**A GLOBAL ASYMPTOTIC EXPANSION
FOR THE SOLUTION TO THE GENERALIZED
RIEMANN PROBLEM**

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A GLOBAL ASYMPTOTIC EXPANSION FOR THE SOLUTION TO THE GENERALIZED RIEMANN PROBLEM*

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Abstract. We study the generalized Riemann problem for quasilinear hyperbolic systems of conservation laws. Assuming that this problem has an entropy weak solution globally defined on $t \geq 0$, we construct an asymptotic expansion of this solution, which is also globally defined in time. For technical reasons, we treat the case where the solution is only composed of shock waves or contact discontinuities. We expect that this expansion gives an approximation to the exact solution, which remains valid uniformly in time. Moreover, some improvements of our general results are obtained for the isentropic gas dynamics system.

1. Introduction. Let us consider entropy weak solutions to quasilinear hyperbolic systems of conservation laws (Lax [4] or Smoller [12])

$$(1.1) \quad \partial_t u + \partial_x f(u) = g(u), \quad u(x, t) \in \mathbb{R}^p, \quad x \in \mathbb{R}, \quad t > 0,$$

where the flux-function $f : \mathbb{R}^p \rightarrow \mathbb{R}^p$ and the source-term $g : \mathbb{R}^p \rightarrow \mathbb{R}^p$ are smooth. Assume that system (1.1) is strictly hyperbolic with genuinely nonlinear or linearly degenerate characteristic fields. That is, for each u in $\mathbb{R}^p$, the Jacobian matrice

$$A(u) = Df(u)$$

admits p distinct and real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_p(u),$$

a basis of corresponding right eigenvectors

$$r_1(u), r_2(u), \dots, r_p(u),$$

and a basis of corresponding left eigenvectors

$$l_1(u)^T, l_2(u)^T, \dots, l_p(u)^T.$$

As usual ([4], [12]), for $i = 1, 2, \dots, p$, we choose the following normalization

$$(1.2) \quad l_m(u)^T r_i(u) = \delta_{mi}, \quad \forall u \in \mathbb{R}^p, \quad \forall m = 1, 2, \dots, p,$$

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and, for a genuinely nonlinear characteristic i -field,

$$(1.3a) \quad \nabla \lambda_i(u) r_i(u) \equiv 1, \quad \forall u \in \mathbb{R}^p,$$

or, for a linearly degenerate i -field,

$$(1.3b) \quad \nabla \lambda_i(u) r_i(u) \equiv 0, \quad \forall u \in \mathbb{R}^p.$$

We are interested in the *generalized Riemann problem* for (1.1), which is a Cauchy problem with a piecewise smooth initial data of the form

$$(1.4) \quad u(x, 0) = \begin{cases} u_L(x), & x \leq 0, \\ u_R(x), & x \geq 0. \end{cases}$$

In (1.4), $u_L : \mathbb{R}_- \rightarrow \mathbb{R}^p$ and $u_R : \mathbb{R}_+ \rightarrow \mathbb{R}^p$ are given smooth functions. This problem has been investigated in great details by many authors: Li Tatsien - Yu Wenci [7], [8], Harabetian [3], Le Floch - Raviart [6], Bourgeade - Le Floch - Raviart [1]. Roughly speaking, one proves that in a neighborhood of the origin the entropy weak solution of (1.1), (1.4) has the same structure as the one of the associated *classical Riemann problem*:

$$(1.5) \quad \partial_t u^0 + \partial_x f(u^0) = 0, \quad u^0(x, t) \in \mathbb{R}^p,$$

$$(1.6) \quad u^0(x, 0) = \begin{cases} u_L(0) \stackrel{\text{def}}{=} u_L^0, & x \leq 0 \\ u_R(0) \stackrel{\text{def}}{=} u_R^0, & x \geq 0. \end{cases}$$

The classical Riemann problem (1.5), (1.6) was solved by P.D. Lax in [4]. For $|u_R^0 - u_L^0|$ small enough, there exists a weak solution which is self-similar, i.e. a function of the variable $\xi = \frac{x}{t}$, and consists of at most $(p + 1)$ constant states denoted by

$$(1.7) \quad u_L^0 \equiv u_0^0, u_1^0, \dots, u_{p-1}^0, u_p^0 \equiv u_R^0,$$

separated by rarefaction waves, shock waves or contact discontinuities ([4] and Li - Yu [7]).

We observe that the results of [1,3,6,7,8] only concern local solutions to the generalized Riemann problem in a neighborhood of the origin $(x, t) = (0, 0)$. On the other hand, let us briefly recall one of the global existence results given by Li Tatsien and Zhao Yanchun [9,10]. Assuming that the classical Riemann problem (1.5), (1.6) contains exactly p shock waves and the initial data u_L and u_R are sufficiently closed to u_L^0 and u_R^0 respectively with some decay properties at infinity, they prove that the generalized Riemann problem (1.1), (1.4) (with $g \equiv 0$) admits an entropy weak solution $u = u(x, t)$ defined for *all* $x \in \mathbb{R}$ and *all* $t \geq 0$ with the same global structure as the one of the solution u^0 to the classical Riemann problem (1.1), (1.6).

In this paper, motivated by the previous work of Li Tatsien and Zhao YanChun, we consider the generalized Riemann problem under the assumption that it admits an entropy weak solution u globally defined in time with the same global structure as the solution u^0 to the classical Riemann problem (1.5), (1.6). Our purpose is to derive an asymptotic expansion of the solution u for small oscillations $|u_L(\cdot) - u_L^0|, |u_R(\cdot) - u_R^0| \ll 1$. of course, we expect to construct an expansion defined globally in time, which remains valid (in some sense to specify) uniformly in time.

Let us introduce a small parameter ε and consider the following *generalized Riemann problem*:

$$(1.8) \quad \partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon g(u^\varepsilon), \quad u^\varepsilon(x, t) \in \mathbb{R}^p,$$

$$(1.9) \quad u^\varepsilon(x, 0) = \begin{cases} u_L^0 + \sum_{k=1}^{\infty} \varepsilon^k u_L^k(x), & x \leq 0, \\ u_R^0 + \sum_{k=1}^{\infty} \varepsilon^k u_R^k(x), & x \geq 0. \end{cases}$$

In (1.9), u_L^k and u_R^k are given smooth functions. For technical reasons, we assume that:

HYPOTHESIS 1. *The classical Riemann problem (1.5), (1.6) contains only shock waves or contact discontinuities but no centered rarefaction waves or sonic waves.*

For $i = 1, 2, \dots, p$, the speed of the i -wave will be denoted by σ_i^0 (see Figure 1.1). The constant states separated by these waves are denoted by $u_0^0, u_1^0, \dots, u_{p-1}^0, u_p^0$ (see (1.7)).

HYPOTHESIS 2. *For any fixed ε small enough, the generalized Riemann problem (1.8), (1.9) admits an entropy weak solution $u^\varepsilon(x, t)$ defined for all $x \in \mathbb{R}$ and all $t \geq 0$, which has the similar structure to the solution u^0 to the classical Riemann problem (1.5), (1.6), i.e. this solution contains only shocks or contact discontinuities of the same family as in hypothesis 1.*

Precisely speaking, we assume the existence of p smooth curves $t \geq 0 \mapsto \varphi_i^\varepsilon(t) \in \mathbb{R}$ ($i = 1, 2, \dots, p$):

$$(1.10) \quad \varphi_1^\varepsilon(t) < \varphi_2^\varepsilon(t) < \dots < \varphi_p^\varepsilon(t), \quad \forall t > 0,$$

such that

$$(1.11) \quad \varphi_i^\varepsilon(0) = 0, \quad \lim_{t \rightarrow 0} \frac{d}{dt} \varphi_i^\varepsilon(t) = \sigma_i^0 \quad (i = 1, 2, \dots, p).$$

Moreover, in each one of the domains

$$\begin{cases} D_0^\varepsilon = \{t \geq 0, x \leq \varphi_1^\varepsilon(t)\}, \\ D_i^\varepsilon = \{t \geq 0, \varphi_i^\varepsilon(t) \leq x \leq \varphi_{i+1}^\varepsilon(t)\}, & 1 \leq i \leq p-1, \\ D_p^\varepsilon = \{t \geq 0, x > \varphi_p^\varepsilon(t)\}, \end{cases}$$

the function u^ε is smooth and

$$(1.12) \quad \lim_{\substack{t \rightarrow 0 \\ (x,t) \in D_i^\varepsilon}} u^\varepsilon(x,t) = u_i^0$$

(see Figure 1.2).

