**ON SOLUTIONS TO A QUASILINEAR DIFFUSION
PROBLEM FROM THE STUDY OF SOFT TISSUE**

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ON SOLUTIONS TO A QUASILINEAR DIFFUSION PROBLEM FROM THE STUDY OF SOFT TISSUE

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Abstract. A continuum mechanics model for the uni-directional deformation of soft tissue can be approximated by a quasilinear diffusion equation $u_t = \exp(u_x)u_{xx}$ defined on the scaled spatial interval $(0,1)$, with boundary conditions given by $u(x,0) = 0$, $u(1,t) = 0$, $u(0,t) = g(t)$. We give conditions on $g(t)$ under which the problem has no global solutions, and conditions when such a solution exists. We also study the behavior of the non-global solution near the time $t = T$ when it ceases to exist.

Key words. Soft tissue, mixture theory, existence of solutions, nonexistence.

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§1. Introduction. Articular cartilage is a multiphasic, nonlinearly permeable viscoelastic material consisting of two principle phases; a solid organic matrix, predominately collagen fibrils and proteoglycan macromolecules; and a motile interstitial fluid phase, predominately water. During tissue compression the permeability of the solid matrix changes significantly. Experiments have been conducted for the purpose of determining the permeability dependence on the strain tensor of the solid matrix. Lai, et al [7] formulated a theoretical model involving nonlinear strain-dependent permeability, and solved the model numerically under certain conditions. The simplest problem is to consider one-dimensional motion, where the quantity of interest, $u(z,t)$, is the axial component of the displacement vector describing the deformation of the solid matrix. A biophasic theory of cartilage assumes it is a mixture of incompressible fluid and a compressible, isotropic elastic solid. Mixture theory (see [1], [5]) is used to obtain an equation of motion

$$(1.1) \quad \lambda \frac{\partial u}{\partial \tau} = k \frac{\partial \sigma_E}{\partial z}$$

where λ is the stretch in the z direction given by $\lambda = 1 + u_z$, and k is a material constant. Here σ_E is the elastic stress which is determined from a constitutive law for the Helmholtz free energy density of the solid. After imposing various assumptions on σ_E , Holmes [5] reduces (1.1), after rescaling time and space, to

$$(1.2) \quad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(\exp \left(\varepsilon \frac{\partial u}{\partial x} \right) \right) = \varepsilon \exp \left(\varepsilon \frac{\partial u}{\partial x} \right) \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < 1, t > 0.$$

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Here ε is a positive constant representing the absolute equilibrium strain. For a more detailed derivation of (1.2) see [5], [8].

A fixed rigid impermeable surface will be considered at $x = 1$, with the displacement of the solid phase being prescribed at $x = 0$. This is referred to as the stress relaxation boundary condition [4]. Hence, boundary conditions to be considered for (1.1) are

$$(1.3) \quad \begin{aligned} u(1, t) &\equiv 0, & u(0, t) &= g(t) \quad \text{for } t > 0, \\ u(x, 0) &\equiv 0 & & \text{for } 0 < x < 1. \end{aligned}$$

In this theory a stress-strain law is assumed so that one quantity of interest is the surface stress, which in our normalized variables is $\sigma_0 = -u_x(0, t)$. We show below that the surface stress in this theory is unbounded.

Holmes [3] did an asymptotic analysis of the model under conditions of slow and fast rates of compression. He also established some properties via maximum principle arguments of solutions to the problem with the diffusive drag coefficient $K(u_x)$ left more general than the exponential form $\exp(u_x)$ [4]. (In the remainder of the paper subscripts will denote partial derivatives, where appropriate.) He also obtained some stability properties with the stress relaxation boundary condition $u(0, t) = g(t)$ replaced by a creep condition $u_x(0, t) = -1$. For a review of this approach to the theoretical, as well as experimental, background to fluid transport in cartilage, see [5], [8].

