**LINEAR DEFORMATIONS AS GLOBAL MINIMIZERS
IN NONLINEAR ELASTICITY**

By

Scott J. Spector

IMA Preprint Series # 596

October 1989

LINEAR DEFORMATIONS AS GLOBAL MINIMIZERS IN NONLINEAR ELASTICITY

by

Scott J. Spector
Department of Mathematics
Southern Illinois University
Carbondale, Illinois 62901-4408

1. Introduction

In his fundamental paper on cavitation BALL [3] considered the problem of minimizing the total energy,

$$E(f) = \int_B W(\nabla f(x)) dx,$$

of a ball $B \subset \mathbb{R}^3$, composed of a homogeneous, isotropic, compressible hyperelastic material, among those deformations $f \in W^{1,1}(B)$ that satisfy $f(x) = \lambda x$ for $x \in \partial B$. BALL showed that for λ sufficiently large the smooth solution of the corresponding equilibrium (Euler-Lagrange) equations, $f_0(x) := \lambda x$, is *not* the global minimizer of the total energy E for a large class of stored energy functions W . BALL also showed that among those deformations that are radial, orientation preserving, and one-to-one, a global minimizer exists and exhibits a spherical hole at the center of the deformed body. A result of KNOPS & STUART [11] shows that $f_0(x) = \lambda x$ is the global minimizer among smooth deformations. Thus functions that are not smooth must be considered if one wants to determine the global minimizer for such problems.

Motivated by BALL'S work others¹ have shown that both radial minimizers and radial solutions of the equilibrium equations can exhibit spherical cavities under a wide range of constitutive hypotheses. However, it is not known whether any such radial deformation is an absolute minimizer of the energy. In fact one does not even know that an absolute minimizer does indeed exist.²

In this paper we give constitutive hypotheses under which the linear deformations $f_0(x) := F_0 x$ are minimizers of the total energy among those deformations that are one-to-one and have the same boundary-values and orientation as f_0 . In particular we assume that

$$W(F) = g(F) + h(\det F),$$

where g and h are convex and $h'(H) = 0$, and show that³

$$(1.1) \quad \int_B W(F_0 + \nabla \tilde{u}(x)) dx \geq \int_B W(F_0) dx$$

whenever $0 < \det F_0 \leq H$ and $\tilde{u} \in W_0^{1,1}(B)$ is such that the deformation $f_0 + \tilde{u}$ is one-to-one and satisfies the containment constraint⁴ $(f_0 + \tilde{u})(B) \subset f_0(B)$. Thus such materials will not

¹ See JAMES & SPECTOR [10] for an extensive list of references.

² The only cases for which it is known that an absolute minimizer exists are trivial cases (e.g. $W(F) = h(\det F)$) in which a minimizer of the integrand, W , that satisfies the boundary condition, $f(x) = \lambda x$, can be exhibited.

³ If $g = g(F, \operatorname{adj} F)$ is convex then this result remains true provided $\tilde{u} \in W_0^{1,p}(B)$, $p \geq 2$.

⁴ The competitors need only satisfy these constraints almost everywhere. See Section 3.

exhibit any material instabilities in compression, but can exhibit cavitation or other such instabilities in tension.

Results of BALL & MURAT [4] and JAMES & SPECTOR [10] show that the injectivity hypothesis is crucial to our result. If the growth of g is slow enough to permit cavitation then BALL & MURAT have shown that h must be constant in order for (1.1) to be satisfied for all $\tilde{u} \in W_0^{1,1}(B)$ and all matrices F_0 , while JAMES & SPECTOR have shown that if instead $h(\det F) \rightarrow +\infty$ when $\det F \rightarrow 0^+$, as is expected from the physics, then (1.1) will not be satisfied, for all $\tilde{u} \in W_0^{1,1}(B)$, at any F_0 with sufficiently small determinant. The failure of (1.1) in [10] is due to the ability of the body to relieve severe compressive strains and lower the total energy by overlapping material.

In this paper we also investigate the containment constraint $f(B) \subset d(B)$. We note that each equivalence class $\tilde{f} \in W^{1,1}(B)$ has a representative $f \in \tilde{f}$ that is absolutely continuous on almost every line segment parallel to the coordinate axes (see Section 4). We show that a one-to-one function that satisfies such a continuity property and assumes the same boundary values as a continuous one-to-one function d will either satisfy $f(B-N) \subset d(B)$ or $f(B-N) \subset \mathbb{R}^3 - d(\bar{B})$ for some N with measure zero. Thus such a function must assume one of only two possible orientations.

An unresolved problem is whether the sign of the determinant of the gradient of a deformation determines its orientation, *i.e.*, suppose that $d \in C^1(\bar{B}, \mathbb{R}^3)$ is one-to-one and satisfies

$\det \nabla d > 0$ on $\overline{B}$ and let $f \in \tilde{\mathbf{f}} \in W^{1,p}(B)$ be one-to-one and satisfy $f = d$ on ∂B and the aforementioned absolute continuity property. If $\det \nabla \tilde{\mathbf{f}} > 0$ almost everywhere does it follow that $f(B-N) \subset d(B)$ for some N with measure zero? If this is true then the containment constraint in (1.1) can be replaced by the natural hypothesis that $W(F) = +\infty$ whenever $\det F \leq 0$.

