# A STRICTLY HYPERBOLIC SYSTEM OF CONSERVATION LAWS ADMITTING SINGULAR SHOCKS

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**IMA Preprint Series # 602**

February 1990

This paper will appear in the IMA
Volumes in Mathematics and its
Applications Volume 27 on Nonlinear
Evolution Equations that Change Type

# A STRICTLY HYPERBOLIC SYSTEM OF CONSERVATION LAWS ADMITTING SINGULAR SHOCKS

HERBERT C. KRANZER† AND BARBARA LEE KEYFITZ‡

**1. Introduction.** The system

$$
\left\{ \begin{array}{l} u_t + (u^2 - v)_x = 0 \\ v_t + (\frac{1}{3}u^3 - u)_x = 0 \end{array} \right. \tag{1}
$$

is an example of a strictly hyperbolic, genuinely nonlinear system of conservation laws. Usually the Riemann problem for such a system is well-posed: centered weak solutions consisting of combinations of simple waves and admissible jump discontinuities (shocks) exist and are unique for each set of values of the Riemann data [1-3]. The characteristic speeds $\lambda_1$ and $\lambda_2$ of system (1), however, do not conform to the usual pattern for strictly hyperbolic, genuinely nonlinear systems: although locally separated, they overlap globally (cf. Keyfitz [4] for a more general discussion of the significance of overlapping characteristic speeds). In particular, $\lambda_1 = u - 1$ and $\lambda_2 = u + 1$ are real and unequal at any particular point $U = (u, v)$ of state space (as strict hyperbolicity requires), and $\lambda_2 - \lambda_1 = 2$ is even bounded away from zero globally, but $\lambda_1$ at one point $U_1$ may be equal to $\lambda_2$ at a different point $U_2$. The corresponding right eigenvectors $\mathbf{r}_1 = (1, u + 1)$ and $\mathbf{r}_2 = (1, u - 1)$ of the gradient matrix for (1) display genuine nonlinearity, since $\mathbf{r}_i \cdot \nabla\lambda_i > 0$ for $i = 1, 2$ but the two eigenvalues vary in the same direction: $\mathbf{r}_i \cdot \nabla\lambda_j > 0$ for $i \neq j$, rather than the usual "opposite variation" $\mathbf{r}_i \cdot \nabla\lambda_j < 0$ familiar from (say) gas dynamics. As a result, classical global existence and uniqueness theorems [3,5] no longer apply.

In Section 2, we investigate the Riemann problem for system (1). We find that the rarefaction curves cover the $u, v$-plane smoothly, but that the Hugoniot loci are compact curves. As a result, for each fixed left-hand state $U_L$ there is a large region of the plane which cannot be reached from $U_L$ by any combination of rarefaction waves and admissible shocks, even if we were to admit as shocks jump discontinuities which violate the Lax entropy condition.

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In Section 3, we introduce a new type of solution to (1), called a *singular shock*, which might be used to connect the left state $U_L$ to right states $U_R$ in this inaccessible region. We discuss how singular shocks may appear as limits of solutions to the Dafermos-DiPerna viscosity approximation

$$\begin{cases} u_t + (u^2 - v)_x = \epsilon t u_{xx} \\ v_t + (\frac{1}{3}u^3 - u)_x = \epsilon t v_{xx} \end{cases} \tag{2}$$

Solutions to (2) do not always remain uniformly bounded as $\epsilon \to 0^+$, but may instead approach singular distributions similar to modified Dirac $\delta$-functions. (Singular solutions of this type were first found by Korchinski [6] for a nonstrictly hyperbolic system.) We investigate the asymptotic behavior of these solutions for small $\epsilon$ and discuss how their limits may be regarded as shocks with internal structure. We derive a generalized form of the Rankine-Hugoniot condition for these singular shocks and introduce two additional admissibility conditions for them. These conditions allow us to prove our principal result (Theorem 2), which asserts that the Riemann problem for (1) becomes well-posed for all Riemann data when the category of solutions is enlarged to include admissible singular shocks.

Finally, in Section 4 we attempt to show that the admissible singular shocks which we have defined are actually limits of solutions to the Dafermos-DiPerna approximation (2). We present some analytic results and some numerical calculations as supporting evidence for this conjecture, and we describe what would be needed to convert the conjecture into a theorem.

**2. The classical solution and its limitations.** Since system (1) is strictly hyperbolic and genuinely nonlinear, the Riemann problem

$$U(x,0) = \begin{pmatrix} u \\ v \end{pmatrix}(x,0) = \begin{cases} U_L, & x < 0 \\ U_R, & x > 0 \end{cases} \tag{3}$$

for (1) has a classical solution when $U_R$ is sufficiently close to $U_L$. To describe this classical solution, we rewrite (1) as

$$U_t + F_x \equiv U_t + AU_x = 0 \tag{4}$$

with

$$F = \begin{pmatrix} u^2 - v \\ \frac{1}{3}u^3 - u \end{pmatrix}$$

and

$$A = \frac{\partial F}{\partial U} = \begin{pmatrix} 2u & -1 \\ u^2 - 1 & 0 \end{pmatrix}.$$

The eigenvalues of $A$ are the characteristic speeds $\lambda_1 = u - 1$ and $\lambda_2 = u + 1$ with corresponding right eigenvectors

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ u+1 \end{pmatrix}, \qquad \mathbf{r}_2 = \begin{pmatrix} 1 \\ u-1 \end{pmatrix}.$$

The rarefaction curves $R_i$ are the integral curves of the $\mathbf{r}_i$, namely

$$R_1 : v = \frac{1}{2}u^2 + u + c_1$$

and

$$R_2 : v = \frac{1}{2}u^2 - u + c_2.$$

The Rankine-Hugoniot condition $s[U] = [F]$ defines the *Hugoniot locus* $H(U_0)$, the set of $U$-points which can be connected across a jump discontinuity to $U_0$, as

(5) $$[v] = [u]\left(\frac{u + u_0}{2} \pm \sqrt{1 - [u]^2/12}\right).$$

where $[u] = u - u_0, [v] = v - v_0$; the Rankine-Hugoniot condition also determines the propagation speed $s$ of the discontinuity as

(6) $$s = u_0 + [u]/2 \mp \sqrt{1 - [u]^2/12}.$$

We note that the Hugoniot locus is restricted to the strip $|u - u_0| \leq \sqrt{12}$ and consists of four branches in the neighborhood of $(u_0, v_0)$ which join to form a figure eight (see Figure 2.1). In particular, the locus is compact, and its compactness places a finite upper bound on the strength of any discontinuity which can occur as part of a weak solution of (4).