In this paper we study the global existence of the solution of (1.2), (1.3) taking for simplicity $\varepsilon = 1$. In §2 we establish the existence (and uniqueness) of global smooth solution provided $-u_x(0, t)$ is a priori bounded from below. In §3 we establish global existence for a specific function g with growth

$$(1.4) \quad g(t) = \log t + o(\log t) \quad \text{as } t \rightarrow \infty.$$

In §4 we prove that if a global smooth solution exists then

$$(1.5) \quad g(t) \leq 2 + \log t \quad \text{for all } t > 1.$$

In §5 we consider the maximal function g^* of all g for which a smooth solution exists. (The solution corresponding to g^* is not smooth at $x = 0$). From (1.4), (1.5) it follows that

$$g^*(t) \begin{cases} \geq \log t + o(\log t), & t \rightarrow \infty \\ \leq 2 + \log t, & t > 1. \end{cases}$$

We derive in §5 more refined estimates on g^* ; for instance,

$$g^*(t) \geq 1 + \log t - \log \log t \quad \text{as } t \rightarrow \infty.$$

Finally, in §6 we assume that the smooth solution exists in a maximal interval $0 \leq t < T$, and prove that

$$\lim_{t \rightarrow T} u_x(x, t) = \begin{cases} -\infty & \text{if } x = 0 \\ \text{a smooth function} & \text{for } 0 < x \leq 1. \end{cases}$$

Although we shall deal only with equation (1.2), our methods can be extended to more general equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(K(u_x)).$$

§2. Local existence and a priori estimates. For simplicity we take $\varepsilon = 1$ in (1.2), i.e.,

$$(2.1) \quad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} e^{u_x} = e^{u_x} u_{xx} \quad \text{for } 0 < x < 1, t > 0.$$

For any $0 < T < \infty$ set

$$\Omega_T = \{0 < x < 1, \quad 0 < t < T\}.$$

We assume that $g(t)$ satisfies:

$$(2.2) \quad \begin{cases} g \in C[0, \infty], \quad g' \text{ is piecewise continuous,} \\ g(t) \text{ is monotone increasing and positive for } t > 0, \\ g(0) = 0. \end{cases}$$

DEFINITION 2.1. A function u satisfying (2.1), (1.3) for $t < T$ such that

$$(2.3) \quad u_t, u_x, u_{xx} \quad \text{belong to } L^p(\Omega_T) \quad \forall p > 1$$

will be called a smooth solution (or briefly a solution) of (2.1), (1.3) in Ω_T .

Recall [6; Chap. 2, Lemma 3.3] that for any function $w(x, t)$,

$$(2.4) \quad \begin{aligned} |w|_{C^\alpha(\Omega_T)} + |w_x|_{C^\alpha(\Omega_T)} &\leq C_p \left\{ |w|_{L^p(\Omega_T)} + |w_x|_{L^p(\Omega_T)} \right. \\ &\quad \left. + |w_{xx}|_{L^p(\Omega_T)} + |w_t|_{L^p(\Omega_T)} \right\} \\ &\text{for } \alpha = 1 - \frac{3}{p}. \end{aligned}$$

Consequently the solution u must satisfy:

$$(2.5) \quad |u_x|_{C^\alpha(\Omega_T)} \leq C < \infty \quad \forall \quad 0 < \alpha < 1.$$

LEMMA 2.1. *There exists a unique smooth solution of (2.1), (1.3) in Ω_T for some small $T > 0$.*

Proof. Define a mapping $S : \tilde{u} \rightarrow u$ where u is the solution of

$$u_t = e^{\tilde{u}_x} u_{xx}$$

satisfying (1.3); here $\tilde{u}$ belongs to $C^\beta(\Omega_T)$ for some $0 < \beta < 1$. Using the L^p theory for parabolic equation [6] one can show that S is a contraction if T is small enough, and the lemma follows.

LEMMA 2.2. *Suppose that for some $T > 0$ the solution u of (2.1), (1.3) exists and*

$$(2.6) \quad u, u_x, u_{xx}, u_t \quad \text{are in } L^p(\Omega_T) \text{ for any } p > 1.$$

Then there exists a $\delta > 0$ such that the solution can be extended into $\Omega_{T+\delta}$ and (2.6) holds in $\Omega_{T+\delta}$.

Proof. We simply apply the proof of Lemma 2.1 with initial time $t = T - \varepsilon$ and ε small enough.