Finally, we note that if inequality (1.1) is satisfied at every $\tilde{\mathbf{u}} \in W_0^{1,p}(B)$ then W is said to be $W^{1,p}$ -quasiconvex at F_0 . This terminology is due to BALL & MURAT who show that if $E(\cdot)$ is sequentially weakly lower semicontinuous on $W^{1,p}(B)$, as is usually required when one minimizes via the direct method of the calculus of variations, then W must be $W^{1,p}$ -quasiconvex at every matrix F_0 . The additional constraints on $\tilde{\mathbf{u}}$ that we use were previously used by JAMES & SPECTOR who show that if F_0 is the deformation gradient of a one-to-one deformation $f \in \tilde{\mathbf{f}} \in W^{1,p}(B)$ at a point x_0 at which f is continuously differentiable and if f is a local minimizer of the total energy E in the $W^{1,p}$ -topology then (1.1) is satisfied by every $\tilde{\mathbf{u}} \in W_0^{1,p}(B)$ that obeys these constraints.

2. Weak Derivatives, Approximate Differentiability, and the Area Formula

Let

$\text{Lin} :=$ space of all linear transformations from $\mathbb{R}^3$ into $\mathbb{R}^3$ with norm

$$|H| := [\text{trace } (HH^T)]^{1/2},$$

where H^T denotes the transpose of H . Define

$$\begin{aligned}\text{Lin}^> &:= \{H \in \text{Lin}: \det H > 0\}, \\ \text{Lin}^{\leq} &:= \{H \in \text{Lin}: \det H \leq 0\},\end{aligned}$$

where $\det$ denotes the determinant and denote by adj the unique continuous function $\text{adj}: \text{Lin} \rightarrow \text{Lin}$ that satisfies

$$(\text{adj } H)H = (\det H)I.$$

The usual Lebesgue measure on $\mathbb{R}^n$ will be denoted by $\mathcal{L}^n$ and, unless stated otherwise, all references to measurability will be with respect to the three-dimensional Lebesgue measure $\mathcal{L} := \mathcal{L}^3$. The complement of a set $E \subset \mathbb{R}^n$ will be denoted $E^c := \mathbb{R}^n - E$.

Let $A \subset \mathbb{R}^3$ be measurable. A function $f: A \rightarrow \mathbb{R}^3$ is said to be *approximately differentiable* at $x \in A$ provided that there is an $L \in \text{Lin}$ such that for every $\epsilon > 0$

$$0 = \lim_{\rho \rightarrow 0^+} \frac{\mathcal{L}(\{z \in B_\rho(x): |f(x) - f(z) - L(z-x)| > \epsilon|z-x|\})}{\mathcal{L}(B_\rho(x))},$$

where $B_\rho(x) := \{z \in \mathbb{R}^3: |x - z| < \rho\}$. If such an L exists it is unique and it will be denoted $Df(x)$; the *approximate derivative* of f at the point x . An important property of approximately differentiable functions is the *Area Formula*:

Lemma 2.1 (SVERAK [14, Theorem 2], FEDERER [9, Theorems 2.10.43, 3.1.8, and 3.2.3]). *Let $A \subset \mathbb{R}^3$ be measurable and suppose that $f: A \rightarrow \mathbb{R}^3$ is approximately differentiable at every point of A . Then the set $f(A)$ is measurable, the function $y \mapsto \text{card}\{f^{-1}(y)\}$ is measurable, and*

$$(2.1) \quad \int_A |\det Df(x)| dx = \int_{f(A)} \text{card}\{f^{-1}(y)\} dy.$$

Here $\text{card } E$ denotes the cardinality of a set $E \subset \mathbb{R}^3$.

Remark 2.2. Formula (2.1) is well-known for Lipschitz functions. SVERAK observed that a result of FEDERER implies that, for an approximately differentiable f , $A = \bigcup_{i=1}^{\infty} A_i$ where $f|_{A_i}$ is Lipschitz and hence (2.1) is valid for such functions.

A nonempty bounded open region $\Omega \subset \mathbb{R}^3$ will be called *regular* provided that $\partial\Omega$, the boundary of Ω , has measure zero. For $1 \leq p < \infty$, $\|\cdot\|_{1,p}$ will denote the $W^{1,p}$ -norm on Ω ; i.e. for $f \in C^\infty(\Omega, \mathbb{R}^3)$

$$\|f\|_{1,p} := \left[\int_{\Omega} (|f(x)|^p + |\nabla_c f(x)|^p) dx \right]^{1/p},$$

where ∇_c denotes the (classical) gradient operator in $\mathbb{R}^3$: for a vector field f , $\nabla_c f$ is the tensor field with components

$$(\nabla_c f)_{ij} = \partial f_i / \partial x_j. \quad \text{Define}$$

$$W^{1,p}(\Omega) := \text{completion of } \{f \in C^\infty(\Omega, \mathbb{R}^3) : \|f\|_{1,p} < +\infty\},$$

with respect to the $W^{1,p}$ -norm, and

$$W_0^{1,p}(\Omega) := \text{closure of } C_0^\infty(\Omega, \mathbb{R}^3) \text{ in } W^{1,p}(\Omega),$$

and note that each element $\tilde{f} \in W^{1,p}(\Omega)$ is an *equivalence class* of functions $f: \Omega \rightarrow \mathbb{R}^3$ and weak derivatives $\nabla f: \Omega \rightarrow \text{Lin}$. We write $f \in \tilde{f}$ in order to specify one such function and one such weak derivative.

The following result shows that each $\tilde{f} \in W^{1,p}(\Omega)$ is approximately differentiable almost everywhere.

