RICHNESS AND THE CLASSIFICATION OF QUASILINEAR HYPERBOLIC SYSTEMS

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IMA Preprint Series # 597

October 1989

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Abstract. Rich quasilinear hyperbolic systems are those which possess the largest possible set of entropies. Such systems have a property of global existence of weak solutions, whatever large is the bounded initial data. Although the full gas dynamics is not rich, many physically meaningful systems are. One gives below new examples and properties of the fully linearly degenerate case.

Résumé: Un essai de classification des systèmes quasilinéaires hyperboliques conduit à considérer ceux dont l' ensemble d' entropies est aussi grand que le permettent des considérations immédiates. Ces systèmes, dits riches, ont des solutions faibles globales pour des données initiales bornées. Bien que la dynamique des gaz n' entre pas dans cette catégorie, de nombreux systemes ayant un sens physique sont riches. On donne ci - dessous de nouveaux exemples et on étudie dans cette famille la dégénérescence linéaire des champs caractéristiques.

I. Classification of quasilinear systems. Given an integer $n \geq 1$, one studies systems of the form $u_t + A(u)u_x = 0$, where t is a time-variable, x a space-variable, $u(x,t)$ the unknown belonging pointwise to $\mathbb{R}^n$, and $A(u)$ is an $n \times n$ matrix, depending smoothly on u.

Classically, one is interested with the Cauchy problem for this system. In order to get accurate properties, we shall often restrict to smooth, local in time solutions. In some cases, we shall deal with weak solutions, which means that we choose a conservative form $v_t + (f(v))_x = 0$, even though we do not specify it; it requires that such a form exists.

1. Diagonalization. Hyperbolicity implies that the matrix $A(u)$ is diagonalizable for any value of u. In the linear case (A is constant), the system can thus be reduced to a diagonal form, consisting of uncoupled transport equations. When nonlinearity occurs, the most that one can expect is an equivalent coupled system of transport equations

$$D_i w_i = 0 \ , \ 1 \leq i \leq n \ , \ D_i = \partial_t + \lambda_i(w)\partial_x \ . \tag{1.1}$$

In that event, the functions $w_i(u)$ are called strict Riemann invariants. The speeds $\lambda_i(w(u))$ are nothing but the eigenvalues of $A(u)$, for which $\operatorname{grad}_u w_i$ is a left eigenvector.

It turns out that not all the quasilinear hyperbolic systems (QLH) diagonalize. Assuming for simplicity that Spec $(A(u))$ consists of n distinct real values, a necessary and sufficient condition for the existence of w_i is the well-known Frobenius condition $l_i\{r_j, r_k\} = 0$ for any $j, k \neq i$. Here above, l_i and r_i denotes the left and right eigenvectors related to λ_i, and $\{.,.\}$ is the Poisson bracket of vector fields in u-space.

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When $n = 2$, a QLH system is always diagonalizable, but the Frobenius criterion becomes non trivial as $n \geq 3$.

A qualitative way for to introduce diagonalizable system is to consider the interaction of two incident weak shocks for conservative cases. For $n = 2$, two transmitted waves are outgoing. For $n \geq 3$, the interaction produces waves belonging to the other characteristic fields. These waves have strength $O(\alpha\beta)$ where α and β are the strength of incident waves. The necessary and sufficient condition in order that the strength should be weaker is that strict Riemann invariants exist. In that event, the strength turns out to be $O(\alpha\beta(|\alpha|+|\beta|))$. A special case is given by B. Temple's class of systems, for which an i-th and a j-th incident shocks produce only an i-th and a j-th transmitted shock (see [1] and below II.6. for a description of this class).

2. Physical, over-physical systems. Systems which are given by physics, mechanics, chemistry, . . . are often in a conservative form and possess an extra conservation law $E(v)_t + F(v)_x = 0$ where E is a strictly convex function. Following Godunov [2], those systems are hyperbolic and symmetrizable in variables $Q = \text{grad}_u E$.

Some of them may possess many more additional conservation laws. Functions as E are called "entropies" by mathematicians, no matter with their physical sense or their lack of sense; they are not required to be all convex. Trivial entropies are the components v_i and the constants. An example of infinite dimensional set of entropies is provided by Eulerian gas dynamics, with $E = \rho f(S), \rho$ being the mass density, S the physical entropy and f any real-valued function.

It turns out that entropies $E(u)$ are the solutions of a linear partial differential system of the form

$$D_j D_k E = \text{ lower order terms}, \qquad j \neq k \;, \tag{1.2}$$

where $D_j = r_j \cdot \text{grad}_u$. For $n = 2$, one can choose $D_j = \partial/\partial w_j$ and the system reduces to a single equation for which the Goursat problem is well-posed. Thus the set of entropies is infinite dimensional and parametrized by two arbitrary functions of one variable $F(w_1), G(w_2)$.

3. Rich systems. For $n \geq 3$, the system (1.2) consists of $n(n-1)$ equations, which can give rise to other equations by combination of their derivatives. Especially, the difference between the equations for the choice (j,k) and (k,j) is a first order equation. Thus generically, a QLH system has not any entropy, except trivial ones in the conservative case.

A simplification occurs in the diagonal case, where there are only $n(n-1)/2$ equations:

$$\partial(\lambda_i \partial E/\partial w_i)/\partial w_j = \partial(\lambda_j \partial E/\partial w_j)/\partial w_i, \; i \neq j \;. \tag{1.3}$$

Nevertheless, even this system turns out to be overdetermined and generally prevent the existence of non trivial entropy. Beside the constant coefficient case, where an entropy is uniquely specified by its values on an orthogonal set of reference axis $\mathbb{R}e^i$, one may search for nonlinear systems which endow this property. One calls them "rich hyperbolic systems". An algebraic characterization is given by (see Serre [3] for a description and details about those results which will be given below without proof):

$$D_k((\lambda_j - \lambda_i)^{-1} D_j \lambda_i) = D_j((\lambda_k - \lambda_i)^{-1} D_k \lambda_i) \,, \tag{1.4}$$

for distinct i, j and k. All along this article, D_i will denote the derivative with respect to w_i.