Then, in each regular domain D_i^ε ($0 \leq i \leq p$), we formally write the following asymptotic expansion for u^ε :

$$(1.13) \quad u^\varepsilon(x,t) = u_i^0 + \sum_{k=1}^{\infty} \varepsilon^k u_i^k(x,t), \quad (x,t) \in D_i^\varepsilon, \quad 0 \leq i \leq p,$$

while, for each curve of discontinuity, we set

$$(1.14) \quad \varphi_i^\varepsilon(t) = \sigma_i^0 t + \sum_{k=1}^{\infty} \varepsilon^k \varphi_i^k(t), \quad 1 \leq i \leq p, \quad t \geq 0.$$

In (1.13), (1.14), u_i^0 , $0 \leq i \leq p$, and σ_i^0 , $1 \leq i \leq p$, are given by the solution of the classical Riemann problem (1.5), (1.6) (see also (1.7)).

In Sections 2 and 3, we shall prove the global existence and uniqueness of the functions $u_i^k : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}^p$, $0 \leq i \leq p$, and $\varphi_i^k : \mathbb{R}_+ \rightarrow \mathbb{R}$, ($1 \leq i \leq p$), $k \geq 1$, provided that the initial amplitude in (1.6) is small enough, i.e.

$$(1.15) \quad |u_R^0 - u_L^0| \ll 1.$$

Notice that assumption (1.15) is the usual assumption to solve the classical Riemann problem. Then, Section 4 is devoted to the isentropic gas dynamics system, for which we shall prove the existence and uniqueness of the asymptotic expansion without condition (1.15).

To end this introduction, let us recall that DiPerna - Majda [2] have considered a similar asymptotic expansion in the case that u^0 is constant, i.e. $u_L^0 = u_R^0$ in (1.6), and they shown that this expansion is valid uniformly in time.

2. Construction of the asymptotic expansion. This section is devoted to the formal derivation of the equations satisfied by the functions u_i^k ($0 \leq i \leq p$) and φ_i^k ($1 \leq i \leq p$) for $k \geq 1$ in the expansions (1.13), (1.14).

Consider an index i in $\{0, 1, \dots, p\}$ and a positive integer k in what follows. In the domain of smoothness of u^0

$$(2.1) \quad D_i^0 = \{(x,t) / \sigma_i^0 t < x < \sigma_{i+1}^0 t\}$$

where, as a convention, we set $\sigma_0^0 = -\infty$, $\sigma_{p+1}^0 = +\infty$, we first determine the structure of the function u_i^k (see Figure 2).

PROPOSITION 2.1. In the domain D_i^0 , the function u_i^k is a solution to the following linear hyperbolic system with constant coefficients:

$$(2.2) \quad \partial_t u_i^k + A(u_i^0) \partial_x u_i^k = h_i^k,$$

where $h_i^k = h_i^k(x, t)$ is a regular function depending only on $u_i^0, u_i^1, \dots, u_i^{k-1}$. Hence, the function u_i^k may be written in the form

$$(2.3) \quad u_i^k(x, t) = \sum_{\partial=1}^P \alpha_{j,i}^k(x - t\lambda_j(u_i^0)) r_j(u_i^0) + t v_i^k(x, t), \text{ in } D_i^0,$$

where v_i^k is a regular function depending only on $u_i^0, u_i^1, \dots, u_i^{k-1}$, and $\alpha_{j,i}^k$ are functions from $\mathbb{R}$ to $\mathbb{R}$ to be determined later on.

In particular, the functions h_i^k and v_i^k for $k = 1$ are explicitly given from u_i^0 by

$$(2.4) \quad h_i^1 = v_i^1 = g(u_i^0).$$

Proof. We plug expansions (1.13) into system (1.8). In domain $D_i^0 (1 \leq i \leq p)$, we have

$$\begin{aligned} \partial_t u^\varepsilon &= \partial_t u_i^0 + \sum_{k=1}^{\infty} \varepsilon^k \partial_t u_i^k, \\ \partial_x f(u^\varepsilon) &= \partial_x f(u_i^0 + \sum_{k \geq 1} \varepsilon^k u_i^k) \\ &= \partial_x f(u_i^0) + \sum_{k \geq 1} \varepsilon^k (A(u_i^0) \partial_x u_i^k + a_i^k) \end{aligned}$$

and

$$\varepsilon g(u^\varepsilon) = \varepsilon g(u_i^0) + \varepsilon \sum_{k \geq 1} \varepsilon^k (Dg(u_i^0) u_i^k + b_i^k),$$

where, for each $k \geq 1$, the functions a_i^k and b_i^k depend only on $u_i^0, u_i^1, \dots, u_i^{k-1}$. Thus, noting that u_i^0 is constant in D_i^0 , we find

$$\sum_{k \geq 1} \varepsilon^k \{ \partial_t u_i^k + A(u_i^0) \partial_x u_i^k + a_i^k \} = \varepsilon g(u_i^0) + \sum_{k \geq 2} \varepsilon^k \{ Dg(u_i^0) u_i^{k-1} + b_i^{k-1} \},$$

so that each function u_i^k is the solution to

$$\partial_t u_i^k + A(u_i^0) \partial_x u_i^k + a_i^k = \begin{cases} g(u_i^0), & \text{for } k = 1, \\ Dg(u_i^0) u_i^{k-1} + b_i^{k-1}, & \text{for } k \geq 2, \end{cases}$$

which proves (2.2).

The expression (2.3) follows immediately from the diagonalization of system (2.2) on the basis $\{r_j(u_i^0)\}_{1 \leq j \leq p}$ and the following remark: the general solution to a scalar equation

$$\partial_t \alpha + c \partial_x \alpha = b(x, t) \quad (c \text{ constant})$$

is of the form

$$\alpha(x, t) = \beta(x - ct)_+ \int_0^t b(x - c(t - \tau), \tau) d\tau.$$

□

We next determine the jump conditions satisfied by the function u^k at the discontinuity lines of u^0 : $x = \sigma_i^0 t$ ($i = 1, \dots, p$).

PROPOSITION 2.2. *For each i in $\{1, \dots, p\}$ and each integer $k \geq 1$, the function u_{i-1}^k, u_i^k and φ_i^k satisfy the following relation on $x = \sigma_i^0 t$:*

$$(2.5) \quad [(\sigma_i^0 - A(u^0)u^k] + \frac{d}{dt} \varphi_i^k[u^0] = w_i^k,$$

where the jump $[a]$ of a function $a(x, t)$ on $x = \sigma_i^0 t$ is defined by

$$[a](t) = a(\sigma_i^0 t +, t) - a(\sigma_i^0 t -, t), \quad t \geq 0,$$

and $w_i^k = w_i^k(t)$ depends only on $u_i^0, u_i^1, \dots, u_i^{k-1}$ and $u_{i-1}^0, u_{i-1}^1, \dots, u_{i-1}^{k-1}$.