We remark that the solution u is in $C^\infty\{0 < x \leq 1, 0 < t \leq T\}$; its smoothness at $x = 0$ depends on the smoothness of $g(t)$. If for example, $g \in C^{1+\alpha}(t_0, t_1)$ then u_x, u_{xx} and u_t belong to C^α in a neighborhood of $\{(0, t), t_0 < t < t_1\}$, by the Schauder estimates [2].

We also note that since the compatibility condition for u is satisfied at $(0, 1)$,

$$(2.7) \quad u = u_x = u_{xx} = u_t = 0 \quad \text{at } (0, 1).$$

LEMMA 2.3. *If u is a smooth solution in Ω_T then*

$$(2.8) \quad 0 < u(x, t) \leq g(t) \quad \text{in } \Omega_T,$$

$$(2.9) \quad u_x(x, t) < 0 \quad \text{in } \overline{\Omega_T} \setminus \{t = 0\},$$

$$(2.10) \quad 0 \leq u_t(x, t) \leq \sup_{0 \leq s \leq t} g'(s) \quad \text{in } \Omega_T,$$

and

$$(2.11) \quad u_{xx}(x, t) \geq 0 \quad \text{in } \Omega_T.$$

Proof. The assertion (2.11) follows from (2.10). The assertions (2.8), (2.9) follow by applying the strong maximum principle to u and u_x respectively. To prove (2.10) we differentiate (2.1) with respect to t and obtain, for $z = u_t$,

$$z_t = az_{xx} + bz_x \quad (a = e^{u_x}, \quad b = e^{u_x}u_{xx}).$$

Since $z = 0$ for $x = 1, t > 0$ and for $t = 0, 0 < x < 1$, and $z = g'(t)$ for $x = 1, t > 0$, and since z is continuous in $\overline{\Omega}_T$ (except at the points on $x = 0$ where $g'(t)$ is discontinuous), the assertion (2.10) follows by the maximum principle (which is valid also if b is just in L^p ; for instance, by approximation).

LEMMA 2.4. *If u is a smooth solution in $\Omega_{T-\delta}$ for any $\delta > 0$ and if*

$$(2.12) \quad u_x(0, t) \geq -C \quad \text{for all } 0 < t < T$$

thus u has an extension to a smooth solution in $\Omega_{T+\delta}$, for some $\delta > 0$.

Proof. By (2.12) and (2.9), (2.11),

$$|u_x| \leq C \quad \text{in } \Omega_T$$

By (2.10) and (2.1) we thus also have

$$|u_t| \leq C_1, \quad |u_{xx}| \leq C_2 \quad \text{in } \Omega_T,$$

and the assertion follows by Lemma 2.2.

REMARK 2.1. From the proof of Lemma 2.4 it follows that the smooth solution satisfies not only (2.6) but also:

$$(2.13) \quad u_x, \quad u_{xx}, \quad u_t \quad \text{are bounded in } \Omega_T.$$

§3. Sufficient condition for global existence.

Let v be the solution to

$$(3.1) \quad \begin{aligned} v_t &= Kv_{xx}, \quad 0 < t < \delta, \quad 0 < x < 1, \\ v_x(0, t) &= -p\sqrt{t}, \quad 0 < t < \delta, \\ v(1, t) &= 0, \quad 0 < t < \delta, \\ v(x, 0) &= 0 \end{aligned}$$

for some constant $p > 0$. Then v is a lower solution of (2.1) (1.3) taking $g(t) = v(0, t)$, provided $K \leq e^{-p}$; also $g' \in C^\alpha[0, \delta]$.

Define a function $h(t)$ for $t \geq \delta$ by

$$h' = m(m-1)he^{-mh} \quad \text{for some } m > 1$$

with

$$h(\delta) = \inf_{0 \leq x \leq 1} \frac{v(x, \delta)}{(1-x)^m}.$$

Letting $\eta \geq \delta$ be given by $h(\eta) = v(0, \delta)$, we choose

$$(3.2) \quad g(t) = \begin{cases} v(0, t) & \text{for } 0 \leq t \leq \delta \\ v(0, \delta) & \text{for } \delta \leq t \leq \eta \\ h(t) & \text{for } t \geq \eta \end{cases}$$

and

$$(3.3) \quad w(x, t) = \begin{cases} v(x, t) & \text{for } 0 \leq t \leq \delta \\ v(x, \delta) & \text{for } \delta \leq t \leq \eta \\ h(t)(1-x)^m & \text{for } t \geq \eta. \end{cases}$$

It is easily verified that $w(x, t)$ is a lower solution so that $u_x(0, t) \geq w_x(0, t) \geq -C$ in any interval $0 \leq t \leq T$, where u is the solution corresponding to $g(t)$ as in (3.2). Appealing to Lemma 2.4, we deduce the existence of a global solution to (2.1), (1.3).