We again remark that 2×2 systems are trivially rich, so that the theory of rich systems appears to be an attempt of generalization of the theory of 2×2 systems; and actually it is for almost all points of view. An essential one is the method of compensated compactness, developed by L. Tartar [4], R. DiPerna [5], M. Rascle and D. Serre [6], [7],[8] in order to solve the Cauchy problem and to describe the propagation and the interaction of large oscillations. A fundamental consequence holds for genuinely non linear strictly hyperbolic rich systems: a bounded sequence of approximate solutions (namely through artificial viscosity or Godunov's scheme) should converge strongly in L^p for any finite $p \geq 1$ to a weak solution of the system, provided it is given in a conservative form.

The plan of the paper is as follows. The second part is a list of examples and counter-examples arising in physics, mechanics and chemistry. I am particularly grateful to Pr. C. Dafermos for to have driven my attention to a system arising in electrophoresis. It has given me motivation to include the B. Temple's class. The third part deals with the case where all the characteristic fields are linearly degenerate. It contains a short list of formula describing the construction of rich systems and of their entropies, as an abstract of my previous article [3]. A global existence of smooth solutions is given in that case. Its proof is essentially different from the genuine nonlinear case. Finally we discuss the commutability of resolvant operators.

In a forthcoming paper* I shall discuss in more details the linearly degenerate case, including a description of the propagation of large oscillations as in Serre [8] and M. Bonnefille [9]. A group action of resolvant operators on the set of solutions of special ODE systems will be studied. It gives rise to results about (x, t)-almost periodicity or periodicity of solutions of the hyperbolic system, and x-almost periodicity or periodicity for the ODE system.

An alternate approach to rich hyperbolic systems is Tsarev's work [10], where hamiltonian systems are considered. However his class is more restrictive than the rich one. It is essentially due to the definition of symplectic structure: it turns out that naturally hamiltonian 2×2 systems (e.g. isentropic gas dynamics) do not belong to Tsarev's class. Subsequent papers developing the Tsarev's hodograph method or computing higher order

*Systèmes d' EDO invariants sous l'action de systèmes hyperboliques d' EDP. To appear in Annales de l'Institut Fourier.

densities are due to Y. Kodama [11], Y. Kodama and J.Gibbons [12], P.J. Olver and al. [13].

II. Examples of rich and non rich hyperbolic systems. As discussed in the first section, all the 2×2 systems are rich, so that we shall only consider the case $n \geq 3$. We begin with a well-known non rich example.

1) The gas dynamics. This system consists of balance laws of mass density ρ, momentum ρu and total energy $\rho e + \rho u^2/2$. Thus $n = 3$. A classical result describe all the extra conserved densities by the formula $\rho f(S)$ where f is any real-valued function of one variable, and S is the physical entropy given by the equality

$$TdS = de + pd(1/\rho).$$

Here above, $p(\rho, e)$ is the hydrodynamic pressure.

On the other hand, the full gas dynamics possesses only one strictly Riemann invariant, namely S. Thus this system appears to be only "one third" rich. In particular, the compensated compactness method is not efficient and the Cauchy problem for large initial data is still open.

Let me give however a consequence of this partial richness related to a linearly degenerate field: because of $\rho_t + (\rho u)_x = 0$ and $S_t + uS_x = 0$, one finds an infinite sequence of quantities satisfying the same transport equation than S. They are inductively defined by $S_{n+1} = \rho^{-1}\partial_x S_n$, $S_0 = S$. Consequently, any function $\rho f(S_0, S_1, \dots)$ is a conserved density. Finally the breakdown of smooth solutions in finite time gives rise to a shock development, without contact discontinuity. In fact, contacts may appear at positive time only as a byproduct of the interaction of two shocks.

2) Nonlinear electromagnetic plane waves. The Maxwell's equations involve four vector fields B, D, E, H depending on time and on a 3-d space variable:

$$B_t + \operatorname{curl} E = 0,\ D_t - \operatorname{curl} H = 0\ . \qquad (2.1)$$

Following Coleman and Dill [14], the constitutive laws involve an electromagnetic energy density $W(D, B)$:

$$E = \partial W/\partial D\ , \quad H = \partial W/\partial B\ .$$

Hyperbolicity of (2.1) corresponds to the convexity of W. An important conserved density is W.

$$W_t + \operatorname{div}(E \times H) = 0\ . \qquad (2.2)$$

The system (2.1) is translationally invariant and admit plane waves depending only on t and one space variable, e.g. $x = x_1$. The components B_1 and D_1 are then constant and (2.1) reduces to 4 equations. A natural choice for W is to assume axisymmetry:

$$W(D,B) = W(r) \ , \ r^2 = B^2 + D^2 \ . \tag{2.3}$$

With that choice, the plane waves obey to a closed 2×2 system, supplemented by two transport equations of the form

$$(pv)_t + (qv)_x = 0 \ , \quad (v \text{ is the unknown}) \tag{2.4}$$

where it is known that $p_t + q_x = 0$. Thus this system is endowed with the entropies of the sub-2×2 system and of entropies of the form $pf(v)$ for any f. On the other hand, it has a complete set of strict Riemann invariants: the ones of the subsystem and the unknowns of the transport equations. We conclude that this system is rich.

A complete description may be found in Serre [15], where a global existence theorem is proved for large Cauchy data.

Let us point out that two characteristic fields are linearly degenerate, while the two others may be genuinely nonlinear although the whole system is linear for the most of electromagnetic media as vacuum.

3) Elastic strings. An elastic string lying in a 2 or 3-d physical space can be described in Lagrangian coordinates by $y_{tt} = (T(r)y_x/r)_x$ where the stretching r is the norm of y_x and T is the (scalar) tension. By using $u = y_x$, $v = y_t$, the balance laws become a QLH system of 4 or 6 equations. It is one of the worst of the hyperbolic theory: its entropy set is finite dimensional (6 or 8 independent entropies), while not any strict Riemann invariant exist. There are two linearly degenerate fields, but the interaction of two contact discontinuities produces shocks, yielding to energy dissipation! Even a linear stress-strain relation $T(r) = r - 1$ has been of no help for the existence theory in spite of the linear degeneracy of all the fields.

