**NUMERICAL COMPUTATION AND
CONTINUATION OF INVARIANT
MANIFOLDS CONNECTING FIXED POINTS**

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Numerical computation and continuation of invariant manifolds connecting fixed points

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Abstract: We give a numerical method for the computation of an invariant manifold that connects two fixed points of a vector field in $\mathbb{R}^n$, extending the results of an earlier paper [1989] by the authors. Basically, we truncate a boundary value problem on the real line to a finite interval. The method applies, in particular, to the computation of heteroclinic orbits. The emphasis is on the systematic computation of such orbits by using continuation. The method is incorporated in the software package AUTO. Using the fact that the linearized operator of our problem is Fredholm in appropriate Banach spaces, we employ the general theory of approximation of nonlinear problems to show that the errors in the approximate solution decay exponentially with the length of the approximating interval. Several applications are considered, including the computation of traveling wave solutions to reaction diffusion problems.

Key words. heteroclinic orbits, computation and continuation, Banach spaces, error analysis.

AMS(MOS) subject classification. 34B15, 34C30, 34C35, 65J15, 65L10, 68-04.

1. Introduction.

The problem of finding traveling wave front solutions of constant speed to nonlinear parabolic partial differential equations is equivalent to the problem of finding trajectories that connect two fixed points of an associated system of autonomous ordinary differential equations. Such a trajectory is an example of a *heteroclinic orbit*, i. e. an orbit with several fixed points on it. The period of such orbits is necessarily infinite.

In an earlier paper [6] we gave an accurate, robust, and systematic method for computing branches of orbits connecting two saddle points in $\mathbb{R}^2$. In this paper we extend the results in [6] to the case of manifolds connecting fixed points in $\mathbb{R}^n$. Practically we limit our computations to manifolds of dimension 1 and 2. In applications we use the software package AUTO [5]. This package incorporates algorithms for the numerical bifurcation analysis of differential equations. Some of the preliminary results of this paper were communicated in [7].

Specifically, we are interested in finding a branch of solutions of the system

$$(1.1) \quad \begin{aligned} a) & \quad v'(t) - f(v(t), \lambda) = 0, \quad v(\cdot), f(\cdot, \cdot) \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^{n_\lambda}, \\ b) & \quad \lim_{t \rightarrow -\infty} v(t) = w_0, \quad \lim_{t \rightarrow \infty} v(t) = w_1, \end{aligned}$$

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In Section 2 we describe our numerical method and give several examples. The reduction of the problem to the finite interval is similar to that in Lentini and Keller [13]. Essentially, in the neighborhoods of the fixed points we use linear approximation of the stable and unstable manifolds, respectively, so that outside of a finite interval the problem can be solved analytically. In future we intend to use higher order approximations of stable and unstable manifolds developed by Hassard [8, 9].

Section 3 contains error analysis. In analyzing the exact problem we use the Fredholm theory in a way similar to that in Palmer [14] and Hale and Lin [10]. This is combined with the general theory of approximation of nonlinear problems in Descloux and Rappaz [4].

Recently, similar results were obtained in Beyn [2]. Differences between Beyn's results and ours are as follows. We use weighted Banach spaces, and therefore some cases with nonhyperbolic fixed points are allowed, such as zero eigenvalues of the linearization of f at a fixed point with the corresponding eigenvectors pointing in the direction along an orbit other than the one being computed. Our algorithm emphasizes continuation, and the error estimates are obtained for the branches of orbits, which are assumed to be computed using the pseudo- arclength continuation [12]. Our phase condition is more general. Our use of the Fredholm theory seems to be somewhat more straightforward than in [2], and we do not utilize the exponential dichotomies. We allow the connecting manifolds of dimension higher than one and show how to estimate this dimension in some cases. In Corollary 1 we prove the superconvergence estimate for the parameter first observed in [2]. An adaptive strategy for choosing the truncation interval as in [2] will not be discussed here. However we intend to utilize an adaptive strategy in the final version of our algorithm.

2. Numerical method.

The algorithm is based on the following equations:

$$(2.1) \quad v'(t) - Tf(v(t), \lambda) = 0, \quad 0 < t < 1,$$

$$(2.2) \quad \begin{aligned} a) \quad & f(w_0, \lambda) = 0, \\ b) \quad & f(w_1, \lambda) = 0. \end{aligned}$$

For $v_{0i}, v_{1i} \in \mathbb{R}^n$, $\mu_{0i}, \mu_{1i} \in \mathbb{R}$ defined, respectively, by

$$(2.3) \quad \begin{aligned} a) \quad & f_u(w_0, \lambda) v_{0i} = \mu_{0i} v_{0i}, \quad i = 1, \dots, n_0, \\ b) \quad & f_u(w_1, \lambda) v_{1i} = \mu_{1i} v_{1i}, \quad i = 1, \dots, n_1, \end{aligned}$$

$$(2.4) \quad \begin{aligned} a) \quad & |v_{0i}| = 1, \quad i = 1, \dots, n_0, \\ b) \quad & |v_{1i}| = 1, \quad i = 1, \dots, n_1, \end{aligned}$$

$$(2.5) \quad \int_0^1 (f(v, \lambda) - f(v_0, \lambda^0)) \cdot f_v(v, \lambda) f(v, \lambda) dt = 0,$$

$$\begin{aligned}
(2.6) \quad & a) \quad v(0) = w_0 + \epsilon_0 \sum_{i=1}^{n_0} c_{0i} v_{0i} \quad \sum_{i=1}^{n_0} c_{0i}^2 = 1, \\
& b) \quad v(1) = w_1 + \epsilon_1 \sum_{i=1}^{n_1} c_{1i} v_{1i} \quad \sum_{i=1}^{n_1} c_{1i}^2 = 1.
\end{aligned}$$

Equation (2.1) is the differential equation. The independent variable t has been scaled so that it varies from 0 to 1. The actual period T therefore appears explicitly in (2.1). There are n_λ problem parameters, viz., λ_i , $i = 1, \dots, n_\lambda$. Equation (2.2) defines two fixed points of the vector field. In (2.3a) we assume that the Jacobian $f_v(w_0, \lambda)$ has n_0 distinct real positive eigenvalues μ_{0i} with eigenvectors v_{0i} , and $n - n_0$ real nonpositive eigenvalues. Similarly, in (2.3b) we assume that the Jacobian $f_v(w_1, \lambda)$ has n_1 distinct real negative eigenvalues μ_{1i} with eigenvectors v_{1i} , and $n - n_1$ real nonnegative eigenvalues. Under appropriate assumptions on f stated in the next section, by the stable manifold theorem [11] the fixed point w_0 has a (strongly) unstable manifold of dimension n_0 to which the linear subspace $S_0 \equiv \text{Span}(\{v_{0i}\}_{i=1}^{n_0})$ is tangent at w_0 . The fixed point w_1 has a (strongly) stable manifold of dimension n_1 to which the linear subspace $U_1 \equiv \text{Span}(\{v_{1i}\}_{i=1}^{n_1})$ is tangent at w_1 . In Section 3 we derive (2.6) analytically.

Remark 1. The extension of the above to the case of distinct complex eigenvalues is straightforward. Say, $\mu_{01} = \alpha + i\beta$, $\mu_{02} = \alpha - i\beta$, $\alpha, \beta \in \mathbb{R}$. Then we choose $v_{01} = u + iv$, $v_{02} = u - iv$, $u, v \in \mathbb{R}^n$; $c_{02} = \bar{c}_{01}$, $c_{01} = ae^{i\delta}$, $a, \delta \in \mathbb{R}$. This implies

$$\begin{aligned}
c_{01}v_{01}e^{\mu_{01}t} + c_{02}v_{02}e^{\mu_{02}t} &= 2\text{Re} \left(c_1 e^{(\alpha+i\beta)t} v_{01} \right) \\
&= 2ae^{\alpha t} [u \cos(\beta t + \delta) - v \sin(\beta t + \delta)].
\end{aligned}$$

As is well known, $u, v, v_{03}, \dots, v_{0,n_0}$ form a linearly independent set. It is also easy to extend our results to the case when the eigenvalues are not necessarily distinct. •

We have $v_{0i}, v_{1i}, w_0, w_1 \in \mathbb{R}^n$. Equation (2.6a) then requires that the starting point $v(0)$ of the orbit $v(t)$ lie in the tangent manifold S_0 at “distance” ϵ_0 from the fixed point w_0 . Similarly, equation (2.6b) requires the endpoint $v(1)$ to lie in U_1 at “distance” ϵ_1 from the fixed point w_1 . Finally (2.5) represents the “*phase condition*”. Notice that since the system (1.1) is autonomous, every translation of its solution is also a solution. Hence to remove this “phase” indeterminacy we need to add a constraint. Equation (2.5) seems to be, computationally, the most appropriate way to do this (see [6] for the discussion).