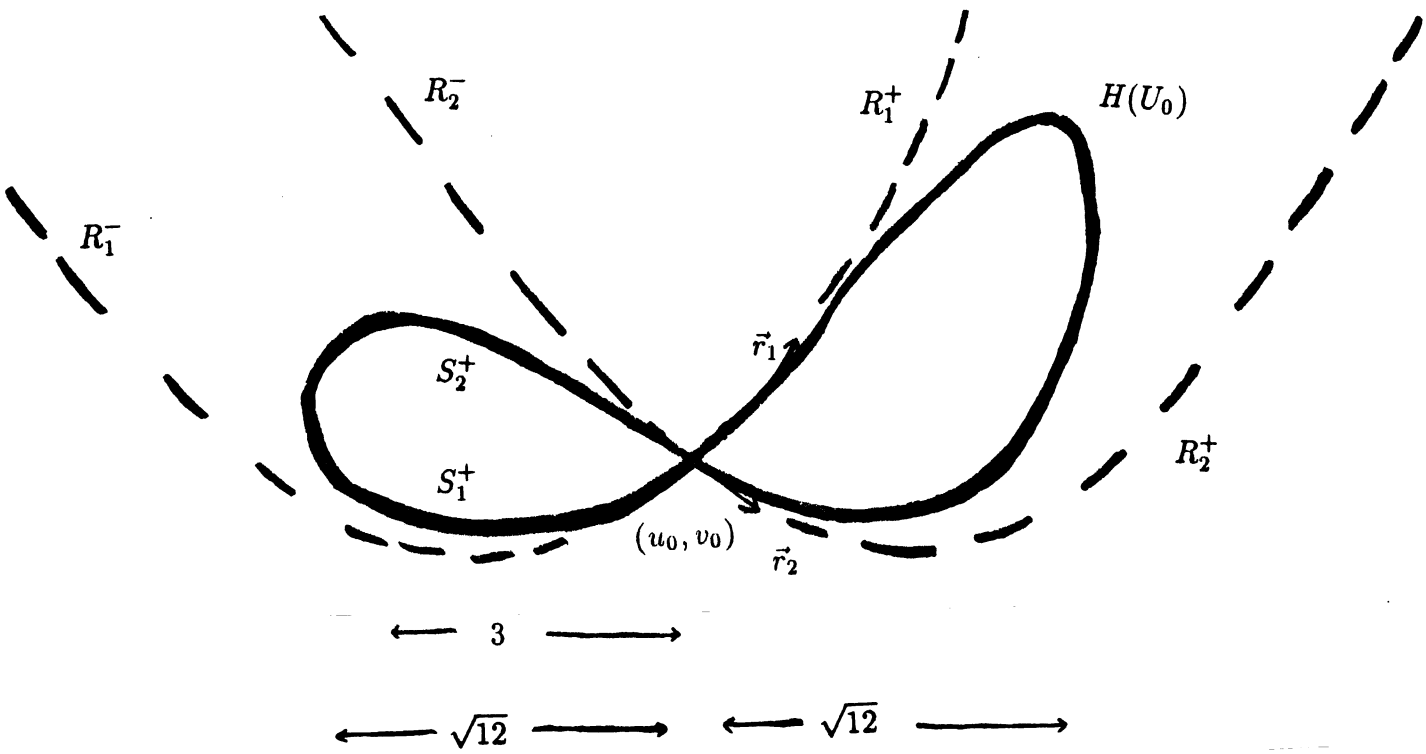


Figure 2.1

To solve the Riemann problem, we need to identify the admissible (+) portions of $H$ and $R_i$ with respect to any state $U_0$, considered to be on the left. For the rarefaction curves, we have

$$
(7)\qquad \begin{aligned} R_1^+(U_0) &: v = v_0 + u^2/2 + u - u_0^2/2 - u_0, u_0 \le u; \\ R_2^+(U_0) &: v = v_0 + u^2/2 - u - u_0^2/2 + u_0, u_0 \le u. \end{aligned}
$$

The shock curves $S_i^+(U_0)$ are classically defined as those portions of $H(U_0)$ which satisfy the Lax entropy condition for discontinuities of the $i$-th family. A straightforward calculation identifies the $S_i^+$ as those portions of the curves (5) which lie within the narrower strip $u_0 - 3 \le u \le u_0$; $S_1^+$ has the upper sign in (5) and $S_2^+$ the lower sign. For a classical Riemann solution to exist with $U_L = U_0$, the right-hand state $U_R$ must lie on either $R_2^+(U_m)$ or $S_2^+(U_m)$ for some intermediate state $U_m$ which, in turn, lies on either $R_1^+(U_0)$ or $S_1^+(U_0)$, leading to the standard partition of the $U$-plane into four regions. However, in the present case the four "standard" regions do not fill the entire $U$-plane. Instead, they cover only the domain $Q = Q(U_0)$ described by

$$v_-(u) \le v \le v_+(u), \qquad u_0 - \sqrt{12} \le u \le +\infty,$$

with

$$(8) \qquad v_\pm(u) = \begin{cases} v_0 + [u](\frac{1}{2}(u+u_0) \mp \sqrt{1-[u]^2/12}), & u_0 - \sqrt{12} \le u \le u_0 - 3; \\ v_0 + u^2/2 - u_0^2/2 \pm [u] \pm \frac{9}{2}, & u_0 - 3 \le u < +\infty \end{cases}$$

(See Figure 2.2). When $U_R$ lies in the complement of $Q(U_L)$ (which in the curvilinear $R_1 - R_2$ coordinate system occupies approximately three fourths of the plane), no classical solution to the Riemann problem (1), (3) exists.

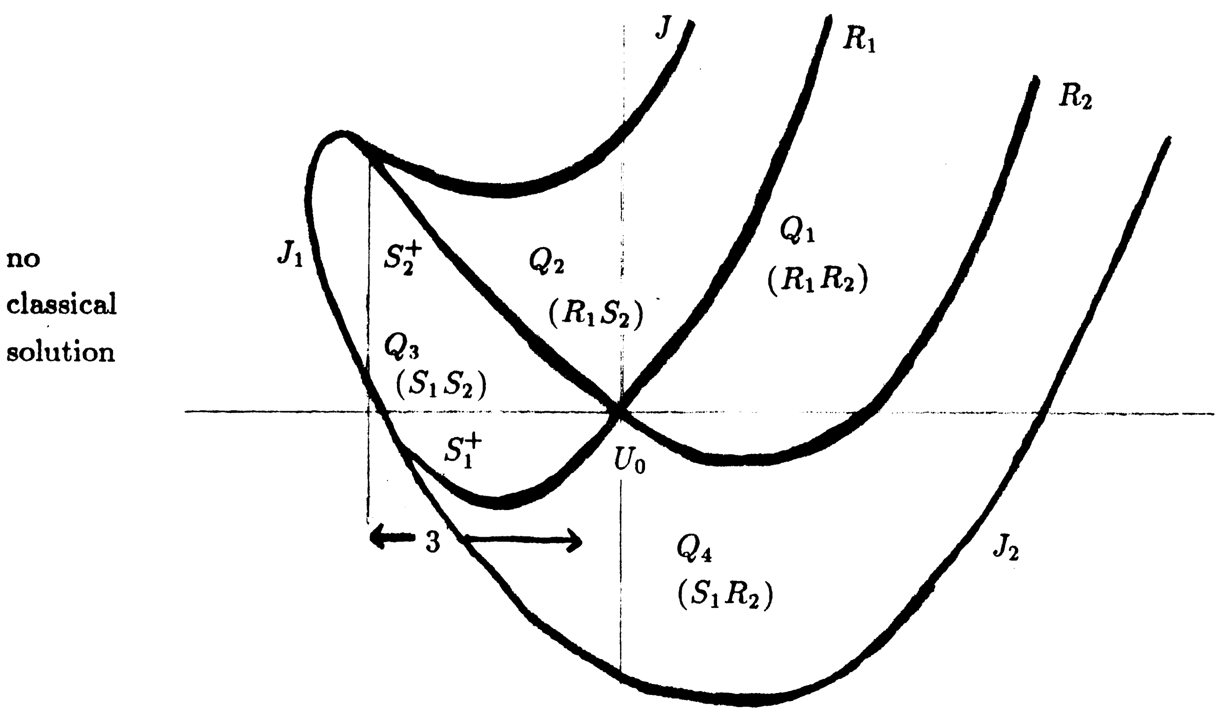


Figure 2.2

**3. Singular shocks.** Following the ideas of Dafermos [7] and Dafermos and DiPerna [8], we consider the viscosity approximation (2), i.e.

$$U_t + F_x = \epsilon t U_{xx}, \tag{9}$$

which reduces to (4) when $\epsilon = 0$. This approximation was designed for looking at solutions to the Riemann problem because it has centered solutions of the form $U = U(x/t)$ which satisfy the system of ordinary differential equations

$$\epsilon \ddot{U} = (A(U) - \xi)\dot{U}, \tag{10}$$

where $\xi = x/t$ and $\dot{} = d/d\xi$. In contrast to the viscosity approximation generally used in studying classical shocks, system (10) is not autonomous; it depends explicitly on $\xi$ (and also on $\epsilon$). The approximate Riemann problem for (9) provides boundary conditions

$$U(-\infty) = U_L, \qquad U(+\infty) = U_R \tag{11}$$

for (10).

