**AN EXISTENCE THEOREM FOR MODEL EQUATIONS
RESULTING FROM KINETIC THEORIES
OF POLYMER SOLUTIONS**

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1. Introduction

General existence theorems for the partial differential equations of continuum mechanics are relatively recent, even for the case of elasticity. The initial value problem for compressible elastic materials occupying all of space was solved in [6], and the Dirichlet initial-boundary value problem is treated in [7], [2] and [3]. Existence results for incompressible materials were obtained in [4] and [11] for the initial value problem on all of space and in [5] for the Dirichlet problem.

Viscoelastic fluids such as polymers are characterized by a constitutive relation which gives the stress as a functional of the deformation history. Most popular models fall into two categories: integral and differential models. Integral models give the stress by an integral expression involving the history of the deformation gradient; differential models give the stress as the solution of a differential equation which involves the stress and velocity gradient. There is, however, a class of models motivated by considerations of kinetic theory, which can in general not be represented in either integral or differential form. Instead, such models require the solution of a diffusion equation in order to determine the stress [1]. Because of the practical impossibility - except in special cases - of solving this diffusion equation, such models have only had limited applications in solving flow problems. For the case of dumbbell models, however, computers may soon have the capacity to obtain solutions, at least in two-dimensional flow problems.

For materials with instantaneous elasticity, the terms of leading differential order in the equations of motion are like those for elasticity. A natural approach to a mathematical existence theory is therefore to regard the problem as essentially “elastic” and to treat the memory as a perturbation. This has been successfully carried out for integral models, see e.g. Chapter III of [8], and also for differential models [10]. In this paper, we shall discuss the models from kinetic theory in a similar fashion. We shall in part be able to rely on the results of [10], but the solution of the diffusion equation raises new issues.

The equations of motion for an incompressible fluid are

$$\begin{aligned} \rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) &= \operatorname{div} \mathbf{T} - \nabla p + \mathbf{f}, \\ \operatorname{div} \mathbf{v} &= 0. \end{aligned} \tag{1}$$

Here $\mathbf{v}$ denotes the velocity, p the pressure, $\mathbf{T}$ the extra stress tensor, ρ the density (assumed constant), and $\mathbf{f}$ is a given body force. We investigate molecular models which are based on treating polymer molecules as dumbbells consisting of two beads connected by a spring and floating in a Newtonian solvent. We shall assume that the solvent contribution to the stress is small relative to the polymer contribution and can be neglected; if this is not the case, the equations of motion can be treated as a perturbation of the Stokes equations, and existence results can be obtained along the lines of [9]. Let $\mathbf{R}$ denote the vector between the two ends of the dumbbell, and let $\psi(\mathbf{R}, \mathbf{x}, t)$ denote the probability density for dumbbells in $\mathbf{R}$ -space (this probability density depends on the point $\mathbf{x}$ in space and the time t). The spring force is given by $\mathbf{F}(\mathbf{R}) = -\gamma(|\mathbf{R}|^2)\mathbf{R}$, and the stress is given by

$$T_{ij}(\mathbf{x}, t) = -n \int R_i F_j(\mathbf{R}) \psi(\mathbf{R}, \mathbf{x}, t) d\mathbf{R}. \tag{2}$$

Here n is the number density of dumbbells and the integral extends over all possible values of $\mathbf{R}$. We shall consider infinitely extensible as well as finitely extensible dumbbells. For infinitely extensible dumbbells, $\mathbf{R}$ can be any vector in $\mathbb{R}^3$. For finitely extensible dumbbells, the potential associated with $\mathbf{F}(\mathbf{R})$ becomes infinite at a finite value R_0 of $|\mathbf{R}|$, and the integral in (2) extends only over the ball $|\mathbf{R}| \leq R_0$. The probability density $\psi(\mathbf{R}, \mathbf{x}, t)$ obeys the diffusion equation

$$\left(\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\right)\psi = \alpha \Delta_{\mathbf{R}}\psi + \operatorname{div}_{\mathbf{R}} (-(\nabla_{\mathbf{x}}\mathbf{v}) \cdot \mathbf{R}\psi - \beta \mathbf{F}(\mathbf{R})\psi). \quad (3)$$

Here we have used subscripts $\mathbf{R}$ and $\mathbf{x}$ to clarify with respect to which variables derivatives are being taken. Our convention for the gradient of a vector is that the first index refers to the component of the vector and the second index to the direction of differentiation. The quantities α and β are positive constants. The reader is referred to [1] for the derivation of (2) and (3) and for a discussion of possible generalizations.

We seek solutions of (1)-(3) for $\mathbf{x} \in \Omega$, $t > 0$, where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary. We impose Dirichlet boundary conditions,

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{0}, \quad \mathbf{x} \in \partial\Omega, \quad t \geq 0, \quad (4)$$

and initial conditions,

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \quad \psi(\mathbf{R}, \mathbf{x}, 0) = \psi_0(\mathbf{R}, \mathbf{x}). \quad (5)$$

Concerning the spring force, we shall make one of the following two assumptions:

- (F) The function γ is C^∞ -smooth from $[0, \infty)$ to $(0, \infty)$, $\gamma' \geq 0$, and there exist numbers $\sigma \geq 0$ and $k > 0$ such that $\lim_{|\mathbf{R}| \rightarrow \infty} \gamma(|\mathbf{R}|^2)/|\mathbf{R}|^\sigma = k$. Moreover, $\limsup_{|\mathbf{R}| \rightarrow \infty} \gamma'(|\mathbf{R}|^2)/|\mathbf{R}|^{\sigma-2}$ is finite and higher derivatives of γ have at most polynomial growth as $|\mathbf{R}| \rightarrow \infty$.
- (F') For some $R_0 > 0$, the function γ is C^∞ -smooth from $[0, R_0^2)$ to $(0, \infty)$, $\gamma' \geq 0$, and there exist numbers $\sigma > 1$ and $k > 0$ such that $\lim_{|\mathbf{R}| \rightarrow R_0} \gamma(|\mathbf{R}|^2)(R_0 - |\mathbf{R}|)^\sigma = k$. Moreover, $\limsup_{|\mathbf{R}| \rightarrow R_0} \gamma'(|\mathbf{R}|^2)(R_0 - |\mathbf{R}|)^{\sigma+1}$ is finite and higher derivatives of γ grow at most like powers of $(R_0 - |\mathbf{R}|)^{-1}$ as $|\mathbf{R}| \rightarrow R_0$.