The unknowns w_0 and w_1 can be eliminated entirely from (2.1)-(2.5) by using (2.6). Then (2.1)-(2.6) represent n coupled differential equations subject to $n_c = 2n + (n+1)(n_0 + n_1) + 3$ constraints, of which (2.5) is an integral condition. In addition to the vector function variable $v(t) \in \mathbb{R}^n$ we have scalar variables

$$\begin{aligned}
(2.7) \quad & \lambda \in \mathbb{R}^{n_\lambda}, \quad \epsilon_0, \epsilon_1 \in \mathbb{R}, \\
& \mu_{0i}, c_{0i} \in \mathbb{R}, \quad v_{0i} \in \mathbb{R}^n, \quad i = 1, \dots, n_0, \\
& \mu_{1i}, c_{1i} \in \mathbb{R}, \quad v_{1i} \in \mathbb{R}^n, \quad i = 1, \dots, n_1.
\end{aligned}$$

The total number of scalar variables equals $n_v \equiv n_\lambda + (n + 2)(n_0 + n_1) + 2$. Formally we need $n_v = n_c - n$ for a single heteroclinic connection. Usually we shall be interested in computing an entire *branch* (one dimensional continuum) of orbits, in which case $n_v = n_c - n + 1$. This requirement is equivalent to setting the number of free problem parameters

$$(2.8) \quad n_\lambda = n - (n_0 + n_1) + 2.$$

In Section 3 we show that indeed this choice of the number of free problem parameters leads to the unique solution for both for the exact problem and the truncated one.

The period T is kept fixed in this continuation. For T large and ϵ_0 and ϵ_1 small, each solution on the branch represents an approximate heteroclinic connection. If we want to increase the period T , then we can replace one of the problem parameters, say λ_1 , by T (see [6] for an example of such a computation).

If $n_\lambda = n - (n_0 + n_1) + 2 \leq 1$, then the dimension of the invariant manifold connecting fixed points already equals $|n_\lambda|$ when all problem parameters are fixed.

We now mention some particular cases. First consider the heteroclinic connection of two saddle points in the plane. It arises, for example, when finding traveling wave front solutions to the Huxley equation, which is done in [6], and the combustion fronts, which is done in Section 4. For this case $n = 2$ and $n_0 = n_1 = 1$ so that $n_\lambda - 1 = 1$, which reflects the fact that the codimension of such orbit is 1, and a branch of such orbits requires two free problem parameters. Another case is the connection of a saddle to a node in the plane. Here $n = 2$, $n_0 = 1$, and $n_1 = 2$. This gives $n_\lambda = 1$. Thus a branch of heteroclinic connections now requires one free problem parameter. The problem of finding traveling wave front solutions to the Fisher equation is of this type. This is also done in Section 4. Another case is when $n = 3$ with $n_0 = 3$ and $n_1 = 2$. This gives $n_\lambda = 0$ so that a two dimensional manifold connecting the two fixed points is already possible with all problem parameters fixed. As a last special case we mention the situation where the two fixed points are actually the same. This is the well known *homoclinic orbit*. In this case $n_\lambda - 1$ always equals 1, which is the codimension of a homoclinic orbit. Such orbits can also be effectively be computed as the limit of periodic orbits as the period $T \rightarrow \infty$ (see [5]).

Our basic assumption is that the equation is generic, i.e., that the problem parameters perturb the heteroclinic connection transversally. This assumption is expressed analytically in Section 4 as an assumption on the range of the appropriate linearized operator. This is not always the case. For example, if the differential equation is Hamiltonian, then the branch of homoclinic orbits requires only one problem parameter. Very complicated schemes have been proposed, even for the simpler problem of computing periodic orbits to integrable systems. But the numerical computation of periodic and homoclinic orbits to such systems becomes no different from the generic case if one introduces an unfolding parameter (see Section 3.4 in [5]). The same can be done for the current problem of computing heteroclinic connections in a Hamiltonian system, and thus there is no need to treat this case separately.

3. Error Analysis.

Let X, Y be Banach spaces. For a linear operator $A \in L(X, Y)$ we denote $\|A\|_{L(X, Y)}$ the norm of A , $\mathcal{N}(A) = \text{Null space of } A$, $\mathcal{R}(A) = \text{Range of } A$. We shall use the notation $L_\infty = L_\infty(\mathbb{R})$. We denote by C, c generic positive constants.

We consider the problem of finding a branch of solutions $(v(t, s), \lambda(s))$, for some parametrization s , given a solution $(v_0(t), \lambda^0) \equiv (v(t, 0), \lambda(0))$ of the system

$$(3.1) \quad \begin{aligned} a) & \quad v'(t) - f(v(t), \lambda) = 0, \quad v(\cdot), f(\cdot, \cdot) \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^{n_\lambda}, \\ b) & \quad \lim_{t \rightarrow -\infty} v(t) = w_0, \quad \lim_{t \rightarrow \infty} v(t) = w_1, \end{aligned}$$

$$(3.2) \quad \begin{aligned} a) & \quad f(w_0, \lambda) = 0, \\ b) & \quad f(w_1, \lambda) = 0, \end{aligned}$$

$$(3.3) \quad \int_{T_-}^{T_+} (v'(t) - v_0'(t)) \cdot v_0''(t) dt = 0, \quad T_- < 0 < T_+.$$

where (3.3) is a “phase condition”. Here for $u, v \in \mathbb{R}^n$ $u \cdot v$ denotes the inner product in $\mathbb{R}^n$. In what follows we suppress the dependence of $\tilde{u}$ on T_+, T_- .

We next specify our assumptions on f . Assume that f is C^3 , $|f_u|, |f_{uu}|, |f_{uu\lambda}|$ are bounded uniformly in u, λ , and f satisfies the Lipschitz condition

$$(3.4) \quad |f(u_1, \lambda) - f(u_2, \lambda)| \leq \gamma |u_1 - u_2|, \quad \forall u_1, u_2 \in \mathbb{R}^n,$$

where $|\cdot|$ is the l^2 norm in $\mathbb{R}^n$. From (3.2) and boundedness of $|f_{uu}|$ we also have

$$(3.5) \quad \begin{aligned} a) & \quad |f(w_0 + u, \lambda) - f_u(w_0, \lambda)u| \leq C|u|^2, \quad \forall u \in \mathbb{R}^n, \\ b) & \quad |f(w_1 + u, \lambda) - f_u(w_1, \lambda)u| \leq C|u|^2, \quad \forall u \in \mathbb{R}^n. \end{aligned}$$

Remark 2. The assumption $|f_u|, |f_{uu}| \leq C$, can be relaxed. We show below, without using $|f_u|, |f_{uu}| \leq C$, that the solution of (3.1)-(3.3) is bounded. Hence for large $|u|, |\lambda|$, f can be modified to satisfy $|f_{uu}| \leq C$, without changing the solution, i.e., it is sufficient to assume only that f is C^3 .•

For $v_{0i}, v_{1i} \in \mathbb{R}^n$, $\mu_{0i}, \mu_{1i} \in \mathbb{R}$ defined, respectively, by (2.3), (2.4) :

$$(3.6) \quad \begin{aligned} a) & \quad f_u(w_0, \lambda)v_{0i} = \mu_{0i}v_{0i}, \quad i = 1, \dots, n, \\ b) & \quad f_u(w_1, \lambda)v_{1i} = \mu_{1i}v_{1i}, \quad i = 1, \dots, n, \end{aligned}$$

$$(3.7) \quad \begin{aligned} a) & \quad |v_{0i}| = 1, \quad i = 1, \dots, n_0, \\ b) & \quad |v_{1i}| = 1, \quad i = 1, \dots, n_1, \end{aligned}$$

we assume that

$$(3.8) \quad \begin{aligned} \mu_{01} & > \dots > \mu_{0n_0} > 0 \geq \mu_{0, n_0+1} \geq \dots \geq \mu_{0n}, \\ \mu_{11} & < \dots < \mu_{1n_1} < 0 \leq \mu_{1, n_1+1} \leq \dots \leq \mu_{1n}. \end{aligned}$$

We also assume that (3.2) can be solved to find $w_0 = w_0(\lambda)$, $w_1 = w_1(\lambda)$, and denote $w_0^0 = w_0(\lambda^0)$, $w_1^0 = w_1(\lambda^0)$. If those solutions are not unique (in the case of zero eigenvalues) we assume that appropriate branches of solutions of (3.2a) and (3.2b) are chosen.