The classical solutions to (10), (11) can be divided into *rarefactions*, for which $U$ and $\dot{U}$ remain uniformly bounded as $\epsilon \to 0$, and *shocks*, for which $U$ remains bounded but $\dot{U}$ approaches infinity. As we saw in Section 2, we should not expect a classical solution unless $U_R \in Q(U_L)$. We consider the possibility that *singular solutions* of (10) exist, in which $U$ itself becomes unbounded as $\epsilon \to 0$ for $\xi$ in the vicinity of some value $s$, which represents the speed of propagation of the singularity. Following the route which we previously took in [9], we may try the substitution

$$U(\xi) = \begin{pmatrix} \frac{1}{\epsilon^p}\tilde{u}(\frac{\xi - s}{\epsilon^q}) \\ \frac{1}{\epsilon^r}\tilde{v}(\frac{\xi - s}{\epsilon^q}) \end{pmatrix} \tag{12}$$

with undetermined positive constants $p, q$ and $r$. For nontrivial solutions to exist we must balance at least two terms in each equation (after expansion); this leads to the relations

$$q = 1 + p, \quad r = 2p. \tag{13}$$

If we then expand $\tilde{u}$ and $\tilde{v}$ as series in $\epsilon$, their lowest-order terms $\tilde{u}_0$ and $\tilde{v}_0$ can be shown (cf. [9]) to satisfy the autonomous system

$$\begin{aligned} \tilde{u}_0' &= \tilde{u}_0^2 - \tilde{v}_0 \\ \tilde{v}_0' &= \tfrac{1}{3}\tilde{u}_0^3 \end{aligned} \tag{14}$$

where $' = d/d\eta$, $\eta = (\xi - s)/\epsilon^q$. This system was found in [9] to have a one-parameter family of closed trajectories beginning and ending at $(0, 0)$ (cf. Figure 3.1).

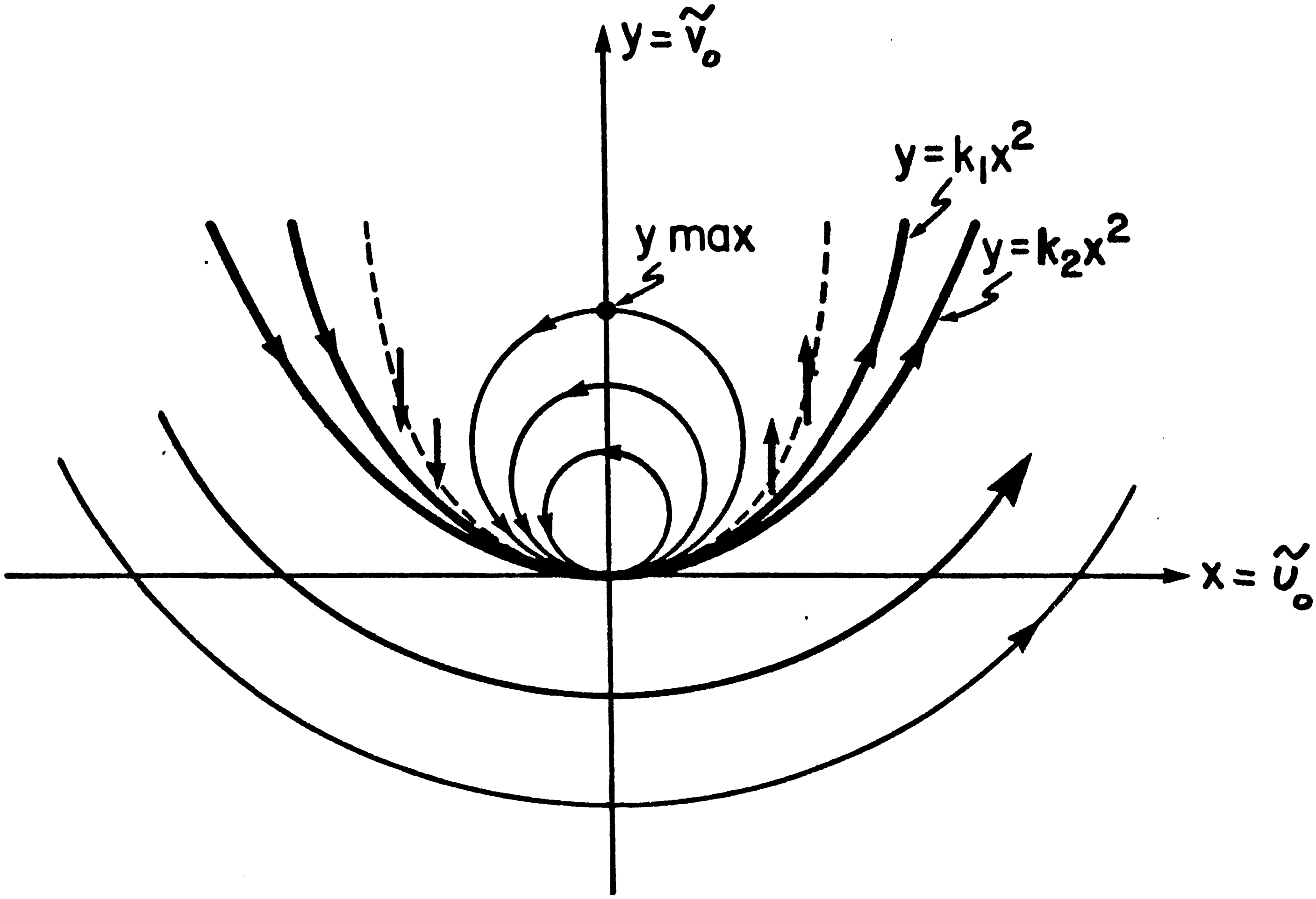


Figure 3.1

These trajectories represent functions whose essential support lies in a layer of width $|\xi - s| = O(\epsilon^q) = o(\epsilon)$ by (13). Thus they do not by themselves solve Riemann problems, since necessarily $\tilde{U}(\pm\infty) = 0$. However, their singularities lie in a zone narrower than a conventional shock profile, which has width $O(\epsilon)$, so that we are naturally led to the idea of embedding a singular solution within a shock profile of conventional type (Figure 3.2 and Figure 3.3). We shall call such a combination, if it exists, a *singular shock*.