In the first case, the dumbbell is infinitely extensible, in the second case it is finitely extensible. The assumption $\gamma' \geq 0$ means that the spring gets stiffer as the dumbbell is extended; this assumption is usually made in molecular models. The analysis that follows can be modified to allow faster growth of γ as $|\mathbf{R}| \rightarrow \infty$ or, respectively, $|\mathbf{R}| \rightarrow R_0$, but slower growth would pose difficulties. In particular, we have to exclude $\sigma = 1$ in (F'). This case corresponds to the most popular model of finitely extensible dumbbells (see [1]). There is, however, no specific reason to assume $\sigma = 1$ rather than a larger value of σ .

The goal of this paper is an existence and uniqueness theorem for solutions of the problem defined by (1)-(5). The paper is organized as follows. In Section 2, we define an iteration scheme which is used to construct the solution. The scheme alternates between solving an equation of the same type as encountered in incompressible elasticity and solving a linear diffusion equation such as (3). In Section 3 we define the function spaces used to carry out the analysis and state a precise theorem. The proof is given in Section 4.

2. Iterative construction of solution

We apply the operation $\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) + (\nabla \mathbf{v})^T$ to the equation of motion (1). The resulting equation, written in components, reads

$$\begin{aligned} \rho \left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right]^2 v_i &= \frac{\partial}{\partial x_j} \left[\left(\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right) T_{ij} \right] - \frac{\partial q}{\partial x_i} \\ &\quad - \frac{\partial v_k}{\partial x_j} \frac{\partial T_{ij}}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \frac{\partial T_{kj}}{\partial x_j} - \rho \frac{\partial v_j}{\partial x_i} \left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right] v_j \\ &\quad + \left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right] f_i + \frac{\partial v_j}{\partial x_i} f_j, \end{aligned} \quad (6)$$

where $q = \frac{\partial p}{\partial t} + (\mathbf{v} \cdot \nabla)p$. Using equation (2), we find that

$$\left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right] T_{ij} = -n \int R_i F_j(\mathbf{R}) \left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla_{\mathbf{x}}) \right] \psi(\mathbf{R}, \mathbf{x}, t) d\mathbf{R}, \quad (7)$$

and using equation (3), we find further

$$\begin{aligned} \left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right] T_{ij} &= -n \int R_i F_j(\mathbf{R}) \left[\alpha \Delta_{\mathbf{R}} \psi + \text{div}_{\mathbf{R}} (-(\nabla_{\mathbf{x}} \mathbf{v}) \cdot \mathbf{R} \psi - \beta \mathbf{F}(\mathbf{R}) \psi) \right] d\mathbf{R} \\ &= -n\alpha \int \Delta_{\mathbf{R}} (R_i F_j(\mathbf{R})) \psi d\mathbf{R} - n\beta \int \frac{\partial}{\partial R_k} (R_i F_j(\mathbf{R})) F_k(\mathbf{R}) \psi d\mathbf{R} \\ &\quad - n \int \frac{\partial}{\partial R_k} (R_i F_j(\mathbf{R})) \frac{\partial v_k}{\partial x_l} R_l \psi d\mathbf{R}. \end{aligned} \quad (8)$$

Next we note that

$$\frac{\partial}{\partial R_k} (R_i F_j(\mathbf{R})) = -\frac{\partial}{\partial R_k} (R_i R_j \gamma(|\mathbf{R}|^2)) = -\gamma(|\mathbf{R}|^2) (R_j \delta_{ik} + R_i \delta_{jk}) - 2R_i R_j R_k \gamma'(|\mathbf{R}|^2). \quad (9)$$

By combining (8) and (9), we find

$$\begin{aligned} \frac{\partial}{\partial x_j} \left[\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) T_{ij} \right] &= n \frac{\partial^2 v_k}{\partial x_j \partial x_l} \int (\gamma(|\mathbf{R}|^2) R_j R_l \delta_{ik} + 2\gamma'(|\mathbf{R}|^2) R_i R_j R_k R_l) \psi d\mathbf{R} \\ &\quad - n\alpha \int \Delta_{\mathbf{R}} (R_i F_j(\mathbf{R})) \frac{\partial \psi}{\partial x_j} d\mathbf{R} - n\beta \int \frac{\partial}{\partial R_k} (R_i F_j(\mathbf{R})) F_k(\mathbf{R}) \frac{\partial \psi}{\partial x_j} d\mathbf{R} \\ &\quad - n \frac{\partial v_k}{\partial x_l} \int \frac{\partial}{\partial R_k} (R_i F_j(\mathbf{R})) R_l \frac{\partial \psi}{\partial x_j} d\mathbf{R}. \end{aligned} \quad (10)$$

By inserting (10) into (6), we finally obtain

$$\rho \left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right]^2 v_i = n \frac{\partial^2 v_k}{\partial x_j \partial x_l} \int (\gamma(|\mathbf{R}|^2) R_j R_l \delta_{ik} + 2\gamma'(|\mathbf{R}|^2) R_i R_j R_k R_l) \psi d\mathbf{R} - \frac{\partial q}{\partial x_i}$$

$$\begin{aligned}
& -n\alpha \int \Delta_{\mathbf{R}}(R_i F_j(\mathbf{R})) \frac{\partial \psi}{\partial x_j} d\mathbf{R} - n\beta \int \frac{\partial}{\partial R_k}(R_i F_j(\mathbf{R})) F_k(\mathbf{R}) \frac{\partial \psi}{\partial x_j} d\mathbf{R} \\
& -n \frac{\partial v_k}{\partial x_l} \int \frac{\partial}{\partial R_k}(R_i F_j(\mathbf{R})) R_l \frac{\partial \psi}{\partial x_j} d\mathbf{R} \\
& -\frac{\partial v_k}{\partial x_j} \frac{\partial T_{ij}}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \frac{\partial T_{kj}}{\partial x_j} - \rho \frac{\partial v_j}{\partial x_i} \left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right] v_j \\
& + \left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right] f_i + \frac{\partial v_j}{\partial x_i} f_j.
\end{aligned} \tag{11}$$