We are interested in those solutions of (1.1) which decay exponentially to their limits at $\pm\infty$. We shall formulate our problem as a nonlinear operator equation in weighted Banach spaces with exponential weights. When considered on such spaces, the linearized operator has “nice” spectral properties: it is Fredholm. We assume that the range $I_\lambda \subset \mathbb{R}^{n_\lambda}$ of change of the parameter λ (in our continuation procedure) is small enough so that for some fixed numbers μ_0 and μ_1 we have

$$(3.9) \quad \mu_{0n_0} > \mu_0 > 0, \quad \mu_{1n_1} < \mu_1 < 0.$$

For $\mu = (\mu_0, \mu_1)^T$ we then introduce a weight function

$$(3.10) \quad w_\mu(t) = \begin{cases} e^{-\mu_0 t}, & t \leq 0, \\ e^{-\mu_1 t}, & t > 0, \end{cases}$$

and for $u = (u_1, \dots, u_n)^T$ with $w_\mu u \in (L_\infty)^n$ denote by $\|\cdot\|_{\mu,0}$ the norm $\|u\|_{\mu,0} = \max_{1 \leq i \leq n} \|w_\mu u_i\|_{L_\infty}$. Define $\|\cdot\|_{\mu,j}$ by

$$\|u\|_{\mu,j} = \|u\|_{\mu,0} + \|du/dt\|_{\mu,0} + \dots + \|d^j u/dt^j\|_{\mu,0}, \quad j \geq 0,$$

and let $B_{\mu,j}$ be the Banach space of vector valued functions on $\{-\infty < t < \infty\}$ with finite $\|\cdot\|_{\mu,j}$ nor

Let $\chi \in (C^\infty(\mathbb{R}))^n$ be such that for some $T_- < 0 < T_+$

$$(3.11) \quad \chi(t, \lambda) = \begin{cases} w_0(\lambda) & \text{for } t < T_-, \\ w_1(\lambda) & \text{for } t > T_+. \end{cases}$$

The solution $v(t)$ can be written as

$$(3.12) \quad v(t) = u(t) + \chi(t, \lambda),$$

where we may assume that $|w_\mu(t)u(t)|$ is bounded. This suggests that we search for a solution $v(t)$ of the form (3.12) with $u \in B_{\mu,1}$. Substituting (3.12) into (3.1), we replace (3.1), (3.2) by the equation

$$(3.13) \quad F(\cdot, u, \lambda) \equiv u' - f(u + \chi(\cdot, \lambda), \lambda) + \chi_t(\cdot, \lambda) = 0,$$

which we consider as an operator equation, with $F : B_{\mu,1} \times \mathbb{R}^{n_\lambda} \rightarrow B_{\mu,0}$. Let $(u_0, \lambda^0) \in B_{\mu,1} \times \mathbb{R}^{n_\lambda}$ be a solution of (3.13) with

$$(3.14) \quad u_0(t) = v_0(t) - \chi(t, \lambda^0),$$

and define the linearized operator $F_u^0 : B_{\mu,1} \rightarrow B_{\mu,0}$ by

$$(3.15) \quad F_u^0 \equiv D_u F(\cdot, u_0, \lambda^0) = \frac{d}{dt} - A(\cdot),$$

where

$$(3.16) \quad A(t) = f_u(v_0(t), \lambda^0).$$

Let A_- , A_+ be the matrices $\lim_{t \rightarrow -\infty} A(t) = f_u(w_0^0, \lambda^0)$, $\lim_{t \rightarrow \infty} A(t) = f_u(w_1^0, \lambda^0)$, respectively, and define the matrix operator A by

$$(3.19) \quad A_0(t) = \begin{cases} A_-, & t \leq 0, \\ A_+, & t > 0. \end{cases}$$

The following lemma is similar to Lemma 4.2 in Palmer [13].

Lemma 1. (i) $v'_0 \in \mathcal{N}(F_u^0)$, and

$$(3.18a) \quad \max\{1, n_0 + n_1 - n\} \leq \dim \mathcal{N}(F_u^0) \leq \min\{n_0, n_1\}.$$

(ii) Moreover, the operator $F_u^0 : B_{\mu,1} \rightarrow B_{\mu,0}$ is a Fredholm operator of index $n_0 + n_1 - n$, and hence

$$(3.18b) \quad \max\{1, n_0 + n_1 - n\} \leq \operatorname{codim} \mathcal{R}(F_u^0) + n_0 + n_1 - n \leq \min\{n_0, n_1\}.$$

Proof. Consider the system

$$(3.19) \quad (F_u^0 u)(t) \equiv u'(t) - A(t)u(t) = 0.$$

By the definitions of $A(t)$, $A_0(t)$ and $\chi(t)$ and using that $|f_{uu}|$ is bounded for $|t|$ large enough, we have

$$(3.20) \quad |A_0(t) - A(t)| < C|u_0(t)|.$$

Taking into account that $|u_0(t)|$ decays exponentially as $|t| \rightarrow \infty$, we have

$$(3.21) \quad \int_{-\infty}^{\infty} |A_0(t) - A(t)| dt < \infty.$$

Then in view of (3.6) by [3, p. 90, Th. 2] equation (3.19) has a fundamental system of solutions $v_1(t), \dots, v_n(t)$ satisfying

$$(3.22) \quad \begin{aligned} v_i(t) &\sim e^{\mu_{0i}t} v_{0i}, \quad \text{as } t \rightarrow -\infty, \quad i = 1, \dots, n, \\ v_{\sigma(i)}(t) &\sim e^{\mu_{1i}t} v_{1i}, \quad \text{as } t \rightarrow \infty, \quad i = 1, \dots, n, \end{aligned}$$

for some permutation σ . Introduce the subspaces of $\mathbb{R}^n$:

$$\begin{aligned} V_u &= \operatorname{span}\{v_i(0), i = 1, \dots, n_0\}, \\ V_u^c &= \operatorname{span}\{v_i(0), i = n_0 + 1, \dots, n\}, \\ V_s &= \operatorname{span}\{v_{\sigma(i)}(0), i = 1, \dots, n_1\}, \\ V_s^c &= \operatorname{span}\{v_{\sigma(i)}(0), i = n_1 + 1, \dots, n\}. \end{aligned}$$

Then the space of initial values (at $t = 0$) of solutions $u \in B_{\mu,1}$ of (3.19) is $V_u \cap V_s$. And hence

$$(3.23) \quad \dim \mathcal{N}(F_u^0) = \dim V_u \cap V_s.$$

Next notice that if $v_0(t)$ is a solution of (3.1) then so is $v_0(t+c)$, $c \in \mathbb{R}$. Hence v_0' is a null function of the linearized operator (3.1) or F_u^0 :

$$(3.26) \quad (F_u^0 v_0')(t) = v_0''(t) - f_u(v_0(t), \lambda^0) v_0'(t) = 0.$$

Thus the first part of Lemma 1 follows from (3.23) and (3.24).