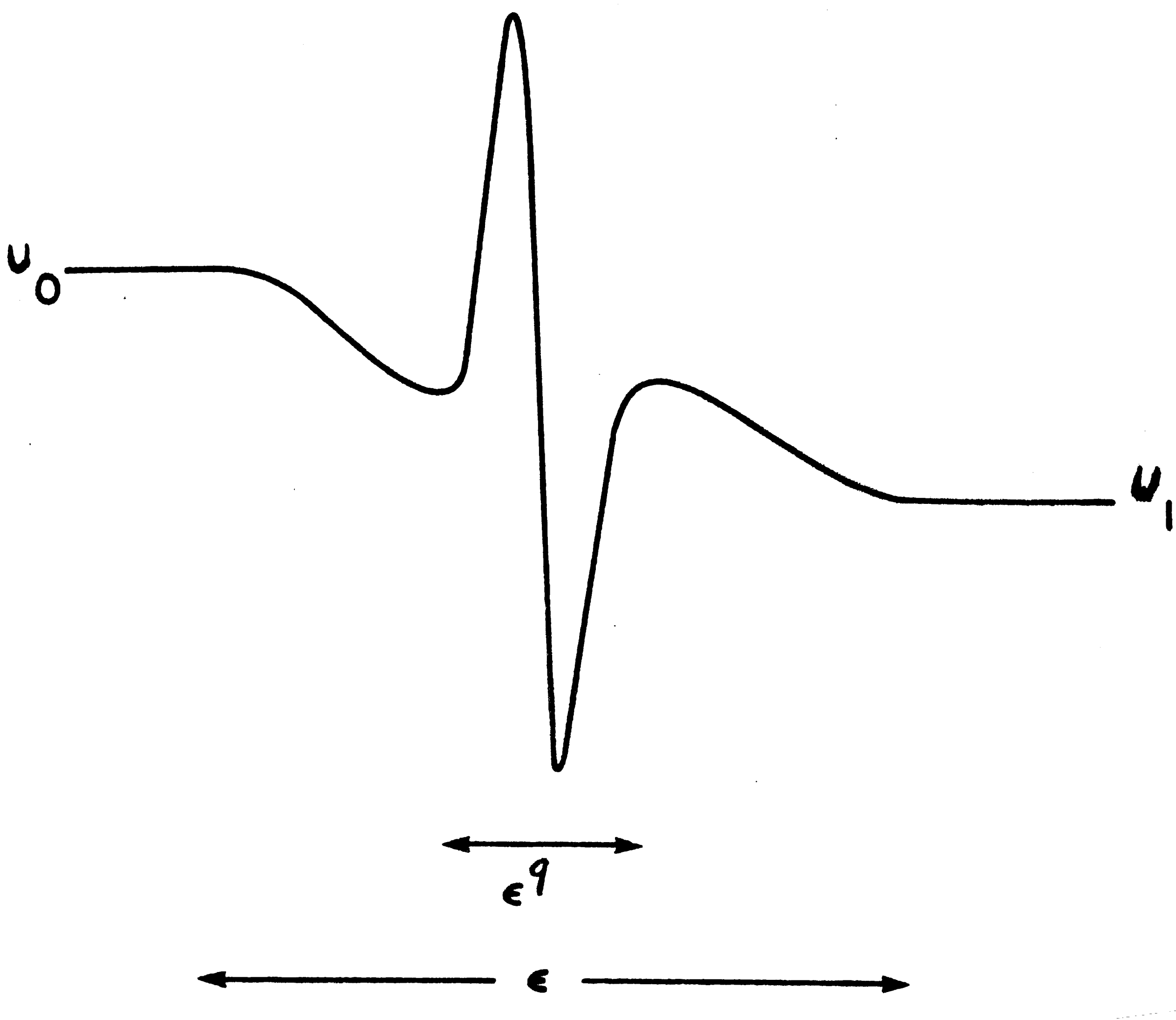


Figure 3.2

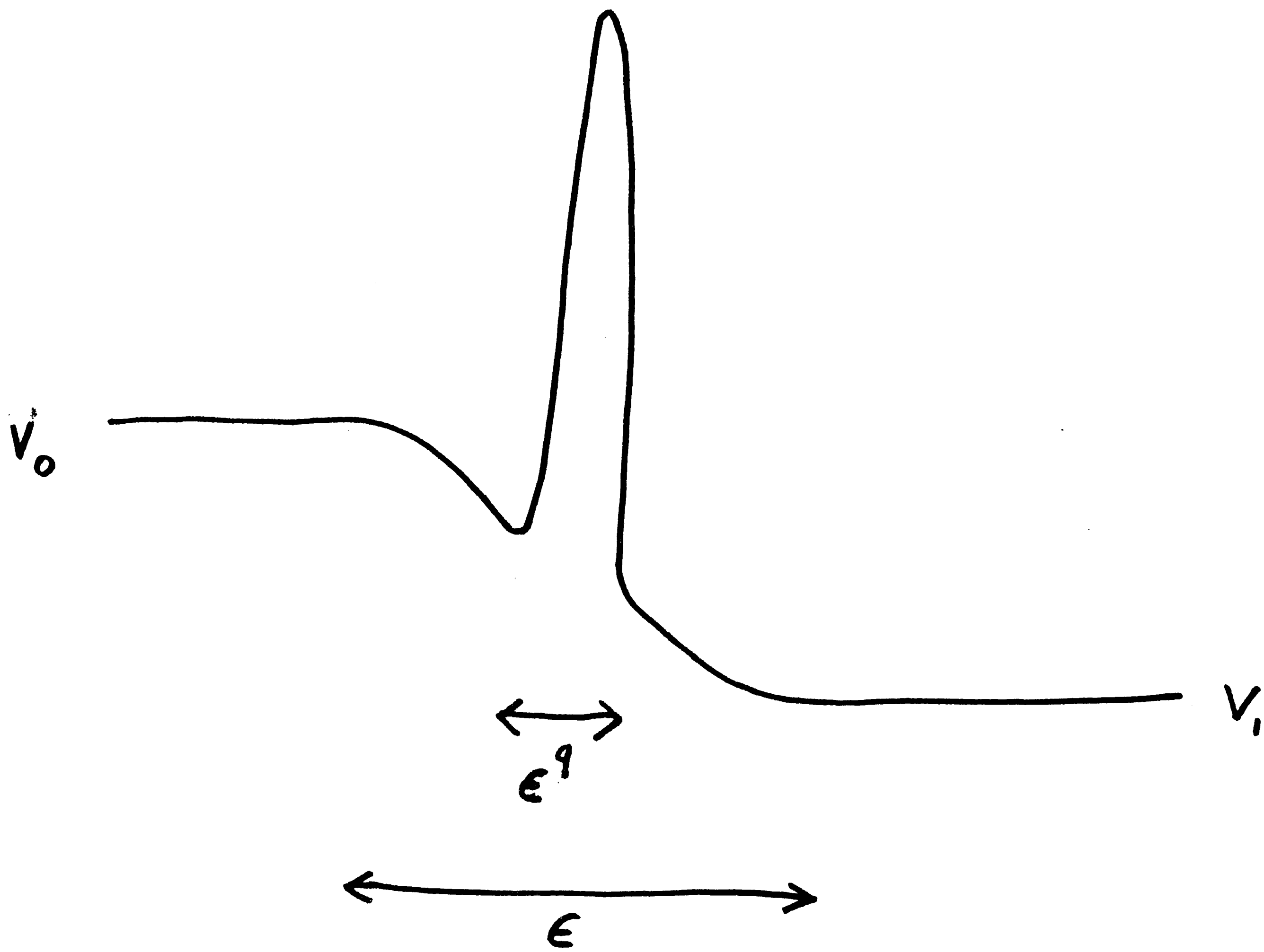


Figure 3.3

The following theorem helps us to determine the precise degree $p$ of the singularity as well as the propagation speed $s$.