In particular, for $k = 1$, we have

$$(2.6) \quad w_i^1 = 0.$$

Proof. The solution u^ε to (1.8) satisfies the Rankine-Hugoniot relations ([12])

$$(2.7) \quad \frac{d}{dt} \varphi_i^\varepsilon(t) \{u^\varepsilon(\varphi_i^\varepsilon(t) +, t) - u^\varepsilon(\varphi_i^\varepsilon(t) -, t)\} = f(u^\varepsilon(\varphi_i^\varepsilon(t) +, t)) - f(u^\varepsilon(\varphi_i^\varepsilon(t) -, t)).$$

We want to plug the expansion (1.13) and (1.14) into (2.7). We have

$$(2.8) \quad \frac{d}{dt} \varphi_i^\varepsilon(t) = \sigma_i^0 + \sum_{k \geq 1} \varepsilon^k \frac{d}{dt} \varphi_i^k(t),$$

and in each zone $\{x < \varphi_i^\varepsilon(t)\}$ or $\{x > \varphi_i^\varepsilon(t)\}$:

$$u^\varepsilon(\varphi_i^\varepsilon(t) +, t) = u_i^0 + \sum_{k \geq 1} \varepsilon^k u_i^k(\varphi_i^\varepsilon(t), t),$$

or

$$u^\varepsilon(\varphi_i^\varepsilon(t)-, t) = u_{i-1}^0 + \sum_{k \geq 1} \varepsilon^k u_{i-1}^k(\varphi_i^\varepsilon(t), t).$$

Thus, we get respectively

$$(2.9) \quad \begin{cases} u_i^\varepsilon(\varphi_i^\varepsilon(t)+, t) = u_i^0 + \sum_{k \geq 1} \varepsilon^k \{u_i^k(\sigma_i^0 t, t) + a_i^k\}, \\ u_{i-1}^\varepsilon(\varphi_i^\varepsilon(t)-, t) = u_{i-1}^0 + \sum_{k \geq 1} \varepsilon^k \{u_{i-1}^k(\sigma_i^0 t, t) + a_{i-1}^k\}, \end{cases}$$

and

$$(2.10) \quad \begin{cases} f(u_i^\varepsilon(\varphi_i^\varepsilon(t)+, t)) = f(u_i^0) + \sum_{k \geq 1} \varepsilon^k \{A(u_i^0)u_i^k(\sigma_i^0 t, t) + b_i^k\}, \\ f(u_{i-1}^\varepsilon(\varphi_i^\varepsilon(t)-, t)) = f(u_{i-1}^0) + \sum_{k \geq 1} \varepsilon^k \{A(u_{i-1}^0)u_{i-1}^k(\sigma_i^0 t, t) + b_{i-1}^k\}, \end{cases}$$

where a_i^k and b_i^k (respectively a_{i-1}^k and b_{i-1}^k) denote, for each k , functions depending only on the functions $u_i^0, u_i^1, \dots, u_i^{k-1}$ (resp. $u_{i-1}^0, u_{i-1}^1, \dots, u_{i-1}^{k-1}$) and their derivatives.

Formulae (2.8) - (2.10) together with the jump relation (2.7) yield for $k = 0$ the usual Rankine-Hugoniot relation

$$\sigma_i^0(u_i^0 - u_{i-1}^0) = f(u_i^0) - f(u_{i-1}^0).$$

And, for $k \geq 1$, we obtain

$$\frac{d}{dt} \varphi_i^k(t)[u^0] + \sigma_i^0[u^k] = [A(u^0)u^k] + w_i^k, \quad \text{on } x = \sigma_i^0 t,$$

with w^k depending only on $u_i^0, u_i^1, \dots, u_i^{k-1}, u_{i-1}^0, u_{i-1}^1, \dots, u_{i-1}^{k-1}$ and their derivatives. $\square$

To be complete, we have to use the initial condition (1.9) to identify the function u^k in the domains D_0^0 and D_p^0 .

PROPOSITION 2.3. *For each $k \geq 1$, suppose that the functions $\alpha_{j,L}^k$ and $\alpha_{j,R}^k$ ($1 \leq j \leq p$) are defined from the initial data u_L^k and u_R^k by*

$$(2.11) \quad \begin{cases} u_L^k(x) = \sum_{j=1}^p \alpha_{j,L}^k(x) r_j(u_L^0), & x \leq 0, \\ u_R^k(x) = \sum_{j=1}^p \alpha_{j,R}^k(x) r_j(u_R^0), & x \geq 0. \end{cases}$$

Then the function $\alpha_{j,0}^k$ and $\alpha_{j,p}^k$ ($k \geq 1$ and $1 \leq j \leq p$) introduced in (2.3) are explicitly given by

$$(2.12) \quad \begin{cases} \alpha_{j,0}^k(x) = \alpha_{j,L}^k(x), & x \leq 0, \\ \alpha_{j,p}^k(x) = \alpha_{j,R}^k(x), & x \geq 0. \end{cases}$$

Proof. Consider for instance the case $x \leq 0$. It suffices to note that for $t = 0$, the formula (2.3) for u^k in D_0^0 becomes

$$u^k(x, 0) = \sum_{j=1}^p \alpha_{j,0}^k(x) r_j(u_L^0), \quad x \leq 0,$$

which must be equal to

$$u_L^k(x) = \sum_{j=1}^p \alpha_{j,L}^k(x) r_j(u_L^0).$$

□

By Proposition 2.1 and 2.3, for each integer $k \geq 1$, in order to determine u_i^k ($0 \leq i \leq p$) and φ_i^k ($1 \leq i \leq p$), we only need to compute the following p^2 functions

$$x \longrightarrow \alpha_{j,i}^k(x), \quad 1 \leq j \leq p, \quad 1 \leq i \leq p-1,$$

and

$$t \longrightarrow \varphi_i^k(t), \quad 1 \leq i \leq p.$$

However, from Propositions 2.1 - 2.2, we can get exactly p^2 scalar equations for these unknowns. In the next section, we shall prove the well-posedness of this system.

We end this section by rewriting equations (2.5) in a form which will be more convenient in what follows. Define the new unknowns $\beta_{j,i}^k$ as follows. Setting

$$(2.13a) \quad \tilde{\alpha}_{j,i}^k(x) = \int_0^x \alpha_{j,i}^k(x') dx',$$

we define for $t \geq 0$ and $1 \leq i \leq p-1$:

$$(2.13b) \quad \beta_{j,i}^k(t) = \begin{cases} \tilde{\alpha}_{j,i}^k((\sigma_i^0 - \lambda_j(u_i^0))t), & \text{for } j \geq i+1, \\ \tilde{\alpha}_{j,i}^k((\sigma_{i+1}^0 - \lambda_j(u_i^0))t), & \text{for } j \leq i. \end{cases}$$

We introduce the numbers $\theta_{j,i}^0$ ($1 \leq i \leq p-1$) by

$$(2.14) \quad \begin{cases} \theta_{j,i}^0 = \frac{\sigma_i^0 - \lambda_j(u_i^0)}{\sigma_{i+1}^0 - \lambda_j(u_i^0)}, & \text{for } j \leq i, \\ \theta_{j,i}^0 = \frac{\sigma_{i+1}^0 - \lambda_j(u_i^0)}{\sigma_i^0 - \lambda_j(u_i^0)}, & \text{for } j \geq i+1. \end{cases}$$

Observe that (2.14) makes sense even in the case that u^0 contains contact discontinuities.

PROPOSITION 2.4. For each positive integer k , the p^2 scalar functions

$$\begin{aligned}\beta_{j,i}^k : \mathbb{R}_+ &\rightarrow \mathbb{R}, \quad 1 \leq j \leq p, 1 \leq i \leq p-1, \\ \varphi_i^k : \mathbb{R}_+ &\rightarrow \mathbb{R}, \quad 1 \leq i \leq p,\end{aligned}$$

satisfy the following system of p (vectorial) equations:

$$(2.15)_1 \quad \varphi_1^k(t)(u_1^0 - u_L^0) + \sum_{j=2}^p \beta_{j,1}^k(t)r_j(u_1^0) + \beta_{1,1}^k(\theta_{1,1}^0 t)r_1(u_1^0) = z_1^k(t), t \geq 0,$$

$$(2.15)_i \quad \begin{cases} \varphi_i^k(t)(u_i^0 - u_{i-1}^0) + \sum_{j=i+1}^p \beta_{j,i}^k(t)r_j(u_i^0) - \sum_{j=1}^{i-1} \beta_{j,i-1}^k(t)r_j(u_{i-1}^0) + \\ \sum_{j=1}^i \beta_{j,i}^k(\theta_{j,i}^0 t)r_j(u_i^0) - \sum_{j=i}^p \beta_{j,i-1}^k(\theta_{j,i-1}^0 t)r_j(u_{i-1}^0) = z_i^k(t), \\ t \geq 0, \quad 2 \leq i \leq p-1, \end{cases}$$

$$(2.15)_p \quad \begin{cases} \varphi_p^k(t)(u_R^0 - u_{p-1}^0) - \sum_{j=1}^{p-1} \beta_{j,p-1}^k(t)r_j(u_{p-1}^0) - \beta_{p,p-1}^k(\theta_{p,p-1}^0 t)r_p(u_{p-1}^0) \\ = z_p^k(t), \quad t \geq 0. \end{cases}$$