Lemma 2.3⁵. *Let $\tilde{f} \in W^{1,p}(\Omega)$. Then there is a measurable set $A \subset \Omega$ with $\mathcal{L}(\Omega - A) = 0$ such that every representative $f \in \tilde{f}$ is approximately differentiable at each $x \in A$ and the approximate derivative is independent of the representative. Moreover, for each $f \in \tilde{f}$,*

$$\nabla f(x) = Df(x) \text{ at almost every } x \in A.$$

3. Linear Deformations as Global Minimizers

Consider a homogeneous body that, for convenience, will be identified with the region $\bar{\Omega}$ that it occupies in a fixed homogeneous reference configuration. Let $1 \leq p < 3$. An equivalence class $\tilde{f} \in W^{1,p}(\Omega)$ will be called a deformation of Ω provided that $\tilde{f}$ is *one-to-one almost everywhere*, that is, each $f \in \tilde{f}$ is one-to-one on a measurable $D_f \subset \Omega$ with $\mathcal{L}(\Omega - D_f) = 0$. Define

$$\text{Def}^p(\Omega) := \{\tilde{f} \in W^{1,p}(\Omega) : \tilde{f} \text{ is one-to-one a.e.}\}.$$

Remark 3.1. Additional restrictions are needed to insure that a function $f \in \tilde{f} \in \text{Def}^p(\Omega)$ corresponds to a reasonable physical notion of a deformation. One such restriction is that for every measurable $D \subset \Omega$, $f(D)$ has measure zero if and only if D has measure zero. Without such a restriction one can construct (cf. BESICOVITCH [5]) a function that is equal to any given function except on a flat surface, which is mapped onto the unit

⁵ Cf., e.g., EVANS & GARIEPY [8, Section 6.1.3].

cube. If $\tilde{f}$ satisfies $\det \nabla \tilde{f} \neq 0$ almost everywhere then SVERAK [14, Lemma 2] has shown that D must have measure zero if $f(D)$ has measure zero and $D \subset \Omega$ is measurable.

Remark 3.2. BALL [2] and CIARLET & NECAS [7] define a deformation $\tilde{f} \in W^{1,p}(\Omega)$, $p > 3$, to be one-to-one almost everywhere provided that the continuous representative f of $\tilde{f}$ satisfies $\text{card}\{f^{-1}(y)\} = 1$ for almost every $y \in f(\Omega)$. Since this definition necessitates a choice of representative (cf. Remark 3.1) and since it is not clear which representative is appropriate when $p < 3$ we cannot use this definition. If $p > 2$ and $\text{adj } \nabla \tilde{f} \in L^{p/p-1}(\Omega)$ then SVERAK [14] has identified representatives that he shows are one-to-one almost everywhere in an appropriate sense (see also TANG [15]). However, as SVERAK notes the requirement $\text{adj } \nabla \tilde{f} \in L^{p/p-1}(\Omega)$ prohibits cavitation.

Assume that the body is hyperelastic with continuous stored energy function $W: \text{Lin} \rightarrow \mathbb{R}^{\geq} \cup \{+\infty\}$. W gives the energy stored per unit volume in Ω ,

$$W(\nabla_c f(x))$$

at any point $x \in \Omega$ when the body is deformed by a smooth deformation f . Further, assume that $W = +\infty$ on $\text{Lin}^{<}$. Let

$$E(\tilde{f}, \Omega) = \int_{\Omega} W(\nabla \tilde{f}(x)) dx$$

denote the total energy when the body is deformed by $\tilde{f} \in \text{Def}^p(\Omega)$.

Let $\tilde{f} \in \text{Def}^p(\Omega)$ and $E \subset \mathbb{R}^3$. We write

$$\tilde{f}(\Omega) \subset E \text{ a.e.}$$

provided that for each $f \in \tilde{f}$,

$$f(x) \in E \text{ for a.e. } x \in \Omega.$$

Remark 3.3. This notation may be misleading since $\tilde{f}(\Omega) \subset E$ almost everywhere does *not* imply that $\mathcal{L}(E - f(\Omega)) = 0$ for every representative $f \in \tilde{f}$ (cf. Remark 3.1).

Let $d \in \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^3)$ be one-to-one with $\det \nabla d > 0$ on $\bar{\Omega}$. We are interested in deformations that have the same boundary values and orientation as d . Therefore define

$$\text{Kin}_d^p(\Omega) := \{\tilde{f} \in \text{Def}^p(\Omega) : \tilde{f} - d \in W_0^{1,p}(\Omega) \text{ and } \tilde{f}(\Omega) \subset d(\Omega) \text{ a.e.}\},$$

the set of *kinematically admissible* deformations.

We now assume that

$$(3.1) \quad W(F) = g(F, \text{adj } F) + h(\det F)$$

for all $F \in \text{Lin}^>$, where $g: \text{Lin} \times \text{Lin} \rightarrow \mathbb{R}^>$ and $h \in \mathcal{C}^1(\mathbb{R}^>, \mathbb{R}^>)$ are convex and

$$(3.2) \quad h \text{ is monotone decreasing on } (0, H)$$

for some $H > 0$.

Remark 3.4. Assumption (3.2) is consistent with the usual hypothesis that it would require an infinite amount of energy to compress a given finite amount of material to zero volume. See,

e.g., CIARLET [6, Section 4.6] for a discussion of this constitutive hypothesis.