We next show that $\mathcal{R}(F_u^0)$ is closed and has finite codimension. Let us denote by $B_{-\mu,-j}$ the dual space of $B_{\mu,j}$, with the norm $\|v\|_{-\mu,-j} = \sup_{\|u\|_{\mu,j}=1} |(u,v)|$, where $(\cdot, \cdot)$ is the L_2 inner product. By $\langle u, v \rangle$ we shall mean the duality pairing between the spaces $B_{\mu,0}$ and $B_{-\mu,0}$. Let $e^* \in B_{-\mu,0}$ be in $\mathcal{R}(F_u^0)^\perp$. Then we have

$$(3.25) \quad \langle F_u^0 u, e^* \rangle = 0, \quad \forall u \in C_0^\infty(\mathbb{R}^n).$$

Note that the linear functional $u \rightarrow \langle F_u^0 u, e^* \rangle$ is a distribution in the sense of Schwarz. Since the coefficients of F_u^0 are continuous functions, F_u^0 has a well defined adjoint $(F_u^0)^*$, and (3.25) is a statement of the fact that

$$(3.26) \quad (F_u^0)^* e^* = 0$$

in the sense of distributions. Since (3.26) is a system of ordinary differential equations, however, e^* can be identified with a smooth function ψ which satisfies (3.26) in a classical sense:

$$(3.27) \quad (F_u^0)^* \psi \equiv \psi' + A^T(t) \psi = 0.$$

Note also that if

$$\int_{-\infty}^0 e^{\mu_0 t} |\psi(t)| dt + \int_0^\infty e^{\mu_1 t} |\psi(t)| dt < \infty$$

i. e. $w_{-\mu} \psi \in (L_1)^n$ then $\psi \in B_{-\mu,0}$. Hence by (3.9), (3.11) and (3.24) the subspace of initial values (at $t = 0$) of solutions of (3.29) is $V_u^c \cap V_s^c$. And we have shown that $\mathcal{R}(F_u^0)$ is the subspace of $B_{\mu,0}$ annihilated by a finite dimensional subspace of the dual space $B_{-\mu,0}$ and so $\mathcal{R}(F_u^0)$ is closed and

$$\text{codim } \mathcal{R}(F_u^0) = \dim(V_u^c \cap V_s^c).$$

Hence F_u^0 is Fredholm as asserted. Now the index of F_u^0 is

$$\begin{aligned} \text{Ind}(F_u^0) &= \dim V_u \cap V_s - \dim V_u^c \cap V_s^c \\ &= \dim V_u \cap V_s - \dim (V_u \cup V_s)^c \\ &= \dim V_u \cap V_s - (n - \dim V_u \cup V_s) \\ &= \dim V_u \cap V_s - [n - (\dim V_u + \dim V_s - \dim V_u \cap V_s)] \\ &= \dim V_u + \dim V_s - n. \end{aligned}$$

This completes the proof of Lemma 1.●

We want to view our “phase condition” (3.3) as an operator equation. We hence define a linear functional $l : B_{\mu,1} \rightarrow \mathbb{R}$ by

$$(3.28) \quad l(u) = \int_{T_-}^{T_+} u'(t) \cdot v_0''(t) dt, \quad T_- < 0 < T_+.$$

l is clearly well defined and satisfies

$$(3.29) \quad l_u^0 v_0' \equiv (D_u l(v_0)) v_0' = \int_{T_-}^{T_+} v_0''(t) \cdot v_0''(t) dt \neq 0.$$

We next rewrite (3.13), (3.3) in the operator form as

$$(3.30) \quad \Phi(\cdot, u, \lambda) \equiv (F(\cdot, u, \lambda), l(u + \chi(\cdot, \lambda) - v_0(\cdot))) = 0,$$

where $\Phi : B_{\mu,1} \times \mathbb{R}^{n_\lambda} \rightarrow B_{\mu,0} \times \mathbb{R}$. Recall that in Section 2 we required $n_\lambda = n - (n_0 + n_1) + 2$. Hence by Lemma 1 we have

$$(3.31) \quad \text{Ind } F_u^0 \equiv \dim \mathcal{N}(F_u^0) - \text{codim } \mathcal{R}(F_u^0) = n_0 + n_1 - n = 2 - n_\lambda.$$

Lemma 2. (i) *The linearized operator $\Phi'(\cdot, u_0, \lambda^0) : B_{\mu,1} \times \mathbb{R}^{n_\lambda} \rightarrow B_{\mu,0} \times \mathbb{R}$ satisfies*

$$(3.32) \quad \text{codim } \mathcal{R}(\Phi'(\cdot, u_0, \lambda^0)) \leq \text{codim } \mathcal{R}(F_u^0) \leq \min\{n_0, n_1\} - n_0 - n_1 + n.$$

(ii) *Moreover, if $n_\lambda \geq 0$ is defined by (3.31) then $\Phi'(\cdot, u_0, \lambda^0)$ is a Fredholm operator of index 1.*

Proof. Φ is obviously differentiable with

$$\Phi'(\cdot, u_0, \lambda^0) = \begin{bmatrix} F_u^0 & F_\lambda^0 \\ l_u^0 & l_\lambda^0 \end{bmatrix},$$

where $F_\lambda^0 : \mathbb{R}^{n_\lambda} \rightarrow B_{\mu,0}$ and $l_\lambda^0 : \mathbb{R}^{n_\lambda} \rightarrow \mathbb{R}$ are defined by

$$(3.33) \quad \begin{aligned} F_\lambda^0 &\equiv D_\lambda F(\cdot, u_0, \lambda^0) \\ &= -f_u(v_0(\cdot), \lambda^0) \chi_\lambda(\cdot, \lambda^0) - f_\lambda(v_0(\cdot), \lambda^0) + \chi_{t\lambda}(\cdot, \lambda^0), \\ l_\lambda^0 &\equiv \int_{T_-}^{T_+} \chi_{t\lambda}(t, \lambda^0) \cdot v_0''(t) dt. \end{aligned}$$

By an application of the bordering Lemma below with $N = \text{Ind}(F_u^0)$, $p = 0$, $q = 1$ we have

$$\text{Ind} \left(\begin{bmatrix} F_u^0 \\ l_u^0 \end{bmatrix} \right) = \text{Ind}(F_u^0) - 1.$$

Taking into account that by part (i) of lemma 1 and (3.31)

$$\mathcal{N} \left(\begin{bmatrix} F_u^0 \\ l_u^0 \end{bmatrix} \right) = \mathcal{N}(F_u^0) - 1,$$

we obtain

$$\text{codim}\mathcal{R}\left(\begin{bmatrix} F_u^0 \\ l_u^0 \end{bmatrix}\right) = \text{codim}\mathcal{R}(F_u^0).$$

Together with an obvious inequality

$$\text{codim}\mathcal{R}(\Phi'(\cdot, u_0, \lambda^0)) \leq \text{codim}\mathcal{R}\left(\begin{bmatrix} F_u^0 \\ l_u^0 \end{bmatrix}\right)$$

and (3.18b), this implies (3.32). Now the second part of Lemma 2 follows by an application of the bordering Lemma with $N = n_0 + n_1 - n$, $p = n_\lambda$, $q = 1$.

Lemma 3. (*bordering Lemma, see e.g. [2, Lemma 2.4]*). *Let X, Y be Banach spaces and consider the operator*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \times \mathbb{R}^p \rightarrow Y \times \mathbb{R}^q$$

with bounded linear operators $A : X \rightarrow Y$, $B : \mathbb{R}^p \rightarrow Y$, $C : X \rightarrow \mathbb{R}^q$, $D : \mathbb{R}^p \rightarrow \mathbb{R}^q$. If A is Fredholm of index N then S is Fredholm of index $N + p - q$.

We approximate (3.30) by

$$(3.34) \quad \Phi_T(t, u, \lambda) \equiv (F_T(t, u, \lambda), l(u + \chi(\cdot, \lambda) - v_0)) = 0,$$

where $\Phi_T : B_{\mu,1} \times \mathbb{R}^{n_\lambda} \rightarrow B_{\mu,0} \times \mathbb{R}$, and the operator $F_T : B_{\mu,1} \times \mathbb{R}^{n_\lambda} \rightarrow B_{\mu,0}$ is defined by

$$(3.35) \quad F_T(t, u, \lambda) = \begin{cases} u' - f_u(w_0, \lambda)u & \text{for } t < T_-, \\ F(\cdot, u, \lambda) & \text{for } T_- \leq t \leq T_+, \\ u' - f_u(w_1, \lambda)u & \text{for } t > T_+, \end{cases}$$

Remark 3. The boundary conditions (2.6) follow easily from (3.34). For example, the solution of

$$(3.36) \quad u' - f_u(w_0, \lambda)u = 0, \quad -\infty < t < T_-,$$

in $B_{\mu,1}$ has the form

$$(3.37) \quad u_-(t) = \sum_{i=1}^{n_0} c'_{0i} v_{0i} e^{\mu_{0i} t}, \quad t < T_-,$$

for some constants c'_{0i} . Setting in (3.37)

$$(3.37a) \quad \epsilon_0 = \sqrt{\sum_{i=1}^{n_0} |c'_{0i} e^{\mu_{0i} T_-}|^2}, \quad c_{0i} = c'_{0i} e^{\mu_{0i} T_-} / \epsilon_0$$

gives (2.6a). •

To establish existence, uniqueness and error estimates for the approximate problem we shall use a version of the implicit function theorem in Descloux and Rappaz [4].