The iteration is based on equations (3) and (11). Given an iterate $\mathbf{v}^m$, we determine ψ^m by solving equation (3),

$$\left(\frac{\partial}{\partial t} + (\mathbf{v}^m \cdot \nabla_{\mathbf{x}}) \right) \psi^m = \alpha \Delta_{\mathbf{R}} \psi^m + \operatorname{div}_{\mathbf{R}} (-(\nabla_{\mathbf{x}} \mathbf{v}^m) \cdot \mathbf{R} \psi^m - \beta \mathbf{F}(\mathbf{R}) \psi^m), \tag{12a}$$

$$\psi^m(\mathbf{R}, \mathbf{x}, 0) = \psi_0(\mathbf{R}, \mathbf{x}). \tag{12b}$$

We define $\mathbf{T}^m$ in terms of ψ^m by equation (2) and then determine a new velocity field $\mathbf{v}^{m+1}$ by solving the problem

$$\begin{aligned}
& \rho \left[\frac{\partial}{\partial t} + (\mathbf{v}^m \cdot \nabla) \right]^2 v_i^{m+1} \\
& = n \frac{\partial^2 v_k^{m+1}}{\partial x_j \partial x_l} \int (\gamma(|\mathbf{R}|^2) R_j R_l \delta_{ik} + 2\gamma'(|\mathbf{R}|^2) R_i R_j R_k R_l) \psi^m d\mathbf{R} - \frac{\partial q^{m+1}}{\partial x_i} \\
& -n\alpha \int \Delta_{\mathbf{R}}(R_i F_j(\mathbf{R})) \frac{\partial \psi^m}{\partial x_j} d\mathbf{R} - n\beta \int \frac{\partial}{\partial R_k}(R_i F_j(\mathbf{R})) F_k(\mathbf{R}) \frac{\partial \psi^m}{\partial x_j} d\mathbf{R} \\
& -n \frac{\partial v_k^m}{\partial x_l} \int \frac{\partial}{\partial R_k}(R_i F_j(\mathbf{R})) R_l \frac{\partial \psi^m}{\partial x_j} d\mathbf{R} \\
& -\frac{\partial v_k^m}{\partial x_j} \frac{\partial T_{ij}^m}{\partial x_k} + \frac{\partial v_k^m}{\partial x_i} \frac{\partial T_{kj}^m}{\partial x_j} - \rho \frac{\partial v_j^m}{\partial x_i} \left[\frac{\partial}{\partial t} + (\mathbf{v}^m \cdot \nabla) \right] v_j^m \\
& + \left[\frac{\partial}{\partial t} + (\mathbf{v}^m \cdot \nabla) \right] f_i + \frac{\partial v_j^m}{\partial x_i} f_j,
\end{aligned} \tag{13a}$$

$$\operatorname{div} \mathbf{v}^{m+1} = 0, \tag{13b}$$

$$\mathbf{v}^{m+1} = \mathbf{0}, \quad x \in \partial\Omega, \quad t \geq 0. \tag{13c}$$

$$\mathbf{v}^{m+1}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}). \tag{13d}$$

3. Definition of function spaces and statement of results

Since ψ has the meaning of a probability density, L^1 -spaces are natural for the $\mathbf{R}$ -dependence. However, the definition of $\mathbf{T}$ in equation (2) involves integrals of ψ against increasing functions of $\mathbf{R}$, and we shall therefore use weighted L^1 -spaces. We define

$$X_n = \{\psi : \mathbb{R}^3 \rightarrow \mathbb{R} \mid \int (1 + |\mathbf{R}|^n) |\psi(\mathbf{R})| d\mathbf{R} < \infty\}, \quad (14a)$$

in the case of infinitely extensible dumbbells, and

$$X_n = \{\psi : B \rightarrow \mathbb{R} \mid \int (R_0 - |\mathbf{R}|)^{-n} |\psi(\mathbf{R})| d\mathbf{R} < \infty\}, \quad (14b)$$

in the case of finitely extensible dumbbells. Here B denotes the ball of radius R_0 . Moreover, we let $X_{n,k}$ be the space of all ψ whose derivatives up to order k lie in X_n . Finally, let

$$X_k = \bigcap_{n=0}^{\infty} X_{n,k}, \quad (15)$$

with the natural topology of a Fréchet space.

We make the following smoothness assumptions:

- (S1) The domain $\Omega \subset \mathbb{R}^3$ is bounded and $\partial\Omega$ is of class C^5 .
- (S2) $\mathbf{v}_0 \in H^4(\Omega)$.
- (S3) $\psi_0 \in \bigcap_{k=0}^4 H^k(\Omega; X_{8-2k})$, where $H^k(\Omega; X_l)$ stands for $\bigcap_{n=0}^{\infty} H^k(\Omega; X_{n,l})$. Moreover, $\psi_0 \geq 0$ and $\int \psi_0(\mathbf{R}, \mathbf{x}) d\mathbf{R} = 1$ for every $\mathbf{x} \in \Omega$.
- (S4) For some $T > 0$, we have $\mathbf{f} \in \bigcap_{k=0}^4 W^{k,1}([0, T]; H^{4-k}(\Omega))$.

In addition, we need compatibility conditions between the initial data and the incompressibility and boundary conditions. Note that by applying the Hodge projection operator to equation (1), we can obtain an initial value for $\partial \mathbf{v} / \partial t$, and, after differentiating (1) with respect to t , we can obtain initial values of higher time derivatives in an analogous fashion. We shall denote the initial value of $\partial^i \mathbf{v} / \partial t^i$ by $\mathbf{v}_i$. These initial values for time derivatives of $\mathbf{v}$ satisfy the incompressibility condition because of their construction, but we still need to require that they satisfy the boundary conditions. We assume the following compatibility conditions:

- (C1) $\operatorname{div} \mathbf{v}_0 = 0$ and $\mathbf{v}_0 = \mathbf{0}$ on $\partial\Omega$.
- (C2) $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ vanish on $\partial\Omega$.