Theorem 3.5. *Let W be given by (3.1) and (3.2). Suppose that $F_0 \in \text{Lin}^>$ satisfies $\det F_0 \leq H$. Then, for $p \geq 2$,*

$$(3.3) \quad \int_{\Omega} W(F_0 + \nabla \tilde{u}(x)) dx \geq \int_{\Omega} W(F_0) dx$$

for every $\tilde{u} \in W_0^{1,p}(\Omega)$ with $f_0 + \tilde{u} \in \text{Kin}_{f_0}^p(\Omega)$, where $f_0(x) := F_0 x$. Moreover, if g is independent of its second argument then this result is valid for $p \geq 1$.

Thus the linear deformation $f_0(x) := F_0 x$ is the global minimizer of the total energy whenever $\det F_0 \leq H$.

Proof of Theorem 3.5. Since g and h are convex there exist $A \in \text{Lin}$, $B \in \text{Lin}$, and $c \in \mathbb{R}$ such that

$$(3.4) \quad \begin{aligned} W(F_0 + L) \geq & W(F_0) + A \cdot L + B \cdot [\text{adj}(F_0 + L) - \text{adj } F_0] \\ & + c[\det(F_0 + L) - \det F_0] \end{aligned}$$

for every $L \in \text{Lin}$ that satisfies $\det(F_0 + L) > 0$.

Let $u \in \tilde{u} \in W_0^{1,p}(\Omega)$ with $f_0 + \tilde{u} \in \text{Kin}_{f_0}^p(\Omega)$. Then

$$(3.5) \quad \int_{\Omega} \nabla u(x) dx = 0,$$

and, if $p \geq 2$,

$$(3.6)^6 \quad \int_{\Omega} [\text{adj}(F_0 + \nabla u(x)) - \text{adj } F_0] dx = 0.$$

Since the product of measurable functions is measurable

⁶ Cf., e.g., BALL [1, Lemma 3.3a].

$N_u := \{x \in \Omega: \det(F_0 + \nabla u(x)) \leq 0\}$ is measurable. If N_u has positive measure then (3.3) follows since W is bounded below and $W = +\infty$ on $\text{Lin}^<$. Otherwise let $L = \nabla u(x)$ in (3.4) and integrate over Ω to conclude, with the aid of (3.5) and (3.6), that

$$\int_{\Omega} W(F_0 + \nabla u(x)) dx \geq \int_{\Omega} W(F_0) dx + c \int_{\Omega} [\det(F_0 + \nabla u(x)) - \det F_0] dx.$$

Since (3.2) implies that $c \leq 0$ the desired result will follow from the inequality

$$(3.7) \quad \int_{\Omega} (\det \nabla \tilde{f}(x)) dx \leq \int_{\Omega} (\det F_0) dx,$$

which we will now establish for all $\tilde{f} \in \text{Kin}_{f_0}^p(\Omega)$.

Let $f \in \tilde{f} \in \text{Kin}_{f_0}^p(\Omega)$. Then by Lemma 2.1, Lemma 2.3 and the definitions of $\text{Kin}_{f_0}^p(\Omega)$ and $\text{Def}^p(\Omega)$ there is a measurable set $A \subset \Omega$ with $\mathcal{L}(\Omega - A) = 0$ such that f_A , the restriction of f to A , has the following properties:

- (i) f_A is one-to-one;
- (ii) $f_A(A) \subset f_0(\Omega)$;
- (iii) f is approximately differentiable at every $x \in A$ with $Df(x) = \nabla f(x)$;

$$(iv) \quad \int_A |\det Df(x)| dx = \int_{f(A)} \text{card}\{f_A^{-1}(y)\} dy.$$

Thus, by (i)-(iv),

$$\begin{aligned}
\int_{\Omega} \det \nabla \tilde{f}(x) dx &\leq \int_A |\det \nabla f(x)| dx = \int_A |\det Df(x)| dx \\
&= \int_{f(A)} \text{card}\{f_A^{-1}(y)\} dy = \int_{f(A)} 1 dy \\
&= \mathcal{L}(f(A)) \leq \mathcal{L}(f_0(\Omega)) \\
&= \int_{\Omega} (\det F_0) dx,
\end{aligned}$$

which establishes (3.7) and hence (3.3) for $p \geq 2$. Finally, we note that when g is independent of its second argument (3.6) is not needed and so the restriction $p \geq 2$ can be dropped. $\square$

Remark 3.6. Suppose that instead of (3.1) one assumes that W is polyconvex, i.e.,

$$W(F) = g(F, \text{adj } F, \det F)$$

for all $F \in \text{Lin}^>$, where $g: \text{Lin} \times \text{Lin} \times \mathbb{R}^> \rightarrow \mathbb{R}^>$ is convex. Then, for each $F_0 \in \text{Lin}^>$, there exist $A \in \text{Lin}$, $B \in \text{Lin}$, and $c \in \mathbb{R}$ such that (3.4) is satisfied for every $L \in \text{Lin}$ with $\det(F_0 + L) > 0$. If $c = c(F_0) \leq 0$ it is clear from the proof that Theorem 3.5 is valid for such an F_0 . However, the physics that yield $c(F_0) \leq 0$ for certain F_0 is not clear.

4. An Orientation Property for Weak Deformations

In the previous section a deformation was identified with an equivalence class of functions rather than a single (vector-valued) function. Within the context of Theorem 3.5 there was no need to specify which elements of an equivalence class correspond to one's physical notion of a deformation. However, it will now be shown that the use of the unusual constraint $\tilde{f}(\Omega) \subset f_0(\Omega)$

almost everywhere was partially necessitated by the failure to identify a specific representative. In order to motivate a choice of representative we note that the usual definition of the spaces $W^{1,p}$ is equivalent to the definition that was first used by BEPPO LEVI.