Theorem 1. *Let X and Y be two real Banach spaces, $\Phi, \Phi_n : X \rightarrow Y$ be a C^p mapping and a family of mappings, respectively, with $p \geq 1$, and $x_0 \in X$ be such that $\Phi(x_0) = 0$. Assume that x_0 is regular point in the following sense:*

$$(3.38) \quad \begin{aligned} a) & \quad \Phi'(x_0) : X \rightarrow Y \text{ is a Fredholm operator of index } 1, \\ b) & \quad \mathcal{R}(\Phi'(x_0)) = Y, \end{aligned}$$

and assume that $\mathcal{N}(\Phi'(x_0))$ is spanned by some vector $\varphi_0 \in X$, $\varphi_0 \neq 0$. And let $\bar{l} : X \rightarrow \mathbb{R}$ be a linear functional that satisfies

$$(3.39) \quad \bar{l}(\varphi_0) \neq 0.$$

We also assume that for any $0 \leq k \leq p-1$ and for any fixed $x, \xi_1, \dots, \xi_k \in X$, we have

$$(3.40) \quad \lim_{n \rightarrow \infty} \left\| \left(\Phi^{(k)}(x) - \Phi_n^{(k)}(x) \right) (\xi_1, \dots, \xi_k) \right\| = 0.$$

and that

$$(3.41) \quad \lim_{n \rightarrow \infty} \left\| \Phi'(x_0) - \Phi_n'(x_0) \right\| = 0;$$

Then there exist positive constants n_0, s_0, α, K and two unique C^1 maps $x(s), x_n(s) \in X$, $|s| < s_0$, satisfying for $|s| < s_0$, respectively, the conditions:

$$(3.42) \quad (\Phi(x(s)), \bar{l}(x(s) - x_0) - s) = 0, \quad \|x(s) - x_0\| < \alpha,$$

$$(3.43) \quad (\Phi_n(x_n(s)), \bar{l}(x_n(s) - x_0) - s) = 0, \quad \|x_n(s) - x_0\| < \alpha, \quad n > n_0.$$

Moreover $x(0) = x_0$ and for $|s| < s_0$, $n > n_0$, $0 \leq k \leq p-1$ we have an error estimate

$$(3.44) \quad \left\| x^{(k)}(s) - x_n^{(k)}(s) \right\| \leq C \sum_{l=0}^k \left\| \frac{d^l}{ds^l} \Phi_n(x(s)) \right\|,$$

where C is a positive constant independent of n .

Theorem 1 requires a consistency condition (a pointwise convergence) and a stability condition. These conditions are given by the following lemma.

Lemma 4. *Let $x = (u, \lambda) \in B_{\mu,1} \times \mathbb{R}^{n_\lambda}$, where $x = (u_0, \lambda^0)$ solves (3.30), and assume that f is C^p , $p \geq 1$.*

Then for $-T_-, T_+ \rightarrow \infty$ we have: for any $0 \leq k \leq p-1$ and for any fixed $x, \xi_1, \dots, \xi_k \in B_{\mu,1} \times \mathbb{R}^{n_\lambda}$,

$$(3.45) \quad \begin{aligned} & \left\| \left(\Phi^{(k)}(x) - \Phi_T^{(k)}(x) \right) (\xi_1, \dots, \xi_k) \right\|_{\mu,0} \\ & \leq C \|u\|_{\mu,1} \left(e^{T-\mu_0} + e^{T+\mu_1} \right) \rightarrow 0, \end{aligned}$$

$$(3.46) \quad \|\Phi'(x_0) - \Phi'_T(x_0)\|_{\mu,0} \leq C \left(e^{T-\mu_0} + e^{T+\mu_1} \right) \rightarrow 0,$$

where $\mu_0 > 0$ and $\mu_1 < 0$ are as in (3.9), and C does not depend on λ , T_- and T_+ .

Proof. We verify (3.45) for $k = 0$. For $k > 0$ the proof is similar. From the definitions (3.13) and (3.35) of F and F_T , respectively, we have

$$(3.47) \quad \begin{aligned} & \|F(x) - F_T(x)\|_{\mu,0} \\ & \leq \left(\|f_u(w_0, \lambda)u - f(w_0 + u, \lambda)\|_{B_{\mu,0}(-\infty, T_-]} \right. \\ & \quad \left. + \|f_u(w_1, \lambda)u - f(w_1 + u, \lambda)\|_{B_{\mu,0}[T_+, \infty)} \right). \end{aligned}$$

By (3.5) we have

$$(3.48) \quad \begin{aligned} & \| (f_u(w_1, \lambda)u - f(w_1 + u, \lambda)) \|_{B_{\mu,0}[T_+, \infty)} \\ & = \sup_{t \geq T_+} |w_\mu(t) (f_u(w_1, \lambda)u(t) - f(w_1 + u(t), \lambda))| \\ & \leq C \sup_{t \geq T_+} |w_\mu(t) u^2(t)| \leq C \|u\|_{\mu,0}^2 e^{T+\mu_1}, \end{aligned}$$

where in the last inequality we used that $u \in B_{\mu,0}$ and the definition (3.10) of $w_\mu(t)$. A similar estimate holds for $\|(f_u(w_1, \lambda)u - f(w_1 + u, \lambda))\|_{B_{\mu,0}(-\infty, T_-]}$. Hence (3.45) follows from (3.47), (3.48), the definitions (3.30) and (3.34) of Φ and Φ_T , respectively.

Similarly, (3.46) follows from the estimates of the type

$$(3.49) \quad \begin{aligned} & \| (D_u F(\cdot, u_0, \lambda^0) - D_u F_T(\cdot, u_0, \lambda^0)) v \|_{\mu,0} \\ & \leq C \left(\sup_{t \leq T_-} |w_\mu(t) (f_u(w_0, \lambda^0) - f_u(w_0 + u_0(t), \lambda^0)) v(t)| \right. \\ & \quad \left. + \sup_{t \geq T_+} |w_\mu(t) (f_u(w_1, \lambda^0) - f_u(w_1 + u_0(t), \lambda^0)) v(t)| \right), \forall v \in B_{\mu,1}, \end{aligned}$$

$$(3.50) \quad \begin{aligned} & \sup_{t \geq T_+} |w_\mu(t) (f_u(w_1, \lambda^0) - f_u(w_1 + u_0(t), \lambda^0)) v(t)| \\ & \leq C \sup_{t \geq T_+} |w_\mu(t) u_0(t) v(t)| \leq C \|v\|_{\mu,0} \sup_{t \geq T_+} |u_0(t)| \\ & \leq C \|v\|_{\mu,1} \|u_0\|_{\mu,0} e^{T+\mu_1}, \forall v \in B_{\mu,1}, \end{aligned}$$

$$(3.51) \quad \begin{aligned} & \| D_\lambda F(\cdot, u_0, \lambda^0) - D_\lambda F_T(\cdot, u_0, \lambda^0) \|_{\mu,0} \\ & \leq \sup_{t \leq T_-} |w_\mu(t) (f_{\lambda u}(w_0, \lambda^0) u_0(t) - f_\lambda(w_0 + u_0(t), \lambda^0))| \\ & \quad + \sup_{t \geq T_+} |w_\mu(t) (f_{\lambda u}(w_1, \lambda^0) u_0(t) - f_\lambda(w_1 + u_0(t), \lambda^0))| \\ & \leq C \left(\sup_{t \leq T_-} w_\mu(t) |u_0(t)|^2 + \sup_{t \geq T_+} w_\mu(t) |u_0(t)|^2 \right) \\ & \leq C \|u_0\|_{\mu,0}^2 \left(e^{T-\mu_0} + e^{T+\mu_1} \right). \end{aligned}$$

where in (3.50) we used that $|f_{uu}| \leq C$; and in (3.51) we used (3.5) and $|f_{uu\lambda}| \leq C \bullet$

From Lemmas 2, 4 and Theorem 1 with $X = B_{\mu,1} \times \mathbb{R}^{n_\lambda}$, $Y = B_{\mu,0} \times \mathbb{R}$ we have the following theorem.