In (2.15), the functions z_i^k ($1 \leq i \leq p$) depend only on $u^0, u^1, \dots, u^{k-1}$. For $k = 1$, they are explicitly given by

$$(2.16)_1 \quad \begin{cases} z_1^1(t) = \sum_{j=1}^p \tilde{\alpha}_{j,L}^1((\sigma_1^0 - \lambda_j(u_L^0))t)r_j(u_L^0) - \frac{t^2}{2} \{(\sigma_1^0 - A(u_1^0))g(u_1^0) \\ - (\sigma_1^0 - A(u_L^0))g(u_L^0)\}, \quad t \geq 0, \end{cases}$$

$$(2.16)_i \quad \begin{cases} z_i^1(t) = -\frac{t^2}{2} \{(\sigma_i^0 - A(u_i^0))g(u_i^0) - (\sigma_i^0 - A(u_{i-1}^0))g(u_{i-1}^0)\}, \\ t \geq 0, \quad 2 \leq i \leq p-1, \end{cases}$$

$$(2.16)_p \quad \begin{cases} z_p^1(t) = -\sum_{j=1}^p \tilde{\alpha}_{j,R}^1((\sigma_p^0 - \lambda_j(u_R^0))r_j(u_R^0) - \frac{t^2}{2} \{(\sigma_p^0 - A(u_R^0))g(u_R^0) \\ - (\sigma_p^0 - A(u_{p-1}^0))g(u_{p-1}^0)\}, \quad t \geq 0, \end{cases}$$

where $\tilde{\alpha}_{j,L}^1$ and $\tilde{\alpha}_{j,R}^1$ are the primitive functions of $\alpha_{j,L}^1$ and $\alpha_{j,R}^1$ defined similarly to (2.13a).

Proof. By (2.3) and (2.5), we get on $x = \sigma_i^0 t$ ($1 \leq i \leq p$)

$$\begin{aligned}& \sum_{j=1}^p (\sigma_i^0 - \lambda_j(u_i^0))\alpha_{j,i}^k((\sigma_i^0 - \lambda_j(u_i^0))t)r_j(u_i^0) \\ & - \sum_{j=1}^p (\sigma_i^0 - \lambda_j(u_{i-1}^0))\alpha_{j,i-1}^k((\sigma_i^0 - \lambda_j(u_{i-1}^0))t)r_j(u_{i-1}^0) \\ & + \frac{d}{dt}\varphi_i^k(t) \cdot (u_i^0 - u_{i-1}^0) = w_i^k(t) - [t(\sigma_i^0 - A(u^0))\theta^k].\end{aligned}$$

Then, integrating with respect to t and using (2.13a) for $t \geq 0$ and $1 \leq i \leq p$, we obtain

$$(2.17) \quad \left\{ \begin{array}{l} \sum_{j=1}^p \tilde{\alpha}_{j,i}^k((\sigma_i^0 - \lambda_j(u_i^0))t)r_j(u_i^0) - \sum_{j=1}^p \tilde{\alpha}_{j,i-1}^k((\sigma_i^0 - \lambda_j(u_{i-1}^0))t)r_j(u_{i-1}^0) + \\ + \varphi_i^k(t) \cdot (u_i^0 - u_{i-1}^0) = \int_0^t w_i^k(s)ds - \int_0^t s[(\sigma_i^0 - A(u^0))v^k]ds, \text{ on } x = \sigma_i^0 t. \end{array} \right.$$

Using (2.13b) and (2.16) for $1 \leq i \leq p-1$ and $t \geq 0$, we have

$$\sum_{j=1}^p \tilde{\alpha}_{j,i}^k((\sigma_i^0 - \lambda_j(u_i^0))t)r_j(u_i^0) = \sum_{j=1}^i \beta_{j,i}^k(\theta_i^0 t)r_j(u_i^0) + \sum_{j=i+1}^p \beta_{j,i}^k(t)r_j(u_i^0),$$

and similarly for $2 \leq i \leq p$ and $t \geq 0$,

$$\begin{aligned} & \sum_{j=1}^p \tilde{\alpha}_{j,i-1}^k((\sigma_i^0 - \lambda_j(u_{i-1}^0))t)r_j(u_{i-1}^0) \\ &= \sum_{j=1}^{i-1} \beta_{j,i-1}^k(t)r_j(u_{i-1}^0) + \sum_{j=i}^p \beta_{j,i-1}^k(\theta_{i-1}^0 t)r_j(u_{i-1}^0). \end{aligned}$$

Hence, for $i = 2, \dots, p-1$ and $t \geq 0$, (2.17) becomes

$$\begin{aligned} & \sum_{j=i+1}^p \beta_{j,i}^k(t)r_j(u_i^0) - \sum_{j=1}^{i-1} \beta_{j,i-1}^k(t)r_j(u_{i-1}^0) + \sum_{j=1}^i \beta_{j,i}^k(\theta_{j,i}^0 t)r_j(u_i^0) \\ & - \sum_{j=1}^p \beta_{j,i-1}^k(\theta_{i-1}^0 t)r_j(u_{i-1}^0) + \varphi_i^k(t)(u_i^0 - u_{i-1}^0) \\ &= \int_0^t w_i^k(s)ds - \int_0^t s[(\sigma_i^0 - A(u^0))v^k]ds, \text{ on } x = \sigma_i^0 t, \end{aligned}$$

while for $i = 1$ and $i = p$, noting (2.11), we find

$$\begin{aligned} & \sum_{j=2}^p \beta_{j,1}^k(t)r_j(u_1^0) + \beta_{1,1}^k(\theta_{1,1}^0 t)r_1(u_1^0) + \varphi_1^k(t)(u_1^0 - u_L^0) \\ &= \sum_{j=1}^p \tilde{\alpha}_{j,0}^k((\sigma_1^0 - \lambda_j(u_L^0))t)r_j(u_L^0) + \int_0^t w_1^k(s)ds - \int_0^t s[(\sigma_1^0 - A(u^0))v^k]ds, \text{ on } x = \sigma_1^0 t, \end{aligned}$$

and

$$\begin{aligned} & - \sum_{j=1}^{p-1} \beta_{j,p-1}^k(t)r_j(u_{p-1}^0) - \beta_{p,p-1}^k(\theta_{p,p-1}^0 t)r_p(u_{p-1}^0) + \varphi_p^k(t)(u_R^0 - u_{p-1}^0) \\ &= - \sum_{j=1}^p \tilde{\alpha}_{j,p}^k((\sigma_p^0 - \lambda_j(u_R^0))t)r_j(u_R^0) + \int_0^t w_p^k(s)ds - \int_0^t s[(\sigma_p^0 - A(u^0))v^k]ds, \text{ on } x = \sigma_p^0 t. \end{aligned}$$

That proves (2.15). Moreover, (2.16) is a consequence of (2.4), (2.5) and (2.12), since

$$\begin{aligned} - \int_0^t s[(\sigma_i^0 - A(u_i^0))v^1]ds &= - \int_0^t s \{(\sigma_i^0 - A(u_i^0))g(u_i^0) - (\sigma_i^0 - A(u_{i-1}^0))g(u_{i-1}^0)\} ds \\ &= -\frac{t^2}{2} \{(\sigma_i^0 - A(u_i^0))g(u_i^0) - (\sigma_i^0 - A(u_{i-1}^0))g(u_{i-1}^0)\}. \end{aligned}$$

□

3. Existence and uniqueness of the asymptotic expansion. We now prove that the system of equations found in Section 2 for the functions u_i^k ($0 \leq i \leq p$) and φ_i^k ($1 \leq i \leq p$), $k \geq 1$, admits a unique solution globally defined in time.

THEOREM 3.1. *Suppose that, for $k \geq 1$, the initial data $u_L^k : \mathbb{R}_- \rightarrow \mathbb{R}^p$ and $u_R^k : \mathbb{R}_+ \rightarrow \mathbb{R}^p$ are functions of class $\mathcal{C}^N$, $N \geq 0$. Assume furthermore that $|u_R^0 - u_L^0|$ is small enough. Then, for each $k \geq 1$, there exists a unique solution $\beta_{j,i}^k$ and φ_i^k in $\mathcal{C}^{N+1}(\mathbb{R}_+, \mathbb{R})$ ($1 \leq j \leq p, 1 \leq i \leq p-1$) to the system of equations (2.15).*

Hence, for each $k \geq 1$, the functions $u^k : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}^p$ piecewise $\mathcal{C}^N$ and φ_i^k in $\mathcal{C}^{N+1}(\mathbb{R}_+, \mathbb{R})$ can be uniquely determined by (2.3), (2.11) - (2.15).