For $t > \eta$ we have $g' = m(m-1)ge^{-mg}$, so that, as $t \rightarrow \infty$,

$$t \sim \frac{e^{mg}}{m^2(m-1)g} \quad \text{and} \quad g \sim \frac{1}{m} \log t + \frac{1}{m} \log \log t + \frac{1}{m} \log(m(m-1)) + o(1).$$

Taking m arbitrarily close to 1, we deduce:

THEOREM 3.1. *There exists a function g satisfying (2.2) and*

$$(3.4) \quad g(t) = \log t + o(\log t) \quad \text{as } t \rightarrow \infty,$$

for which a global solution of (2.1), (1.3) exists.

In the next section we prove that if g is such that a global solution exists then g cannot grow faster than $\log t$.

§4. Necessary condition on g for global existence.

THEOREM 4.1. *If there exists a smooth solution u of (2.1), (1.3) in Ω_T , then*

$$(4.1) \quad g(t) \leq \begin{cases} 2\sqrt{t} & \text{for } 0 \leq t \leq 1, \\ 2 + \log t & \text{for } 1 < t < T. \end{cases}$$

Proof. We shall compare u with the function

$$v(x, t) = At + (\log A - 1)x + x \log x + B.$$

It is easily seen that v is a solution of (2.1); further

$$v_x(x, t) \rightarrow -\infty \quad \text{if } x \rightarrow 0.$$

If $A \leq 1$ and $B = 1 - \log A$ then

$$v_x = \log A + \log x \leq 0$$

and

$$v(1, t) \geq \log A - 1 + B = 0$$

It then follows, by comparison, that $u \leq v$ in Ω_T ; in particular,

$$(4.2) \quad u(0, t) \leq At + 1 - \log A.$$

Given $t > 1$, we minimize the right-hand side by choosing $A = 1/t$, and thus conclude that

$$g(t) = u(0, t) \leq 2 + \log t.$$

To estimate $g(t)$ for $0 < t < 1$ we use the same function $v(x, t)$ with $A \geq 1$ and compare with $u(x, t)$ in $\widetilde{\Omega}_1 = \{0 < x < \frac{1}{A}, 0 < t < 1\}$. In $\widetilde{\Omega}_1$, $v_x \leq 0$ and on $x = \frac{1}{A}$

$$v_x = 0 \geq u_x.$$

In order for v to be an upper solution to u it remains to require that $v(x, 0) \geq 0$ for $0 \leq x \leq \frac{1}{A}$, i.e., $B \geq 1/A$. We conclude that

$$u(x, t) \leq At + (\log A - 1)x + x \log x + \frac{1}{A}$$

in $\widetilde{\Omega}_1$ and, in particular,

$$g(t) = u(0, t) \leq At + 1/A.$$

For fixed t we can choose $A = 1/\sqrt{t}$ and thus obtain $g(t) \leq 2\sqrt{t}$ if $t < 1$.

§5. Refined conditions for global existence. Theorem 4.1 gives a bound from above on how fast $g(t)$ may grow while admitting a smooth solution to (2.1), (1.3).

Taking an alternative boundary condition on $x = 0$, $u_x = -f(t)$ with $f(t) \geq 0$, it is clear from the maximum principle and Lemma 2.3 that the maximal $g(t) = u(0, t)$, say $g^*(t)$, will be given by taking $f(t) \equiv \infty$; equivalently $u_x \rightarrow -\infty$ as $x \rightarrow 0$ for $t > 0$. Thus, if we denote by u^* the solution of (2.1) satisfying

$$\begin{aligned} u_x(0, t) &= -\infty, \quad u(1, t) = 0 \quad \text{for } t > 0, \\ u(x, 0) &= 0 \quad \text{for } 0 < x < 1 \end{aligned}$$

then

$$g^*(t) = u^*(0, t).$$

It is clear that for any $t_0 > 0$

$$(5.1) \quad g^*(t_0) = \sup_g g(t_0)$$

where g varies on all functions defined for $0 \leq t \leq t_0$ satisfying (2.2) for which a smooth solution exists in Ω_{t_0} .