The Riemann problem for elastic strings has been solved by Keyfitz–Kranzer [16] and Carasso–Rascle–Serre [17]. Analysis and numerical simulations via the Glimm's and Godunov's schemes when the natural type changing occurs have been performed by H. Gilquin, R. Pego and Serre [18], [19], [20].

4) The KdV limit. The behaviour of the Cauchy problem for the Korteweg-de Vries equation $u_t + uu_x = \epsilon u_{xxx}$, as ϵ goes to zero, has been studied independently by P. Lax–D. Levermore [21] by means of the inverse scattering method, and by H.F. Flashka, G. Forest, D. McLaughlin [22] via the formalism of modulated waves. It turns out that the solution sequence u^ϵ does not converge in any norm of Lebesque space in general. But weak convergence holds, which is described by a finite piecewise constant odd number of functions $w_i(x,t)$, $0 \le i \le 2p$. These functions obey to a diagonal system

$$\partial_t w_i + \lambda_i(w)\partial_x w_i = 0 \ , \quad 0 \le i \le 2p \ ,$$

where the speeds are defined in a complicated way by means of hyperelliptic integrals.

Because the KdV equation is known to have infinite sets of independent conserved densities, it is expected that its limit should be rich, and actually it is. The criterion (1.4) follows trivially from a formula of D. Levermore [23] which motivated me in this research. However the relation between the conserved densities of the KdV equation and the ones of its limit had not been yet enlighted.

These systems were proved to have genuinely nonlinear characteristic fields [23] so that the convergence result cited in I.3 would be valid and provide global weak solutions for convenient conservation forms. However such weak solutions have no meaning for the description of KdV phenomena as soon as shocks develop.

An other hierarchy of rich systems, consisting of an even number $N = 2p$ of equations, is the N-reduction of Benney's moment equation (the dispersionless KP hierarchy), see D. Benney [24]. It has been studied in details by Kodama.

5) Electrophoresis. The following system arises in electrophoresis

$$\partial_t u_i + \partial_x(m^{-1} a_i u_i) = 0 \ , \ 1 \le i \le N \ , \tag{2.5}$$

where a_i is a positive constant and $m = u_1 + \cdots + u_N$. The unknowns $u_i(x,t)$ should be non-negative functions. Clearly, (2.5) satisfies the corresponding minimum principle. Moreover, it will be asked that $m(x,t) > 0$. This is easily achieved due to the equality

$$\left(\sum_i u_i/a_i\right)_t = 0 \ .$$

We shall assume without generality that $a_1 < a_2 < \cdots < a_N$, otherwise if $a_i = a_{i+1}$, we should replace N by $N-1$ and (u_i, u_{i+1}) by $u_i + u_{i+1}$. Thus we may define nonlinear functions $d_1(u), \ldots, d_{N-1}(u)$ by the formulas

$$\sum_i u_i/(a_i - d_k(u)) = 0 \ , \quad a_k < d_k < a_{k+1} \ . \tag{2.6}$$

Finally, $d_0(u)$ will denote $\left(\left(\prod_i a_i\right)\sum_i u_i/a_i\right)^{-1}$. The d_k's are strict Riemann invariants of the system, which is transformed to

$$\begin{cases} \partial_t d_0 = 0 \\ \partial_t d_k + m^{-1} d_k \partial_x d_k = 0 \ , \qquad 1 \le k \le N-1 \ . \end{cases} \tag{2.7}$$

In order to close the system, it remains to relate m to the d_k's. Let us define the rational fraction of one variable

$$F(X) = X \sum_i \frac{u_i}{a_i - X} = \sum_i \frac{a_i u_i}{a_i - X} - m \ .$$

One may rewrite F as a ratio P/Q of two polynomials of degrees not greater than N. But we know the poles a_i and the zeros 0 and $d_k, k \geq 1$, of F. Thus there exists a real constant α such that

$$F(X) = \alpha X \prod_{k=1}^{N-1} (d_k - X) \cdot \prod_{k=1}^{N} (a_i - X)^{-1}$$

Because of $F(\infty) = -m, \quad \alpha = m$. Finally

$$F'(0) = \sum_i \frac{u_i}{a_i} = m \prod_{k=1}^{N-1} d_k \left(\prod_{i=2}^{N} a_i \right)^{-1} ,$$

so that $m^{-1} = d_0 \prod_{k=1}^{N-1} d_k$, and $\lambda_i = d_0 d_i \prod_{k=1}^{N-1} d_k$. It is now easy to check by hand the criterion (1.4).

6) The B. Temple's class. The electrophoresis system, as the one of chromatography, actually belongs to the B. Temple's class. Let me give a rather rigid definition. A conservative system will be said to belong to the Temple's class if any point of the u-space is the intersection of $(N-1)$-dimensional linear characteristic manifolds. The existence of such (linear or nonlinear) manifolds is nothing but the existence of strict Riemann invariants. Their linearity is related to the fact that they will be invariant for weak solutions.

An alternate criterion of algebraic type is as follow. Consider the equation $l.u = c$ of a linear manifold. Then this manifold is characteristic for the conservative system $u_t + f(u)_x = 0$ if and only if the function $l.f(u)$ assumes a constant value b on it. In electrophoresis, $l_i = (a_i - d)^{-1}$ is convenient for any d (take $c = 0$), and this gives N distinct families of linear manifolds, depending on the position of d with respect to the a_i's.

For another such system, let w_i the Riemann invariants and $l^i(w_i)u = c_i(w_i)$ corresponding equations of the characteristic linear manifolds. Then it is easily seen that $(l^i.u - c_i)^+$ is a non trivial (i.e. non affine) entropy for any choice of i and w_i. Thus we can construct N infinite families of entropies, parametrized by $1-d$ bounded measures

(2.8) $$E_{i,\sigma}(u) = \int (l^i(w).u - c_i(w))^+ d\sigma(w) \ ,$$

with corresponding fluxes

$$F_{i,\sigma}(u) = \int (l^i(w).f(u) - b_i(w)) \operatorname{sgn}(l^i.u - c_i) d\sigma(w) \ .$$

We have thus proved the

THEOREM. *Any conservative systems, such that all its left eigenvectors $l_i(u)$ are normal vector fields to linear real manifold, is rich.*

□

For such systems, we shall not pay attention to the nonlinearity of characteristic fields because the compensated compactness method is superfluous. A global existence theorem has been proved in the BV class for large initial data by Serre [25], Leveque and Temple [26]. See Temple [1] for a systematic construction in case $N = 2$.