Lemma.⁷ *Let $\tilde{f} \in W^{1,p}(\Omega)$. Then $\tilde{f}$ has a representative $f^* \in \tilde{f}$ for which each of the functions $t \mapsto f^*(t, x_2, x_3)$, $t \mapsto f^*(x_1, t, x_3)$, $t \mapsto f^*(x_1, x_2, t)$ is absolutely continuous on each line segment in Ω for $\mathcal{L}^2$ almost every value of the other independent variables. Moreover, f^* tends to a limit at the endpoint of any such segment.*

Suppose that $d \in C^0(\bar{\Omega}, \mathbb{R}^3)$ is a given one-to-one function. The main result of this section shows that a one-to-one function f that satisfies the aforementioned absolute continuity property and assumes the same boundary values as d will either satisfy $f(\Omega) \subset d(\Omega)$ or $f(\Omega) \subset \mathbb{R}^3 - d(\bar{\Omega})$ almost everywhere.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a nonempty connected open set and suppose that $f: \bar{\Omega} \rightarrow \mathbb{R}^3$ satisfies the following properties:*

- (i) *f is one-to-one on $\bar{\Omega}$;*
- (ii) *$f(x) = d(x)$ for $x \in \partial\Omega$;*

⁷ Cf., e.g., EVANS & GARIEPY [8, Section 4.9.2] or MORREY [12, Lemma 3.1.1].

(iii) Each of the functions $t \mapsto f(t, x_2, x_3)$, $t \mapsto f(x_1, t, x_3)$, $t \mapsto f(x_1, x_2, t)$ is continuous on each line segment in Ω for $\mathcal{L}^2$ almost every value of the other independent variables.

Then

$$f(\Omega) \subset d(\Omega) \text{ a.e. or } f(\Omega) \subset \mathbb{R}^3 - d(\bar{\Omega}) \text{ a.e.}$$

Proof of Theorem 4.1. First note that since d is continuous and one-to-one d is an open mapping.⁸ In particular $d(\Omega)$ is open and hence⁹ $\partial d(\Omega) \subset d(\partial\Omega)$. Thus the continuous image of any line segment contained in Ω must either be (a) contained in $d(\Omega)$; (b) contained in $d(\bar{\Omega})^c$; or (c) contain a point $y = d(x)$, where $x \in \partial\Omega$. Since f is one-to-one on $\bar{\Omega}$ and satisfies $f = d$ on $\partial\Omega$ the f image of any subset of Ω cannot satisfy (c). Therefore the f image of each line segment upon which f is continuous is either contained in $d(\Omega)$ or $\mathbb{R}^3 - d(\bar{\Omega})$.

Next, let $a \in \Omega$, $\delta \in \mathbb{R}^>$ be such that the open cube

$$\mathcal{C} := (a_1, a_1 + \delta) \times (a_2, a_2 + \delta) \times (a_3, a_3 + \delta) \subset \Omega$$

and define

$$\mathcal{C}_{ij} := (a_i, a_i + \delta) \times (a_j, a_j + \delta)$$

for $i, j = 1, 2, 3$, $i < j$, the projection of $\mathcal{C}$ onto the i, j -plane.

Then, in particular, by hypothesis (iii) there is an

$\mathcal{L}^2$ -measurable set $S_{23} \subset \mathcal{C}_{23}$ with $\mathcal{L}^2(S_{23}) = \mathcal{L}^2(\mathcal{C}_{23}) = \delta^2$ such that the function

⁸ Cf., e.g., SCHWARTZ [13, Corollary 3.22].

⁹ Cf., e.g., CIARLET [6, Theorem 1.2-7].

Figure 1 : The cube C .

$$t \mapsto f(t, z_2, z_3), \quad t \in (a_1, a_1 + \delta)$$

is continuous for each $z \in S_{23}$. Let $S_I(S_E)$ be the subset of S_{23} such that the f image of $(a_1, a_1 + \delta) \times S_I$ ($(a_1, a_1 + \delta) \times S_E$) is contained in $d(\Omega)(d(\bar{\Omega})^c)$. Then

$$S_{23} = S_I \cup S_E, \quad S_I \cap S_E = \emptyset.$$

We will now show that S_I and S_E are $\mathcal{L}^2$ -measurable and that one of these sets has $\mathcal{L}^2$ -measure zero while the other has $\mathcal{L}^2$ -measure δ^2 and hence that

$$(4.1) \quad f(\mathcal{C}) \subset d(\Omega) \text{ a.e. or } f(\mathcal{C}) \subset d(\bar{\Omega})^c \text{ a.e.}$$

Let L_I (L_E) be the projection of S_I (S_E) on the x_2 -axis and define (see Figure 1)

$$A_{12} := (a_1, a_1 + \delta) \times (L_I \cap L_E) \subset \mathcal{C}_{12}.$$

Let $z = (z_1, z_2) \in A_{12}$. Then by construction there are $z_3^I, z_3^E \in (a_3, a_3 + \delta)$ such that $(z_2, z_3^I) \in S_I$ and $(z_2, z_3^E) \in S_E$. Thus the f image of the line segment $\{(t, z_2, z_3^I) : a_1 < t < a_1 + \delta\}$ is contained in $d(\Omega)$ while the f image of the line segment $\{(t, z_2, z_3^E) : a_1 < t < a_1 + \delta\}$ is contained in $d(\bar{\Omega})^c$. We note that the line segment $\{(z_1, z_2, t) : a_3 < t < a_3 + \delta\}$ intersects both of these line segments and hence that f cannot be continuous on this line. Therefore, by hypothesis (iii), $\mathcal{L}^2(A_{12}) = 0$ and hence $\mathcal{L}^1(L_I \cap L_E) = 0$.