Observe that (5.1) is valid also if we admit solutions u which are not smooth at $x = 0$, i.e. solutions with

$$u_x(0, t) = -f(t) \quad \text{when } 0 \leq f(t) \leq \infty$$

The estimate of Theorem 4.1 gives

$$(5.2) \quad g^*(t) \leq \begin{cases} 2\sqrt{t} & \text{for } 0 \leq t \leq 1 \\ 2 + \log t & \text{for } t > 1. \end{cases}$$

This implies that g^* and u^* are finite valued; g^* can be shown to be continuous for all $t \geq 0$, but it is not Lipschitz continuous at $t = 0$ (see Theorem 5.2). Observe next that Theorem 3.1 yields the asymptotic estimate

$$(5.3) \quad g^*(t) \geq \log t + o(\log t) \quad \text{as } t \rightarrow \infty.$$

In this section we derive sharper lower estimates on $g^*(t)$, as well as some more upper estimates.

THEOREM 5.1. *The function g^* satisfies:*

$$(5.4) \quad \begin{aligned} g^*(t) &\geq t \quad \text{if } 0 \leq t \leq 1, \\ g^*(t) &\geq 1 + \log t - \log(1 + \log t), \quad t \geq 1. \end{aligned}$$

Proof. As in §4 we shall work with the solution to the differential equation

$$v(x, t) = At + (\log A - 1)x + x \log x + B,$$

for $0 \leq t \leq \tau$, for some $\tau \geq 1$ and set $g(t) = At + B$; however we shall choose A and B so that v is a lower solution. Clearly

$$v(1, t) = At + (\log A - 1) + B \leq 0 \quad \text{for } 0 \leq t \leq \tau$$

if we take $B = 1 - \log A - A\tau$.

Also, choosing $A \leq 1$, we see that $v_x = \log(Ax) \leq 0$ so

$$v(x, 0) \leq v(x, \tau) = B = 1 - \log A - A\tau \leq 0,$$

on taking $A = A(\tau)$ to satisfy $A\tau + \log A = 1$.

For $\tau = 1, A = 1$ while for $\tau > 1$ it is clear that $A = (1 - \log A)/\tau < 1$, so then

$$A > \frac{1}{\tau}, \quad A < (1 + \log \tau)/\tau.$$

For $t \geq 1$ take $\tau = t, A = A(t)$ to get

$$\begin{aligned} g^*(t) &\geq g(t) = v(0, t) = At + (1 - \log A - At) \\ &= 1 - \log A \geq 1 + \log t - \log(1 + \log t). \end{aligned}$$

For $t \leq 1$ we may choose $A = \tau = 1$ to give

$$g^*(t) \geq g(t) = v(0, t) = t.$$

REMARK 5.1. It is possible to improve slightly the lower bound for g^* in the case of large t by getting the better estimate

$$A(\tau) < \{1 + \log \tau - \log[1 + \log \tau - \log(1 + \log \tau)]\}/\tau,$$

but this still leaves us with

$$(5.5) \quad g^*(t) \geq \log t - \log \log t + o(\log \log t) \quad \text{for } t \rightarrow \infty.$$

REMARK 5.2. We can modify our comparison function $v(x, t)$ to take

$$\begin{aligned} v(x, t) &= A(t - \tau) + (x + \varepsilon) \log(x + \varepsilon) + (\log A - 1)x + 1 - \log A \\ &\quad - (1 + \varepsilon) \log(1 + \varepsilon) \end{aligned}$$

with $0 < \varepsilon < 1, A = (1 + \log \tau)/\tau$ for τ sufficiently large (to ensure that $A < \frac{1}{1 + \varepsilon}$). This v is a lower solution for $0 \leq t \leq \tau$ with both v_t, v_x bounded. We then get a lower bound on g^* by taking $t = \tau$:

$$\begin{aligned} g^*(t) &\geq g(t) = v(0, t) = 1 + \varepsilon \log \varepsilon - (1 + \varepsilon) \log(1 + \varepsilon) \\ &\quad + \log t - \log(1 + \log t), \end{aligned}$$

which again gives (5.5).