The following subsection is devoted to a special 2×2 example, for which we use the formula (2.8) for to exhaust the list of entropies.

7) An example in Temple's class.

The system $u_t + (uv)_x = 0$, $v_t + (v^2 + u)_x = 0$ belongs to the Temple's class, because the trivial entropy $E_a = u - av - a^2$ is an algebraic divisor of its flux $F_a = (v - a)E_a$, for any choice of the real parameter a. Actually we have to restrict the above assertion, because the system is hyperbolic only on the zone $v^2 + 4u > 0$. It becomes elliptic as the dependent variables enter inside the parabola $v^2 + 4u < 0$. Thus the aforementioned linear manifolds (straight lines in this 2×2 case) cover only the domain $v^2 + 4u \geq 0$. An important consequence of this remark is that the formula (2.8) will not be usable for constructing entropies inside the elliptic zone. We thus shall restrict to the hyperbolic one.

The Riemann invariants are clearly the two (real) roots w and z of the quadratic equation

$$X^2 + vX - u = 0 \ .$$

Then $E_a = (w - a)(a - z)$, where we assume $w \geq z$. A similar idea to (2.8) is that $(w - a)^+(a - z)$ is again an entropy, so that the following formulae define an entropy and its flux for any choice of a bounded measure m:

$$E = \int_{-\infty}^{w} (w - a)(a - z)dm(a)$$

$$F = \int_{-\infty}^{w} (v - a)(w - a)(a - z)dm(a) \ .$$

We shall keep in mind that $v = -w - z$. Defining functions $f, g, h, k(w)$ as the antiderivatives of $a^p dm(a)$, $0 \leq p \leq 3$, we rewrite E and F as

$$E(w, z) = -wzf(w) + (w + z)g(z) - h(w) \ ,$$
$$F(w, z) = (w + z)wzf(w) - (w^2 + wz + z^2)g(w) + k(w) \ .$$

The definition of f, g, h and k allows us to introduce a function $T(w)$ satisfying the following four equalities:

$$
\begin{aligned}
f &= T''' & k &= w^3T''' - 3w^2T'' + 6wT' - 6T \\
g &= wT''' - T'' & h &= w^2T''' - 2wT'' + 2T'
\end{aligned}
$$

So that we get an infinite family of pairs entropy-flux, parametrized by a real function T of one variable:

$$
\left\{
\begin{array}{l}
E = (w-z)T''(w) - 2T'(w) \\
F = (z-w)(z+2w)T''(w) + 6wT'(w) - 6T(w)
\end{array}
\right. \tag{2.9}
$$

The above formula actually does not give all the entropies of our system because w and z did not play the smale role in our calculations. We need to supplement it by the symmetric formula depending on a real function $S(z)$, so that the general entropy would be

$$
(w-z)(T''(w) - S''(z)) - 2T'(w) - 2S'(z) \ . \tag{2.10}
$$

It turns out that this formula gives all the entropies of the system, as it can be checked by hand, using the entropy equation

$$
((2w+z)E_w)_z = ((2z+w)E_z)_w \ .
$$

Conversely, (2.10) does not give any information about entropies in the elliptic zone, except the case where $S = T$ is a polynomial. Then the formula makes sense and defines a smooth function on the whole plane. For instance, the choice $S = T = X^5/10$ gives $E = v^4 + 6v^2u + 6u^2$.

REMARK. In the formula (2.8), the special entropy $(l_i.u - c_i)^+$ appears to be an extremal one in the cone of convex entropies, so that $E_{i,\sigma}$ will be convex if and only if σ is a non-negative measure. This fact relies to the scalar example where the entropies $|u-k|$ or $(u-k)^+$, used by Kruzkhov [27] and Tartar [4] generate all the convex functions of one variable by means of the integrals

$$
\int_{-\infty}^{u} (u-k)d\sigma(k) \ .
$$

Coming back to the more explicit formula (2.10), we get a convex entropy if and only if $T = -S$ and T^{IV} is non-negative. The condition $T = -S$ comes from the fact that $(w-a)^+(a-z)$ is not convex, so that we have to apply formula (2.8).

The next subsection is devoted to the compensated compactness theory, applied to this system, and we shall pay attention to the elliptic zone.

8) Compensated compactness with an elliptic zone. The compensated-compactness theory is a tool which has been powerful in the study of the convergence of the artificial viscosity method for the Cauchy problem:

$$(2.11) \qquad \begin{cases} u_t^\epsilon + f(u^\epsilon)_x = \epsilon \, u_{xx}^\epsilon \\ u^\epsilon(x,0) = u_0(x) \end{cases}$$

There are essentially two *a priori* requirements in order to be allowed to apply this theory:

i) to have an L^∞ uniform estimate for u^ϵ. In all the known examples, it has been proved by means of the positively invariant domains of Chuey–Conley–Smoller [28].

ii) to have an entropy-flux pair (η, q) where η is uniformly strictly convex on every compact set of the u-space.

It is important to notice that none of these requirements occurs for the system introduced in the former section, so that I shall not prove the convergence of (u^ϵ, v^ϵ) to a solution, but only the following alternative:

THEOREM. *Let (u^ϵ, v^ϵ) a solution of the Cauchy problem*

$$u_t^\epsilon + (u^\epsilon v^\epsilon)_x = \epsilon u_{xx}^\epsilon \ , \ v_t^\epsilon + ((v^\epsilon)^2 + u^\epsilon)_x = \epsilon v_{xx}^\epsilon$$
$$u^\epsilon(x,0) = u_0(x) \ , \ v^\epsilon(x,0) = v_0(x) \ .$$

