THE STRUCTURE OF SOLUTIONS TO THE PSEUDO-CONFORMALLY INVARIANT NONLINEAR SCHRÖDINGER EQUATION

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1. Introduction. In this paper we study solutions of the nonlinear Schrödinger equation

$$iu_t + \Delta u = \lambda|u|^\gamma u. \tag{1.1}$$

Here $u = u(t,x)$ is a complex valued function defined for t in some subset of the real numbers and all $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}$, and γ is the fixed power 4/n. We frequently write u(t) for the spatial function $u(t,\cdot)$. One natural point of view in studying solutions of (1.1) is to study the associated Cauchy problem, where the initial value $\phi = u(0)$ is specified. This Cauchy problem is formally equivalent to the integral equation

$$u(t) = S(t)\phi - i\lambda \int_0^t S(t-\tau)[|u(\tau)|^\gamma u(\tau)]d\tau, \tag{1.2}$$

where S(t) is the unitary group $e^{it\Delta}$ determined by the linear Schrödinger equation, i.e. when $\lambda = 0$. S(t) is given by the well-known complex Gauss kernel,

$$[S(t)\phi](x) = (4\pi it)^{-n/2} \int_{\mathbf{R}^n} \exp[-|x-y|^2/4it]\phi(y)\, dy. \tag{1.3}$$

The particular power $\gamma = 4/n$ is especially interesting for several reasons. First, the pseudo-conformal conservation law assumes a particularly simple form for equation (1.1) with this power; and solutions of this equation satisfy an additional invariance not satisfied by the equation with other powers. Moreover, this power is the smallest for which the equation (1.1) has non-global solutions (if $\lambda < 0$); and the blow-up behavior of these solutions seems qualitatively different from such behavior when γ is replaced by a larger power. Finally, this power appears in several important physical models, in particular the self-focusing of laser beams; and the self-focusing corresponds mathematically to the blowing up of solutions. See [15] for a review of some of the theoretical and numerical work done on the nonlinear Schrödinger equation, as well as some of its physical applications.

The invariance properties referred to in the previous paragraph give a correspondence between global and non-global solutions. Our goal is to use this correspondence to learn more about the asymptotic behavior of global solutions and the blow-up behavior of non-global solutions. Other authors - Ginibre and Velo [8], Weinstein [30], and Merle [22] - have made important use of the invariance properties of (1.1). However, it seems that these properties have not yet been exploited to the fullest extent possible. We propose here a step in that direction.

We begin by recalling the various conservation laws satisfied by solutions (1.1) and (1.2). See [30] (and [8]) for a more complete discussion, including historical notes. Let E and F denote the following functionals:

$$E(\phi) = \frac{1}{2}\left\{ \|\nabla\phi\|_{L^2(\mathbf{R}^n)} \right\}^2 + \frac{\lambda}{\gamma+2}\left\{ \|\phi\|_{L^{\gamma+2}(\mathbf{R}^n)} \right\}^{\gamma+2}, \tag{1.4}$$

$$F(\phi) = \mathrm{Im} \int_{\mathbf{R}^n} x\overline{\phi(x)}\cdot\nabla\phi(x)\, dx. \tag{1.5}$$

Then, formally at least, solutions of (1.1) satisfy the following conservation laws.

Conservation of charge: $\frac{d}{dt}\|u(t)\|_{L^2} = 0$ (1.6)

Conservation of energy: $\frac{d}{dt}E(u(t)) = 0$ (1.7)

Pseudo-conformal conservation:

$$\begin{cases} \frac{d}{dt}\|xu(t)\|_{L^2}^2 = 4F(u(t)) & (1.8) \\ \frac{d}{dt}F(u(t)) = 4E(u(t)) & (1.9) \end{cases}$$

(Actually, what is usually considered as the law of pseudo-conformal conservation is not formulas (1.8) and (1.9), but rather something which follows from these two relations, and which in fact, given (1.8), is equivalent to (1.9). We hope the reader is not too disturbed by this departure from convention.)

Conservation of charge is formally derived by taking the (complex $L^2(\mathbf{R}^n)$) inner product of equation (1.1) with u and then taking the imaginary part of the resulting equation. Conservation of energy is derived by taking the inner product of (1.1) with u_t and then taking the real part of the resulting equation. Formula (1.8) is obtained from (1.1) by taking the inner product with $|x|^2u$ and considering the imaginary part of the resulting equation. Finally, (1.9) follows from taking the inner product of (1.1) with $(n/2)u + x\cdot\nabla u$, and then taking the real part of the resulting equation. We remark that it is formula (1.9) that distinguishes the power $\gamma = 4/n$. For other powers, the right hand side of this formula has an additional term.

The following theorem summarizes some of the known rigorous results concerning the conservation laws, and the Cauchy problem associated with (1.1). For convenience, we consider only positive values of t. Since the equation is reversible, an analagous statement holds for negative time. (For proofs of the following facts, see [4,5,6,7,8,9,12,23].)

Theorem A. Suppose $\phi\in L^2(\mathbf{R}^n)$. There exists $T^* = T^*(\phi) > 0$ and a solution $u\in C([0,T^*),L^2(\mathbf{R}^n)) \cap L^{\gamma+2}_{loc}(0,T^*;L^{\gamma+2}(\mathbf{R}^n)$ of the integral equation (1.2) satisfying:

(i) $u\in L^{\gamma+2}(0,T;L^{\gamma+2}(\mathbf{R}^n))$ for all $T < T^*$, and is the unique solution of (1.2) in that space;

(ii) $\|u(t)\|_{L^2(\mathbf{R}^n)} = \|\phi\|_{L^2(\mathbf{R}^n)}$ for all $t\in[0,T^*)$, i.e. u respects the conservation of charge law (1.6);

(iii) If $\|\phi\|_{L^2(\mathbf{R}^n)}$ is sufficiently small then $T^* = \infty$ and $u\in L^{\gamma+2}(0,\infty;L^{\gamma+2}(\mathbf{R}^n))$;

(iv) If $T^* < \infty$, then $\|u(t)\|_{L^{\gamma+2}(0,T^*;L^{\gamma+2}(\mathbf{R}^n))} = \infty$;

(v) If $T^* < \infty$, u(t) does not have a strong L^2 limit as $t \to T^*$, not even along some subsequence;

(vi) If $T < T^*$ and if ϕ_k is a sequence in $L^2(\mathbf{R}^n)$ such that $\phi_k \to \phi$ in $L^2(\mathbf{R}^n)$, then for sufficiently large k, $T < T^*(\phi_k)$; also $u_k \to u$ in $C([0,T];L^2)$ and $L^{\gamma+2}(0,T,L^{\gamma+2})$, where u_k is the solution of (1.2) with ϕ replaced by ϕ_k.

Suppose in addition that $\phi\in H^1(\mathbf{R}^n)$. Then the solution u stays in $H^1(\mathbf{R}^n)$ throughout its trajectory and has the following additional properties:

(vii) $u\in C([0,T^*);H^1(\mathbf{R}^n)) \cap C^1([0,T^*);H^{-1}(\mathbf{R}^n))$ and satisfies the differential equation (1.1) in the sense of H^{-1};

(viii) $E(u(t)) = E(\phi)$ for all $t\in[0,T^*)$, i.e. u respects the conservation of energy law (1.7);

(ix) If $T^* < \infty$, then $\lim_{t\to T^*} \|\nabla u(t)\|_{L^2(\mathbf{R}^n)} = \infty$ and there exists a positive number C such that

$$\|\nabla u(t)\|^2_{L^2(\mathbf{R}^n)} \geq C(T^* - t)^{-1}; \tag{1.10}$$

in particular, if $\lambda > 0$ then $T^* = \infty$ for all $\phi \in H^1$;

(x) If $T < T^*$ and if ϕ_k is a sequence in $H^1(\mathbf{R}^n)$ such that $\phi_k \to \phi$ in $H^1(\mathbf{R}^n)$, then $u_k \to u$ in $C([0,T];H^1)$, where u_k is the solution of (1.2) with ϕ replaced by ϕ_k.