Let L_I^* (L_E^*) be the projection of S_I (S_E) on the x_3 -axis. Then by the argument used in the previous paragraph $\mathcal{L}^1(L_I^* \cap L_E^*) = 0$. Now,

$$S_I \subset L_I \times L_I^* \subset \mathcal{C}_{23}, \quad S_E \subset L_E \times L_E^* \subset \mathcal{C}_{23},$$

and hence

$$S_{23} = S_I \cup S_E \subset (L_I \times L_I^*) \cup (L_E \times L_E^*) \subset \mathcal{C}_{23}.$$

Since $\mathcal{L}^2(S_{23}) = \mathcal{L}^2(\mathcal{C}_{23}) = \delta^2$ we conclude that the set $(L_I \times L_I^*) \cup (L_E \times L_E^*)$ is $\mathcal{L}^2$ -measurable with measure δ^2 and hence, in particular, that

$$\mathcal{L}^2(L_I \times L_I^*) = 0.$$

A standard result from real analysis (*e.g.*, Fubini's Theorem) then implies that at least one of the sets L_I or L_E^* is $\mathcal{L}^1$ -measurable with measure zero. Consequently, either S_I or S_E has $\mathcal{L}^2$ -measure zero which gives us (4.1).

Next, define

$$\Omega_I := \bigcup \{ \mathcal{C} \subset \Omega : \mathcal{C} \text{ is an open cube}^{10} \text{ and } f(\mathcal{C}) \subset d(\Omega) \text{ a.e.} \},$$

$$\Omega_E := \bigcup \{ \mathcal{C} \subset \Omega : \mathcal{C} \text{ is an open cube}^{10} \text{ and } f(\mathcal{C}) \subset d(\bar{\Omega})^c \text{ a.e.} \}.$$

Then Ω_I and Ω_E are open subsets of Ω and, by the above argument, $\Omega = \Omega_I \cup \Omega_E$. We note that $\Omega_I \cap \Omega_E$ is open. If it were nonempty then it would contain an open cube $\hat{\mathcal{C}}$ that would satisfy $f(\hat{\mathcal{C}}) \subset d(\Omega)$ a.e. and $f(\hat{\mathcal{C}}) \subset d(\bar{\Omega})^c$ a.e., which is not possible. Thus $\Omega_I \cap \Omega_E = \emptyset$. However, Ω is connected and hence either Ω_I or Ω_E must be empty.

Finally, let $\mathcal{C}_i$, $i = 1, 2, 3, \dots$ be an ordering of those open cubes¹⁰ in Ω that have edges of rational length and whose centers have rational coordinates. Then by the density of the rationals

¹⁰More precisely, an open cube with each of its edges parallel to some coordinate axis.

$\Omega = \bigcup_{i=1}^{\infty} C_i$. Now, by the conclusion of the previous paragraph there are $M_i \subset C_i$ such that $\mathcal{L}(M_i) = 0$ and either $f(C_i - M_i) \subset d(\Omega)$ for every $i \in \mathbb{I}^>$ or $f(C_i - M_i) \subset d(\bar{\Omega})^c$ for every $i \in \mathbb{I}^>$. Thus if we define $M := \bigcup_{i=1}^{\infty} M_i$ we conclude that $\mathcal{L}(M) = 0$ and either $f(\Omega - M) \subset d(\Omega)$ or $f(\Omega - M) \subset d(\bar{\Omega})^c$, which is the desired result. $\square$

Acknowledgement. I give special thanks to RICHARD JAMES for many helpful discussions. I also thank JOHN GREENE for several useful comments. The research upon which this paper is based was initiated while the author was on sabbatical leave at the Institute for Mathematics and its Applications, University of Minnesota. This work was also supported by the National Science Foundation (DMS-8810653) and the Air Force Office of Scientific Research (AFOSR-88-0200).

REFERENCES

- [1] BALL, J. M., Constitutive inequalities and existence theorems in nonlinear elastostatics, *Nonlinear Analysis and Mechanics, Vol. I* (R. J. KNOPS, Ed.) London: Pitman, 1977.
- [2] BALL, J. M., Global invertibility of Sobolev functions and the interpenetration of matter. Proc. Royal Soc. Edinburgh 88A (1981), 315-328.
- [3] BALL, J. M., Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. Phil. Trans. Roy. Soc. London A 306 (1982), 557-611.
- [4] BALL, J. M. & F. MURAT, $W^{1,p}$ -quasiconvexity and variational problems for multiple integrals. J. Func. Anal. 58 (1984), 225-253.
- [5] BESICOVITCH, A. S., Parametric Surfaces. Bull. Amer. Math. Soc. 56 (1950), 228-296.
- [6] CIARLET, P. G., *Mathematical Elasticity, Vol. I: Three-Dimensional Elasticity*. Amsterdam: North-Holland, 1988.
- [7] CIARLET, P. G. & J. NECAS, Injectivity and self-contact in nonlinear elasticity. Arch. Rational Mech. Anal. 97 (1987), 171-188.
- [8] EVANS, L. C. & R. E. GARIEPY, *Lecture Notes on Measure Theory and Fine Properties of Functions*. Preprint.
- [9] FEDERER, H., *Geometric Measure Theory*. New York: Springer, 1969.
- [10] JAMES, R. D. & S. J. SPECTOR, The formation of filamentary voids in solids. IMA Preprint #572.
- [11] KNOPS, R. J. & C. A. STUART, Quasiconvexity and uniqueness of equilibrium solutions in nonlinear elasticity. Arch. Rational Mech. Anal. 86 (1984), 233-249.
- [12] MORREY, C. B., *Multiple Integrals in the Calculus of Variations*. New York: Springer, 1966.
- [13] SCHWARTZ, J., *Nonlinear Functional Analysis*. New York: Gordon and Breach, 1969.
- [14] SVERAK, V., Regularity properties of deformations with finite energy. Arch. Rational Mech. Anal. 100 (1988), 105-127.