THEOREM 1. *Let $W = W(\xi)$ be a two-vector defined by*

$$W(\xi) = \begin{cases} U_L, & \xi < s \\ U_R, & \xi > s \end{cases} \tag{15}$$

*Let $U = U(\xi)$ be any solution of (10) and (11). Then*

$$\int_{-\infty}^{\infty} (U - W) d\xi = s(U_R - U_L) - (F(U_R) - F(U_L)).$$

*Proof.* Integrating (10) from $-\infty$ to $\infty$, we find

$$\int_{-\infty}^{\infty} A(U)\dot{U} d\xi - \int_{-\infty}^{\infty} \xi \dot{U} d\xi = \varepsilon \int_{-\infty}^{\infty} \ddot{U} d\xi = \varepsilon \dot{U}\Big|_{-\infty}^{\infty} = 0.$$

Rewriting $\xi = s + (\xi - s)$ in the second integral yields

$$\int_{-\infty}^{\infty} (\dot{F} - s\dot{U}) d\xi = \int_{-\infty}^{\infty} (\xi - s)\dot{U} d\xi$$

or (since $\dot{W} = 0$ except at $\xi = s$)

$$\begin{aligned} (F(U(\xi)) - sU(\xi))\Big|_{-\infty}^{\infty} &= \int_{-\infty}^{\infty} (\xi - s)(\dot{U} - \dot{W}) d\xi \\ &= (\xi - s)(U - W)\Big|_{-\infty}^{s} + (\xi - s)(U - W)\Big|_{s}^{\infty} - \int_{-\infty}^{\infty} (U - W)\frac{d}{d\xi}(\xi - s) d\xi \\ &= 0 + 0 - \int_{-\infty}^{\infty} (U - W) d\xi. \end{aligned}$$

Interchanging right and left hand sides then yields the theorem. □

Suppose now that a singular shock directly connects two states $U_L$ and $U_R$, where $U_R \notin Q(U_L)$. Theorem 1 then tells us that

$$\int_{\infty}^{\infty} (U - W) d\xi = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = C = s[U] - [F],$$

a constant two-vector independent of $\varepsilon$. Now, $U_R \notin H(U_L) \subseteq Q(U_L)$, so $C \neq 0$. However, by letting $\varepsilon \to 0$ we find from (12) and (13) that

$$c_1 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^p} \int_{-\infty}^{\infty} \tilde{u}\left(\frac{\xi - s}{\varepsilon^q}\right) d\xi = \lim_{\varepsilon \to 0} \varepsilon^{q-p} \int_{-\infty}^{\infty} \tilde{u}(\eta) d\eta = 0, \tag{16}$$

so that the speed of a singular shock must satisfy the *generalized Rankine-Hugoniot condition*

$$s(u_R - u_L) = f(U_R) - f(U_L), \tag{17}$$

where $f(u,v) = u^2 - v$ is the first component of the flux function $F(U)$. A similar calculation applied to the second component shows that

$$c_2 = \lim_{\varepsilon\to 0} \varepsilon^{q-r} \int_{-\infty}^{\infty} \tilde{v}(\eta)d\eta.$$

But $c_2$ must be a finite nonzero constant (since $C \neq 0$ and $c_1 = 0$), which implies that $q = r$, and therefore from (13) we get

$$p = 1, \quad q = r = 2. \tag{18}$$

How large a family of right states can be connected through a singular shock to a given left state $U_0$? For conventional shocks, satisfying the full vector Rankine-Hugoniot condition, $U_0$ together with the shock speed $s$ determines $U_R$, so the standard theory yields a one-parameter family of right states, those on the Hugoniot locus $H(U_0)$. For singular shocks, the "singular shock strength"

$$c_2 = \int_{-\infty}^{\infty} \tilde{v}(\eta)d\eta = \int_{-\infty}^{\infty} (v - W_2)d\xi, \tag{19}$$

which measures the amount by which the full Rankine-Hugoniot condition fails to hold, becomes a second, independent parameter. Hence the states connectible to $U_0$ by singular shocks form a two-parameter family, which can fill a two-dimensional subset (region) of the $U$-plane.

We next attempt to identify this "singular region" more precisely. In doing so, we temporarily turn our attention away from the internal structure of singular shocks and introduce a definition of admissibility that depends only on the two states which are connected. We shall return to the internal structure in Section 4.

DEFINITION. A jump discontinuity propagating with speed $s$ in the $x,t$-plane is called an *admissible singular shock* if its right and left states

$$U_{\pm} = \lim_{x\to st\pm 0} U(x,t)$$

satisfy the following three conditions:

1) The generalized Rankine-Hugoniot condition

$$s(u_+ - u_-) = (u_+^2 - u_-^2) - (v_+ - v_-).$$

2) The singular shock strength condition

$$c_2 \doteq s(v_+ - v_-) - \frac{1}{3}(u_+^3 - u_-^3) + (u_+ - u_-) > 0.$$

3) The characteristic speeds condition

$$\lambda_2(U_-) > \lambda_1(U_-) \geq s \geq \lambda_2(U_+) > \lambda_1(U_+).$$

This definition can be motivated as follows. Condition 1 is just (17). Condition 2 is suggested by the first equality in (19) and the asymptotic behavior of $\tilde{v}$ as $\varepsilon \to 0$: the closed trajectories for the leading term $\tilde{v}_0$ all lie in the half-plane $\tilde{v}_0 \geq 0$ (see Figure 3.1). Condition 3 might be thought of as a requirement of "over-compression" (cf. Shearer [10]). It states that four characteristic curves enter the discontinuity and none leave. Since a conventional (internally bounded) shock has three incoming characteristic curves and one outgoing, it seems plausible that an additional incoming characteristic should be needed to provide "energy" (integrated $|\dot{U}|$) to maintain a singularity.

Let us denote by $Q_S(U_0)$ the region of the $u, v$-plane consisting of those right states $U_+$ which can be joined to the left state $U_- = U_0$ by an admissible singular shock. An easy calculation shows that $Q_S(U_0)$ is disjoint from $Q(U_0)$, shares with $Q(U_0)$ as common boundary the portion $J$ of $H(U_0)$ lying in the strip $u_0 - \sqrt{12} \leq u \leq u_0 - 3$, and has as its other two boundaries the ray

$$E : v = v_0 + (u_0 - 1)(u - u_0), \quad -\infty < u \leq u_0 - 3$$

and the parabolic segment

$$D : v = v_0 + u^2 + (1 - u_0)u - u_0, \quad -\infty < u \leq u_0 - 3$$

(see Figure 3.4). The remainder of the plane, not contained in either $Q_S$ or $Q$, falls into two regions, $Q'(U_0)$ lying below $E$ and $Q''(U_0)$ lying above and to the right of $D$.