Suppose further that $x\phi \in L^2(\mathbf{R}^n)$. (We still suppose $\phi \in H^1$.) Then $xu(t) \in L^2(\mathbf{R}^n)$ throughout the trajectory. Moreover:

(xi) $xu \in C([0,T^*);L^2(\mathbf{R}^n))$;

(xii) The pseudo-conformal conservation laws (1.8) and (1.9) hold throughout the trajectory;

(xiii) If either $\|\phi\|_{L^2(\mathbf{R}^n)}$ is sufficiently small or $\lambda > 0$, then ($T^* = \infty$ and) $\|u(t)\|^{\gamma+2}_{L^{\gamma+2}(\mathbf{R}^n)} \leq Ct^{-2}$.

(xiv) If $T < T^*$ and if ϕ_k is a sequence such that $\phi_k \to \phi$ in $H^1(\mathbf{R}^n)$ and $x\phi_k \to x\phi$ in L^2, then $u_k \to u$ in $C([0,T];H^1)$ and $xu_k \to xu$ in $C([0,T];L^2)$, where u_k is the solution of (1.2) with ϕ replaced by ϕ_k.

Finally, if we suppose that $\phi \in L^2$ and $x\phi \in L^2$, but not necessarily that $\phi \in H^1$, then statement (xiii) is still correct.

We remark that (v) has been proved for H^1 solutions by Merle and Tsutsumi [23]. For L^2 solutions, the result has not appeared explicitly, but follows immediately from continuous dependence, statement (vi), which has appeared [6]. Indeed, if $u(t_k) \to \psi$ in L^2 as $t_k \to T^* < \infty$, then $\liminf_{k\to\infty} T^*(u(t_k)) \geq T^*(\psi) > 0$. Thus, u(t) exists for values of t arbitrarily close to $t_k + T^*(\psi)$, which is eventually larger than T^*.

The time $T^*(\phi)$ is the maximal existence time of the (forward) solution starting at ϕ; and if $T^*(\phi)$ is finite, it is called the (forward) blow-up time of the solution. We denote by $T_*(\phi) < 0$ the existence time of the backward solution starting at ϕ, called the backward blow-up time if $T_*(\phi) > -\infty$.

In addition to the conservation laws, the solutions to (1.1) exhibit other invariance properties. For example, if u(t,x) is a solution of (1.1) and if

$$v(t,x) \; = \; \overline{u(-t,x)}, \tag{1.11}$$

then v is a solution of (1.1). In particular, $T_*(\phi) = -T^*(\overline{\phi})$. Also, if $u = u(t,x)$ is a solution of (1.1) and if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{R})$, i.e. if $a,b,c,d \in \mathbf{R}$ and $ad - bc = 1$, then $u_A(t,x)$ is likewise a solution, where

$$u_A(t,x) \; = \; (a+bt)^{-n/2}\, u\Big(\frac{c+dt}{a+bt}\,,\, \frac{x}{a+bt}\Big) \exp\Big(\frac{ib|x|^2}{4(a+bt)}\Big)\,. \tag{1.12}$$

(See [30] for this precise formulation.) The subgroup $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, $c \in \mathbf{R}$, simply gives time translations of the solution u. The subgroup $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$, $a > 0$, gives a well-known group of dilation transformations, which has analogues for other powers. It is the subgroup $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in \mathbf{R}$, which is unique to this particular equation; and the corresponding invariance was discovered independently by Ginibre and Velo [8] and Weinstein [30]. (For the linear equation, i.e. with $\lambda = 0$, this invariance has been known by physicists for some time.) Also, if $c = 0$, then time $t = 0$ is fixed by the transformation (1.12). In this case if $u(0,x) = \phi(x)$, we denote $u_A(0,x)$ by

$$\phi_A(x) \; = \; a^{-n/2}\, \phi(x/a) \exp(ib|x|^2/4a). \tag{1.13}$$

If $\phi \in H^1$ and $x\phi \in L^2$, then the same is true for ϕ_A and also $u_A(t)$. (Indeed $u(t)$ has these same properties.) One checks easily that, as long as it is defined, $u_A(t)$ satisfies the equation (1.1) in the sense of H^{-1}, and thus also the integral equation. Consequently, $u_A(t)$, as long as it is defined, agrees with the solution starting at ϕ_A. By standard limit arguments, using the L^2 continuity result above, we may draw the same conclusion if $\phi \in L^2$.

Throughout this paper we will use the following notation and conventions. The complex space $L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, is denoted simply by L^p and its norm by $\|\cdot\|_p$. Similarly, H^1 denotes $H^1(\mathbf{R}^n)$, and its norm is $(\|\cdot\|_2^2 + \|\nabla(\cdot)\|_2^2)^{1/2}$. We let $X = \{\phi \in H^1 : x\phi \in L^2\}$. Then X is a Banach space with the norm $(\|\cdot\|_2^2 + \|\nabla(\cdot)\|_2^2 + \|x(\cdot)\|_2^2)^{1/2}$.

In Section 2 we set up the basic correspondence between global and non-global solutions of (1.1), and thereby derive decay estimates, asymptotic limits, and stability results for global solutions. In Section 3 we apply these ideas to scattering theory for equation (1.1). We greatly simplify and slightly extend the known theory in the space X, and we develop an L^2 theory, which is new. In addition, we derive explicit formulas for the wave operators in terms of the Fourier transform. Finally, in Section 4, we reformulate certain aspects of blow up behavior for (1.1) in terms of asymptotic behavior of global solutions to the same equation. Also, we make two modest additions to Weinstein's blow-up theorem [30] for equation (1.1), and generalize a weakened version of a result of Merle and Tsutsumi [23]. Moreover, we discuss two conflicting conjectures concerning blow up and propose a specific numerical experiment whose outcome would necessarily show one of the conjectures to be false.

2. The interplay between global and non-global solutions. In this section we draw some consequences from the transformation defined by (1.12) and (1.13). For most of what we wish to do, it suffices to consider this transformation with the matrix $A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. For a matrix A of this form we modify the notation above to make explicit the dependence on the real parameter b. Thus we have

$$u_b(t,x) = (1+bt)^{-n/2}\, u\left(\frac{t}{1+bt}, \frac{x}{1+bt}\right) \exp\left(\frac{ib|x|^2}{4(1+bt)}\right), \tag{2.1}$$

and

$$\phi_b(x) = \phi(x) \exp(ib|x|^2/4). \tag{2.2}$$

We consider $u_b(t)$ to be defined *a priori* starting at $t = 0$, and continuing in both time directions, as long as u is defined at $t/(1+bt)$, until the singularity at $t = -1/b$ is reached. In particular, $1+bt > 0$ throughout the domain of definition. Also, in this section we will systematically write $s = t/(1+bt)$, and so $t = s/(1-bs)$, as well as $(1+bt) = 1/(1-bs)$, and similarly for S and T. Simple calculations show that

$$\|u_b(t)\|_2^2 = \|u(s)\|_2^2 = \|\phi\|_2^2 = \|\phi_b\|_2^2 , \tag{2.3}$$

$$\|u_b(t)\|_{\gamma+2}^{\gamma+2} = (1 + bt)^{-2}\|u(s)\|_{\gamma+2}^{\gamma+2} = (1 - bs)^2\|u(s)\|_{\gamma+2}^{\gamma+2} , \tag{2.4}$$

$$\|u_b\|_{L^{\gamma+2}(0,T;L^{\gamma+2})} = \|u\|_{L^{\gamma+2}(0,S;L^{\gamma+2})} . \tag{2.5}$$

We easily obtain the following information about asymptotic behavior of global solutions.

Theorem 2.1. (a) Let $\phi \in X$, ϕ not identically zero, with $T^*(\phi) = \infty$. Then

$$\liminf_{s\to\infty} (1+s)^2 \|u(s)\|_{\gamma+2}^{\gamma+2} > 0,$$

where u(t) is the solution to the integral equation (1.2) with initial value ϕ.