The goal of the paper is the following result.

Theorem:

Assume that (S1)-(S4), (C1), (C2) and (F) or, respectively, (F') hold. Then there is a $T' \in (0, T]$ such that the problem (1)-(5) has a unique solution with the regularity

$$\mathbf{v} \in \bigcap_{k=0}^4 C^k([0, T']; H^{4-k}(\Omega)); \quad \mathbf{T} \in \bigcap_{k=0}^3 C^k([0, T']; H^{3-k}(\Omega));$$

$$\psi \in \bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} C^k([0, T']; H^l(\Omega; X_{8-2k-2l})). \quad (16)$$

The proof will be based on showing that the mapping $\Sigma : \mathbf{v}^m \rightarrow \mathbf{v}^{m+1}$ defined by (12) and (13) is a contraction in an appropriate complete space of functions. This space of functions, denoted by $Z(M, T')$, is defined as the set of all functions $\mathbf{v} : \Omega \times [0, T'] \rightarrow \mathbb{R}^3$ with the following properties:

$$\mathbf{v} \in \bigcap_{k=0}^4 W^{k,\infty}([0, T']; H^{4-k}(\Omega)), \quad (17a)$$

$$\|\mathbf{v}\|_{0,4} + \|\mathbf{v}\|_{1,3} + \|\mathbf{v}\|_{2,2} + \|\mathbf{v}\|_{3,1} + \|\mathbf{v}\|_{4,0} \leq M, \quad (17b)$$

$$\operatorname{div} \mathbf{v} = 0, \quad (17c)$$

$$\mathbf{v}|_{\partial\Omega} = \mathbf{0}, \quad (17d)$$

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \quad \frac{\partial \mathbf{v}}{\partial t}(\mathbf{x}, 0) = \mathbf{v}_1(\mathbf{x}), \quad \frac{\partial^2 \mathbf{v}}{\partial t^2}(\mathbf{x}, 0) = \mathbf{v}_2(\mathbf{x}), \quad \frac{\partial^3 \mathbf{v}}{\partial t^3}(\mathbf{x}, 0) = \mathbf{v}_3(\mathbf{x}). \quad (17e)$$

Here $\|\cdot\|_{k,l}$ denotes the norm in $W^{k,\infty}([0, T']; H^l(\Omega))$. In $Z(M, T')$, we introduce the metric

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|_{0,3} + \|\mathbf{v} - \mathbf{w}\|_{1,2} + \|\mathbf{v} - \mathbf{w}\|_{2,1} + \|\mathbf{v} - \mathbf{w}\|_{3,0}. \quad (18)$$

It is easy to see that $Z(M, T')$ with this metric is complete. We refer to [10] for a proof that it is not empty (provided M is large enough).

We note that equation (13) is of the same type as the problem (16) considered in [10]. The quantity

$$\int (\gamma(|\mathbf{R}|^2) R_j R_l \delta_{ik} + 2\gamma'(|\mathbf{R}|^2) R_i R_j R_k R_l) \psi^m d\mathbf{R} \quad (19)$$

assumes the role of C_{ijkl} in [10]. We note that the symmetry condition

$$C_{ijkl} = C_{klij} \quad (20)$$

and the strong ellipticity condition

$$C_{ijkl} \zeta_i \zeta_k \eta_j \eta_l \geq \kappa |\zeta|^2 |\eta|^2, \quad \forall \zeta, \eta \in \mathbb{R}^3, \quad (21)$$

with $\kappa > 0$ depending continuously on $\psi \in \{\psi \in L^1 \mid \psi \geq 0, \int \psi d\mathbf{R} = 1\}$, are satisfied, because we have assumed γ to be an increasing function of $|\mathbf{R}|^2$. Hence the results of [10] are applicable to equation (13), and most of the proof in the following Section will be concerned with solving the diffusion equation (12).

4. Proof of the theorem

In order to show that Σ is a contraction in $Z(M, T')$, we shall have to choose M sufficiently large and T' sufficiently small. The estimates used in the proof will involve bounds of the form $K(M, T', a_1, a_2, \dots)$, where the size of a_1, a_2 etc. can be kept within given bounds. It is important that the size of K for large M can be controlled by choosing T' small enough. This leads us to the following definition.

Definition:

A continuous function $K(M, T', a_1, a_2, \dots) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \dots \rightarrow \mathbb{R}^+$ is called controllable, if there are continuous positively valued functions $\tau(M, a_1, a_2, \dots)$ and $\omega(a_1, a_2, \dots)$ such that $K(M, T', a_1, a_2, \dots) \leq \omega(a_1, a_2, \dots)$ as long as $T' \leq \tau(M, a_1, a_2, \dots)$.

We are concerned with the solution of equations of the form (12), i.e. for given $\mathbf{v} \in Z(M, T')$, we have to find ψ satisfying

$$\left(\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\right)\psi = \alpha \Delta_{\mathbf{R}}\psi + \operatorname{div}_{\mathbf{R}}(-(\nabla_{\mathbf{x}}\mathbf{v}) \cdot \mathbf{R}\psi - \beta \mathbf{F}(\mathbf{R})\psi),$$

$$\psi(\mathbf{R}, \mathbf{x}, 0) = \psi_0(\mathbf{R}, \mathbf{x}). \quad (22)$$

The following lemma holds:

Lemma 1:

Given $\mathbf{v} \in Z(M, T')$, there exists a unique solution of (22) which has the regularity

$$\psi \in \bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} C^k([0, T']; H^l(\Omega; X_{8-2k-2l})). \quad (23)$$

Let $\|\cdot\|_{(n)}$ be the norm in

$$\bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} C^k([0, T']; H^l(\Omega; X_{n,8-2k-2l})). \quad (24)$$

For every n , we have an estimate of the form

$$\|\psi\|_{(n)} \leq C_n \cdot K(M, T', n), \quad (25)$$

where C_n depends only on the initial data and K is controllable.