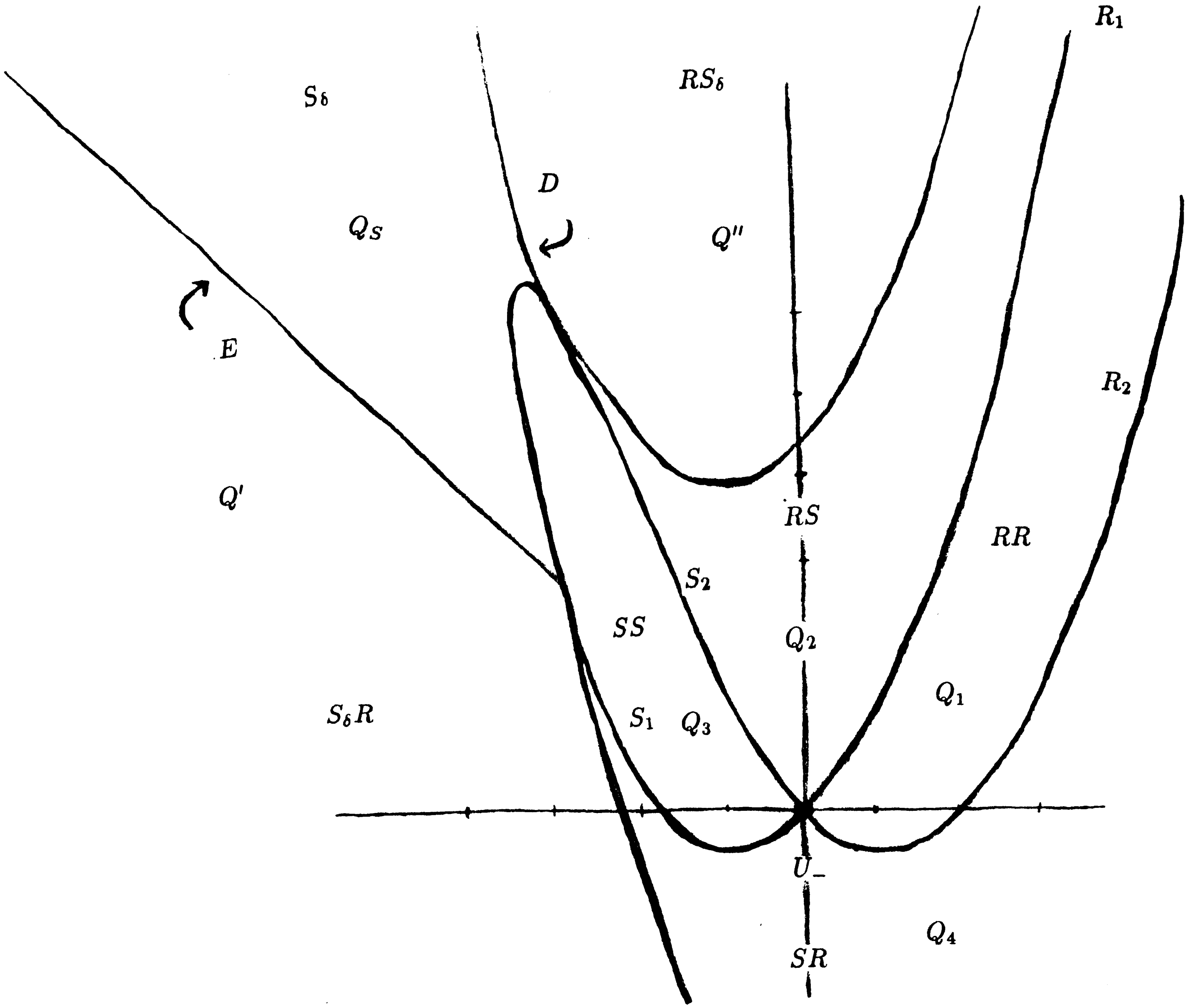


Figure 3.4

For a singular shock which connects $U_0$ to a state in the interior of $Q_S$, both inequalities in Condition 3 are strict. Such a shock is therefore not continuable on either the right or the left by a conventional wave (shock or rarefaction) of either family. However, a singular shock connecting to a point (say $U_m$) on $E$ has $s = \lambda_2(U_+)$ and can be continued on the right by a rarefaction wave along $R_2^+(U_m)$. Note that the curves $R_2^+(U_m), U_m \in E$, exactly fill $Q'$, so that all points in $Q'$ can be reached from $U_0$ by a singular shock followed by a rarefaction of the second family. Similarly, a singular shock connecting $U_0$ to a point on $D$ has $s = \lambda_1(U_-)$ and can be preceded on the left by a rarefaction wave of the first family; solutions of this type exactly fill $Q''$. [More precisely, such solutions go from $U_0$ to a point $U_m \in R_1^+(U_0)$, then to $U_R \in Q_S(U_m)$. ]

We have accordingly obtained the following result.

THEOREM 2. *A unique solution to the Riemann problem (1), (3), composed of constant states, centered rarefaction waves, Lax entropy shocks and admissible singular shocks, exists for every pair $U_L, U_R$ of Riemann data.*

**4. Admissible singular shocks as Dafermos-DiPerna limits.** In this section, we investigate more closely the connection between admissible singular shocks and solutions of the Dafermos-DiPerna viscosity approximation (10), (11).

*Conjecture.* If $U_R \in Q_S(U_L)$, then for sufficiently small positive $\varepsilon$ equations (10), (11) possess solutions

$$U = U_\varepsilon(\xi) = \begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix}$$

whose asymptotic behavior as $\varepsilon \to 0$ is described by

$$(20) \qquad \begin{cases} u_\varepsilon(\xi) \approx W_1(\xi) + \frac{1}{\varepsilon}\tilde{u}_0\left(\frac{\xi - s}{\varepsilon^2}\right) \\ v_\varepsilon(\xi) \approx W_2(\xi) + \frac{1}{\varepsilon^2}\tilde{v}_0\left(\frac{\xi - s}{\varepsilon^2}\right) \end{cases}$$

here $(\tilde{u}_0, \tilde{v}_0)$ is a closed-trajectory solution of (14) and $s$ is determined by the generalized Rankine-Hugoniot relation (17), while $W_1$ and $W_2$ are the components of the Heaviside function $W(\xi)$ defined by (15).

We present three types of evidence in support of this conjecture, and we indicate what is still needed to convert the conjecture into a theorem.

First of all, we observe that the conjecture would follow from Theorem 3.1 of Dafermos [7] provided we could verify the hypotheses of that theorem. To do that, we must embed (10), (11) in a two-parameter family of boundary-value problems

$$(21) \qquad \begin{cases} \varepsilon\ddot{U} = (aA(U) - \xi)\dot{U}, \\ U(-b) = aU_L, \;\; U(b) = aU_R, \end{cases}$$

where $0 \leq a \leq 1$ and $b \geq 1$. Dafermos shows that, if all solutions of (21) satisfy an a priori bound

$$(22) \qquad |U(\xi)| < M \text{ for } -b \leq \xi \leq b,$$

where $M$ may depend on $U_L, U_R, \varepsilon$ and $A$ but not on $a$ or $b$, then (10), (11) does have a solution for each $\varepsilon > 0$ which satisfies the same bound. Once we knew that solutions to

(10), (11) existed, we could apply the asymptotic analysis described in section 3 to obtain (20). Thus an a priori estimate on the maximum norm of $U_\varepsilon$ for each fixed $\varepsilon$ would be sufficient to establish the conjecture. While we do not yet have a maximum-norm estimate, it is suggestive in this regard to note that our Theorem 1 does provide an $L^1$ estimate, which is even uniform in $\varepsilon$.

The second bit of supporting evidence arises when we look at the actual maxima (and minima) of solutions of (21). The Riemann invariants $\pi = v - \frac{1}{2}u^2 + u$ and $\rho = v - \frac{1}{2}u^2 - u$ corresponding to these solutions satisfy the equations

$$\begin{aligned} \varepsilon\ddot{\pi} &= [a(u-1) - \xi]\dot{\pi} - \varepsilon\dot{u}^2, \\ \varepsilon\ddot{\rho} &= [a(u+1) - \xi]\dot{\rho} - \varepsilon\dot{u}^2. \end{aligned} \tag{23}$$

Hence, for nonconstant solutions of (21), $\dot{\pi} = 0$ implies $\ddot{\pi} < 0$ and $\dot{\rho} = 0$ implies $\ddot{\rho} < 0$. Thus each Riemann invariant has no (relative) minimum and at most one maximum on any solution trajectory. From this it can be shown (see [11] for details) that $u$ has at most one maximum point $\xi = \xi_1$ and one minimum point $\xi = \xi_2$, while $v$ has only a maximum at $\xi = \xi_3$, and that $\xi_1 < s < \xi_3 < \xi_2$. Thus the trajectory of a solution to the Dafermos-DiPerna viscosity approximation in the $u, v$-plane should look like Figure 4.1. Observe that Figure 4.1 is related to the asymptotic picture of Figure 3.1 in exactly the same manner as (20) is related to (12), namely by addition of the Heaviside function $W(\xi)$.