- [15] TANG, Qi, Almost-everywhere injectivity in nonlinear elasticity. Proc. Royal Soc. Edinburgh 109A (1988), 79-95.

Recent IMA Preprints

#	Author/s	Title
516	P. Singh, Ph. Caussignac, A. Fortes, D.D. Joseph and T. Lundgren,	Stability of Periodic Arrays of Cylinders Across the Stream by Direct Simulation
517	Daniel D. Joseph,	Generalization of the Foscolo-Gibilaro Analysis of Dynamic Waves
518	A. Narain and D.D. Joseph,	Note on the Balance of Energy at a Phase Change Interface
519	Daniel D. Joseph,	Remarks on inertial radii, persistent normal stresses, secondary motions , and non-elastic extensional viscosities
520	D. D. Joseph,	Mathematical Problems Associated with the Elasticity of Liquids
521	Henry C. Simpson and Scott J. Spector,	Some Necessary Conditions at an Internal Boundary for Minimizers in Finite Elasticity
522	Peter Gritzmann and Victor Klee,	On the 0-1 Maximization of Positive Definite Quadratic Forms
523	Fu-Cho Pu and D.H. Sattinger,	The Yang-Baxter Equations and Differential Identities
524	Avner Friedman and Fernando Reitich,	A Hyperbolic Inverse Problem Arising in the Evolution of Combustion Aerosol
525	E.G. Kalnins, Raphael D. Levine and Willard Miller, Jr.,	Conformal Symmetries and Generalized Recurrences for Heat and Schrödinger Equations in One Spatial Dimension
526	Wang Jinghua and Gerald Warnecke,	On Entropy Consistency of Large Time Step Godunov and Glimm Schemes
527	C. Guillopé and J.C. Saut,	Existence Results for the Flow of Viscoelastic Fluids with a Differential Constitutive Law
528	H.L. Bodlaender, P. Gritzmann, V. Klee and J. Van Leeuwen	Computational Complexity of Norm-Maximization
529	Li Ta-tsien (Li Da-qian) and Yu Xin,	Life-Span of Classical Solutions to Fully Nonlinear Wave Equations
530	Jong-Shenq Guo,	A Variational Inequality Associated with a Lubrication Problem
531	Jong-Shenq Guo,	On the Semilinear Elliptic Equation $\Delta w - \frac{1}{2}y \cdot \nabla w + \lambda w - w^{-\beta} = 0$ in R^n
532	Andrew E. Yagle,	Inversion of the Bloch transform in magnetic resonance imaging using asymmetric two-component inverse scattering
533	Bei Hu,	A Fiber Tapering Problem
534	Peter J. Olver,	Canonical Variables for BiHamiltonian Systems
535	Michael Renardy,	A Well-Posed Boundary Value Problem for Supercritical Flow of Viscoelastic Fluids of Maxwell Type
536	Michael Renardy,	Ill-Posedness Resulting from Slip As a Possible Explanation of Melt Fracture
537	Michael Renardy,	Compatibility Conditions at Corners Between Walls and Inflow Boundaries for Fluids of Maxwell Type
538	Rolf Rees,	The Spectrum of Restricted Resolvable Designs with $r = 2$
539	D. Lewis and J.C. Simo,	Nonlinear stability of rotating pseudo-rigid bodies
540	Robert Hardt and David Kinderlehrer,	Variational Principles with Linear Growth
541	San Yih Lin and Yisong Yang,	Computation of Superconductivity in Thin Films
542	A. Narain,	Pressure Driven Flow of Pure Vapor Undergoing Laminar Film Condensation Between Parallel Plates
543	P.J. Vassiliou,	On Local Equivalence for Vector Field Systems
544	Brian A. Coomes,	On Conditions Sufficient for Injectivity of Maps
545	Yanchun Zhao,	A Class of Global Smooth Solutions of the One Dimensional Gas Dynamics System
546	H. Holden, L. Holden and N.H. Risebro,	Some Qualitative Properties of 2×2 Systems of Conservation Laws of Mixed Type
547	M. Slemrod,	Dynamics of Measured Valued Solutions to a Backward-Forward Heat Equation
548	Avner Friedman and Jürgen Sprekels,	Steady States of Austenitic–Martensitic–Domains in the Ginzburg–Landau Theory of Shape Memory Alloys
549	Avner Friedman and Bei Hu,	Degenerate Hamilton–Jacobi–Bellman Equations in a Bounded Domain
550	E.G. Kalnins, Willard Miller, Jr., and M.V. Tratnik,	Families of Orthogonal and Biorthogonal Polynomials on the N-Sphere
551	Heinrich Freistühler,	On Compact Linear Degeneracy
552	Matthew Witten,	Quantifying the Concepts of Rate and Acceleration/Deceleration of Aging
553	J.P. Albert and J.L. Bona,	Total Positivity and the Stability of Internal Waves in Stratified Fluids of Finite Depth
554	Brian Coomes and Victor Zurkowski,	Linearization of Polynomial Flows and Spectra of Derivations