We note that it is in general neither possible nor necessary to obtain a bound for $K(M, T', n)$ which is independent of n . By using (23) and equation (22), we find, moreover, that

$$\frac{\partial \psi}{\partial t} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\psi \in \bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} W^{k,\infty}([0, T']; H^l(\Omega; X_{8-2k-2l})). \quad (26)$$

In addition to (22), let us consider a second equation of the same form;

$$\begin{aligned} \left(\frac{\partial}{\partial t} + (\tilde{\mathbf{v}} \cdot \nabla_{\mathbf{x}})\right)\tilde{\psi} &= \alpha \Delta_{\mathbf{R}} \tilde{\psi} + \operatorname{div}_{\mathbf{R}}((-\nabla_{\mathbf{x}} \tilde{\mathbf{v}}) \cdot \mathbf{R} \tilde{\psi} - \beta \mathbf{F}(\mathbf{R}) \tilde{\psi}), \\ \tilde{\psi}(\mathbf{R}, \mathbf{x}, 0) &= \psi_0(\mathbf{R}, \mathbf{x}). \end{aligned} \quad (27)$$

The following result holds.

Lemma 2:

Let $\mathbf{v}, \tilde{\mathbf{v}} \in Z(M, T')$ be given. Let $\|\cdot\|_{[n]}$ denote the norm in

$$\bigcap_{k=0}^2 \bigcap_{l=0}^{2-k} C^k([0, T']; H^l(\Omega; X_{6-2k-2l})). \quad (28)$$

Then for every n , we have an estimate of the form

$$\|\psi - \tilde{\psi}\|_{[n]} \leq K(M, T', n) d(\mathbf{v}, \tilde{\mathbf{v}}). \quad (29)$$

Here $d(\mathbf{v}, \tilde{\mathbf{v}})$ is as defined in (18). The function $K(M, T', n)$ tends to zero as $T' \rightarrow 0$.

Using the equations, we find, as a consequence, a similar estimate for

$$\left(\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla_{\mathbf{x}})\right)\psi - \left(\frac{\partial}{\partial t} + (\tilde{\mathbf{v}} \cdot \nabla_{\mathbf{x}})\right)\tilde{\psi} \quad (30)$$

in the space

$$\bigcap_{k=0}^2 \bigcap_{l=0}^{2-k} C^k([0, T']; H^l(\Omega; X_{4-2k-2l})). \quad (31)$$

We can apply Lemmas 3 and 4 of [10] to the solution of equation (13). By combining the resulting estimates with Lemmas 1 and 2 above, it follows easily that Σ is a contraction in $Z(M, T')$ if M is chosen sufficiently large and T' is chosen sufficiently small. The theorem follows immediately. The rest of the paper will therefore be concerned with the proofs of Lemmas 1 and 2.

We introduce Lagrangian coordinates, which we shall denote by $\mathbf{X}$. Let $\mathbf{x}(\mathbf{X}, t)$ denote the Eulerian coordinate corresponding to $\mathbf{X}$, i.e. the solution of the problem

$$\frac{\partial}{\partial t} \mathbf{x}(\mathbf{X}, t) = \mathbf{v}(\mathbf{x}(\mathbf{X}, t), t), \quad \mathbf{x}(\mathbf{X}, 0) = \mathbf{X}. \quad (32)$$

Let $\phi(\mathbf{R}, \mathbf{X}, t) = \psi(\mathbf{R}, \mathbf{x}(\mathbf{X}, t), t)$. Equation (22) can then be written in the alternative form

$$\begin{aligned} \frac{\partial}{\partial t} \phi(\mathbf{R}, \mathbf{X}, t) &= \alpha \Delta_{\mathbf{R}} \phi + \operatorname{div}_{\mathbf{R}}(-(\nabla_{\mathbf{x}} \mathbf{v}(\mathbf{x}(\mathbf{X}, t), t)) \cdot \mathbf{R} \phi - \beta \mathbf{F}(\mathbf{R}) \phi), \\ \phi(\mathbf{R}, \mathbf{X}, 0) &= \psi_0(\mathbf{R}, \mathbf{X}). \end{aligned} \quad (33)$$

This is a parabolic equation for ϕ , in which the only derivatives are with respect to $\mathbf{R}$ and t , while $\mathbf{X}$ only appears as a parameter in the coefficients and initial conditions. We note that the transformation $(\mathbf{X}, t) \rightarrow (\mathbf{x}(\mathbf{X}, t), t)$ is smooth enough so that if ψ has the regularity claimed by Lemma 1, then so does ϕ and vice versa.

The fact that solutions of (33) with the regularity claimed by Lemma 1 are unique follows from a straightforward application of the maximum principle. It also follows from the maximum principle that positivity is preserved, and by integrating both sides of (33) we find that $\int \phi(\mathbf{R}, \mathbf{X}, t) d\mathbf{R} = \int \psi(\mathbf{R}, \mathbf{x}, t) d\mathbf{R} = 1$ for all t . To show the existence of solutions, we cannot use standard results available in the literature because the coefficients of the equation are unbounded. To get around this difficulty, we use a sequence of approximating problems with bounded coefficients, for which we derive uniform estimates. The cases of infinitely, and respectively, finitely extensible dumbbells have to be treated somewhat differently, and we shall first deal with infinitely extensible dumbbells.