Then, as $\epsilon \to 0_+$, for a suitable subsequence,

i) *either the maximal time-interval $(0, T(\epsilon))$ of existence shrinks to zero,*

ii) *either the solution does not remain bounded in L^∞,*

iii) *either $\epsilon^{1/2}(u_x^\epsilon, v_x^\epsilon)$ does not remain bounded in L^2 ,*

iv) *either strong convergence in L^p_{loc} , $1 \le p < \infty$, holds, so that the limit (u,v) is a solution of $u_t + (uv)_x = 0$, $v_t + (v^2 + u)_x = 0$*

□

Proof. If i), ii) and iii) do not hold, then we are allowed to apply the compensated compactness theory. So that the Young measure $\nu_{x,t}$ which describes the oscillations of the sequence of approximated solutions has a compact support and satisfies almost everywhere (x,t) the Tartar's equality

$$(2.12) \qquad \nu(e_1 f_2 - e_2 f_1) = \nu(e_1)\nu(f_2) - \nu(e_2)\nu(f_1) \ ,$$

for all entropy-flux pairs (e_i, f_i), $i = 1,2$. We shall only deal with the entropies $E_a = E_a^+ - E_a^-$ and $|E_a| = E_a^+ + E_a^-$.

We begin by assuming that for any real number a, Supp ν is not contained in the straight line $u - av - a^2 = 0$. It implies that either $\nu(E_a^+)$ or $\nu(E_a^-)$ is positive, so that $\nu(|E_a|) > 0$. Due to the equality $E_a^+ F_a^- = E_a^- F_a^+$, we get ($F_a^\pm$ are not equal to positive and negative parts of F_a)

$$\nu(E_a^+)\nu(F_a^-) = \nu(E_a^-)\nu(F_a^+) \ ,$$

so that we may define uniquely a real number $c(a)$ by the formulae

$$\nu(F_a^\pm) = c(a)\nu(E_a^\pm) \ .$$

By linearity, $\nu(F_a) = \nu(E_a)c(a)$. Furthermore, $c(\cdot)$ is clearly a continuous function defined on the real line, and $\nu(F_a^+ + F_a^-) = c(a)\nu(|E_a|)$.

We then apply (2.12) to $|E_a|$ and $|E_b|$, with b different from a:

$$(a-b)\nu(|E_a|\ |E_b|) = (c(b) - c(a))\nu(|E_a|)\nu(|E_b|)$$

Letting now $b \to a$, we conclude that $c(\cdot)$ is continuously differentiable, with

$$c'(a) = -\nu(E_a^2)\nu(|E_a|)^{-2} \tag{2.13}$$

By Cauchy-Schwarz inequality, $c' \leq -1$. The second step is to remark that $c(\cdot)$ is a rational fraction, expanding at $\pm\infty$ as

$$c(a) = -a + \text{ constant } + O\left(\frac{1}{a}\right) \ ,$$

so that the non-increasing function $c(a) + a$ assumes equal values at $\pm\infty$. Thus it is a constant, and $c'(a) \equiv -1$. Returning now to (2.13), we conclude that for any real value a, the support of the Young measure is contained in the union of three parallel lines $u - av = \text{constant}$. Using two different values of a, we find a set consisting of 9 points, which contains Supp ν. Then almost all choice of a third value of a will reduce this set to three points. Now remarking that the aforementioned parallel lines are $u - av - a^2 = 0$ or $\pm\sigma$, a fourth choice of a reduces the support to one point. Thus the preclude assumption is false.

We thus now deal with the case where Supp ν is contained in one line $u - av - a^2 = 0$. Then the analysis becomes one-dimensional and is nothing more than that Tartar did in [4]. Thus Supp ν reduces to one point and $\nu_{x,t}$ is a Dirac mass for almost all x, t.

□

REMARK. The first part of this proof can be mimic for any system belonging to the Temple's class, until a formula similar to (2.13). It would be interesting to consider the general case. The essential question is to understand the role of the genuine nonlinearity and to decide how many inflections are allowed in order to get the result that the Young measure is a Dirac mass.

III. Systems with linearly degenerate characteristic fields.

1) Facts about rich systems. This subsection gathers some facts about the construction of rich systems and their entropies. All of them are proved in [3].

a) Given a rich system, the condition (1.4) gives us a set of positive functions $N_i(w)$, $1 \le i \le N$, such that

$$D_j\lambda_i = (\lambda_j - \lambda_i)D_j(\operatorname{Log} N_i) \ , \qquad i \neq j \ . \tag{3.1}$$

Let us define quantities $c_{ij} = N_j^{-1} D_j N_i$ and $y_i = \lambda_i N_i$. Then one gets the relations

$$D_k c_{ij} = c_{ik} c_{kj} \qquad , \ i \neq j \neq k, \ i \neq k \ , \tag{3.2}$$

$$D_j y_i = c_{ij} y_j \qquad , \ i \neq j \ . \tag{3.3}$$

Reciprocally, let us choose functions $\gamma_{ij}(w_i, w_j)$, $i \neq j$, then the system (3.2) has a unique local solution $\{c_{ij} \ , \ i \neq j\}$ assuming the Goursat data

$$c_{ij}(w) = \gamma_{ij}(w_i, w_j) \ , \text{ as } \ w_k = w_k^* \ , \ k \neq i, j \ , \tag{3.4}$$

where w_k^* are given constants. This solution is global provided the following norms are small enough:

$$\|\gamma\| = \operatorname*{Max}_{i,j} \int_{\mathbb{R}} \operatorname{Sup}\{|\gamma_{ij}|; w_j \in \mathbb{R}\} \ dw_k$$

The next step is to solve (3.3) and the similar system $D_j N_i = c_{ij} N_j$, $i \neq j$. It turns out that (3.2) is the set of compatibility conditions for this overdetermined system, so that a global existence and uniqueness result holds for Goursat data:

$$N_i(w) = \text{ given } n_i(w_i) \text{ as } w_k = w_k^* \ , k \neq i \ , \tag{3.5}$$

and similarly for y. A rich system is then completely defined by the formula $\lambda_i = y_i / N_i$.

b) The derivation of entropies is as follow. Clearly the transposed $\mathbf{c}'$ satisfies (3.2) too, so that the following system has a global existence and uniqueness property:

$$D_j p_i = c_{ji} p_j \tag{3.6}$$

$$p_i(w) = \text{ given } q_i(w_i) \text{ as } w_k = w_k^* \ , \ k \neq i \ . \tag{3.7}$$