(b) Let $\phi \in X$, not identically zero, with $T^*(\phi) = \infty$, and suppose that $\liminf_{s\to\infty} (1+s) \|u(s)\|_{\gamma+2}^{\gamma+2} = 0$. Then $\lim_{t\to\infty} (1+s)^2 \|u(s)\|_{\gamma+2}^{\gamma+2}$ exists, is finite and non-zero. Also, there exists $\xi \in X$, not identically zero, such that

$$X - \lim_{s\to\infty} (1+s)^{n/2} u(s,x(1+s)) \exp(-i(1+s)|x|^2/4) = \xi(x). \tag{2.6}$$

(c) Let $\phi \in L^2$, not identically zero, with $T^*(\phi) = \infty$. If $\|u\|_{L^{\gamma+2}(0,\infty;L^{\gamma+2})} < \infty$, then $\liminf_{s\to\infty} (1+s)^2 \|u(s)\|_{\gamma+2}^{\gamma+2} > 0$, and there exists $\xi \in L^2$, not identically zero, such that

$$L^2 - \lim_{s\to\infty} (1+s)^{n/2} u(s,x(1+s)) \exp(-i(1+s)|x|^2/4) = \xi(x). \tag{2.7}$$

(d) The set of ξ achieved as the limit in (2.6) or (2.7) is precisely the set of ξ in X or L^2 respectively such that $T_*(\xi) < -1$.

Proof. Let $\phi \in X$ with $T^*(\phi) = \infty$, and suppose $\liminf_{s\to\infty} (1+s)^2 \|u(s)\|_{\gamma+2}^{\gamma+2} = 0$. We consider $v = u_{-1}$, i.e. we fix $b = -1$, and we denote $\psi = v(0) = \phi_{-1}$. It follows from (2.4) that $\liminf_{t\to 1} \|v(t)\|_{\gamma+2} = 0$, and so, by conservation of energy, $\liminf_{t\to 1} \|\nabla v(t)\|_2 < \infty$. Therefore, $T^*(\psi) > 1$, and $v(t)$ has a limit in X as $t \to 1^-$, which we call ξ. On the other hand, since, along a subsequence, $v(t) \to 0$ in $L^{\gamma+2}$, we must have $\xi \equiv 0$. By uniqueness of the (backwards) solution, it follows that $v(t) \equiv 0$, which is impossible since ϕ is not identically zero.

Suppose next that $\phi \in X$, ϕ not identically zero, $T^*(\phi) = \infty$, and $\liminf_{s\to\infty} (1+s) \|u(s)\|_{\gamma+2}^{\gamma+2} = 0$. Using the same notation as just above, it follows that $\liminf_{t\to 1} (1-t) \|v(t)\|_{\gamma+2}^{\gamma+2} = 0$. Again by conservation of energy, it follows that $\liminf_{t\to 1} (1-t) \|\nabla v(t)\|_2^2 = 0$. Therefore, by part (ix) of Theorem A, $T^*(\psi) > 1$; and so $v(t)$ has a limit in X as $t \to 1^-$, which we call ξ. (ξ is not identically zero by backwards uniqueness.) In particular, $\|v(t)\|_{\gamma+2}^{\gamma+2} \to \|\xi\|_{\gamma+2}^{\gamma+2}$; and so $(1+s)^2 \|u(s)\|_{\gamma+2}^{\gamma+2} \to \|\xi\|_{\gamma+2}^{\gamma+2}$. Formula (2.6) now follows from the definition of $v = u_{-1}$, i.e. formula (2.1).

Finally, suppose $\phi \in L^2$, ϕ not identically zero, $T^*(\phi) = \infty$, and $\|u\|_{L^{\gamma+2}(0,\infty;L^{\gamma+2})} < \infty$. It follows from (2.5) that $\|v\|_{L^{\gamma+2}(0,1;L^{\gamma+2})} < \infty$, and so by part (iv) of Theorem A that $T^*(\psi) > 1$. Thus, $v(t)$ has a limit in L^2 as $t \to 1^-$, which we call ξ. If $\liminf_{s\to\infty} (1+s)^2 \|u(s)\|_{\gamma+2}^{\gamma+2} = 0$, then $\liminf_{t\to 1} \|v(t)\|_{\gamma+2}^{\gamma+2} = 0$, which means that $\xi \equiv 0$, which is impossible. Thus $\liminf_{s\to\infty} (1+s)^2 \|u(s)\|_{\gamma+2}^{\gamma+2} > 0$, and formula (2.7) follows as just above.

The last statement in the theorem follows easily from the construction of ξ.

Definition 2.2. Let $\phi \in L^2$ with $T^*(\phi) = \infty$, and let $u(t)$ be the solution of (1.2) with initial value ϕ. We say $u(t)$ *decays rapidly* or has *rapid (forward) decay* if $u \in L^{\gamma+2}(0,\infty;L^{\gamma+2})$. We denote by $\mathcal{R}$ the set of such ϕ giving rise to rapidly decaying solutions; and $\mathcal{R}_X$ denotes $\mathcal{R} \cap X$. If $\phi \in L^2$ and $T^*(\phi) = \infty$, but $\|u\|_{L^{\gamma+2}(0,\infty;L^{\gamma+2})} = \infty$, we say $u(t)$ *does not decay rapidly*. (Such a solution does not necessarily even decay, as in the case of a stationary state, so we can not say it decays slowly.) The set of such ϕ is denoted by $\mathcal{N}$, and $\mathcal{N}_X$ denotes $\mathcal{N} \cap X$. Finally, $\mathcal{B}$ denotes the set of $\phi \in L^2$ with $T^*(\phi) < \infty$, and $\mathcal{B}_X$ denotes $\mathcal{B} \cap X$.

The above theorem gives the precise asymptotic behavior for rapidly decaying solutions, and incidently shows that the decay rate given by part (xiii) of Theorem A is optimal. Also, for $\phi \in X$, in order that $\phi \in \mathcal{R}$, it suffices that $\liminf_{s\to\infty} (1+s) \|u(s)\|_{\gamma+2}^{\gamma+2} = 0$, which is slightly weaker than the criterion given in the definition. Theorem A says that if $\|\phi\|_2$ is sufficiently small, then $\phi \in \mathcal{R}$. Also, it follows easily from the next propostion that $\mathcal{R}$ contains elements with arbitrarily large L^2 norm and that $\mathcal{R}_X$ contains elements with arbitrarily large X norm. Finally, if $\lambda > 0$, except perhaps if $\|\phi\|_2$ is large and $x\phi \notin L^2$, then $\phi \in \mathcal{R}$. In particular, if $\lambda > 0$, $\mathcal{R}_X = X$.

The proof of the previous theorem uses the fact that if u(s) is a (forward) global solution, then $u_b(t)$ might or might not blow up as $t \to -1/b$ (where $b < 0$). The following proposition shows, among other things, that this is one way to distinguish between $\mathcal{R}$ and $\mathcal{N}$.

Proposition 2.3. (a) Let $\phi \in L^2$. Then either $T^*(\phi_b) = \infty$ for all real b, or there exists $b_0 \in \mathbf{R}$ such that $T^*(\phi_b) = \infty$ for all $b \geq b_0$ and $T^*(\phi_b) < \infty$ for all $b < b_0$. In the former case, $\phi_b \in \mathcal{R}$ for all real b. In the latter case $\phi_b \in \mathcal{R}$ for all $b > b_0$ and $\phi_{b_0} \in \mathcal{N}$.

(b) $\mathcal{N}$ is the set of all $\phi \in L^2$ such that $T^*(\phi_b) = \infty$ for all $b \geq 0$ and $T^*(\phi_b) < \infty$ for all $b < 0$. If $\phi \in \mathcal{N}$, then $T^*(\phi_b) = -1/b$ for all $b < 0$. If $\phi \in \mathcal{R}$, then $T^*(\phi_b) > -1/b$ for all $b < 0$.

(c) If $T^*(\phi) = \infty$, then $\phi_b \in \mathcal{R}$ for all $b > 0$.

(d) Suppose $\phi \in \mathcal{B}$. If $b < 1/T^*(\phi)$, then $T^*(\phi_b) = [T^*(\phi)^{-1} - b]^{-1} < \infty$; if $b = 1/T^*(\phi)$, then $\phi_b \in \mathcal{N}$; if $b > 1/T^*(\phi)$, then $\phi_b \in \mathcal{R}$.