We state two preliminary lemmas which are useful for the proof of Theorem 3.1.

LEMMA 3.2. *Let $C = (C_{nm})_{1 \leq n, m \leq q}$ and $\tau = (\tau_{nm})_{1 \leq n, m \leq q}$ be two given $q \times q$ matrices and $h = (h_i)_{1 \leq i \leq q}$ a function defined from $\mathbb{R}_+$ into $\mathbb{R}^q$.*

Consider the following system of equations with the unknowns $g = (g_1, g_2, \dots, g_q) : \mathbb{R}_+ \rightarrow \mathbb{R}^q$:

$$(3.1) \quad g_n(x) + \sum_{m=1}^q C_{nm} g_m(\tau_{nm}x) = h_n(x), \quad 1 \leq n \leq q, x \geq 0.$$

Then

1) *If the matrix C satisfies the property*

$$(3.2) \quad \|C\| \stackrel{\text{def}}{=} \max_{1 \leq n \leq q} \sum_{m=1}^q |C_{nm}| < 1,$$

and the function h belongs to $\mathcal{C}^0(\mathbb{R}_+, \mathbb{R}^q)$, then system (3.1) has one and only one solution g in $\mathcal{C}^0(\mathbb{R}_+, \mathbb{R}^q)$.

2) *Suppose that the matrix τ satisfies the property*

$$(3.3) \quad 0 \leq \tau_{nm} < 1, \quad (1 \leq n, m \leq q),$$

and let N_0 denote the smallest integer such that

$$(3.4) \quad \|(\tau_{nm}^{N_0} C_{nm})_{1 \leq n, m \leq q}\| < 1.$$

Assume furthermore that

$$(3.5) \quad \det(Id + (\tau_{nm}^N C_{nm})_{1 \leq n, m \leq q}) \neq 0, \quad 0 \leq N \leq N_0 - 1.$$

Then, for every function h in $\mathcal{C}^{N_0}(\mathbb{R}_+, \mathbb{R}^q)$, there exists one and only one solution g to (3.1) in $\mathcal{C}^{N_0}(\mathbb{R}_+, \mathbb{R}^q)$.

3) Moreover, when the function h is of class $\mathcal{C}^M$, $M \geq 0$, suppose that (3.4) (3.5) hold for $N_0 = M$. Then, there exists one and only one solution g to (3.1) in $\mathcal{C}^M(\mathbb{R}_+, \mathbb{R}^q)$.

We emphasize that in (3.5), the condition is automatically satisfied for N large enough (see also Li-Yu [7] and [8]).

LEMMA 3.3. The coefficients $\theta_{j,i}^0$ defined by (2.14) satisfy

$$0 < \theta_{j,i}^0 < 1.$$

Moreover, for $u_R^0 = u_L^0 = u_*$, we have

$$\begin{cases} \theta_{j,i}^0 = \frac{\lambda_i(u_*) - \lambda_j(u_*)}{\lambda_{i+1}(u_*) - \lambda_j(u_*)}, & j \leq i, \\ \theta_{j,i}^0 = \frac{\lambda_{i+1}(u_*) - \lambda_j(u_*)}{\lambda_i(u_*) - \lambda_j(u_*)}, & j \geq i + 1. \end{cases}$$

Thus, for a given state u_* in $\mathbb{R}^p$, there exists a constant $K = K(u_*)$ such that the inequalities

$$0 < \sup_{j,i} \theta_{j,i}^0 \leq K < 1$$

hold uniformly for u_L^0 and u_R^0 in a neighborhood of u_* .

Lemma 3.3 is an easy consequence of Lax's entropy inequalities (Lax [4] and also Li-Yu [7]). We now demonstrate Theorem 3.1 and then give the proof of Lemma 3.2.

Proof of Theorem 3.1. We start from the system of equations (2.15) given by Proposition 2.4. Let us multiply each equation of (2.15) by the left eigenvectors l_m^T ($1 \leq m \leq p$). Precisely we use, in $(2.15)_i$, $l_m(u_{i-1}^0)^T$ for $1 \leq m \leq i$ and $l_m(u_i^0)^T$ for $i + 1 \leq m \leq p$, except for $m = i = 1$ where we use $l_1(u_1^0)^T$. Thanks to (1.2), for $t \geq 0$ we obtain

$$(3.6a)_1 \quad \varphi_1^k(t) l_m(u_1^0)^T (u_1^0 - u_L^0) + \beta_{m,1}^k(\theta_{1,1}^0 t) = l_m(u_1^0) z_1^k(t), \quad m = 1,$$

$$(3.6b)_1 \quad \varphi_1^k(t) l_m(u_1^0)^T (u_1^0 - u_L^0) + \beta_{m,1}^k(t) = l_m(u_1^0)^T z_1^k(t), \quad 2 \leq m \leq p,$$

$$(3.7a)_i \quad \left\{ \begin{array}{l} \varphi_i^k(t) l_m(u_{i-1}^0)^T (u_i^0 - u_{i-1}^0) + \sum_{j=i+1}^p \beta_{j,i}^k(t) l_m(u_{i-1}^0)^T r_j(u_i^0) - \beta_{m,i-1}^k(t) \\ + \sum_{j=1}^i \beta_{j,i}^k(\theta_{j,i}^0 t) l_m(u_{i-1}^0)^T r_j(u_i^0) = l_m(u_{i-1}^0)^T z_i^k(t), \quad 2 \leq i \leq p-1, \\ 1 \leq m \leq i-1, \end{array} \right.$$

$$(3.7b)_i \quad \left\{ \begin{array}{l} \varphi_i^k(t) l_m(u_{i-1}^0)^T (u_i^0 - u_{i-1}^0) + \sum_{j=i-1}^p \beta_{j,i}^k(t) l_m(u_{i-1}^0)^T r_j(u_i^0) \\ + \sum_{j=1}^i \beta_{j,i}^k(\theta_{j,i}^0 t) l_m(u_{i-1}^0)^T r_j(u_i^0) - \beta_{m,i-1}^k(\theta_{m,i-1}^0 t) = l_m(u_{i-1}^0)^T z_i^k(t), \\ 2 \leq i \leq p-1, m = i, \end{array} \right.$$

$$(3.7c)_i \quad \left\{ \begin{array}{l} \varphi_i^k(t) l_m(u_i^0)^T (u_i^0 - u_{i-1}^0) + \beta_{m,i}^k(t) - \sum_{j=1}^{i-1} \beta_{j,i-1}^k(t) l_m(u_i^0)^T r_j(u_{i-1}^0) \\ - \sum_{j=i}^p \beta_{j,i-1}^k(\theta_{j,i-1}^0 t) l_m(u_i^0)^T r_j(u_{i-1}^0) = l_m(u_i^0)^T z_i^k(t), \quad 2 \leq i \leq p-1, \\ i+1 \leq m \leq p, \end{array} \right.$$

$$(3.8a)_p \quad \varphi_p^k(t) l_m(u_{p-1}^0)^T (u_R^0 - u_{p-1}^0) - \beta_{m,p-1}^k(t) = l_m(u_{p-1}^0)^T z_p^k(t), \quad 1 \leq m \leq p-1,$$

$$(3.8b)_p \quad \varphi_p^k(t) l_m(u_{p-1}^0)^T (u_R^0 - u_{p-1}^0) - \beta_{m,p-1}^k(\theta_{p,p-1}^0 t) = l_m(u_{p-1}^0)^T z_p^k(t), \quad m = p,$$

where $z_i^k(t)$ ($1 \leq i \leq p$) are all given functions.