Theorem 2. *Assume that f is C^p , $p \geq 1$. Let $x_0 \equiv (u_0, \lambda^0) \in B_{\mu,1} \times \mathbb{R}^{n_\lambda}$, $n_\lambda = 2 - (n_0 + n_1 - n) \geq 0$, be a solution of (3.30). Assume that the operator $\Phi : B_{\mu,1} \times \mathbb{R}^{n_\lambda} \rightarrow B_{\mu,0} \times \mathbb{R}$ defined by (3.30) satisfies:*

$$(3.52) \quad \mathcal{R}(\Phi'(\cdot, x_0)) = B_{\mu,0} \times \mathbb{R},$$

and that $\mathcal{N}(\Phi'(\cdot, x_0))$ is spanned by some nonzero vector $(q, \nu) \in B_{\mu,1} \times \mathbb{R}^{n_\lambda}$. Define a linear functional $\bar{l} : B_{\mu,1} \times \mathbb{R}^{n_\lambda} \rightarrow \mathbb{R}$ by

$$(3.53) \quad \bar{l}(u, \lambda) = \int_{T_-}^{T_+} u(t) \cdot q(t) dt + \lambda \cdot \nu, \quad T_- < 0 < T_+;$$

note that

$$(3.54) \quad \bar{l}(q, \nu) \neq 0.$$

Then there exist positive constants $-T_-^0$, T_+^0 , s_0 , α , C and the two unique C^1 maps $x(s) \equiv (u(s), \lambda(s))$, $x_T(s) \equiv (u_T(s), \lambda_T(s)) \in B_{\mu,1} \times \mathbb{R}^{n_\lambda}$, $|s| < s_0$, satisfying for $|s| < s_0$, respectively, the conditions

$$(3.55) \quad (\Phi(x(s)), \bar{l}(x(s) - x_0) - s) = 0, \quad \|x(s) - x_0\|_{B_{\mu,1} \times \mathbb{R}^{n_\lambda}} < \alpha,$$

$$(3.56) \quad (\Phi_T(x_T(s)), \bar{l}(x_T(s) - x_0) - s) = 0, \quad \|x_T(s) - x_0\|_{B_{\mu,1} \times \mathbb{R}^{n_\lambda}} < \alpha, \\ T_- < T_-^0, \quad T_+ > T_+^0.$$

Moreover, $x(0) = x_0$, and for $|s| < s_0$, $T_- < T_-^0$, $T_+ > T_+^0$, $0 \leq k \leq p-1$ we have an error estimate

$$(3.57) \quad \left| \lambda^{(k)}(s) - \lambda_T^{(k)}(s) \right| + \left\| u^{(k)}(s) - u_T^{(k)}(s) \right\|_{\mu,1} \\ \leq C \sum_{l=0}^k \left\| u^{(l)}(s) \right\|_{\mu,1} \left(e^{T_- \mu_0} + e^{T_+ \mu_1} \right),$$

where $\mu_0 > 0$ and $\mu_1 < 0$ are as in (3.9).

Corollary 1. *Let the assumptions of Theorem 2 hold.*

(i) *For $|s| < s_0$, $T_- < T_-^0$, $T_+ > T_+^0$, $0 \leq k \leq p-1$ we have*

$$(3.58) \quad \left| \lambda^{(k)}(s) - \lambda_T^{(k)}(s) \right| + \left\| u^{(k)}(s) - u_T^{(k)}(s) \right\|_{0,1} \\ \leq C \sum_{l=0}^k \left\| u^{(l)}(s) \right\|_{\mu,1} \left(e^{2T_- \mu_0} + e^{2T_+ \mu_1} \right).$$

(ii) There exists at least one component, call it λ_1 , of the vector λ such that

$$(3.59) \quad \lambda'_1(0) = 0, \text{codim}\mathcal{R}([F_u^0 \ F_{\lambda_1}^0]) < \text{codim}\mathcal{R}(F_u^0).$$

Assume that

$$(3.60) \quad \lambda''_1(0) \neq 0.$$

and that $p \geq 3$. Then there exist positive constants s_1 , $-T_-^1$ and T_+^1 such that for $T_- < T_-^1$, $T_+ > T_+^1$ there exists an unique $s_T \in (-s_1, s_1)$ with $\lambda'_{1T}(s_T) = 0$, $\lambda''_{1T}(s_T) \neq 0$. In several practical cases one can obtain an 'improved' estimate for $|\lambda_{1T}(s_T) - \lambda_1(0)|$. We consider here one such case. Specifically, assume that $F_{\lambda_i} \in \mathcal{R}(F_u^0)$, $i = 2, \dots, n_\lambda$. Furthermore, let $\psi \in B_{-\mu,0}$ be such that

$$(3.61) \quad \begin{aligned} a) \quad & (F_u^0)^\star \psi = 0, \\ b) \quad & \langle F_{\lambda}^0, \psi \rangle \equiv \langle F_{\lambda_1}^0, \psi \rangle = 1; \end{aligned}$$

$$(3.62) \quad \begin{aligned} a) \quad & |\psi(t)| \leq c e^{-\mu_{0i}t}, \text{ for some } i > n_0, t < 0, \\ b) \quad & |\psi(t)| \leq c e^{-\mu_{1j}t}, \text{ for some } j > n_1, t > 0. \end{aligned}$$

Then for $-T_-$, T_+ large enough and $\mu_{0i} \leq 0$, $\mu_{1j} \geq 0$ as in (3.62) we have an 'improved' error estimate

$$(3.63) \quad \begin{aligned} & |\lambda_{1T}(s_T) - \lambda_1(0)| \\ & \leq C \left(e^{(2\mu_0 - \mu_{0i})T_-} + e^{(2\mu_1 - \mu_{1j})T_+} + \left(e^{2T_- \mu_0} + e^{2T_+ \mu_1} \right)^2 \right). \end{aligned}$$

Proof. Examination of our proofs shows that the right hand side of (3.57) can be replaced by

$$C \left(\|u(s)\|_{B_{\mu,1}(-\infty, T_-]} e^{T_- \mu_0} + \|u(s)\|_{B_{\mu,1}[T_+, \infty)} e^{T_+ \mu_1} \right).$$

Also, $B_{\mu,1}$ -norms in (3.57) can be replaced by $B_{\epsilon,1}$ -norm for any $\epsilon = (\epsilon_0, \epsilon_1)^T$ with $\mu_0 \geq \epsilon_0 > 0$, $\mu_1 \leq \epsilon_1 < 0$. Since $u \in B_{\mu,1}$, the above implies estimates of the type

$$\begin{aligned} \|u\|_{B_{\epsilon,0}[T_+, \infty)} &= \sup_{t \geq T_+} |e^{-t\epsilon_1} u(t)| \\ &= \sup_{t \geq T_+} \left| e^{t(\mu_1 - \epsilon_1)} e^{-t\mu_1} u(t) \right| \leq e^{T_+(\mu_1 - \epsilon_1)} \|u\|_{B_{\mu,0}[T_+, \infty)}, \\ \|u\|_{B_{\epsilon,1}[T_+, \infty)} &\leq e^{T_+(\mu_1 - \epsilon_1)} \|u\|_{B_{\mu,1}[T_+, \infty)}. \end{aligned}$$

Together with the obvious inequality $\|u\|_{0,1} \leq \|u\|_{\epsilon,1}$ this leads to an error estimate

$$\begin{aligned} & |\lambda(s) - \lambda_T(s)| + \|u(s) - u_T(s)\|_{0,1} \\ & \leq C \|u\|_{\mu,1} \left(e^{T_-(2\mu_0 - \epsilon_0)} + e^{T_+(2\mu_1 - \epsilon_1)} \right), \end{aligned}$$

where C does not depend on ϵ . Passing in the above to the limit as $\epsilon \rightarrow (0+, 0-)$, we obtain (3.58).

By differentiating the relation $F(u(s), \lambda(s)) = 0$ at $s = 0$ we have

$$(3.64) \quad F_u^0 u'(0) + F_\lambda^0 \lambda'(0) = 0.$$

First notice that

$$\text{codim} \mathcal{R}([F_u^0 F_\lambda^0]) \leq \text{codim} \mathcal{R}(\Phi'(\cdot, u_0, \lambda^0)) = 0$$

implies $\text{codim} \mathcal{R}([F_u^0 F_\lambda^0]) = 0$. And therefore taking into account that by Lemma 1 $\text{codim} \mathcal{R}(F_u^0) \geq 1$, we conclude that $F_\lambda^0 \notin \mathcal{R}(F_u^0)$, and hence (3.59) follows from (3.64).