(e) $\phi \in \mathcal{B}$, $\phi \in \mathcal{N}$, or $\phi \in \mathcal{R}$, according as $T^*(\phi_{-b}) < 1/b$, $T^*(\phi_{-b}) = 1/b$, or $T^*(\phi_{-b}) > 1/b$, for some (or all) $b > 0$.

(f) $\mathcal{B} = \cup_{b<0} \{\phi_b : \phi \in \mathcal{N}\}$.

Proof. We begin with the remark that $(\phi_c)_b = \phi_{c+b}$ and $(u_c)_b = u_{c+b}$ for all real numbers c and b.

(a) Suppose first that $T^*(\phi_b) = \infty$ for all real b. If $\phi_c \in \mathcal{N}$ for some c, then $\|u_c\|_{L^{\gamma+2}(0,\infty;L^{\gamma+2})} = \infty$; and so by (2.5) $\|(u_c)_b\|_{L^{\gamma+2}(0,-1/b;L^{\gamma+2})} = \infty$ for all $b < 0$. Thus, $T^*(\phi_{c+b}) = -1/b$ for all $b < 0$. This contradicts the assumption on ϕ, and therefore shows that $\phi_b \in \mathcal{R}$ for all real b. (This, by the way, also proves the second assertion in part (b).) Suppose next that $T^*(\phi_c) < \infty$ for some real c. Again it follows from (2.5), applied with u_c on the right side and with $S = T^*(\phi_c)$, that $T^*(\phi_{c+b}) = [T^*(\phi_c)^{-1} - b]^{-1} < \infty$ for all $b < 1/T^*(\phi_c)$. Also, from (2.1) with $S = 1/b$, it follows that $T^*(\phi_{c+b}) = \infty$ for all $b \geq 1/T^*(\phi_c)$. This proves the existence of b_0, such that $T^*(\phi_b) = \infty$ for all $b \geq b_0$ and

$T^*(\phi_b) < \infty$ for all $b < b_0$. Moreover, it is clear from (2.5) with $S = 1/b$, that $\|(u_c)_b\|_{L^{\gamma+2}(0,\infty;L^{\gamma+2})} = \infty$ if $b = 1/T^*(\phi_c)$ and $\|(u_c)_b\|_{L^{\gamma+2}(0,\infty;L^{\gamma+2})} < \infty$ if $b > 1/T^*(\phi_c)$. This completes the proof of part (a), as well as part (d).

(b) Suppose $\phi \in L^2$, $T^*(\phi_b) = \infty$ for all $b \ge 0$, and $T^*(\phi_b) < \infty$ for all $b < 0$. In the terminology of part (a), we have $b_0 = 0$. Thus, by part (a), $\phi = \phi_0 \in \mathcal{N}$. On the other hand, suppose $\phi \in \mathcal{N}$. In the proof of part (a) we already proved the second assertion of part (b); and so in particular $T^*(\phi_b) < \infty$ for all $b < 0$. Finally, we can not have $T^*(\phi_c) < \infty$ for some $c > 0$, for that would imply by part (a) that $T^*(\phi) = T^*((\phi_c)_{-c}) < \infty$. The last assertion in part (b) is proved by contradiction. It is clear from the definition of u_b that $T^*(\phi_b) \ge -1/b$ for all $b < 0$. If $T^*(\phi_b) = -1/b$, then part (d), which has already been proved, applied to ϕ_b shows that $\phi = (\phi_b)_{-b} = (\phi_b)_{1/T^*(\phi_b)} \in \mathcal{N}$. This is the desired contradiction.

The proof of (c) uses the same kind of calculations as the above arguments, and can safely be left to the reader. We have already proved part (d); parts (e) and (f) are restatements of some of the earlier parts.

Remark 2.4. The following result shows in a precise way that rapidly decaying global solutions are stable. On the other hand, the previous proposition shows that global solutions which are not rapidly decaying are unstable: multiplying the initial value by $\exp(ib|x|^2/4)$ for arbitrarily small negative b results in a solution which blows up in finite time, and multiplying the initial value by $\exp(ib|x|^2/4)$ for arbitrarily small positive b results in a solution which is global and rapidly decaying. (See Theorem C in Weinstein [29] for an instability result in a special case.) We therefore obtain a complete stability analysis for global solutions.

Theorem 2.5. The set $\mathcal{R}$ of initial values giving rise to (forward) rapidly decaying solutions is open in L^2, and the asymptotic limit ξ defined by (2.7) depends L^2-continuously on $\phi \in \mathcal{R}$. Similarly, the set $\mathcal{R}_X$ is open in X, and the asymptotic limit ξ defined by (2.6) depends X-continuously on $\phi \in \mathcal{R}_X$.

Proof. Note first that the mapping $\eta \to \eta_b$ is a continuous bijection of L^2 with continuous inverse. Let $\phi \in \mathcal{R}$. Choose any convenient $b < 0$, and let $\psi = \phi_b$. Since $\phi \in \mathcal{R}$, $T^*(\psi) > -1/b$. We let v denote the complete solution starting from ψ. In particular, we consider $v(-1/b)$, even though $u_b(-1/b)$ is not defined. By L^2 continuity (part (vi) of Theorem A), there exists an L^2 neighborhood $\mathcal{V}$ of ψ such that $T^*(\eta) > -1/b$ for all $\eta \in \mathcal{V}$ and on which the mapping $\eta \to u_\eta(-1/b)$ is L^2 continuous. ($u_\eta(t)$ denotes the solution to (1.2) with initial value η.) It follows from parts (c) and (d) of the previous proposition that $\eta_{-b} \in \mathcal{R}$ for all $\eta \in \mathcal{V}$. Moreover, if we fix $b = -1$, then ξ is simply $v(1) = u_\psi(1)$. Since $u_\eta(1)$ depends continuously on η, hence on η_{-b}, it follows that ξ depends continuously on ϕ. The corresponding statement about $\mathcal{R}_X$ can be proved in the same way, but in fact it follows from the statement about $\mathcal{R}$. This completes the proof.

Remark 2.6. It is clear that spatial translation of ϕ does not change $T^*(\phi)$, or whether ϕ is in $\mathcal{N}$ or $\mathcal{R}$. The same is true if we multiply ϕ by the function $\exp(-ix\cdot x_0)$ for any $x_0 \in \mathbf{R}^n$. Indeed, by multiplying ϕ by $\exp(ib|x|^2/4)$, translating by x_0, and then multiplying by $\exp(-ib|x|^2/4)$, one can compute that if $u(t)$ is the solution with initial value ϕ, then $w(t)$ is the solution with initial value $\phi\exp(-ix\cdot x_0)$, where $w(t,x) = \exp(-ix\cdot x_0)\exp(-it|x_0|^2)\, u(t,x+2tx_0)$.

The results of this section show that studying the blow-up behavior of non-global solutions of (1.1) and (1.2) is exactly equivalent to studying the long time asymptotic behavior of global solutions whose initial values are in the set $\mathcal{N}$. In particular, formula (2.4) shows how to translate between blow-up rates and asymptotic decay rates.

3. Application to scattering theory.

Scattering for equation (1.1) has already been studied by Ginibre and Velo [7] and Y. Tsutsumi [26]. In particular, if $\lambda > 0$, it is known that the scattering operator is defined on the entire space X, and is continuous and surjective. If $\lambda < 0$, the scattering operator is defined and continuous for small data in X.

In this section we show that the transformation (2.1) yields an explicit formula for the wave operators, $W_\pm(\phi) = \lim_{t\to\pm\infty} S(-t)u(t)$. This simplifies and extends the existing theory in X. In particular, if $\lambda < 0$, the wave operators each have unbounded domains, which raises the possibility, as yet unresolved, that the scattering operator has an unbounded domain. Moreover, our arguments work equally well in X or L^2, and so we develop two parallel theories.