System (3.5) - (3.8) is composed of $p \times p$ scalar equations. We first get the functions $\varphi_i^k(t)$ from the equations (3.6a), (3.7b) and (3.8a):

$$(3.9)_1 \quad \varphi_1^k(t) = \{l_1(u_1^0)^T (u_1^0 - u_L^0)\}^{-1} \cdot \{l_1(u_1^0)^T z_1^k(t) - \beta_{1,1}^k(\theta_{1,1}^0 t)\},$$

$$(3.9)_i \quad \left\{ \begin{array}{l} \varphi_1^k(t) = \{l_i(u_{i-1}^0)^T (u_i^0 - u_{i-1}^0)\}^{-1} \cdot \left\{ l_i(u_{i-1}^0)^T z_i^k(t) + \right. \\ + \beta_{i,i-1}^k(\theta_{i,i-1}^0 t) - \sum_{j=i+1}^p \beta_{j,i}^k(t) l_i(u_{i-1}^0)^T r_j(u_i^0) \\ \left. - \sum_{j=1}^i \beta_{j,i}^k(\theta_{j,i}^0 t) l_i(u_{i-1}^0)^T r_j(u_i^0) \right\}, \quad 2 \leq i \leq p-1, \end{array} \right.$$

$$(3.9)_p \quad \varphi_p^k(t) = \{l_p(u_{p-1}^0)^T(u_R^0 - u_{p-1}^0)\}^{-1} \cdot \{l_p(u_{p-1}^0)^T z_p^k(t) + \beta_{p,p-1}^k(\theta_{p,p-1}^0 t)\}.$$

Let

$$(3.10) \quad \begin{cases} \mu_{m,1} = l_m(u_1^0)^T \cdot (u_1^0 - u_L^0), & 1 \leq m \leq p, \\ \mu_{m,i} = l_m(u_{i-1}^0)^T \cdot (u_i^0 - u_{i-1}^0), & 2 \leq i \leq p, 1 \leq m \leq i, \\ \mu_{m,i} = l_m(u_i^0)^T \cdot (u_i^0 - u_{i-1}^0), & 2 \leq i \leq p, i+1 \leq m \leq p. \end{cases}$$

Putting (3.9) into (3.6b), (3.7a), (3.7c), (3.8a), we find the following system of $p(p-1)$ scalar equations for the $p(p-1)$ unknowns $(\beta_{j,i})_{\substack{1 \leq j \leq p \\ 1 \leq i \leq p-1}}$:

$$(3.11a) \quad \beta_{m,1}^k(t) - \frac{\mu_{m,1}}{\mu_{1,1}} \beta_{1,1}^k(\theta_{1,1}^0 t) = -\frac{\mu_{m,1}}{\mu_{1,1}} l_1(u_1^0)^T z_1^k(t) + l_m(u_1^0)^T z_1^k(t), \quad 2 \leq m \leq p,$$

$$(3.11b) \quad \begin{cases} -\beta_{m,i-1}^k(t) + \sum_{j=i+1}^p l_m(u_{i-1}^0)^T r_j(u_i^0) \beta_{j,i}^k(t) + \sum_{j=1}^i l_m(u_{i-1}^0)^T r_j(u_i^0) \beta_{j,i}^k(0_{j,i}^0 t) \\ + \frac{\mu_{m,i}}{\mu_{i,i}} \left\{ \beta_{i,i-1}^k(\theta_{i,i-1}^0 t) - \sum_{j=i+1}^p l_i(u_{i-1}^0)^T r_j(u_i^0) \beta_{j,i}^k(t) \right. \\ \left. - \sum_{j=1}^i l_i(u_{i-1}^0)^T r_j(u_i^0) \beta_{j,i}^k(\theta_{j,i}^0 t) \right\} = -\frac{\mu_{m,i}}{\mu_{i,i}} l_i(u_{i-1}^0)^T z_i^k(t) + l_m(u_{i-1}^0)^T z_i^k(t), \\ 2 \leq i \leq p-1, 1 \leq m \leq i-1, \end{cases}$$

$$(3.11c) \quad \begin{cases} \beta_{m,i}^k(t) - \sum_{j=1}^{i-1} l_m(u_i^0)^T r_j(u_{i-1}^0) \beta_{j,i}^k(t) - \sum_{j=i}^p l_m(u_i^0)^T r_j(u_{i-1}^0) \beta_{j,i-1}^k(\theta_{j,i-1}^0 t) \\ + \frac{\mu_{m,i}}{\mu_{i,i}} \left\{ \beta_{i,i-1}^k(\theta_{i,i-1}^0 t) - \sum_{j=i+1}^p l_i(u_{i-1}^0)^T r_j(u_i^0) \beta_{j,i}^k(t) \right. \\ \left. - \sum_{j=1}^i l_i(u_{i-1}^0)^T r_j(u_i^0) \beta_{j,i}^k(\theta_{j,i}^0 t) \right\} = -\frac{\mu_{m,i}}{\mu_{i,i}} l_i(u_{i-1}^0)^T z_i^k(t) + l_m(u_i^0)^T z_i^k(t), \\ 2 \leq i \leq p-1, i+1 \leq m \leq p, \end{cases}$$

$$(3.11d) \quad \begin{cases} -\beta_{m,p-1}^k(t) + \frac{\mu_{m,p}}{\mu_{p,p}} \beta_{p,p-1}^k(\theta_{p,p-1}^0 t) = -\frac{\mu_{m,p}}{\mu_{p,p}} l_p(u_{p-1}^0)^T z_p^k(t) + l_m(u_{p-1}^0)^T z_p^k(t), \\ 1 \leq m \leq p-1. \end{cases}$$

To show that Lemma 3.2 applies to system (3.11), we develop the function for a small initial amplitude $|u_R^0 - u_L^0|$. We introduce the usual parametrization (see Lax [4]) for the solution to the classical Riemann problem such that the intermediate states of u^0 satisfy:

$$(3.12) \quad u_i^0 = u_{i-1}^0 + \eta_i r_i(u_{i-1}^0) + 0(\eta_i^2),$$

and the shock speed or the contact discontinuity speed satisfies

$$(3.13) \quad \sigma_i^0 = \lambda_i(u_{i-1}^0) + \theta(\eta_i),$$

where $\eta = (\eta_1, \eta_{21} \dots, \eta_i)$ is a small parameter. (3.12) and (3.13) imply that

$$(3.14) \quad l_m(u_{i-1}^0)^T r_j(u_i^0) = \delta_{mj} + 0(\eta_i^2),$$

and (see (3.10) for the definition of $\mu_{m,i}$)

$$(3.15) \quad \mu_{m,i} = \eta_i \delta_{m,i} + 0(\eta_i^2),$$

then

$$(3.16) \quad \frac{\mu_{m,i}}{\mu_{i,i}} = 0(\eta_i).$$

Using (3.14) and (3.15), it is easy to verify that system (3.11) is of the following form:

$$(3.17a) \quad \beta_{m,1}^k(t) + 0(\eta) \beta_{1,1}^k(\theta_{1,1}^0 t) = \gamma_{m,1}(t), \quad t \geq 0, \quad 2 \leq m \leq p,$$

$$(3.17b) \quad \begin{cases} \beta_{m,i-1}^k(t) + 0(\eta) \sum_{j=i+1}^p \beta_{j,i}^k(t) - \beta_{m,i}^k(\theta_{m,i}^0 t) + 0(\eta) \sum_{j=1}^i \beta_{j,i}^k(\theta_{j,i}^0 t) \\ + \theta(\eta) \beta_{i,i-1}^k(\theta_{i,i-1}^0 t) = \gamma_{m,i}(t), \quad t \geq 0, \quad 2 \leq i \leq p-1, \quad 1 \leq m \leq i-1, \end{cases}$$

$$(3.17c) \quad \begin{cases} \beta_{m,i}^k(t) + 0(\eta) \sum_{j=1}^p \beta_{j,i-1}^k(t) - \beta_{m,i-1}^k(\theta_{m,i-1}^0 t) + 0(\eta) \sum_{j=1}^i \beta_{j,i}^k(\theta_{j,i}^0 t) = \gamma_{m,i}(t), \\ t \geq 0, \quad 2 \leq i \leq p-1, \quad i+1 \leq m \leq p. \end{cases}$$

$$(3.17d) \quad \beta_{m,p-1}^k(t) + 0(\eta) \beta_{p,p-1}^k(\theta_{p,p-1}^0 t) = \gamma_{m,p}(t), \quad t \geq 0, \quad 1 \leq m \leq p-1.$$

In (3.17), the given functions $\gamma_{m,i}(t)$ have the same regularity as the functions $z_i^k(t)$.