The proof of (3.63) is similar to the the proof of Theorem 3.2 in [4]. We use the estimate (3.58) for $k = 1, 2$. Since $\lambda_1'(0) = 0$ we obtain that $\lim_{-T_-, T_+ \rightarrow \infty} \lambda_{1T}'(0) = 0$; by (3.60), there exists $s_1 > 0$ such that for $-T_-, T_+$, large enough, $|\lambda_{1T}''(s)| \geq c$, $|s| \leq s_1$; consequently there exist $-T_-^1, T_+^1 > 0$ and for $-T_- < T_-^1, T_+ > T_+^1$ an unique $s_T \in (-s_1, s_1)$ such that $\lambda_{1T}'(s_T) = 0$. Furthermore, we have the estimates:

$$\begin{aligned} |s_T| &\leq c |\lambda_{1T}'(0)| = C |\lambda_{1T}'(0) - \lambda_1'(0)|, \\ \lambda_{1T}(0) &= \lambda_{1T}(s_T) - (\lambda_{1T}'(s_T)) s_T + O(s_T^2) = \lambda_{1T}(s_T) + O(s_T^2), \end{aligned}$$

and therefore

$$(3.65) \quad |\lambda_{1T}(0) - \lambda_{1T}(s_T)| \leq C \left(e^{2T_- \mu_0} + e^{2T_+ \mu_1} \right)^2$$

By (3.61a) $\langle F_u^0 v, \psi \rangle = 0$, for any $v \in B_{\mu,1}$. Hence taking the value of ψ on the equality

$$\begin{aligned} &F(u(0), \lambda(0)) - F(u_T(0), \lambda_T(0)) \\ &= F_u^0(u(0) - u_T(0)) + F_\lambda^0(\lambda(0) - \lambda_T(0)) \\ &+ O\left(|\lambda(0) - \lambda_T(0)|^2 + \|u(0) - u_T(0)\|_{0,1}^2\right) \end{aligned}$$

and using (3.61b) we have

$$(3.66) \quad \begin{aligned} &|\lambda_1(0) - \lambda_{1T}(0)| \\ &\leq C \left(|\langle F(u_T(0), \lambda_T(0)), \psi \rangle| + \|u(0) - u_T(0)\|_{0,1}^2 \right). \end{aligned}$$

Using that by (3.56) $F_T(u_T(0), \lambda_T(0)) = 0$, (3.5), the fact that $u_T(0) \in B_{\mu,0}$ and (3.62),

we obtain

$$\begin{aligned}
|\langle F(u_T(0), \lambda_T(0)), \psi \rangle| &= |\langle F(u_T(0), \lambda_T(0)) - F_T(u_T(0), \lambda_T(0)), \psi \rangle| \\
&= \left| \int_{-\infty}^{T_-} [f_u(w_0^0, \lambda_T(0)) u_T(t, 0) - f(w_0^0 + u_T(t, 0), \lambda_T(0))] \cdot \psi(t) dt \right. \\
&\quad \left. \int_{T_+}^{\infty} [f_u(w_1^0, \lambda_T(0)) u_T(t, 0) - f(w_1^0 + u_T(t, 0), \lambda_T(0))] \cdot \psi(t) dt \right| \\
&\leq C \left(\int_{-\infty}^{T_-} |u_T(t, 0)|^2 |\psi(t)| dt + \int_{T_+}^{\infty} |u_T(t, 0)|^2 |\psi(t)| dt \right) \\
&\leq C \left(\int_{-\infty}^{T_-} e^{2\mu_0 t} e^{-\mu_{0i} t} dt + \int_{T_+}^{\infty} e^{2\mu_1 t} e^{-\mu_{1j} t} dt \right) \\
&= C \left(\frac{1}{2\mu_0 - \mu_{0i}} e^{(2\mu_0 - \mu_{0i})T_-} + \frac{1}{-(2\mu_1 - \mu_{1j})} e^{(2\mu_1 - \mu_{1j})T_+} \right).
\end{aligned}$$

Substituting this into (3.66) and estimating the second term in the right side of (3.66) by (3.58), gives

$$\begin{aligned}
&|\lambda_1(0) - \lambda_{1T}(0)| \\
(3.67) \quad &\leq C \left(e^{(2\mu_0 - \mu_{0i})T_-} + e^{(2\mu_1 - \mu_{1j})T_+} + \left(e^{2T_- \mu_0} + e^{2T_+ \mu_1} \right)^2 \right).
\end{aligned}$$

Combining (3.65) and (3.67), we finally arrive at (3.63). •

4. Examples.

Example 1. We first apply our method to the problem of finding traveling wave fronts in the Huxley equation. This problem was also considered in [6]. However the analysis considerably simplifies by using the present theory. The equation is given by

$$\begin{aligned}
(4.1) \quad &a) \quad w_t = w_{xx} + f(w, a), \quad -\infty < x < \infty, \quad t > 0, \\
&b) \quad f(w, a) = w(1-w)(w-a), \quad 0 < a < 1.
\end{aligned}$$

We look for the solutions to (4.1) of the form $w(x, t) = v_1(x + ct)$, where c is the *wave speed*. This gives the first order system

$$\begin{aligned}
(4.2) \quad &v_1'(z) = v_2(z), \\
&v_2'(z) = cv_2(z) - f(v_1(z), a), \\
&v_1(-\infty) = 0, \quad v_1(\infty) = 1, \quad v_2(-\infty) = v_2(\infty) = 0,
\end{aligned}$$

where $z = x + ct$. In this case $n = 2$, $w_0 = (0, 0)^T$, $w_1 = (1, 0)^T$, $n_0 = n_1 = 1$, $\lambda = (a, c)$;

$$\mu_{01} = \frac{c + \sqrt{c^2 + 4a}}{2} > 0, \quad \mu_{11} = \frac{c - \sqrt{c^2 + 4a}}{2} < 0.$$

We next rewrite (4.2) in the form (3.13):

$$\begin{aligned}
(4.3) \quad &u_1'(z) - u_2(z) = 0, \\
&u_2'(z) - cu_2(z) + f(u_1(z) + \chi_1(z), a) + \chi_2'(z) - c\chi_2(z) = 0, \\
&u_1(\pm\infty) = u_2(\pm\infty) = 0,
\end{aligned}$$

where $v_1 = u_1 + \chi_1$, $\chi_2 = \chi'_1$, $\chi_1 \in C^\infty$ and satisfies

$$\chi_1(z) = \begin{cases} 0 & \text{for } z \leq T_-, \\ 1 & \text{for } z \geq T_+. \end{cases}$$

We need $n_\lambda = 2 - (n_0 + n_1 - n) = 2$, which we already have. Next by (3.32)

$$0 \leq \text{codim} \mathcal{R}(\Phi'(\cdot, u_0, \lambda^0)) \leq \text{codim} \mathcal{R}(F_u^0) = 1.$$

Hence to ensure that $\text{codim} \mathcal{R}(\Phi'(\cdot, u_0, \lambda^0)) = 0$ we are left to verify that $F_{\lambda_1}^0 \notin \mathcal{R}(F_u^0)$ or $F_{\lambda_2}^0 \notin \mathcal{R}(F_u^0)$. Let ψ be an eigenfunction of the adjoint $(F_u^0)^*$ of the linearized operator (4.3) F_u^0 about a solution (u_0, λ^0) , $u_0 = v_0 - \chi$, of (4.3). Then (see e.g. Sattinger [16]) ψ satisfies $(F_u^0)^* \psi = 0$ in the classical sense and, in fact

$$\psi(z) = \left(e^{-cz} \varphi_1(z), - (e^{-cz} \varphi_1(z))' \right)^T,$$

where $\varphi(z) = (\varphi_1(z), \varphi_1'(z))^T = v'_0 \equiv (v'_{01}(z), v''_0(z))^T$ spans F_u^0 . The transversality condition $F_{\lambda_2}^0 \notin \mathcal{R}(F_u^0)$ by the above takes the form $\langle F_{\lambda_2}^0, \psi \rangle \neq 0$ or, using that $F_{\lambda_2}^0 = (0, -v'_{01})^T = (0, -\varphi_1)^T$,

$$\langle F_{\lambda_2}^0, \psi \rangle = \int_{-\infty}^{\infty} \varphi_1(z) \cdot (e^{-cz} \varphi_1(z))' dz = -\frac{\lambda_2}{2} \int_{-\infty}^{\infty} |\varphi_1(z)|^2 e^{-cz} dz \neq 0,$$

which obviously holds.

The vector $(q, \nu) \in B_{\mu,1} \times \mathbf{R}^2$ in (3.53) which spans $\mathcal{N}(\Phi'(\cdot, x_0))$ has the form $(q, \nu) = (0, \nu)$.