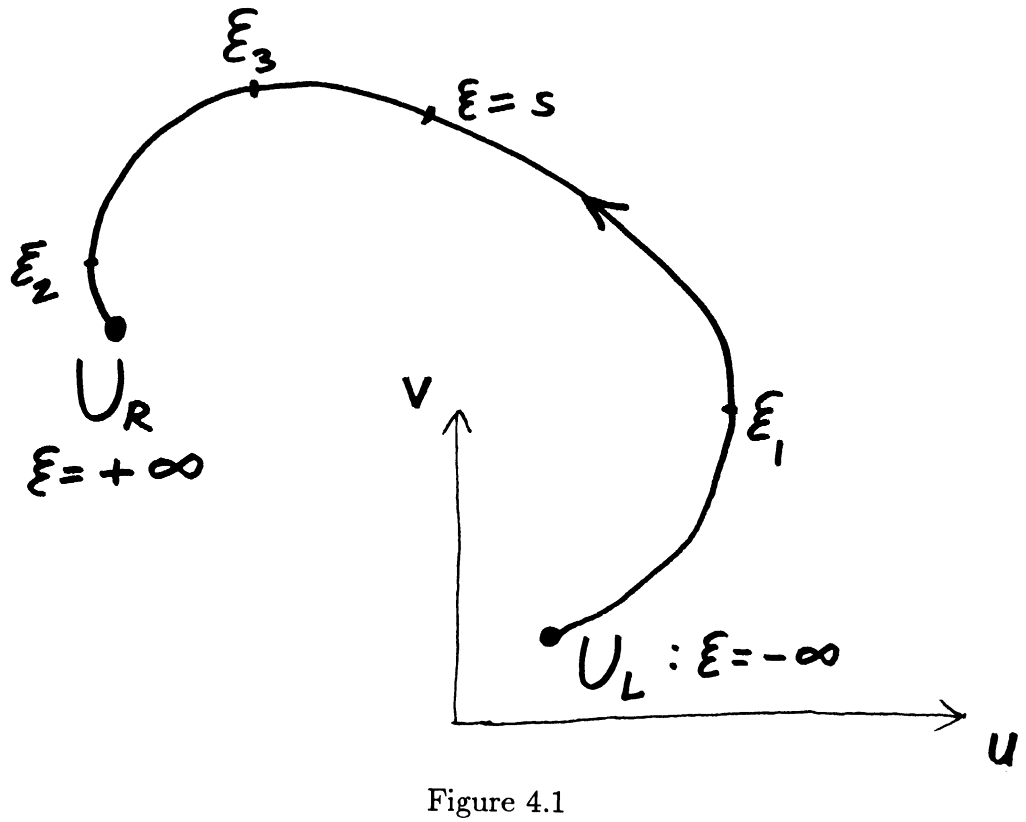


Figure 4.1

As our third piece of evidence, we present the results of a numerical solution of a two-point boundary-value problem similar to (21). We chose $U_L = (0, 0)$ and $U_R = (-4.5, 10.125)$, with the Dafermos parameter $a = 1$ and boundaries symmetrically situated (at $\xi = -4$ and $\xi = -0.5$) with respect to the theoretical singular shock speed $s = -2.25$. With a grid size $\Delta\xi = .0025$, we approximated $\dot{U}$ and $\ddot{U}$ by centered differences. We solved the resulting non-linear system of difference equations by an iteration procedure. At each step we determined the solution of the linear system which results when $A(U)$ is computed from the $U$ found at the previous step. Approximately 80 iterations were required for convergence, and the results did not change significantly when the grid size was halved, nor when the length of the $\xi$-interval was increased or decreased by 50%.

Computations were carried out for three values of $\varepsilon$, namely $\varepsilon = .3, \varepsilon = .2$ and $\varepsilon = .15$, and the numerical solutions are compared in the accompanying figures. Figure 4.2 shows the first component $u(\xi)$, Figure 4.3 shows the second component $v(\xi)$, and Figure 4.4 shows the solution trajectory in the $u, v$-plane. Note the degree of resemblance between the computed Figure 4.4 and the theoretically derived Figure 4.1. Note further that the upward and downward "bumps" in $u$ (Figure 4.2) do seem to roughly double in height and reduce fourfold in width when $\varepsilon$ is halved from .3 to .15, while the height of the single bump in $v$ (Figure 4.3) does seem to more than double (though not quite quadruple). This at least approximately confirms the $1/\varepsilon$ and $1/\varepsilon^2$ scalings of (20). Finally, the theoretical value of the singular shock strength parameter $c_2$ is 3.09375, while the value computed from the numerical solution for $\varepsilon = .2$, using the trapezoidal rule for the second integral in (19), was $c_2 \approx 3.11$.

On all these points, there seems to be sufficient agreement between numerical and theoretical solutions to support the conjecture, at least for the particular values of $U_L$ and $U_R \in Q_s(U_L)$ used in the computation. Computations involving other values of the Riemann data, including cases where $U_R$ is in $Q'(U_L)$ or $Q''(U_L)$, are consistent with this specimen result.

