right| \leq C(t_2 - t_1) . \quad (2.17)$$

Now consider the family $\{f_t(y)\}$ where

$$f_t(y) \equiv u_x(y + s_1(t), t) \quad \text{for} \quad y \in [0, \delta], \ t \in [t_0, 0] .$$

As we observed above, the family $\{f_t(y)\}$ is uniformly bounded and equicontinuous. It therefore contains a uniformly convergent subsequences. However, in view of (2.17), the whole family converges as $t \rightarrow 0$ uniformly with respect to y . In particular

$$\lim_{t \nearrow 0} f_t(0) = \lim_{t \nearrow 0} u_{1x}(s_1(t), t)$$

exists. ■

Lemma 2. Let $v(x, t)$ be a solution of the heat equation $Lv = v_t - a^2 v_{xx} = 0$ in the domain $D = \{(x, t) , t_0 < t < 0 , s_2(t) < x < s_1(t)\}$. Assume that, for $i = 1, 2$, $s_i(t) \in C[t_0, 0] \cap C^\infty[t_0, 0)$,

$$\left| \frac{ds_i}{dt} \right| \leq K \quad \text{in } [t_0, 0) \quad , \quad s_i(0) = 0 \quad , \quad v(x, t) \in C(\overline{D}) ,$$

$$v(s_i(t), t) = 0 \quad , \quad v(x, t) \leq 0 \quad \text{in } \overline{D} .$$

Then

$$\lim_{t \rightarrow 0} \frac{\partial v}{\partial x} \Big|_{x=s_i(t)} = 0 \quad , \quad i = 1, 2 \quad , \quad (2.18)$$

Proof: It follows from the assumptions of the Lemma that $\frac{\partial v}{\partial x} \Big|_{x=s_i(t)}$ exists for $t_0 < t < 0$. Let

$$w(x, t) = (s_1(t) - x)(s_2(t) - x) \quad , \quad x \in [s_2, s_1] .$$

We have

$$Lw = s_1'(s_2 - x) + s_2'(s_1 - x) - 2 < 0$$

for $t \in (\eta, 0)$ and $|\eta|$ small enough. Let $\eta > t_0$. Then for K large enough

$$Kw(x, \eta) \leq v(x, \eta) \quad \text{for } x \in [s_2(\eta), s_1(\eta)]$$

and $Kw(s_i(t), t) = v(s_i(t), t) = 0$. By maximum principle

$$Kw(x, t) \leq v(x, t)$$

for $t \geq \eta$ and hence

$$0 < \frac{\partial v}{\partial x} \Big|_{x=s_1} < K \frac{\partial w}{\partial x} \Big|_{x=s_1} = K(s_1 - s_2) \longrightarrow 0$$

as $t \rightarrow 0$. In the same way $\frac{\partial v}{\partial x} \Big|_{x=s_2} \longrightarrow 0$ as $t \rightarrow 0$. ■

Lemma 2 implies (1.5). Moreover, by Lemmas 1 and 2

$$\lim_{t \nearrow 0} \frac{ds_1}{dt} = \lim_{t \nearrow 0} \left(-\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} \right) = -\lim_{t \nearrow 0} \frac{\partial u_1}{\partial x} = \lambda_1 > -\infty .$$

By strong maximum principle $\frac{\partial u_1}{\partial x}(0, 0) > 0$ therefore $\lambda_1 < 0$. By the same way $\lim_{t \nearrow 0} \frac{ds_2}{dt} = \lambda_2$, $0 < \lambda_2 < \infty$. Thus (1.6) is proved.

Remark. It follows from (1.6) that free boundaries $x = s_i(t)$ ($i = 1, 2$) become eventually monotonic curves.

Estimate (1.4) is a consequence of the next lemma.

Lemma 3. For some constants $C > 0$, $\alpha > 0$, $|u_2(x, t)| \leq Ce^{\frac{\alpha}{t}}$ in D_2 .

Proof: Consider $w(x, t) = (x^2 - \beta t^2)e^{\frac{\alpha}{t}}$ in the triangle $\eta \leq t \leq 0$, $x^2 \leq \beta t^2$, for some constants α, β . We have

$$\begin{aligned} w_t - c_2^2 w_{xx} &= e^{\frac{\alpha}{t}} \left(-2c_2^2 - \frac{\alpha}{t^2}(x^2 - \beta t^2) - 2\beta t \right) = \\ &= e^{\frac{\alpha}{t}} \left(-2c_2^2 + \alpha\beta - 2\beta t - \frac{\alpha x^2}{t^2} \right) \leq \\ &= e^{\frac{\alpha}{t}} (-2c_2^2 + 2\beta|t| + \alpha\beta) . \end{aligned}$$

For fixed β we choose some $\alpha < \frac{c_2^2}{\beta}$ and $\nu \in [\eta, 0)$ with $|\nu| < \frac{c_2^2}{2\beta}$. For these α and ν the function $w(x, t)$ is a subsolution of the heat equation (1.2) for $\nu \leq t < 0$, $x^2 \leq \beta t^2$. Fix $\beta > \max\{|\lambda_1|, |\lambda_2|\}$.

Now we compare $u_2(x, t)$ and $Kw(x, t)$ in the domain $D_2 \cap \{(\eta, 0)\}$. For K large enough

$$Kw(x, \eta) \leq u_2(x, \eta)$$

and by our choice of β

$$Kw(s_i(t), t) < 0 = u_2(s_i(t), t) , \quad i = 1, 2 .$$

Thus

$$Kw(x, t) \leq u_2(x, t) \quad \text{in } D_2 \cap (t, 0)$$

and

$$|u_2(x, t)| \leq Ce^{\frac{\alpha}{t}} . \quad \blacksquare$$

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