Example 2. Finding traveling wave fronts and their singular limits in a simplified combustion modeling the case of a single reaction $A \rightarrow B$ in a premixed laminar gas, with the Lewis number $\Lambda = 1$. The equation for the temperature T is given by

$$\begin{aligned} (4.4) \quad T_t &= T_{xx} + f(T, \epsilon), \quad -\infty < x < \infty, \quad t > 0, \\ f(T, \epsilon) &= \frac{1-T}{\epsilon^2} \varphi(T, \epsilon), \\ \varphi(T, \epsilon) &= \begin{cases} 0, & \text{for } T \leq T_1, \\ \exp\left(\frac{1-T}{\epsilon}\right), & \text{for } T > T_1, \end{cases} \end{aligned}$$

where T_1 with $0 < T_1 < 1$ is the ignition temperature. We look for the solutions to (4.4) of the form $T(x, t) = v_1(x + ct)$, where c is the *wave speed*. This gives the first order system

$$\begin{aligned} (4.5) \quad v'_1(z) &= v_2(z), \\ v'_2(z) &= cv_2(z) - f(v_1(z), \epsilon), \\ v_1(-\infty) &= 0, \quad v_1(\infty) = 1, \quad v_2(-\infty) = v_2(\infty) = 0, \end{aligned}$$

where $z = x + ct$ In this case $n = 2$, $w_0 = (0, 0)^T$, $w_1 = (1, 0)^T$, $n_0 = n_1 = 1$, $n_\lambda = 2$, $\lambda = (a, c)$.

Figure 1 Some orbits that connect the fixed points $(0, 0)$ and $(1, 0)$ of Equation (4.5). Top left: phase plane representation of (4.5), when $\epsilon = 0.1$ and $c = 1.6895$. Top right: a branch of traveling wave solutions to (4.4), starting at the point $(\epsilon, c) = (0.1, 1.6895)$ and terminating at the point $(\epsilon, c) = (0.0001, 1.41440)$. Bottom left: phase plane representation of (4.5), when $\epsilon = 0.0001$ and $c = 1.41440$. Bottom right: (v_1, t) plane representation of (4.5) when $\epsilon = 0.0001$ and $c = 1.41440$.

The solution to (4.5) is obtained by a homotopy from a known solution of the Huxley equation. Specifically, we replace $f(v_1, \epsilon)$ by

$$\beta f(v_1, \epsilon) + (1 - \beta) v_1 (v_1 - 0.5) (1 - v_1), \quad 0 \leq \beta \leq 1,$$

where $\beta = 0$ is used as a starting point. Then for $\beta = 0$ (4.5) becomes (4.2), with $a = 0.5$. In this case the solution of (4.2) is known analytically, in particular, $c = 0$. At the terminal point of the homotopy ($\beta = 1$) we obtain the solution to (4.5). Then we compute the singular limit as $\epsilon \rightarrow 0$. In this case the activation energy $E \rightarrow \infty$, and (see e.g. [1]) it is

known that $c \rightarrow \sqrt{2}$ and $v_1(z) \rightarrow v_1^*(z)$ where for some z_0

$$v_1^*(z) = \begin{cases} 1, & \text{for } z \geq z_0, \\ 1 - \exp(\sqrt{2}(z - z_0)), & \text{for } z < z_0. \end{cases}$$

Calculation was done using AUTO [5] with 100 mesh intervals, 5 orthogonal collocation points per mesh interval, period=10, $T_1 = 0.2$. Some of the computed orbits are shown in Figure 1. Initially $\epsilon = 0.1$, Figure 1, top left. At the end of the homotopy $0 \leq \beta \leq 1$, $c = 1.6895$. In the computation of the approximate singular limit, Figure 1, top right, we obtained $\epsilon = 10^{-4}$, $c = 1.41440$, Figure 1, bottom.

Example 3. Finding traveling wave fronts with speed $c > 2$ in the Fisher equation. The equation has the form (4.1) with

$$(4.6) \quad f(w) = w(1 - w)$$

In this case $n = 2$, $w_0 = (0, 0)^T$, $w_1 = (1, 0)^T$, $n_0 = 2$, $n_1 = 1$, $n_\lambda = 1$, $\lambda = c$;

$$\mu_{01} = \frac{c - \sqrt{c^2 - 4}}{2} > 0, \mu_{02} = \frac{c + \sqrt{c^2 - 4}}{2} > 0, \mu_{11} = \frac{c - \sqrt{c^2 + 4}}{2} < 0.$$

Figure 2. Some orbits that connect the fixed points $(0, 0)$ and $(1, 0)$ of Fisher Equation. From top to bottom the wave speed c of these orbits (wave fronts) equals 2, 3, 5, 10, and 25. For $c < 2$ the eigenvalues at $(0, 0)$ are complex.

By (3.32)

$$0 \leq \text{codim} \mathcal{R}(\Phi'(\cdot, u_0, \lambda^0)) \leq \text{codim} \mathcal{R}(F_u^0) = 0.$$

Hence $\text{codim} \mathcal{R}(\Phi'(\cdot, u_0, \lambda^0)) = 0$. The vector $(q, \nu) \in B_{\mu,1} \times \mathbb{R}$ in (3.53) which spans $\mathcal{N}(\Phi'(\cdot, x_0))$ has the form $(q, \nu) = (0, \nu)$. Some of the computed orbits are shown in Figure 2

Example 4. Computation of a manifold that connects two fixed points of a system

$$(4.7) \quad \begin{aligned} v_1' &= v_2, \\ v_2' &= \gamma v_2 - v_1(1 - v_1) - v_3, \quad \gamma = \text{const} > 2, \\ v_3' &= v_3(1 - v_1), \\ w_0 &\equiv v(-\infty) = (0, 0, 0)^T, \quad w_1 \equiv v(\infty) = (1, 0, 0)^T, \end{aligned}$$

where $v = (v_1, v_2, v_3)^T$. In this case $n = 3$, $n_0 = 3$, $n_1 = 2$. Hence $n_\lambda = 0$, and therefore a branch of solutions already exists when γ is fixed. Again by (3.32)

$$0 \leq \text{codim} \mathcal{R}(\Phi'(\cdot, u_0)) \leq \text{codim} \mathcal{R}(F_u^0) = 0,$$

and hence $\text{codim} \mathcal{R}(\Phi'(\cdot, u_0)) = 0$.

Next note that $\dim \mathcal{N}(F_u^0) = 2$, and hence by Rheinboldt [15, Th. 4.2] the solution set for (4.7) is a 2-dimensional C^∞ -manifold without boundary in $B_{\mu,1}$. Hence $\mathcal{N}(\Phi'(\cdot, u^0))$ is spanned by a vector $q \in B_{\mu,1}$.

On practice it is more convenient to reformulate the problem (4.7) by adding a boundary condition with a parameter λ so that for fixed λ $\dim \mathcal{N}(F_u^0) = 1$, and hence the solution set for the new problem is a 1-dimensional manifold, so that the branch of solutions parametrized by λ coincides with the solution manifold for (4.7). One way to do this is as follows. Let $u^1(z)$ and $u^2(z)$ spanning $\mathcal{N}(F_u^0)$ be such that

$$\lim_{z \rightarrow \infty} u^1(z) e^{-\mu_{11}z} = v_{11}, \quad \lim_{z \rightarrow \infty} u^2(z) e^{-\mu_{12}z} = v_{12}.$$

Then (see Remark 3) for $z > T_+$ the solution of (4.7) can be written as

$$u_+(z) = \epsilon (u^1(z) \cos \lambda + u^2(z) \sin \lambda), \quad \epsilon = \text{const} > 0.$$

Now consider the problem (4.7),

$$(4.8) \quad v(z) = w_1 + \epsilon (u^1(z) \cos \lambda + u^2(z) \sin \lambda) + o(e^{\mu_{12}z}), \quad z \rightarrow \infty.$$

Keeping the same notation F_u^0 for the linearized operator (4.7), (4.8), we have $n_0 = 3$, $n_1 = 1$. Hence $n_\lambda = 1$, $\dim \mathcal{N}(F_u^0) = 1$, $\text{codim} \mathcal{R}(\Phi'(\cdot, u_0)) = 0$, so that $\mathcal{N}(\Phi'(\cdot, u^0))$ is spanned by a vector $(0, \nu) \in B_{\mu,1} \times \mathbb{R}$.

Figure 3. Part of the 2D invariant manifold for Equation (4.7).

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