Finally, for small $|\eta|$, we can rewrite system (3.17) in the following form:

$$(3.18) \quad \begin{cases} \beta_{m,1}^k(t) + 0(\eta) \beta_{1,1}^k(\theta_{1,1}^0 t) = \tilde{\gamma}_{m,1}^k(t), \quad t \geq 0, \quad 2 \leq m \leq p, \\ \beta_{m,i-1}^k(t) - \beta_{m,i}^k(\theta_{m,i}^0 t) + 0(\eta) \sum_{\substack{1 \leq j \leq p \\ 1 \leq n \leq p-1}} \beta_{j,n}^k(\theta_{j,n}^0 t) = \tilde{\gamma}_{m,i}^k(t), \\ t \geq 0, \quad 2 \leq i \leq p-1, \quad 1 \leq m \leq i-1 \\ \beta_{m,i}^k(t) - \beta_{m,i-1}^k(\theta_{m,i-1}^0 t) + 0(\eta) \sum_{\substack{1 \leq j \leq p \\ 1 \leq n \leq p-1}} \beta_{j,n}^k(\theta_{j,n}^0 t) = \tilde{\gamma}_{m,i}^k(t), \quad t \geq 0, \\ 2 \leq i \leq p-1, \quad i+1 \leq m \leq p, \\ \beta_{m,p-1}^k(t) + 0(\eta) \beta_{p,p-1}^k(\theta_{p,p-1}^0 t) = \tilde{\gamma}_{m,p}^k(t), \quad t \geq 0, \quad 1 \leq m \leq p-1. \end{cases}$$

The functions $\tilde{\gamma}_{m,i}^k$ also have the same regularity as z_i^k .

Another way to write (3.18) is

$$(3.19) \quad \left\{ \begin{array}{l} \beta_{1,1}^k(t) - \beta_{1,2}^k(\theta_{1,2}^0 t) + 0(\eta) \sum_{\substack{1 \leq j \leq p \\ 1 \leq n \leq p-1}} \beta_{j,n}^k(\theta_{j,n}^0 t) = \tilde{\gamma}_{1,2}^k(t), \\ \beta_{m,1}^k(t) + 0(\eta) \beta_{1,1}^k(\theta_{1,1}^0 t) = \tilde{\gamma}_{m,1}^k(t), 2 \leq m \leq p, \\ \beta_{m,i}^k(t) - \beta_{m,i+1}^k(\theta_{m,i+1}^0 t) + 0(\eta) \sum_{\substack{1 \leq j \leq p \\ 1 \leq n \leq p-1}} \beta_{j,n}^k(\theta_{j,n}^0 t) = \tilde{\gamma}_{m,i+1}^k(t), \\ \hspace{15em} 2 \leq i \leq p-2, 1 \leq m \leq i, \\ \beta_{m,i}^k(t) - \beta_{m,i-1}^k(\theta_{m,i-1}^0 t) + 0(\eta) \sum_{\substack{1 \leq j \leq p \\ 1 \leq n \leq p-1}} \beta_{j,n}^k(\theta_{j,n}^0 t) = \tilde{\gamma}_{m,i}^k(t), \\ \hspace{15em} 2 \leq i \leq p-2, i+1 \leq n \leq p, \\ \beta_{m,p-1}^k(t) + 0(\eta) \beta_{p,p-1}^k(\theta_{p,p-1}^0 t) = \tilde{\gamma}_{m,p}^k(t), 1 \leq m \leq p-1, \\ \beta_{p,p-1}^k(t) - \beta_{p,p-2}^k(\theta_{p,p-2}^0 t) + 0(\eta) \sum_{\substack{1 \leq j \leq p \\ 1 \leq n \leq p-1}} \beta_{j,n}^k(\theta_{j,n}^0 t) = \tilde{\gamma}_{m,p-1}^k(t). \end{array} \right.$$

Setting

$$\beta_i = (\beta_{m,i}^k)_{1 \leq m \leq p}, \quad \beta = (\beta_i)_{1 \leq i \leq p-1},$$

where β is a q -vector with $q = p(p-1)$, we define the $q \times q$ matrix $D = (D_{n,j}^{m,i})_{\substack{1 \leq m,n \leq p \\ 1 \leq i,j \leq p-1}}$ by

$$(3.20) \quad \left\{ \begin{array}{l} D_{m,i+1}^{m,i} = -1, \text{ for } 1 \leq i \leq p-2, 1 \leq m \leq i, \\ D_{m,i+1}^{m,i} = -1, \text{ for } 2 \leq i \leq p-1, i+1 \leq m \leq p, \\ D_{n,j}^{m,i} = 0, \text{ otherwise.} \end{array} \right.$$

Hence, system (3.19) is exactly of the form (3.1):

$$(3.21) \quad \beta_{m,i}^{(t)} + \sum_{(n,j)} C_{n,j}^{m,i} \beta_{n,j}(\tau_{n,j}^{m,i} t) = \chi_{m,i}(t), \quad 1 \leq m \leq p, 1 \leq i \leq p-1, t \geq 0,$$

where by Lemma 3.3

$$0 < \tau_{n,j}^{m,i} < 1,$$

and the matrix $C(\eta) = (c_{n,j}^{m,i})$ satisfies

$$(3.22) \quad C(\eta) = D + 0(\eta).$$

The function $\chi_{m,i}$ have the same regularity as z_i^k .

We now verify the assumptions of Lemma 3.2. First, condition (3.3) is satisfied due to (3.22). Second, it is a simple matter to see that the norm of the matrix D given by (3.20) is

$$(3.23) \quad \|D\| = 1.$$

Then, for η small enough, condition (3.4) holds with $N_0 = 1$ for the matrix C given by (3.23), since

$$(3.24) \quad \|(\tau_{n,j}^{m,i} C_{n,j}^{m,i})\| \leq \sup_{m,i} |\theta_{m,i}^0| \cdot (1 + 0(\eta)),$$

and by Lemma 3.3

$$\sum_{m,i} |\theta_{m,i}^0| < 1.$$

It remains to check condition (3.5) for $N_0 = 1$. It suffices to prove that the matrix $I + D$ is invertible and (3.5) will follow from (3.23) provided that η is small enough. But, it is clear since the only solution to the algebraic system

$$(3.25) \quad \begin{cases} b_{1,1} &= b_{1,2}, \\ b_{m,1} &= 0, & 2 \leq m \leq p, \\ b_{m,i} &= b_{m,i+1}, & 1 \leq m \leq p, \ 2 \leq i \leq p-2, \\ b_{m,i} &= b_{m,i-1}, & i+1 \leq m \leq p, \ 2 \leq i \leq p-2, \\ b_{m,p-1} &= 0, & 1 \leq m \leq p-1, \\ b_{p,p-1} &= b_{p,p-2} \end{cases}$$

is the null solution

$$b_{i,m} = 0, \ 1 \leq m \leq p, 1 \leq i \leq p-1.$$

Therefore, Lemma 3.2 can be used to yield in a unique way functions $\beta_{j,i}^k$ in $\mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$. Then by (3.9) φ_i^k is obtained in the class $\mathcal{C}^1$.

As soon as $\beta_{j,i}^k$ and φ_i^k are known, u^k is got by (2.3) (2.13).