In addition to the notation of the previous section, we introduce the dilation operator D_β and the multiplication operator M_b defined respectively by

$$(D_\beta\phi)(x) = \beta^{n/2}\phi(\beta x), \qquad (M_b\phi)(x) = \phi(x)\exp(ib|x|^2/4), \tag{3.1}$$

where $\beta\in\mathbf{R}$, $\beta\neq 0$, and $b\in\mathbf{R}$. Using the explicit kernel (1.3), one easily verifies that

$$S(t)D_\beta = D_\beta S(\beta^2 t), \tag{3.2}$$

$$S(t)M_b = M_{b/(1+bt)}D_{1/(1+bt)}S(t/1+bt)), \text{ if } t \neq -1/b. \tag{3.3}$$

In fact, formula (3.3) simply expresses the invariance of solutions of (1.1) with $\lambda = 0$ under the transformation (2.1). Moreover, one can rewrite $u_b(t)$ using the operators D_β and M_b as

$$u_b(t) = M_{b/(1+bt)}D_{1/(1+bt)}u(t/(1+bt)). \tag{3.4}$$

Recall that $1+bt > 0$ throughout the domain of definition of u_b. We mention that in case $t = -1/b$, formula (3.3) is replaced by

$$S(1/b)M_{-b} = i^{-n/2}M_b D_{b/4\pi}\mathcal{F}, \text{ if } b\neq 0, \tag{3.5}$$

where for convenience we have interchanged b and $-b$, and where $\mathcal{F}$ denotes the Fourier transform,

$$\mathcal{F}\phi(x) = \int_{\mathbf{R}^n}\exp(-2\pi i x\cdot y)\phi(y)\,dy. \tag{3.6}$$

Now we wish to calculate $S(-t)u_b(t)$, and for this we use (3.4), (3.3). and (3.2). (We set $a = b/(1+bt)$, and so $a/(1-at) = b$ and $1/(1-at) = 1+bt$.)

$$\begin{aligned} S(-t)u_b(t) &= S(-t)M_{b/(1+bt)}D_{1/(1+bt)}u(t/(1+bt)) \\ &= S(-t)M_a D_{1/(1+bt)}u(t/(1+bt)) \\ &= M_b D_{1+bt}S(-t(1+bt))D_{1/(1+bt)}u(t/(1+bt)) \\ &= M_b D_{1+bt}D_{1/(1+bt)}S(-t/(1+bt))u(t/(1+bt)) \\ &= M_b S(-t/(1+bt))u(t/(1+bt)). \end{aligned} \tag{3.7}$$

Renaming u_b to be u, and hence changing u to u_{-b}, in (3.7) gives

$$S(-t)u(t) = M_b S(-t/(1+bt))u_{-b}(t/(1+bt)), \tag{3.8}$$

as long as $t \neq -1/b$. More precisely, if $b > 0$, then (3.8) is valid for all $t \in (-1/b,\infty)$ for which $u_{-b}(t/(1+bt))$ is defined; and if $b < 0$, then (3.8) is valid for all $t \in (-\infty,-1/b)$ for which $u_{-b}(t/(1+bt))$ is defined.

Theorem 3.1. Let $\phi \in X$ (respectively L^2) with $T^*(\phi) = \infty$, and let u be the solution of (1.2) with initial value ϕ. The wave operator $W_+(\phi) =_{def} X\text{-}\lim_{t\to\infty} S(-t)u(t)$, (respectively $L^2\text{-}\lim_{t\to\infty} S(-t)u(t)$), is defined if and only if $\phi \in \mathcal{R}$, and is given by the formula

$$W_+(\phi) = i^{n/2}\mathcal{F}^{-1}M_{-4\pi}u_{-4\pi}(1/4\pi), \tag{3.9}$$

where, by abuse of notation, $u_{-4\pi}(1/4\pi) = \lim_{t\to 1/4\pi} u_{-4\pi}(t)$, (which exists since $\phi \in \mathcal{R}$). In particular, $\eta \in X$ (respectively L^2) is in the image of W_+ if and only if $\overline{\mathcal{F}\eta} \in \mathcal{R}$, and $W_+ : \mathcal{R}_X \to \mathcal{F}^{-1}(\overline{\mathcal{R}_X})$ (respectively $\mathcal{R} \to \mathcal{F}^{-1}(\overline{\mathcal{R}})$) is a bicontinuous bijection.

Let $\phi \in X$ (respectively L^2) with $T_*(\phi) = -\infty$, and let u be the solution of (1.2) with initial value ϕ. The wave operator $W_-(\phi) =_{def} X\text{-}\lim_{t\to-\infty} S(-t)u(t)$, (respectively $L^2\text{-}\lim_{t\to-\infty} S(-t)u(t)$), is defined if and only if $\phi \in \overline{\mathcal{R}}$, and is given by the formula

$$W_-(\phi) = i^{-n/2}\mathcal{F} M_{4\pi}u_{4\pi}(-1/4\pi), \tag{3.10}$$

where, by abuse of notation, $u_{4\pi}(-1/4\pi) = \lim_{t\to-1/4\pi} u_{4\pi}(t)$, (which exists since $\phi\in\overline{\mathcal{R}}_-$). In particular, $\eta\in X$ (respectively L^2) is in the image of W_- if and only if $\mathcal{F}^{-1}\eta \in \overline{\mathcal{R}}_-$ and $W_- : \overline{\mathcal{R}}_X \to \mathcal{F}(\mathcal{R}_X)$ (respectively $\overline{\mathcal{R}} \to \mathcal{F}(\mathcal{R})$) is a bicontinuous bijection.

Proof. We give here the proof in X. To obtain the proof in L^2, it suffices to replace X by L^2 throughout the following argument.

Let $\phi\in X$ with $T^*(\phi) = \infty$, and let u be the solution of (1.2) with initial value ϕ. If $b > 0$, then $u_{-b}(s)$ is certainly defined for $0 \le s < 1/b$, and so formula (3.8) is certainly valid for all $t \ge 0$. If $\phi\in \mathcal{R}$, then $u_{-b}(1/b) =_{def} \lim_{s\to 1/b} u_{-b}(s)$ exists; and so

$$W_+(\phi) =_{def} X\text{-}\lim_{t\to\infty} S(-t)u(t) = M_bS(-1/b)u_{-b}(1/b)$$

exists. Choosing $b = 4\pi$, we immediately deduce (3.9) from the (inverse of) formula (3.5). On the other hand, if $W_+(\phi) =_{def} X\text{-}\lim_{t\to\infty} S(-t)u(t)$ is well-defined, then from (3.8) we see that $X\text{-}\lim_{s\to 1/b} u_{-b}(s)$ exists. Thus $T^*(\phi_{-b}) > 1/b$, and so $\phi\in \mathcal{R}$.

Suppose $\eta = W_+(\phi)$. Then $\mathcal{F}\eta = i^{n/2}M_{-4\pi}u_{-4\pi}(1/4\pi)$. It is clear that $T_*(u_{-4\pi}(1/4\pi)) < -1/4\pi$, and so $M_{-4\pi}u_{-4\pi}(1/4\pi) \in \overline{\mathcal{R}}$. Since multiplying an initial value by a constant of modulus 1 has the same effect on the entire solution, it follows that $\mathcal{F}\eta \in \overline{\mathcal{R}}$. On the other hand, if $\mathcal{F}\eta \in \overline{\mathcal{R}}$, then $T_*(M_{4\pi}\mathcal{F}\eta) < -1/4\pi$. We let v(t) be the solution of (1.1) with $v(0) = M_{4\pi}\mathcal{F}\eta$, and we set $\phi = i^{-n/2}M_{4\pi}v(-1/4\pi)$. Since $T^*(v(-1/4\pi)) > 1/4\pi$, it follows that $\phi\in \mathcal{R}$. Also, $i^{n/2}u_{-4\pi}(1/4\pi)$ is none other than v(0), which shows that $\eta = i^{n/2}\mathcal{F}^{-1}M_{-4\pi}u_{-4\pi}(1/4\pi)$, i.e. $\eta = W_+(\phi)$.

Continuity of W_+ and its inverse follows from the continuous dependence properties of solutions to (1.1).

The proof of the corresponding facts for W_- is entirely analagous.

Corollary 3.2. $\mathcal{F}^{-1}W_-\mathcal{F}W_+ : \mathcal{R}_X \to \mathcal{R}_X$ (respectively $\mathcal{R} \to \mathcal{R}$) is the identity mapping.