The estimates on g^* for small t , both from above and below, can be improved by comparison with suitable similarly solution to (2.1).

THEOREM 5.2. *There exist constants a_1, a_2 with $1 < a_2 < a_1 < 2$ such that*

$$(5.6) \quad g^*(t) \leq a_1 \sqrt{t} ,$$

$$(5.7) \quad g^*(t) \geq a_2 \sqrt{t} \quad \text{for } 0 \leq t \leq 1,$$

$$(5.8) \quad g^*(t) \sim a_1 \sqrt{t} \quad \text{as } t \rightarrow 0 .$$

Proof. Writing $v = \sqrt{t} f(\eta)$, $\eta = x/\sqrt{t}$, the equation $v_t = (e^{v_x})_x$ for $x, t > 0$ becomes

$$(5.9) \quad (e^{f'})' + \frac{\eta}{2} f' - \frac{1}{2} f = 0 \quad \text{for } \eta > 0 .$$

We drop the boundedness condition on v_x as we are looking for the maximal growth of g (of course the form of the similarity solution shows that we have already dispensed with the boundedness of v_t and g') and impose $v_x(0, t) = -\infty$ for $t > 0$, i.e., $f'(0) = -\infty$. The extra condition $f(0) = a$ is imposed with a to be chosen suitably.

The change of variables $w = e^{f'}$ gives

$$(5.10) \quad 2ww'' + \eta w' = 0 \quad \text{with } w(0) = 0, \quad w'(0) = \frac{a}{2} .$$

This problem has a unique solution.

If we denote by w_1 a solution corresponding to $a = 1$, then

$$w(\eta) = a^2 w_1 \left(\frac{\eta}{a} \right)$$

is the solution of (5.10).

Now $w \geq 0$, $w' > 0$, and $w'' \leq 0$; the concavity of w gives $w \geq \eta w'$. But $w' = -2ww''/\eta$ may be integrated to get

$$w = -2 \int_0^\eta ww'' d\eta / \eta = 2 \{ [-ww'/\eta]_0^\eta + \int_0^\eta (w'^2/\eta - ww'/\eta^2) d\eta \} .$$

So

$$w_1(\eta) = \frac{1}{2} - 2ww'_1/\eta + 2 \int_0^\eta (w'_1\eta - w_1)w'_1 d\eta / \eta^2 < \frac{1}{2} .$$

Thus w_1 is increasing and

$$w_1(\infty) \leq \frac{1}{2} .$$

Also $w''_1 = -\eta w'_1/2w_1 > -\frac{1}{2}$ so $w'_1 > \frac{1}{2} - \frac{\eta}{2}$ for $\eta \leq 1$ while $w'_1 > 0$ for $\eta \geq 1$. It follows that $w_1 \geq \eta/2 - \eta^2/4$ for $\eta \leq 1$ and $w_1 > 1/4$ for $\eta > 1$. We deduce that

$$(5.11) \quad \frac{1}{4} < w_1(\infty) < \frac{1}{2} .$$

Taking $a = a_1 = w_1(\infty)^{-1/2}$ then $w \rightarrow 1$ and $f'(\eta) \rightarrow 0$ as $\eta \rightarrow \infty$ with $w < 1$ and $f'(\eta) < 0$ for $\eta < \infty$. From (5.9) it can be seen that this is only possible if $f(\eta) \rightarrow 0+$ as $\eta \rightarrow \infty$. We then have a solution $v(x, t)$ to (2.1) satisfying $v > 0$ for $x, t > 0$; in particular $v(1, t) > 0$, and v is an upper solution. Thus $g^*(t) \leq v(0, t) = a_1 \sqrt{t}$ with $a_1 = w_1(\infty)^{-1/2} < 2$, and (5.6) follows.

Taking $a < w_1(\infty)^{-1/2}$, w remains bounded away from 1 so that $f'(\eta)$ remains bounded away from 0, and there exists a unique $s = s(a)$ with $f(s) = 0$ and $f(\eta) < 0$ for $\eta > s$.