Recent IMA Preprints (Continued)

#	Author/s	Title
555	Yuriko Renardy , A Couette-Poiseuille Flow of Two Fluids in a Channel	
556	Michael Renardy , Short wave instabilities resulting from memory slip	
557	Daniel D. Joseph and Michael Renardy , Stokes' first problem for linear viscoelastic fluids with finite memory	
558	Xiaxi Ding , Superlinear Conservation Law with Viscosity	
559	J.L. Ericksen , Liquid Crystals with Variable Degree of Orientation	
560	F. Robert Ore, Jr. and Xinfu Chen , Electro-Optic Modulation in an Arbitrary Cross-Section Waveguide	
561	M.V. Tratnik , Multivariable biorthogonal continuous-discrete Wilson and Racah polynomials	
562	Yisong Yang , Existence of Solutions for a Generalized Yang-Mills Theory	
563	Peter Gritzmann, Laurent Habsieger and Victor Klee , Good and Bad Radii of Convex Polygons	
564	Martin Golubitsky, Martin Krupa and Chjan. C. Lim , Time-Reversibility and Particle Sedimentation	
565	G. Yin , Recent Progress in Parallel Stochastic Approximations	
566	G. Yin , On H-Valued SA: Finite Dimensional Approximations	
567	Chien-Cheng Chang , Accurate Evaluation of the Effect of Diffusion and Conductivity in Certain Equations	
568	Chien-Cheng Chang and Ruey-Ling Chern , The Effect of Viscous Diffusion in Discrete Vortex Dynamics for Slightly Viscous Flows	
569	Li Ta-Tsien (Li Da-qian) and Zhao Yan-Chun , Global Existence of Classical Solutions to the Typical Free Boundary Problem for General Quasilinear Hyperbolic Systems and its Applications	
570	Thierry Cazenave and Fred B. Weissler , The Structure of Solutions to the Pseudo-Conformally Invariant Nonlinear Schrödinger Equation	
571	Marshall Slemrod and Athanasios E. Tzavaras , A Limiting Viscosity Approach for the Riemann Problem in Isentropic Gas Dynamics	
572	Richard D. James and Scott J. Spector , The Formation of Filamentary Voids in Solids	
573	P.J. Vassiliou , On the Geometry of Semi-Linear Hyperbolic Partial Differential Equations in the Plane Integrable by the Method of Darboux	
574	Jerome V. Moloney and Alan C. Newell , Nonlinear Optics	
575	Keti Tenenblat , A Note on Solutions for the Intrinsic Generalized Wave and Sine-Gordon Equations	
576	P. Szmolyan , Heteroclinic Orbits in Singularly Perturbed Differential Equations	
577	Wenxiong Liu , A Parabolic System Arising In Film Development	
578	Daniel B. Dix , Temporal Asymptotic Behavior of Solutions of the Benjamin-Ono-Burgers Equation	
579	Michael Renardy and Yuriko Renardy , On the nature of boundary conditions for flows with moving free surfaces	
580	Werner A. Stahel , Robust Statistics: From an Intellectual Game to a Consumer Product	
581	Avner Friedman and Fernando Reitich , The Stefan Problem with Small Surface Tension	
582	E.G. Kalnins and W. Miller, Jr. , Separation of Variables Methods for Systems of Differential Equations in Mathematical Physics	
583	Mitchell Luskin and George R. Sell , The Construction of Inertial Manifolds for Reaction-Diffusion Equations by Elliptic Regularization	
584	Konstantin Mischaikow , Dynamic Phase Transitions: A Connection Matrix Approach	
585	Philippe Le Floch and Li Tatsien , A Global Asymptotic Expansion for the Solution to the Generalized Riemann Problem	
586	Matthew Witten, Ph.D. , Computational Biology: An Overview	
587	Matthew Witten, Ph.D. , Peering Inside Living Systems: Physiology in a Supercomputer	
588	Michael Renardy , An existence theorem for model equations resulting from kinetic theories of polymer solutions	
589	Daniel D. Joseph and Luigi Preziosi , Reviews of Modern Physics: Addendum to the Paper "Heat Waves"	
590	Luigi Preziosi , An Invariance Property for the Propagation of Heat and Shear Waves	
591	Gregory M. Constantine and John Bryant , Sequencing of Experiments for Linear and Quadratic Time Effects	
592	Prabir Daripa , On the Computation of the Beltrami Equation in the Complex Plane	
593	Philippe Le Floch , Shock Waves for Nonlinear Hyperbolic Systems in Nonconservative Form	
594	A.L. Gorin, D.B. Roe and A.G. Greenberg , On the Complexity of Pattern Recognition Algorithms On a Tree-Structured Parallel Computer	
595	Mark J. Friedman and Eusebius J. Doedel , Numerical computation and continuation of invariant manifolds connecting fixed points	
596	Scott J. Spector , Linear Deformations as Global Minimizers in Nonlinear Elasticity	