Let $\chi(\mathbf{R})$ be a C^∞ -function such that $\chi(0) = 1$, χ is a monotone decreasing function of $|\mathbf{R}|$, and $\chi(\mathbf{R}) = |\mathbf{R}|^{-\nu}$ for large $|\mathbf{R}|$, where ν is a sufficiently large number. For $N \in \mathbb{N}$, let $\chi_N(\mathbf{R}) = \chi(\mathbf{R}/N)$. We now replace (33) by the approximate problem

$$\begin{aligned} \frac{\partial \phi_N}{\partial t} &= \alpha \Delta_{\mathbf{R}} \phi_N + \operatorname{div}_{\mathbf{R}} \left[\chi_N(\mathbf{R}) (-(\nabla_{\mathbf{x}} \mathbf{v}(\mathbf{x}(\mathbf{X}, t), t)) \cdot \mathbf{R} \phi_N - \beta \mathbf{F}(\mathbf{R}) \phi_N) \right], \\ \phi_N(\mathbf{R}, \mathbf{X}, 0) &= \psi_0(\mathbf{R}, \mathbf{X}). \end{aligned} \quad (34)$$

We note that equation (34) has bounded coefficients. Hence the existence of solutions and their decay as $|\mathbf{R}| \rightarrow \infty$ are easily established. We now multiply (34) by $|\mathbf{R}|^{2n}$ and integrate. This yields after an integration by parts

$$\begin{aligned} \frac{\partial}{\partial t} \int |\mathbf{R}|^{2n} \phi_N(\mathbf{R}, \mathbf{X}, t) d\mathbf{R} &= \alpha(4n^2 + 2n) \int |\mathbf{R}|^{2n-2} \phi_N(\mathbf{R}, \mathbf{X}, t) d\mathbf{R} \\ &+ 2n \int |\mathbf{R}|^{2n-2} \chi_N(\mathbf{R}) \left[\mathbf{R} \cdot (\nabla_{\mathbf{x}} \mathbf{v}(\mathbf{x}(\mathbf{X}, t), t)) \cdot \mathbf{R} + \beta \mathbf{R} \cdot \mathbf{F}(\mathbf{R}) \right] \phi_N(\mathbf{R}, \mathbf{X}, t) d\mathbf{R}. \end{aligned} \quad (35)$$

The term $\mathbf{R} \cdot \mathbf{F}(\mathbf{R})$ is negative, and all other terms on the right hand side of (35) can be estimated by a constant (depending on n but not on N) times $\int (1 + |\mathbf{R}|^{2n}) \phi_N(\mathbf{R}, \mathbf{X}, t) d\mathbf{R}$. Hence (35) yields a bound of the form

$$\int |\mathbf{R}|^{2n} \phi_N(\mathbf{R}, \mathbf{X}, t) \leq K(n, Mt) \int (1 + |\mathbf{R}|^{2n}) \psi_0(\mathbf{R}, \mathbf{X}) d\mathbf{R}. \quad (36)$$

We next derive estimates for derivatives with respect to $\mathbf{R}$. We differentiate (34) with respect to R_i and obtain

$$\begin{aligned} \frac{\partial^2 \phi_N}{\partial R_i \partial t} &= \alpha \Delta_{\mathbf{R}} \frac{\partial \phi_N}{\partial R_i} + \operatorname{div}_{\mathbf{R}} \left[\chi_N(\mathbf{R}) (-(\nabla_{\mathbf{x}} \mathbf{v}(\mathbf{x}(\mathbf{X}, t), t)) \cdot \mathbf{R} - \beta \mathbf{F}(\mathbf{R})) \frac{\partial \phi_N}{\partial R_i} \right] \\ &+ \operatorname{div}_{\mathbf{R}} \left[\frac{\partial}{\partial R_i} \left(\chi_N(\mathbf{R}) (-(\nabla_{\mathbf{x}} \mathbf{v}(\mathbf{x}(\mathbf{X}, t), t)) \cdot \mathbf{R} - \beta \mathbf{F}(\mathbf{R})) \right) \phi_N \right], \end{aligned}$$

$$\frac{\partial \phi_N}{\partial R_i}(\mathbf{R}, \mathbf{X}, 0) = \frac{\partial \psi_0}{\partial R_i}(\mathbf{R}, \mathbf{X}). \quad (37)$$

The last term on the right hand side of this equation can be further transformed as follows:

$$\begin{aligned} & \operatorname{div}_{\mathbf{R}} \left[\frac{\partial}{\partial R_i} \left(\chi_N(\mathbf{R}) (-(\nabla_{\mathbf{x}} \mathbf{v}(\mathbf{x}(\mathbf{X}, t), t)) \cdot \mathbf{R} - \beta \mathbf{F}(\mathbf{R})) \right) \phi_N \right] \\ &= \phi_N \operatorname{div}_{\mathbf{R}} \left[\frac{\partial}{\partial R_i} \left(\chi_N(\mathbf{R}) (-(\nabla_{\mathbf{x}} \mathbf{v}) \cdot \mathbf{R} - \beta \mathbf{F}(\mathbf{R})) \right) \right] \\ &+ \frac{\partial \phi_N}{\partial R_j} \left[\frac{\partial \chi_N}{\partial R_i} \left(-\frac{\partial v_j}{\partial x_k} R_k - \beta F_j(\mathbf{R}) \right) + \chi_N(\mathbf{R}) \left(-\frac{\partial v_j}{\partial x_i} - \beta \frac{\partial F_j}{\partial R_i} \right) \right]. \end{aligned} \quad (38)$$

Since $\partial \phi_N / \partial R_i$ is not positive, we cannot proceed in exact analogy to equation (35) above. Instead, we first decompose $\partial \phi_N / \partial R_i = \pi_i^+ - \pi_i^-$, where π_i^+ and π_i^- are solutions to the problems,

$$\begin{aligned} \frac{\partial \pi_i^+}{\partial t} &= \alpha \Delta_{\mathbf{R}} \pi_i^+ + \operatorname{div}_{\mathbf{R}} \left[\chi_N(\mathbf{R}) (-(\nabla_{\mathbf{x}} \mathbf{v}) \cdot \mathbf{R} \pi_i^+ - \beta \mathbf{F}(\mathbf{R}) \pi_i^+) \right] + s_i^+, \\ \pi_i^+(\mathbf{R}, \mathbf{X}, 0) &= \max\left(\frac{\partial \psi_0}{\partial R_i}(\mathbf{R}, \mathbf{X}), 0\right), \end{aligned} \quad (39)$$

and, respectively,

$$\begin{aligned} \frac{\partial \pi_i^-}{\partial t} &= \alpha \Delta_{\mathbf{R}} \pi_i^- + \operatorname{div}_{\mathbf{R}} \left[\chi_N(\mathbf{R}) (-(\nabla_{\mathbf{x}} \mathbf{v}) \cdot \mathbf{R} \pi_i^- - \beta \mathbf{F}(\mathbf{R}) \pi_i^-) \right] - s_i^-, \\ \pi_i^-(\mathbf{R}, \mathbf{X}, 0) &= \max\left(-\frac{\partial \psi_0}{\partial R_i}(\mathbf{R}, \mathbf{X}), 0\right). \end{aligned} \quad (40)$$