Such p's are used to construct irrotational fields $g_i = p_i N_i$. The potential E of g is then a generic entropy of the rich system. Its flux F is a potential for the irrotational field $h_i = p_i u_i$.

c) The last feature I want to recall here is the proof of blow up of smooth solutions in the genuinely nonlinear case. It mimics the Lax's proof for 2×2 systems [29]. It appeared in details for the KdV limit in Levermore [23]. Let me first define the operator $L_i = \partial_t + \lambda_i \partial_x$, so that $L_i w_i = 0\ ,\ 1 \leq i \leq N$. Taking the x-derivative, and denoting $\partial_x w_i$ by z_i, it comes

$$L_i z_i + z_i \partial_x \lambda_i = 0\ . \tag{3.8}$$

Let us compute the last term:

$$\partial_x \lambda_i = \sum_j z_j D_j \lambda_i = z_i D_i \lambda_i + \sum_j (\lambda_j - \lambda_i) z_j D_j (\operatorname{Log} N_i)\ .$$

Now, one uses the equality $(\lambda_j - \lambda_i) z_j = -L_i w_j$, which gives

$$\partial_x \lambda_i = z_i D_i \lambda_i - \sum_j (L_i w_j) D_j (\operatorname{Log} N_i) = z_i D_i \lambda_i - L_i (\operatorname{Log} N_i)$$

and finally (3.8) becomes

$$L_i (z_i / N_i) + N_i^{-1} z_i^2 \partial_i \lambda_i = 0 \tag{3.9}$$

This fundamental equality will be used intensively in the linearly degenerate case below. With genuinely nonlinearity, it is a Ricatti-like equation along the i-th characteristic curves of the (x, t) plane, for the unknown $z_i N_i^{-1}$. Because of the transport equations $L_i w_i = 0$, a smooth solution takes its values in a fixed parallelipiped of $\mathbb{R}^N$, so that the coefficients $N_i \partial_i \lambda_i$ are bounded and bounded away from zero. Thus $z_i N_i^{-1}$ becomes infinite in a finite time provided its initial value has the opposite sign to $\partial_i \lambda_i$. It is clearly the case for some x-point if we assume the initial data $w(x)$ to be compactly supported.

2) Construction of fully degenerate examples.

a) Let us consider a rich system satisfying the degeneracy $\partial_i \lambda_i \equiv 0$ for all $1 \leq i \leq n$. We shall assume that none of the λ_i's has a critical point, and also the strict hyperbolicity.

We introduce the diagonal entries c_{ii} of the $\mathbf{C}$ as usual

$$\partial_i N_i = c_{ii} N_i\ .$$

Then deriving the equality $N_i \lambda_i = u_i$ with respect to w_i gives $\partial_i u_i = c_{ii} u_i$. Note that $\lambda_i \neq 0$ and $u_i \neq 0$ almost everywhere.

Let us choose two distinct integers $1 \leq i, j \leq n$ and derive the equality $u_j \partial_j N_i = N_j \partial_j u_i$ with respect to w_j. Using the former, one gets

$$(\lambda_j - \lambda_i)(\partial_j c_{ii} - c_{ij} c_{ji}) = 0 \ ,$$

so that

$$\partial_j c_{ii} - c_{ij} c_{ji} = 0 \ , \qquad \forall\, j \neq i \ . \tag{3.10}$$

Finally, we derive $\partial_i u_i = c_{ii} u_i$ with respect to w_i. It comes

$$u_i(\partial_j c_{ii} - c_{ij} c_{ji}) = u_j(\partial_i c_{ij} - c_{ii} c_{ij}) \ ,$$

so that

$$\partial_i c_{ij} - c_{ii} c_{ij} = 0 \ , \qquad \forall\, j \neq i \tag{3.11}$$

Finally, we gather all the known equalities in:

$$\partial_k c_{ij} = c_{ik} c_{kj} \quad \text{for any } i, j, k \text{ such that } k \neq j \ . \tag{3.12}$$

b) Conversely, we suppose that (3.12) holds, and then we construct a family of fully degenerate systems related to the matrix $\mathbf{C}$. We first construct a convenient filed $(N_1, \ldots, N_n)$. It is uniquely defined by the system $\partial_j N_i = c_{ij} N_j \quad (j \neq i)$ and the prescribed values of each N_i on the i-th axis. In order to satisfy $\partial_i N_i = c_{ii} N_i$ everywhere, it is sufficient to impose this differential equation to the prescribed value of N_i along the i-th axis, thanks to the equality

$$\partial_j(\partial_i N_i - c_{ii} N_i) = 0 \ , \qquad j \neq i \ .$$

We now construct the u_i's. We choose a constant vector $(\alpha_1, \ldots, \alpha_n)$ and prescribe the value $\alpha_i N_i$ to u_i on the i-th axis. Then the equation $\partial_i u_i = c_{ii} u_i$ is again true on the i-th axis and thus everywhere, so that $\partial_i \lambda_i \equiv 0$ for $\lambda_i = u_i / N_i$.

To summarize these results, let us say that each solution of (3.12) gives rise to an n-dimensional real vector space of fields $(N_1, \ldots, N_n)$ satisfying $\partial_j N_i = c_{ij} N_j$ for any i and j. Furthermore, any such N_i gives rise to an n-dimensional vector space of λ's, satisfying

$$\partial_j \lambda_i = (\lambda_j - \lambda_i) \partial_j \operatorname{Log} N_i \quad , \quad \forall\, i, j \ . \tag{3.13}$$

These λ's defined rich hyperbolic systems with linearly degenerate fields. The above construction suggest to parametrize the speeds as λ^α and the systems as (φ^α). Note that the special choice $\alpha = (a, \ldots, a)$ gives the simple decoupled transport equations

$$\partial_t w_i + a \partial_x w_i = 0 \ , \quad 1 \leq i \leq n \ .$$

c) As was shown in [3], the entropies are given explicitly in the linearly degenerate case by the formula

$$E = N_1 f_1(w_1) + \cdots + N_n f_n(w_n) \ ,$$
$$F = u_1 f_1(w_1) + \cdots + u_n f_n(w_n) \ ,$$

where $f_1, \ldots, f_n$ are arbitrary functions of one variable.