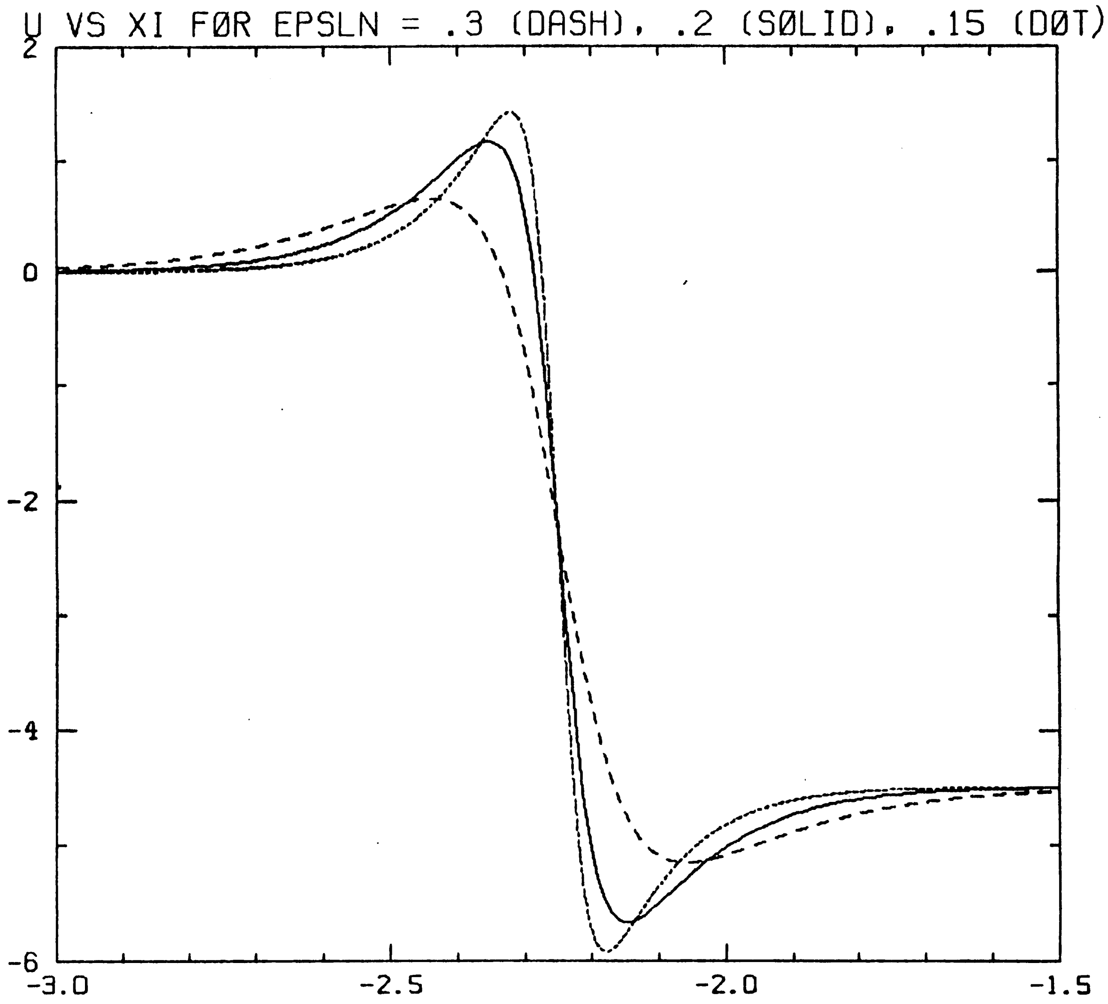


Figure 4.2

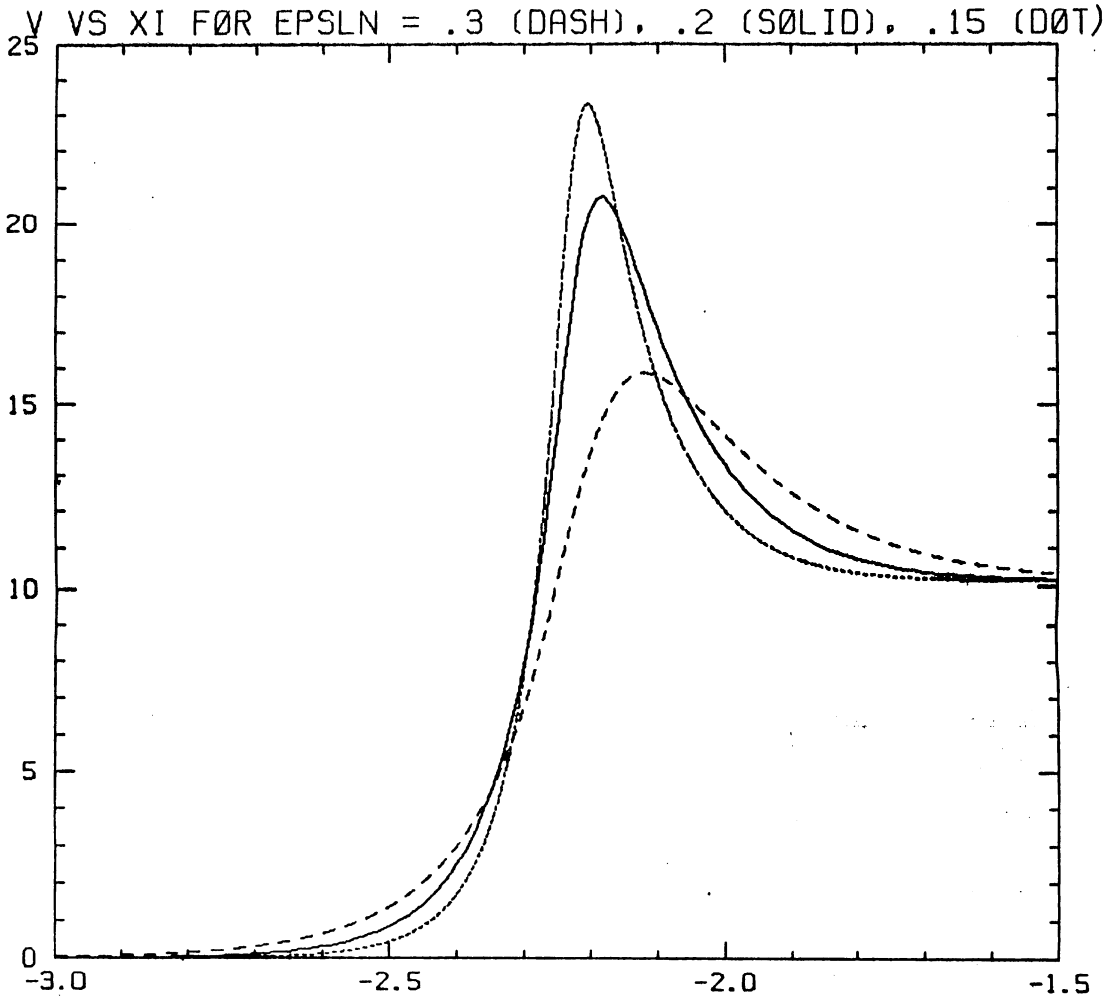


Figure 4.3

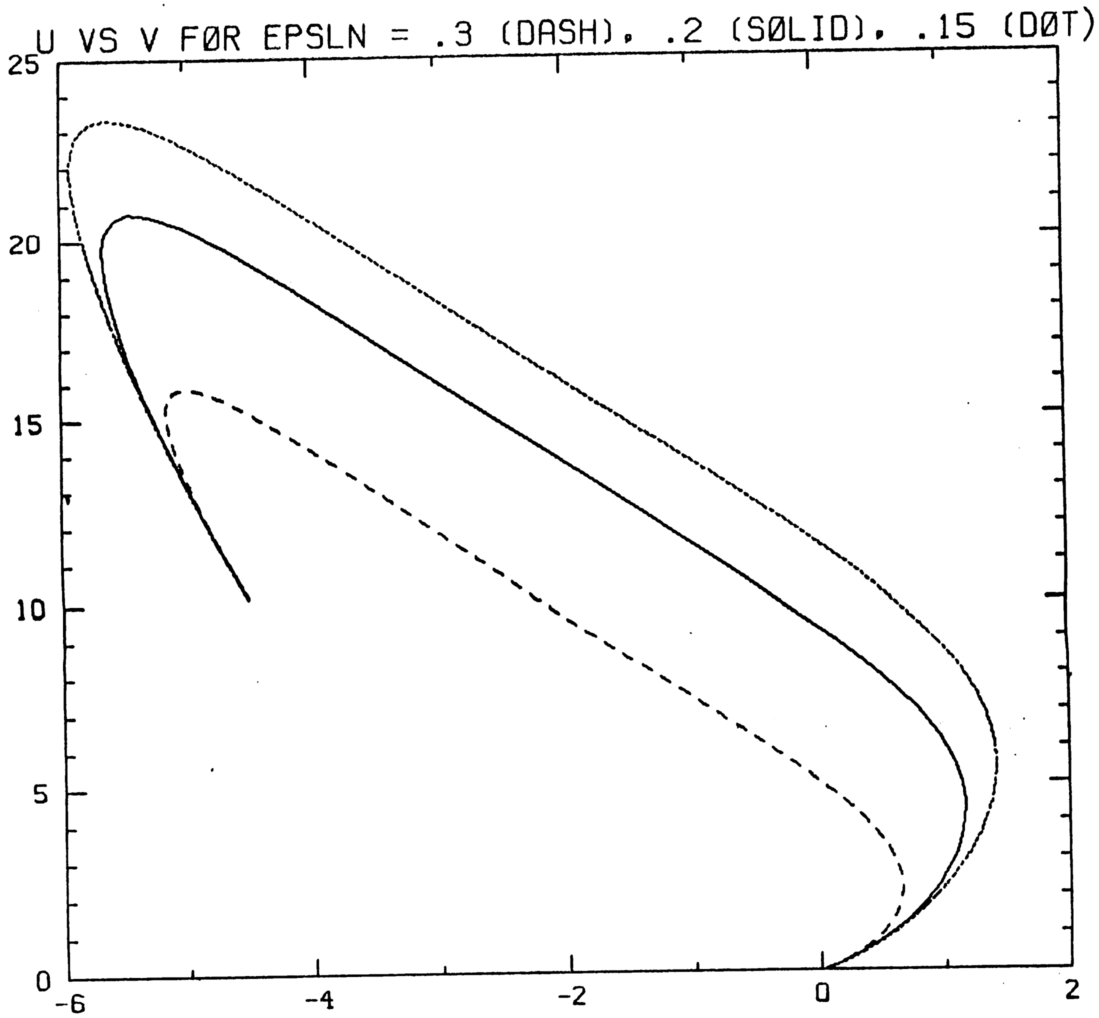


Figure 4.4

**Acknowledgement.** Much of the research described in this article was carried out while the authors were visitors at the Institute for Mathematics and its Applications at the University of Minnesota. It was influenced generally by the stimulating scientific atmosphere at the Institute, and in particular by fruitful conversations with Mitchell Luskin, Bruce Pitman, Bradley Plohr and Michael Shearer.

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