Here s_i^+ and s_i^- denote the positive and negative parts of the expression in (38). Since π_i^+ and π_i^- are positive, we can now proceed as above. In place of (35), we obtain

$$\begin{aligned} & \frac{\partial}{\partial t} \int |\mathbf{R}|^{2n} \pi_i^+(\mathbf{R}, \mathbf{X}, t) d\mathbf{R} = \alpha(4n^2 + 2n) \int |\mathbf{R}|^{2n-2} \pi_i^+(\mathbf{R}, \mathbf{X}, t) d\mathbf{R} \\ & + 2n \int |\mathbf{R}|^{2n-2} \chi_N(\mathbf{R}) \left[\mathbf{R} \cdot (\nabla_{\mathbf{x}} \mathbf{v}(\mathbf{x}(\mathbf{X}, t), t)) \cdot \mathbf{R} + \beta \mathbf{R} \cdot \mathbf{F}(\mathbf{R}) \right] \pi_i^+(\mathbf{R}, \mathbf{X}, t) d\mathbf{R} + \int |\mathbf{R}|^{2n} s_i^+ d\mathbf{R}, \end{aligned} \quad (41)$$

and an analogous equation for π_i^- . (A minor problem in deriving (41) arises from the fact that $\Delta \pi_i^+$ does not necessarily exist at $t = 0$, but we may justify the argument by approximating the initial value of π_i^+ by a sequence of smoother functions and then passing to the limit.) Using (38), we find that

$$\int |\mathbf{R}|^{2n} s_i^+ d\mathbf{R} \leq \int |\mathbf{R}|^{2n} \phi_N \left| \operatorname{div}_{\mathbf{R}} \left[\frac{\partial}{\partial R_i} \left(\chi_N(\mathbf{R}) (-(\nabla_{\mathbf{x}} \mathbf{v}) \cdot \mathbf{R} - \beta \mathbf{F}(\mathbf{R})) \right) \right] \right| d\mathbf{R}$$

$$+ \int |\mathbf{R}|^{2n} (\pi_j^+ + \pi_j^-) \left[\left| \frac{\partial \chi_N}{\partial R_i} \right| \left(\left| \frac{\partial v_j}{\partial x_k} R_k \right| + \beta |F_j(\mathbf{R})| \right) + \chi_N(\mathbf{R}) \left(\left| \frac{\partial v_j}{\partial x_i} \right| + \beta \left| \frac{\partial F_j}{\partial R_i} \right| \right) \right] d\mathbf{R}. \quad (42)$$

The first integral on the right hand side of (42) involves only ϕ_N and no derivatives of ϕ_N ; it can hence be estimated using (36) above. The integrand in the second integral can be estimated for large $|\mathbf{R}|$ by a constant times $\chi_N(\mathbf{R})(M|\mathbf{R}|^{2n} + |\mathbf{R}|^{2n-1}|\mathbf{F}(\mathbf{R})|) \sum_j (\pi_j^+ + \pi_j^-)$. From (41), we therefore obtain an estimate of the form

$$\begin{aligned} \frac{\partial}{\partial t} \int |\mathbf{R}|^{2n} \pi_i^+(\mathbf{R}, \mathbf{X}, t) d\mathbf{R} &\leq K_1(n, M) \int (1 + |\mathbf{R}|^{2n}) \pi_i^+(\mathbf{R}, \mathbf{X}, t) d\mathbf{R} + K_2(n, M) \\ &\quad + 2n \int |\mathbf{R}|^{2n-2} \chi_N(\mathbf{R}) \beta \mathbf{R} \cdot \mathbf{F}(\mathbf{R}) \pi_i^+(\mathbf{R}, \mathbf{X}, t) d\mathbf{R} \\ &\quad + K_3 \int (1 + |\mathbf{R}|^{2n-1} \chi_N(\mathbf{R}) |\mathbf{F}(\mathbf{R})| + M \chi_N(\mathbf{R}) |\mathbf{R}|^{2n}) \sum_j (\pi_j^+ + \pi_j^-)(\mathbf{R}, \mathbf{X}, t) d\mathbf{R}, \end{aligned} \quad (43)$$

Note that K_3 does not depend on n , but that the previous term in (43), which is negative, has a factor n in front. By summing the inequalities corresponding to (43) over all values of i and all choices of $+$ or $-$, we can therefore get bounds for $\int |\mathbf{R}|^{2n} \pi_i^+ d\mathbf{R}$. As a consequence, we have found estimates for the first derivatives of ϕ_N . Bounds for higher derivatives are obtained in an analogous fashion after differentiating (34) with respect to the components of $\mathbf{R}$. The calculation involves higher derivatives of ϕ_N than are known to exist at $t = 0$, but the argument can be justified by approximating the initial datum by smoother functions and passing to the limit. Proceeding in this fashion, we find that

$$\phi_N \in L^\infty([0, T']; L^2(\Omega; X_8)). \quad (44)$$

Derivatives of ϕ_N with respect to $\mathbf{X}$ can be estimated by taking difference quotients in (34), estimating the difference quotients of ϕ_N and then taking limits. Problems near the boundary of Ω are avoided by first extending $\mathbf{v}$ and ψ_0 so that they are defined for $\mathbf{X}$ outside Ω . This can be done in such a fashion that the extended functions have the same level of regularity as the original ones. Mixed derivatives with respect to $\mathbf{R}$ and $\mathbf{X}$ can be estimated by taking difference quotients with respect to $\mathbf{X}$ after first differentiating with respect to $\mathbf{R}$. The calculations are tedious but straightforward and we omit them. We find in this way that

$$\phi_N \in \bigcap_{l=0}^3 L^\infty([0, T']; H^l(\Omega; X_{8-2l})). \quad (45)$$

Finally, temporal derivatives of ϕ_N can be estimated by using the equation itself. This yields

$$\phi_N \in \bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} W^{k,\infty}([0, T']; H^l(\Omega; X_{8-2k-2l})). \quad (46)$$

All these estimates are uniform in N .