But

$$f(\eta) = a + \int_0^\eta \log w(\zeta) d\zeta = a + 2\eta \log a + a \int_0^{\eta/a} \log w_1(\zeta) d\zeta$$

so that choosing $z = a_2$ to give $s(a) = 1$ (i.e., $f(1) = 0$) a_2 is the unique solution of

$$a + 2 \log a + a \int_0^{1/a} \log w_1(\zeta) d\zeta = 0 .$$

Again noting that $w_1 > 0, w'_1 > 0, w''_1 < 0$ so that $w'_1 < \frac{1}{2}$, $w_1 < \frac{\eta}{2}$ for $\eta > 0$, we observe that

$$\begin{aligned} 0 &< a_2 + 2 \log a_2 + a_2 \int_0^{1/a_2} \log \zeta d\zeta - \log 2 \\ &= a_2 + \log a_2 - 1 - \log 2 . \end{aligned}$$

The zero a^* of $a + \log a - 1 - \log 2$ is between 1 and 2, so that $1 < a^* < a_2 < a_1 < 2$.

For $t < 1$, $v(1, t) = \sqrt{t} f(1/\sqrt{t}) < 0$ so that v is now a lower solution and we deduce that $g^*(t) \geq v(0, t) = a_2 \sqrt{t}$, which is (5.7).

The final part of the theorem, (5.8), follows from (5.6) and modifying the result (5.7). In place of $t \leq 1$ we take $t \leq \delta^2$ and choose a such that $s(a) = 1/\delta$. From the continuous dependence of the solution of (5.10) upon a we see that $a \rightarrow a_1$, as $s \rightarrow \infty$ and $\delta \rightarrow 0$. Thus for any $a < a_1$ we may take δ small enough so that

$$a\sqrt{t} < v(0, t) \leq g^*(t) < a_1 \sqrt{t} \quad \text{for } 0 < t < \delta,$$

and (5.8) follows.

§6. The behavior of the smooth solution as it ceases to exist. Consider the case where a global smooth solution does not exist. Then there is a finite maximal $T > 0$ such that the smooth solution u exists for all $t < T$.

From Lemma 2.4 it follows that

$$(6.1) \quad \liminf_{t \rightarrow T} u_x(0, t) = -\infty .$$

THEOREM 6.1. *For any $\varepsilon > 0$, u has a smooth extension to $\{\varepsilon < x \leq 1, 0 \leq t \leq T\}$.*

This means that all the derivatives of u remain bounded as $t \rightarrow T$ provided $x \geq \varepsilon$. Thus, $(0, T)$ is the only singular point of u_x in $\overline{\Omega_T}$.

Proof. By Lemma 2.3, u and u_t remain uniformly bounded in Ω_T . If

$$u_x(x_n, t_n) \rightarrow -\infty$$

for some sequence $(x_n, t_n) \rightarrow (\xi, T)$, $\xi > 0$, then by Lemma 2.3,

$$u_x(x, t_n) \rightarrow -\infty \quad \text{uniformly in } 0 < x < x_n,$$

as $t_n \rightarrow T$, and therefore

$$u(x_n, t_n) - u(0, t_n) \rightarrow -\infty;$$

a contradiction. We have thus proved that u_x remains uniformly bounded in $\Omega_{\varepsilon, T} \equiv \left\{ \frac{\varepsilon}{2} < x \leq 1, 0 \leq t < T \right\}$. It follows that (2.13) holds in $\Omega_{\varepsilon, T}$ and then, by parabolic theory (as in §2) u remains uniformly smooth (i.e., C^∞) in $\Omega_{2\varepsilon, T}$.

By integrating (2.1) in x ,

$$e^{u_x(0, t)} = e^{u_x(1, t)} - \int_0^1 u_t(x, t) dx.$$

Since (by Theorem 6.1) the right-hand side has a limit as $t \rightarrow T$, we deduce that

$$\lim_{t \rightarrow T} e^{u_x(0, t)} \quad \text{exists.}$$

By (6.1), this limit must be equal to zero. Hence:

THEOREM 6.2. *There holds:*

$$(6.2) \quad \lim_{t \rightarrow T} u_x(0, t) = -\infty .$$

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