Proof. This is an immediate consequence of formulas (3.9) and (3.10).

Remark 3.3. The scattering operator $W_+(W_-)^{-1}$ in X is a continuous bijection from $W_-(\mathcal{R}_X \cap \overline{\mathcal{R}}_X)$ onto $W_+(\mathcal{R}_X \cap \overline{\mathcal{R}}_X)$. We have already seen that $\mathcal{R}_X$ and $\overline{\mathcal{R}}_X$ are each unbounded in X. It would be very interesting to know, in the case $\lambda < 0$, if in fact $\mathcal{R}_X \cap \overline{\mathcal{R}}_X$ is also unbounded. (See Proposition 4.10 - part (d) - below for a condition which implies that $\mathcal{R}_X \cap \overline{\mathcal{R}}_X$ is indeed unbounded.) Of course, if $\lambda > 0$, then $\mathcal{R}_X = X$, as we have already remarked just before Proposition 2.3.

In view of the above results, it would certainly be interesting to give an independent characterization of the set $\mathcal{F}(\mathcal{R}_X)$ when $\lambda < 0$. The following result is a step in that direction, and we find its proof at least as interesting as the result itself.

Proposition 3.4. If $\lambda < 0$, then $\mathcal{F}(\mathcal{R}_X)$ is not a subset of $\mathcal{R}_X$, and $\mathcal{F}(\mathcal{R})$ is not a subset of $\mathcal{R}$.

Proof. We first derive a formula involving the Fourier transform. From the definition of S(t) in terms of the Fourier transform, we know that

$$S(t) = \mathcal{F}^{-1} M_{-16\pi^2 t} \mathcal{F}. \tag{3.11}$$

Formula (3.11) with $t = 1/4\pi$ and formula (3.5) with $b = 4\pi$ combine to give

$$i^{-n/2} M_{4\pi} \mathcal{F} M_{4\pi} \mathcal{F} M_{4\pi} \mathcal{F}^{-1} = \text{identity}. \tag{3.12}$$

Now we assume to the contrary that $\mathcal{F}(\mathcal{R}_X) \subset \mathcal{R}_X$. We clearly also have $\mathcal{F}^{-1}(\mathcal{R}_X) \subset \mathcal{R}_X$; and so in fact $\mathcal{F}(\mathcal{R}_X) = \mathcal{R}_X$. Let $\phi \in \mathcal{R}_X$. Since $\mathcal{F}, \mathcal{F}^{-1}$, and $M_{4\pi}$ all preserve $\mathcal{R}_X$, it follows that $\psi =_{\text{def}} i^{-n/2} \mathcal{F} M_{4\pi} \mathcal{F} M_{4\pi} \mathcal{F}^{-1} \phi \in \mathcal{R}_X$. In other words, every $\phi \in \mathcal{R}_X$ is of the form $M_{4\pi}\psi$ for some $\psi \in \mathcal{R}$. This is clearly false since $M_{-4\pi}$ does not preserve $\mathcal{R}_X$. (That $M_{-4\pi}$ does not preserve $\mathcal{R}_X$ follows since if $\lambda < 0$, there exist non-global solutions, and $\mathcal{N}_X$ is non-empty. This is well-known; and we will recall the proof in the next section.) The proof in L^2 is the same.

4. Energy and blow up. With the exception of Proposition 4.9 below, throughout this section we consider solutions to (1.1) only in the space X. Also, we assume that $\lambda < 0$, and to simplify the formulas, that $\lambda = -1$. Let $\phi \in X$ and u(t) the resulting solution of (1.1) with $u(0) = \phi$. We continue with the same notation defined by (2.1), (2.2), and (3.1).

It follows easily from the conservation laws (1.7), (1.8), and (1.9) that

$$\|xu(t)\|_2^2 = \|x\phi\|_2^2 + 4tF(\phi) + 8t^2E(\phi) \tag{4.1}$$

throughout the entire trajectory. This law gives the standard proof of the existence of non-global solutions: indeed, if the quadratic polynomial on the right side of (4.1) is ever negative, which occurs if for example $E(\phi) < 0$, then the solution u(t) must blow up before it reaches that point. (See Glassey [10].) Also, the law (4.1) immediately gives a necessary criterion for a function ϕ to be in $\mathcal{R}$.

Proposition 4.1. If $\phi \in \mathcal{R}_X$ is not identically zero, then $E(\phi) > 0$.

Proof. From the inequality (Weinstein [29], page 573) $\|\psi\|_2^2 \le (2/n)\|\nabla\psi\|_2\|x\psi\|_2$ and the conservation law (1.6), it follows that $\|\nabla u(t)\|_2^2 \ge (n^2/4)\|\phi\|_4^2/\|xu(t)\|_2^2$. If now we also assume that $T^*(\phi) = \infty$ and $E(\phi) = 0$, then $(2/(\gamma+2))\|u(t)\|_{\gamma+2}^{\gamma+2} \ge (n^2/4)\|\phi\|_4^2/[\|x\phi\|_2^2 + 4tF(\phi)]$. In particular,

$$\liminf_{t\to\infty} (1+t)\|u(t)\|_{\gamma+2}^{\gamma+2} > 0,$$

and so u(t) is not rapidly decaying.

It turns out that the polynomial in (4.1) is of interest even for values of t at which the solution u(t) is not defined. Thus, for $\phi \in X$, we define the *associated polynomial* $P[\phi](t)$ by

$$P[\phi](t) = \|x\phi\|_2^2 + 4tF(\phi) + 8t^2E(\phi). \tag{4.2}$$

We note that $F(\overline{\phi}) = -F(\phi)$, and so $P[\overline{\phi}](t) = P[\phi](-t)$. Also, one naturally wonders how this polynomial changes when ϕ is replaced by ϕ_b. Routine calculations show that

$$F(\phi_b) = F(\phi) + \frac{b}{2}\|x\phi\|_2^2 , \tag{4.3}$$

and

$$E(\phi_b) = E(\phi) + \frac{b}{2}F(\phi) + \frac{b^2}{8}\|x\phi\|_2^2 = \frac{b^2}{8}P[\phi](\tfrac{1}{b}), \tag{4.4}$$

where of course the very last equality above only holds for $b \neq 0$. From (4.3) and (4.4), it is a simple matter to verify that

$$P[\phi_b](t) = (1+tb)^2\|x\phi\|_2^2 + 4t(1+tb)F(\phi) + 8t^2E(\phi), \tag{4.5}$$

$$P[\phi_b](t) = (1+tb)^2P[\phi](\tfrac{t}{1+tb}), \text{ if } t \neq -1/b, \tag{4.6}$$

$$P[\phi_b](-1/b) = \frac{8}{b^2}E(\phi). \tag{4.7}$$

We immediately deduce the following consequences.

Proposition 4.2. If $\phi \in X$, not identically zero, is such that $\phi_b \in \mathcal{R}$ for all $b \in \mathbf{R}$, then $P[\phi](t) > 0$ for all $t \in \mathbf{R}$.

Proof. Since $\phi_b \in \mathcal{R}_X$ it follows from Proposition 4.1 that $E(\phi_b) > 0$. Since $P[\phi](0) > 0$, the result now follows from the right most part of (4.4).

This result is interesting in that the hypothesis seemingly concerns only the forward asymptotic behavior of solutions, yet implies something about the associated polynomial for all real t. Since

$P[\overline{\phi}](t) = P[\phi](-t)$, one is led to ask if the condition "$\phi \in X$ is such that $\phi_b \in \mathcal{R}$ for all $b \in \mathbf{R}$" is equivalent to the same condition for $\overline{\phi}$. (See Proposition 4.10 - part (d) - below for a condition which implies just that, at least for radially symmetric functions ϕ.)

Proposition 4.3. (a) Let $\phi \in \mathcal{N}_X$, and let $b < 0$. Then $\|xu_b(t)\|_2^2 \to 0$ as $t \to T^*(\phi_b)$ if and only if $E(\phi) = 0$.

(b) Let $\phi \in \mathcal{B}_X$ and let $b = 1/T^*(\phi)$, the unique real number such that $\phi_b \in \mathcal{N}$. Then $\|xu(t)\|_2^2 \to 0$ as $t \to T^*(\phi)$ if and only if $E(\phi_b) = 0$.