Using the uniform bounds on the ϕ_N , we can extract a subsequence which converges, in the sense of distributions, to a solution ϕ of (33). Unfortunately, it is not a priori clear that

$$\phi \in \bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} W^{k,\infty}([0, T']; H^l(\Omega; X_{8-2k-2l})). \quad (47)$$

To show this, we need a separate argument. We approximate the initial datum ψ_0 by a sequence ψ_0^M such that ψ_0^M has the additional regularity

$$\psi_0^M \in \bigcap_{k=0}^4 H^k(\Omega; X_{10-2k}), \quad (48)$$

and $\psi_0^M \rightarrow \psi_0$ in the topology of

$$\bigcap_{k=0}^4 H^k(\Omega; X_{8-2k}). \quad (49)$$

Let ϕ_N^M be the solution of (34) which is obtained when the initial datum is replaced by ψ_0^M . Then the same argument as above yields

$$\phi_N^M \in \bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} W^{k,\infty}([0, T']; H^l(\Omega; X_{10-2k-2l})); \quad (50)$$

moreover, in the topology of

$$\bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} W^{k,\infty}([0, T']; H^l(\Omega; X_{8-2k-2l})), \quad (51)$$

we have $\phi_N^M \rightarrow \phi_N$ uniformly in N . From (50) and (34), we conclude that

$$\frac{\partial \phi_N^M}{\partial t} \in \bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} W^{k,\infty}([0, T']; H^l(\Omega; X_{8-2k-2l})). \quad (52)$$

From (50) and (52) it follows that

$$\phi_N^M \in \bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} C^k([0, T']; H^l(\Omega; X_{8-2k-2l})). \quad (53)$$

For fixed M , the bounds for ϕ_N^M are again uniform in N . Let now Π_n be a reflexive Banach space such that

$$\left[\bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} W^{k,\infty}([0, T']; H^l(\Omega; X_{n,10-2k-2l})) \right] \cap \left[\bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} W^{k+1,\infty}([0, T']; H^l(\Omega; X_{n,8-2k-2l})) \right]$$

$$\subset \Pi_n \subset \bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} C^k([0, T']; H^l(\Omega; X_{n-1, 8-2k-2l})). \quad (54)$$

Such a space Π_n can be constructed using appropriate Sobolev norms. Let $\Pi = \bigcap_{n \in \mathbb{N}} \Pi_n$. Then, for every M , we have a uniform bound for ϕ_N^M in the topology of Π , hence we can extract a subsequence which converges distributionally to an element ϕ^M of Π . Since, for $M \rightarrow \infty$, ϕ_N^M converges to ϕ_N uniformly in N , we find that the sequence ϕ^M also converges in

$$\bigcap_{k=0}^3 \bigcap_{l=0}^{3-k} C^k([0, T']; H^l(\Omega; X_{8-2k-2l})). \quad (55)$$

By taking $\phi = \lim_{M \rightarrow \infty} \phi^M$, we obtain a solution with the desired regularity. This concludes the proof of Lemma 1 for the case of the infinitely extensible dumbbell.

The proof of Lemma 2 is based on the same type of estimates, applied to the function $\psi - \tilde{\psi}$. We omit the details.

For the case of finitely extensible dumbbells, we use approximation by infinitely extensible dumbbells. We extend the initial datum ψ_0 by zero for $|\mathbf{R}| > R_0$, and we define an approximation γ_ϵ to the spring constant γ as follows. Let χ be a C^∞ -function $[0, 1] \rightarrow [0, 1]$ such that $\chi(s) = 1$ in a neighborhood of 0 and $\chi(s) = 0$ in a neighborhood of 1. Let $\chi_\epsilon(\mathbf{R}) = \chi(\frac{2}{\epsilon}(|\mathbf{R}| - R_0 + \epsilon))$. Then we define

$$\gamma_\epsilon(|\mathbf{R}|^2) = \begin{cases} \gamma(|\mathbf{R}|^2), & \text{if } |\mathbf{R}| \leq R_0 - \epsilon, \\ \gamma((R_0 - \frac{\epsilon}{2})^2), & \text{if } |\mathbf{R}| \geq R_0 - \frac{\epsilon}{2}, \\ \chi_\epsilon(\mathbf{R})\gamma(|\mathbf{R}|^2) + (1 - \chi_\epsilon(\mathbf{R}))\gamma((R_0 - \frac{\epsilon}{2})^2), & \text{if } R_0 - \epsilon < |\mathbf{R}| < R_0 - \frac{\epsilon}{2}. \end{cases} \quad (56)$$

We now consider the problem with spring constant γ_ϵ , which is infinitely extensible and satisfies the assumptions which we required for the infinitely extensible case. We now define

$$w_\epsilon(\mathbf{R}) = \begin{cases} \frac{1}{R_0^2 - |\mathbf{R}|^2}, & \text{if } |\mathbf{R}| \leq R_0 - \epsilon, \\ \frac{1}{\epsilon(2R_0 - \epsilon)} + \frac{2(R_0 - \epsilon)}{\epsilon^2(2R_0 - \epsilon)^2}(|\mathbf{R}| - R_0 + \epsilon), & \text{if } |\mathbf{R}| > R_0 - \epsilon. \end{cases} \quad (57)$$

The function w_ϵ and its gradient are continuous at $|\mathbf{R}| = R_0$, and w_ϵ has linear growth at infinity. We now repeat the estimates above, but instead of multiplying the equation by powers of $|\mathbf{R}|$, we multiply by powers of $w_\epsilon(|\mathbf{R}|)$. The resulting estimates turn out to be uniform in ϵ as $\epsilon \rightarrow 0$, and Lemma 1 follows by passing to the limit.

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