3) An explicit example.

a) A particular solution of (3.12) is

$$(3.14)\qquad \begin{cases} c_{ij} = \prod_{k\neq i,j}(w_i - w_k)\cdot\left[\prod_{k\neq j}(w_j - w_k)\right]^{-1} \quad , \quad j\neq i \\ c_{ii} = \sum_{k\neq i}(w_k - w_i)^{-1} \ . \end{cases}$$

A related field $N(w)$ is

$$(3.15)\qquad N_i = \prod_{k\neq i}(w_i - w_k)^{-1} \ .$$

This field and $\mathbf{C}$ are not globally defined, so that we shall restrict ourselves to a parallelipiped which does not meet any hyperplanes $w_i = w_j$, $j \neq i$. Then we may construct solutions of (3.13) by choosing a symmetric polynomial of $n-1$ variables P, being of partial degree 1 with respect to each variable, next defining $\lambda_i(w) = P(\hat{w}_i)$. Here, $\hat{w}_i$ means that we have removed the i-th component w_i from w.

b) Knowing that, we see that another choice for N is $N_i' = N_i Q(\hat{w}_i)$ where Q has the same properties than P. This yields to new speeds $\lambda_i' = (PQ^{-1})(\hat{w}_i)$. Finally the following system is rich which all its fields being linearly degenerate:

$$(3.16)\qquad \begin{cases} Q(\hat{w}_i)\partial_t w_i + P(\hat{w}_i)\partial_x w_i = 0 \quad , \quad 1\leq i\leq n \\ \text{with symmetric } P \text{ and } Q \text{ s.t. partial degrees} \\ \text{of } P \text{ and } Q \text{ are } \leq 1. \end{cases}$$

Let us remark that for any Q, this gives an n-dimensional vector space of speeds, so that all the fully linearly degenerate systems related to $N_i' = N_i Q(\hat{w}_i)$ belong to the family described by (3.16).

c) Because we know some of the speeds $\mu_i = R(\hat{w}_i)/Q(\hat{w}_i)$ which are related to the same N' than λ', one would try to use Tsarev's hodograph formula [10] to find explicit solutions of (3.16). Here we have to solve

$$(3.17)\qquad P(\hat{w}_i)t = R(\hat{w}_i) + Q(\hat{w}_i)x \ , \quad 1\leq i\leq n \ .$$

In order to keep strict hyperbolicity, we ask for solutions of (3.17) such that $w_i \neq w_j (j\neq i)$. Then the difference of two equations of (3.17) gives

$$P'(\hat{w}_{ij})t = R'(\hat{w}_{ij}) + Q'(\hat{w}_{ij})x \ , \quad i\neq j \ ,$$

where P' denotes the derivative of P with respect to one of its variables, and $\hat{w}_{ij}$ has only $n-2$ components. By induction, we get a non-trivial equation with constant coefficients, which is absurd. Thus Tsarev's hodograph formula does not give the expected solution when applied to speeds μ_i such that $\partial_i\mu_i = 0$. However, other speeds are available among the ones which satisfy $\partial_j\mu_i = (\mu_j - \mu_i)\partial_j \operatorname{Log} N_i'$, and they generally give nontrivial results.

4. Global existence of smooth solutions. The analysis carried out in subsection 1.c gives an opposite result for degenerate systems, namely that smooth initial data yields to globally defined smooth solution, with the same regularity.

The proof consists essentially in four steps: local existence, L^∞ estimate, Lipschitz-type estimates, C^m estimates. We begin with a C^m initial data $w^\circ(x), m \geq 2$, being bounded on the whole line. Boundedness of the derivatives is not required. An essential assumption is that the field N is well-defined on the parallelipiped

$$K = \prod_i [\mathrm{Min}\, w_i^0,\ \mathrm{Max}\, w_i^0] \ ,$$

and that each of the N_i's assumes positive values on K. We require also that the speeds λ_i are smooth functions of w.

The local existence of a smooth solution of the Cauchy problem follows from Kato [30], where a result is stated in the Sobolev class H^m_{loc}. Thus we get a $C^{m-1/2}_{\mathrm{loc}}$ solution. The transport equation $L_i w_i = 0$ shows then that $w(x,t)$ belongs to K, so that each $N_i(w(x,t))$ is bounded by above and bounded away from zero.

Now we use relations (3.9), which reduce here to

$$L_i(z_i/N_i) = 0 \ , \ z_i = \partial_x w_i \tag{3.18}$$

By repeating the argument in (1.c), we find a hierarchy of transport equations involving higher derivatives of w_i, namely:

$$\begin{cases} L_i W_i^k = 0, \\ W_i^{k+1} = N_i^{-1} \partial_x W_i^k \ , \quad W_i^0 = w_i \end{cases} \tag{3.19}$$

This transport equation makes sense for $0 \leq k \leq m$, because it involves no more that products of an $H^{m-\frac{1}{2}}$ function and a $H^{m-k-3/2}$ distribution, and

$$m - \frac{1}{2} + m - k - \frac{3}{2} \geq m - 2 \geq 0 \ .$$

Thus (3.19) propagate the local boundedness of the k-th derivatives of w, up to the m-th. This is the Lipschitz-type estimate.

It remains to prove the continuity of $\partial_x^m w_i$ (estimates of mixed derivatives $\partial_t^k \partial_x^l w_i$ follow trivially from (3.19) and the knowledge about the x-derivatives). We have only to remark that the speeds λ_i are locally Lipschitz continuous, so that the i-th transport equation defines an homeomorphism of the space line at each time $0 \leq t < T$, provided that the solution exists on $(0,T)$. Then $W_i^m(T)$, being the composition of its initial value by the aforementioned homeomorphism, is continuous. Since $W_i^m = N_i^{-m} \partial_x^m w_i + l.o.t$, the continuity of $\partial_x^m w_i$ is proved up to T. Using again the local existence result, we conclude that the solution can be globally defined.