Proof. In view of Proposition 2.3, the two statements are equivalent, and so we prove (a). Since $\phi \in \mathcal{N}_X$, we know that $T^*(\phi_b) = -1/b$. The result is now an immediate consequence of (4.5).

Proposition 4.4. (a) Let $\phi \in \mathcal{N}_X$ be such that $F(\phi) = 0$ and $E(\phi) = 0$, and let $b < 0$. Then there exists a positive number C such that

$$\|\nabla u_b(t)\|_2^2 \geq C(T^*(\phi_b) - t)^{-2}. \tag{4.8}$$

(b) Let $\phi \in \mathcal{B}_X$ and let $b = 1/T^*(\phi)$, the unique real number such that $\phi_b \in \mathcal{N}$. If $F(\phi_b) = E(\phi_b) = 0$, then there exists a positive number C such that

$$\|\nabla u(t)\|_2^2 \geq C(T^*(\phi) - t)^{-2}. \tag{4.9}$$

Proof. As in the previous proposition, the two statements are equivalent, and again we prove (a). We start by repeating the proof of Proposition 4.1, using u_b in place of u. This gives

$$\|\nabla u_b(t)\|_2^2 \geq (n^2/4)\|\phi\|_4^2/\|xu_b(t)\|_2^2.$$

Since $T^*(\phi_b) = -1/b$, it follows from (4.5) that

$$\|xu_b(t)\|_2^2 = (T^*(\phi_b) - t)^2\|x\phi(t)\|_2^2/T^*(\phi_b)^2,$$

from which the result follows.

Note that Proposition 4.4 gives a sufficient condition under which the lower blow-up estimate (1.10) can be significantly improved.

Proposition 4.5. $\mathcal{N}_X \neq \overline{\mathcal{N}}_X$.

Proof. We observe first that if ϕ and $\overline{\phi}$ both belong to $\mathcal{N}_X$, then $T^*(\phi_b) = \infty$ for all $b \geq 0$ and $T_*(\phi_b) = -\infty$ for all $b \leq 0$. Therefore, $E(\phi_b) \geq 0$ for all $b \in \mathbf{R}$.

Now choose $\psi \in X$ with $E(\psi) < 0$. Clearly $T^*(\psi) < \infty$, and so there exists a unique $b > 0$ such that $\psi_b \in \mathcal{N}_X$. If also $\psi_b \in \overline{\mathcal{N}}_X$, then by the observation just above, we would have $E(\psi) = E((\psi_b)_{-b}) \geq 0$. Thus, $\psi_b \in \mathcal{N}_X$, but $\psi_b \notin \overline{\mathcal{N}}_X$.

Next we show that the continuity properties of the blow-up time T* are strongly related to the structure of the set $\mathcal{N}_X$. By Theorem A, $T^*: X \to (0,\infty]$ is lower semicontinuous. Moreover, it is not continuous. Indeed, if $\phi \in X$ has small L^2 norm then by Theorem A, $T^*(\phi) = \infty$. Furthermore, an example of Weinstein [30], which we will discuss below in more detail, shows that there exists $\phi \in \mathcal{B}_X$ of minimal L^2 norm. Clearly, T* is not continuous at such a ϕ. On the other hand, one might hope that $T^*: \mathcal{B}_X \to (0,\infty)$, or perhaps $T^*: \mathcal{B}_X \cup \mathcal{N}_X \to (0,\infty]$, is continuous.

Theorem 4.6. (a) Let B be a closed ball in X such that $W = B \cap \mathcal{N}_X$ is closed. Then $T^*: \cup_{b \in \mathbf{R}}[M_bW] \to (0,\infty]$ is continuous. Furthermore, the rate of blow up is continuous in the following sense. Let $\phi_k \to \phi$ in X as $k \to \infty$, where $\phi_k, \phi \in \cup_{b<0}[M_bW]$; and let u_k and u denote the solutions of (1.1) with initial values ϕ_k and ϕ respectively. Then for all $t \in (0,T^*(\phi))$,

$$\|\nabla u_k(T^*(\phi_k) - t)\|_2^2 \to \|\nabla u(T^*(\phi) - t)\|_2^2 \tag{4.10}$$

as $k \to \infty$.

(b) Suppose $\mathcal{N}_X$ is closed. Let $m \geq 0$ be arbitrary but fixed, and denote by $\mathcal{N}_m$ the closed set $\{\phi \in \mathcal{N}_X : F(\phi) \geq -m\}$. Then $T^*: \cup_{b \in \mathbf{R}} [M_b(\mathcal{N}_m)] \to (0,\infty]$ is continuous; in particular, T^* is continuous on the open set $\{\phi \in X : T^*(\phi) < \infty$ and $P[\phi](t) < 0$ for some $t > 0\}$. Moreover, the rate of blow up is continuous on $\cup_{b<0} [M_b(\mathcal{N}_m)]$ in the sense described in statement (a).

(c) On the other hand, suppose ϕ_k is a sequence in $\mathcal{N}_X$, $\phi_k \to \phi$ in X, but $\phi \notin \mathcal{N}_X$. Then $T^*(\phi) < \infty$, and $T^*: \cup_{b \in \mathbf{R}} [M_b(\mathcal{N}_X)] \to (0,\infty]$ is discontinuous at ϕ_b for all $b < 1/T^*(\phi)$, i.e. for all $b \in \mathbf{R}$ such that $T^*(\phi_b) < \infty$.

Proof. (a) Let $\phi_k, \phi \in W$, and let $b_k, b \in \mathbf{R}$, such that $(\phi_k)_{b_k} \to \phi_b$ in X, as $k \to \infty$. By parts (b) and (c) of Proposition 2.3 , we need to show that $b_k \to b$ as $k \to \infty$. Multiplying by $\exp(-ib|x|^2/4)$, we may assume $b = 0$. Thus, $(\phi_k)_{b_k} \to \phi$ in X. We observe next that the sequence b_k is bounded. Indeed, the sequence ϕ_k is bounded in X, as well as bounded away from 0 in X (since small data are in $\mathcal{R}_X$); and so if (a subsequence of) b_k is unbounded, it follows easily that $\|\nabla(\phi_k)_{b_k}\|_2^2$ is unbounded. This contradicts the assumption that $(\phi_k)_{b_k} \to \phi$ in X.

Since the sequence b_k is bounded, we may assume, by passing to a subsequence, that $b_k \to c$. We must show that $c = 0$, independent of the subsequence. Since $(\phi_k)_{b_k} \to \phi$ in X, it follows that $\phi_k \to \phi_{-c}$ in X. Moreover, since $\phi_k \in W$ and W is closed, we see that $\phi_{-c} \in W \subset \mathcal{N}_X$. However, $\phi \in W \subset \mathcal{N}_X$; and thus by Proposition 2.3, $c = 0$.

Next suppose $\phi_k \to \phi$ in X as $k \to \infty$, where $\phi_k, \phi \in \cup_{b<0}[M_b W]$; and let u_k and u denote the solutions of (1.1) with initial values ϕ_k and ϕ respectively. Let $0 < t < T^*(\phi)$. By what we have just proved, it follows that $T^*(\phi_k) - t$ converges to $T^*(\phi) - t$ as $k \to \infty$. Now $0 < T^*(\phi) - t < T^*(\phi)$, and so there exists $T > 0$ such that $T^*(\phi_k) - t \in [0,T]$ for sufficiently large k. Moreover, by statement (x) of Theorem A, we know that $\nabla u_k \to \nabla u$ in $C([0,T],L^2)$. It follows that $\nabla u_k(T^*(\phi_k) - t) \to \nabla u(T^*(\phi) - t)$ in L^2 as $k \to \infty$, which in particular implies (4.10).