For the result in the class $\mathcal{C}^N$, we need property (3.5) for any integer N_0 . But as before, it suffices to notice the only solution to the system

$$(3.26) \quad \begin{cases} b_{1,1} &= (\theta_{1,2}^0)^N b_{1,2}, \\ b_{m,1} &= 0, \ 2 \leq m \leq p, \\ b_{m,i} &= (\theta_{m,i+1}^0)^N b_{m,i+1}, \ 1 \leq m \leq i, 2 \leq i \leq p-2, \\ b_{m,i} &= (\theta_{m,i-1}^0)^N b_{m,i-1}, \ i+1 \leq m \leq p, 2 \leq i \leq p-2, \\ b_{m,p-1} &= 0, \ 1 \leq m \leq p-1, \\ b_{p,p-1} &= (\theta_{p,p-2}^0)^N b_{p,p-2}, \end{cases}$$

is the null solution. $\square$

Proof of Lemma 3.2. We only indicate the outlines of the proof (see also [7]). First, let $g^0 : \mathbb{R}_+ \rightarrow \mathbb{R}^q$ be an arbitrary continuous function; we construct a sequence of functions $(g^l)_{l \in \mathbb{N}}$ in $\mathcal{C}^0(\mathbb{R}_+, \mathbb{R}^q)$ by

$$(3.27) \quad g_n^{l+1}(x) = h_n(x) \sum_{m=1}^q c_{nm} g_m^l(\tau_{nm} x), \quad 1 \leq n \leq q, x \geq 0, l \geq 0.$$

It is easy to see that

$$(3.28) \quad \|g^{l+2} - g^{l+1}\| \leq \|c\| \|g^{l+1} - g^l\|.$$

Hence, if condition (3.2) holds, then the sequence (g^l) is a Cauchy sequence in the space $\mathcal{C}^0$ and thus converges to a continuous function denoted by g . It is clear that g is the unique solution to (3.1).

Second, consider the case where (3.3) - (3.5) are satisfied. By differentiation, we may obtain a system of the same form

$$(3.29) \quad \tilde{g}_n(x) + \sum_m \tilde{c}_{nm} \tilde{g}_m(\tau_{nm} x) = \tilde{h}_n(x), \quad 1 \leq n \leq q, x \geq 0,$$

with

$$\tilde{g}_n(x) = \frac{d^{N_0}}{dx^{N_0}} g_n(x), \tilde{c}_{nm} = c_{nm} \tau_{nm}^{N_0}, \tilde{h}_n(x) = \frac{d^{N_0}}{dx^{N_0}} h_n(x).$$

By (3.4), the norm of $\tilde{c} = (\tilde{c}_{nm})$ is less than 1, i.e. $\|c\| < 1$, thus the first part of Lemma applies to (3.30). From the solution $\tilde{g}$ to (3.30), we can obtain a unique solution $g \in \mathcal{C}^{N_0}$ to (3.1) by successive integrations. For this purpose, it is sufficient to verify that all the derivatives of g at $x = 0$:

$$b^N \equiv \frac{d^N}{dx^N} g(0), \quad 0 \leq N < N_0$$

can be uniquely determined from system (3.1). thanks to (3.5), the system

$$b_n^N + \sum c_{nm} \tau_{nm}^N b_n^N = \frac{d^N}{dx^N} h_n(0), \quad 1 \leq n \leq q, 0 \leq N < N_0,$$

is invertible. This completes the proof.

The proof of 3) is similar to the proof of 2). $\square$

4. Applications to the system of isentropic gas dynamics. In this section, we restrict ourself to the system of gas dynamics in Lagrangian representation. We want to improve the previous results by proving the existence and uniqueness of an asymptotic

expansion, globally defined in time, *without* the assumption of small initial amplitude in the classical Riemann problem.

We discuss the isentropic 2×2 system

$$(4.1) \quad \begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p(v) = 0, \end{cases}$$

where $v > 0$ is the specific volume, u the velocity and p the pressure. To be specific, we consider the polytropic perfect gas

$$p(v) = Cv^{-\gamma} \quad (C > 0 \text{ and } \gamma > 1),$$

for which we may solve the Riemann problem and the generalized Riemann problem without assumption of small initial amplitude (see Smoller [12] and Li-Yu [7]). Set

$$v = \begin{pmatrix} v \\ u \end{pmatrix}, \quad f(u) = \begin{pmatrix} -u \\ p(v) \end{pmatrix}.$$

Using the same notations as in Sections 1-3 (except u replaced by v), the eigenvalues of $f(v)$ are

$$\lambda_1 = -c \text{ and } \lambda_2 = c,$$

where c is the sound speed given by

$$c(v) = av^{-\frac{\gamma+1}{2}} \quad (a > 0, a = \sqrt{\gamma C}).$$

Moreover, the right eigenvectors of $Df(u)$ are

$$r_1(u) = \begin{pmatrix} 1 \\ c(v) \end{pmatrix}, \quad r_2(u) = \begin{pmatrix} 1 \\ -c(v) \end{pmatrix}.$$

System (4.1) is strictly hyperbolic and genuinely nonlinear.

THEOREM 4.1. *Consider the generalized Riemann problem (1.10) (1.11) specially for the system of isentropic gas dynamics (4.1). Suppose that the initial data u_L^ε and u_R^ε are functions of class $\mathcal{C}^{N_0}$ with $N_0 \geq 0$. Then, the asymptotic expansion (1.13), (1.14) exists globally in time no matter how large the initial amplitude $|u_R^0 - u_L^0|$ is.*

Proof. For system (4.1), (2.15) reduces to

$$(4.2) \quad \begin{cases} \varphi_1^k(t)(U_1^0 - U_L^0) + \beta_{2,1}^k(t)r_2(U_1^0) + \beta_{1,1}^k(\theta_{1,1}^0 t)r_1(U_1^0) = z_1^k(t), \\ \varphi_2^k(t)(U_R^0 - U_1^0) - \beta_{1,1}^k(t)r_1(U_1^0) - \beta_{2,1}^k(\theta_{2,1}^0 t)r_2(U_1^0) = z_2^k(t). \end{cases}$$

Set $b_1(t) = \beta_{1,1}^k(t)$ and $b_2(t) = \beta_{2,1}^k(t)$. Eliminating φ_1^k and φ_2^k in (4.2) yields two equations for b_1 and b_2 :

$$(4.3) \quad \begin{cases} \{(u_1^0 - u_L^0) + c(v_1^0)(v_1^0 - v_L^0)\} b_2(t) + \{u_1^0 - u_L^0 - c(v_1^0)(v_1^0 - v_L^0)\} b_1(\theta_{1,1}^0 t) = a_1(t), \\ \{(u_R^0 - u_1^0) - c(v_1^0)(v_R^0 - v_1^0)\} b_1(t) + \{u_R^0 - u_1^0 + c(v_1^0)(v_R^0 - v_1^0)\} b_2(\theta_{2,1}^0 t) = a_2(t). \end{cases}$$

Since the shock speeds σ_1^0 and σ_2^0 are

$$\sigma_1^0 = -\frac{u_1^0 - u_L^0}{v_1^0 - v_L^0}, \quad \sigma_2^0 = -\frac{u_R^0 - u_1^0}{v_R^0 - v_1^0},$$

we obtain the following system

$$(4.4) \quad \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix} + C \begin{pmatrix} b_1(\theta_{1,1}^0 t) \\ b_2(\theta_{2,1}^0 t) \end{pmatrix} = \begin{pmatrix} d_1(t) \\ d_2(t) \end{pmatrix},$$

where

$$(4.5) \quad C = \begin{pmatrix} 0 & \frac{\sigma_1^0 + c(v_1^0)}{\sigma_1^0 - c(v_1^0)} \\ \frac{\sigma_2^0 - c(v_1^0)}{\sigma_2^0 + c(v_1^0)} & 0 \end{pmatrix}.$$

By the Lax entropy inequalities [4], we have

$$\sigma_1^0 > -c(v_1^0) \text{ and } \sigma_2^0 < c(v_1^0),$$

thus

$$(4.6) \quad \left| \frac{\sigma_1^0 + c(v_1^0)}{\sigma_1^0 - c(v_1^0)} \right|, \quad \left| \frac{\sigma_2^0 - c(v_1^0)}{\sigma_2^0 + c(v_1^0)} \right| < 1.$$

Therefore condition (3.2) of Lemma 3.2 is satisfied.

Concerning the regularity, we only need to verify (3.5) without the assumptions of small initial amplitude. But, it is true because of (4.5) and Lemma 3.2 for all integer $N \geq 0$ the algebraic system

$$\begin{cases} b_1 + \frac{\sigma_1^0 + c(v_1^0)}{\sigma_1^0 - c(v_1^0)} \theta_{1,1}^N b_2 = 0, \\ b_2 + \frac{\sigma_2^0 - c(v_1^0)}{\sigma_2^0 + c(v_1^0)} \theta_{1,2}^N b_1 = 0 \end{cases}$$

has only the null solution

$$b_1 = b_2 = 0.$$

□

- Figure 1.1 -

- Figure 1.2 -

- Figure 2 - (case $p = 3$)

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