In addition, we remark that uniqueness holds thanks to [30], and that the time may be reversed because we deal with classical solutions. Thus we have proved:

THEOREM. *Consider a rich system $\partial_t w_i + \lambda_i(w)\partial_x w_i = 0, \quad 1 \le i \le n$, with smooth speeds λ_i, satisfying $\partial\lambda_i/\partial w_i = 0$. Assume that the N_i's are smooth positive functions on $K = \prod_{i=1}^{n}[w_i^-, w_i^+]$. Then there exists a unique one-parameter group of operators $S(t)$, acting on $C^m(\mathbb{R}; K)$, such that for any Cauchy data $w^0 \in C^m(\mathbb{R}; K)$, the unique smooth solution of the Cauchy problem is given by $w = S(t)w^0$.*

□

REMARK. Let us notice that the strict hyperbolicity is not required, essentially because weak hyperbolicity always holds. However, it is essential that the N_i's do not vanish. For instance, $N_1 = N_2 = 0$ when $w_1 = w_2$ for the system

$$\partial_t w_1 + w_2 \partial_x w_1 = 0 \ , \ \partial_t w_2 + w_1 \partial_x w_2 = 0 \ ,$$

and the Cauchy data $w_1(x,0) = w_2(x,0) = z_0(x)$ gives rise to a solution $z(x,t) = w_1 = w_2$ of the Burgers equation $z_t + z z_x = 0$, for which global existence of smooth solutions does not hold.

5) Commutability of resolvant operators. In this subsection, we shall fix the parallelipiped K and the field N as above. Thanks to 2.b, we know about a system (φ^a), given by speeds λ_i^α, for any $\alpha \in \mathbb{R}^n$, such that the above theorem holds. We denote by $S^\alpha(t)$ its group of resolvant operators and we retain $S^\alpha = S^\alpha(1)$. Any $S^\alpha(t)$ can be reduced to $S^{t\alpha}$, so that it is sufficient to study the S^α's.

THEOREM. *For K and N as above, the map $\alpha \longmapsto S^\alpha$ is a group homomorphism:*

$$S^\alpha \circ S^\beta = S^\beta \circ S^\alpha = S^{\alpha+\beta} \ .$$

□

Proof. We begin with the commutation relation. Let us fix a C^m Cauchy data $w^0, m \ge 3$. We define $w^1 = S^\alpha w^0, w^2 = S^\beta w^1, w^3 = S^\beta w^0, w^4 = S^\alpha w^3$. Our goal is to compare w^2 and w^4 up to the second order as α and β go to zero. Using the fact that $S^\alpha = S^\gamma(t)$ where $t = |\alpha|$ and $\gamma = t^{-1}\alpha$, one derives from $\partial_t w_i = -\lambda_i(w)\partial_x w_i$ the expansion:

$$w_i^1 = w_i^0 - \lambda_i^\alpha(w^0)\partial_x w_i^0 + O(\alpha^2) \ .$$

Combining similar formulae, one gets

$$w_i^2 - w_i^4 = \sum_j [(\lambda_j^\alpha - \lambda_i^\alpha)\partial_j \lambda_i^\beta - (\lambda_j^\beta - \lambda_i^\beta)\partial_j \lambda_i^\alpha]\partial_x w_j^0 \partial_x w_i^0 + O(|\alpha|^3 + |\beta|^3) \tag{3.20}$$

The functions appearing inside the above brackets apply to w^0. We now use the hypothesis that the speeds λ^α and λ^β are related to the same field N, so that the brackets vanish identically. Finally, for $|\alpha|, |\beta| \leq 1$, one finds

$$\text{(3.21)} \qquad \operatorname*{Sup}_x |S^\alpha \circ S^\beta(w^0) - S^\beta \circ S^\alpha(w^0)| \leq C(|\alpha|^3 + |\beta|^3) .$$

Note that the constant in the left hand side depends only on the C^3 norm of w^0, because of the estimates of the former subsection.

The next step is to choose a positive integer r and to write the commutator $[S^\alpha, S^\beta]$ as the sum of r^2 terms of the form $P\Delta Q$, where Δ is the commutator $[S^{\alpha/r}, S^{\beta/r}]$ and P, Q are monomials in $S^{\alpha/r}, S^{\beta/r}$ with $d^0P + d^0Q = 2r - 2$. Using again the estimates of 2), one gets a bound valid for each term, with a constant depending on w^0 but not on r:

$$\operatorname*{Sup}_x |P\Delta Q(w^0)| \leq C\left(|\frac{\alpha}{r}|^3 + |\frac{\beta}{r}|^3\right) .$$

Summing all these inequalities, we then deduce

$$\operatorname*{Sup}_x |[S^\alpha, S^\beta](w^0)| \leq Cr^{-1}(\alpha|^3 + |\beta|^3) ,$$

which gives $[S^\alpha, S^\beta] = 0$ by letting $r \to \infty$.

The last trick is the proof of the additivity formula. To this end, we first note an estimate similar to (3.21):

$$\text{(3.22)} \qquad \operatorname*{Sup}_x |(S^\alpha \circ S^\beta - S^{\alpha+\beta})(w^0)| \leq C(|\alpha|^2 + |\beta|^2) ,$$

where C depends only on the C^2 norm of w^0. We next remark that, due to the commutativity property, $S^\alpha \circ S^\beta - S^{\alpha+\beta}$ can be rewritten as the sum of $r+1$ terms of the form $P\theta$, where $\theta = S^{\alpha/r}S^{\beta/r} - S^{\frac{\alpha+\beta}{r}}$ and P is a monomial in $S^{\alpha/r}, S^{\beta/r}$ and $S^{\frac{\alpha+\beta}{r}}$ with suitable degree. As above, we conclude that

$$\operatorname*{Sup} |(S^\alpha \circ S^\beta - S^{\alpha+\alpha})(w^0)| \leq C(r+1)\left(|\frac{\alpha}{r}|^2 + |\frac{\beta}{r}|^2\right) ,$$

so that $r \to \infty$ gives $S^\alpha \circ S^\beta = S^{\alpha+\beta}$.

□

Acknowledgements. I am grateful to the IMA for its invitation. This work has been significantly improved during the 3 weeks I spend there. I had stimulated conversations on this topic with D. Levermore, D. McLaughlin, P.J. Olver and C. Dafermos.

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