(b) We begin with the same proof as in part (a), with W replaced by $\mathcal{N}_m$. However, we need a different argument to show that the sequence b_k is bounded. Since $(\phi_k)_{b_k} \to \phi$ in X, where ϕ_k and ϕ are all in $\mathcal{N}_X$, lower semicontinuity of T^* implies that $\liminf_{k\to\infty} b_k \geq 0$. Thus, if the b_k do not form a bounded sequence, we may pass to a subsequence and assume that $b_k \to +\infty$. Next, formula

(4.4), first with b replaced by b_k and ϕ replaced by ϕ_k, and then with b replaced by $-b_k$ and ϕ replaced by $(\phi_k)_{b_k}$, gives respectively

$$E((\phi_k)_{b_k}) = E(\phi_k) + (b_k/2)F(\phi_k) + (b_k^2/8)\|x\phi_k\|_2^2.$$
$$E(\phi_k) = E((\phi_k)_{b_k}) - (b_k/2)F((\phi_k)_{b_k}) + (b_k^2/8)\|x(\phi_k)_{b_k}\|_2^2.$$

Adding these two formulas, and noting $\|x\phi_k\|_2^2 = \|x(\phi_k)_{b_k}\|_2^2$, we see that

$$(b_k^2/4)\|x(\phi_k)_{b_k}\|_2^2 = (b_k/2)[-F(\phi_k) + F((\phi_k)_{b_k})] \leq (b_k/2)[m + F((\phi_k)_{b_k})]. \qquad (4.11)$$

Since F is continuous on X and since $(\phi_k)_{b_k} \to \phi$ in X, it follows that $F((\phi_k)_{b_k})$ and $\|x(\phi_k)_{b_k}\|_2^2$ converge as $k \to \infty$. Thus, (4.11) implies that the b_k form a bounded sequence. We then continue with the proof exactly as in part (a).

Finally, we show that if $T^*(\phi) < \infty$ and $P[\phi](t) < 0$ for some $t > 0$, then $\phi \in \cup_{b<0}[M_b(\mathcal{N}_0)]$. Let $b = 1/T^*(\phi)$, so $\phi_b \in \mathcal{N}_X$. We need to show that $F(\phi_b) \geq 0$. By (4.6) $P[\phi_b](t) < 0$ for some $t \in \mathbf{R}$; and since $T^*(\phi_b) = \infty$, this can happen only for negative t. It follows that the polynomial $P[\phi_b](t)$ is non-decreasing for $t \geq 0$, and so its derivative at $t = 0$ is non-negative. In other words, $F(\phi_b) \geq 0$.

(c) Suppose ϕ_k is a sequence in $\mathcal{N}_X$, $\phi_k \to \phi$ in X, but $\phi \notin \mathcal{N}_X$. Since $\mathcal{R}_X$ is open, $\phi \notin \mathcal{R}_X$; and so $\phi \in \mathcal{B}_X$. If $0 \leq b < 1/T^*(\phi)$, then $T^*((\phi_k)_b) = \infty$, but $T^*(\phi_b) < \infty$, so T^* is discontinuous at ϕ_b. If $b < 0$, then $T^*((\phi_k)_b) = -1/b$, but $T^*(\phi_b) < -1/b$, so again T^* is discontinuous at ϕ_b.

It is well-known that (1.1) admits a family of standing wave solutions of the form $u(t,x) = \psi(x)\exp(i\omega t)$, where $\omega > 0$. For a solution u of this form the function ψ must satisfy the semilinear elliptic problem

$$\Delta\psi - \omega\psi + |\psi|^{\gamma}\psi = 0. \qquad (4.12)$$

The reader can consult [1,2,3,11,20,25] for a rather complete accounting of the existence of such standing waves. (One can easily show that H^1 solutions of (4.12) are in X, although we are not sure if this result appears in the literature.) We note in passing that for different values of the parameter ω, the corresponding stationary functions ψ are related by a dilation of the form (3.1) with $\beta = \omega^{1/2}$. One observation which can be made immediately is that a standing wave solution is global in both directions; and since $\|xu(t)\|_2^2$ is constant, it follows that $F(\psi) = E(\psi) = 0$. In particular, $\psi \in \mathcal{N}_X \cap \overline{\mathcal{N}}_X$. By Propositions 4.3 and 4.4, this gives a whole family of non-global solutions such that $\|xu(t)\|_2^2 \to 0$ as t approaches the blow-up time, and which satisfy the estimate (4.9) as well. In fact, since $\|\psi\exp(i\omega t)\|_{\gamma+2}^{\gamma+2}$ is constant, it follows from (2.4) that the corresponding non-global solution satisfies $\|\nabla u(t)\|_2^2 \approx C(T^*(\phi) - t)^{-2}$ at blow up.

Of particular interest are the ground state solutions, i.e. radially symmetric standing wave solutions for which $\psi(x) > 0$ on $\mathbf{R}^n$. For a fixed value of $\omega > 0$, such a solution is unique. (See McLeod and Serrin [19] and Kwong [13].) Since the dilation transformation (3.1) preserves the L^2 norm, all the ground state solutions (for different values of ω) have the same L^2 norm, which we denote by G. It is known (Weinstein [29]) that if $\|\phi\|_2 < G$, then $T^*(\phi) = \infty$. Since, $\|\phi\|_2 = \|\phi_b\|_2$ for all real b, it follows from Proposition 2.3 that if $\|\phi\|_2 < G$, then $\phi \in \mathcal{R} \cap \overline{\mathcal{R}}$. Moreover, starting from the ground state solution, $\psi\exp(i\omega t)$, the solution with initial value ψ_b with $b < 0$, blows up in finite time and has L^2 norm equal to G. (This example first appeared in Weinstein [30].) In addition, Weinstein has proved the following theorem concerning non-global solutions to (1.1) whose L^2 norm is equal to G.

Theorem B. (Weinstein [30], Theorem 1, page 553.) Let $\phi \in H^1$ such that $T^*(\phi) < \infty$, and suppose that $\|\phi\|_2 = G$. Let u(t) be the solution of (1.1) with initial value ϕ. Let ψ be the ground state solution of (4.12) with $\omega = 1$. Set $\beta(t) = \|\nabla\psi\|_2/\|\nabla u(t)\|_2$. Then, there are functions $y(t) \in \mathbf{R}^n$ and $\alpha(t) \in \mathbf{R}$ such that as $t \to T^*(\phi)$, $D_{\beta(t)}u(t,\cdot + y(t))e^{i\alpha(t)}$ tends strongly in H^1 to ψ.

As we have just observed, the example given by multiplying the ground state by $\exp ib|x|^2/4$, for some $b < 0$, blows up at the rate $\|\nabla u(t)\|_2^2 \approx C(T^*(\phi) - t)^{-2}$. For *any* non-global solution satisfying the hypotheses of Weinstein's theorem, we can show at least that the lower bound (1.10) is not sharp.

Proposition 4.7. Suppose ϕ satisfies the hypotheses of Theorem B. Then,

$$\lim_{t\to T^*(\phi)} (T^*(\phi) - t) \, \|\nabla u(t)\|_2^2 = \infty. \tag{4.13}$$

Proof. For notational convenience, we let $v(t) = D_{\beta(t)}u(t,\cdot + y(t))e^{i\alpha(t)}$, as in the statement of Theorem B. On the one hand, we clearly have

$$
\begin{aligned}
T^*(u(t)) &= T^*(\phi) - t, \\
T^*(u(t,\cdot + y(t))) &= T^*(\phi) - t, \\
T^*(D_{\beta(t)}u(t,\cdot + y(t))) &= (T^*(\phi) - t)/\beta^2, \\
T^*(v(t)) &= (T^*(\phi) - t)/\beta^2 = \|\nabla u(t)\|_2^2(T^*(\phi) - t)/\|\nabla\psi\|_2^2.
\end{aligned}
$$

On the other hand, since $v(t) \to \psi$ in H^1 as $t \to T^*(\phi)$, and since $T^*(\psi) = \infty$, it follows from statement (x) of Theorem A (continuous dependence) that $T^*(v(t)) \to \infty$ as $t \to T^*(\phi)$. This proves the result.

In addition, if there is an example of blow up which satisfies the hypotheses of Weinstein's theorem, and which is not the same as the example already noted, it has a rather interesting feature.

Proposition 4.8. Let $\phi \in X$ satisfy the hypotheses of Theorem B. Suppose that ϕ_b, where $b = 1/T^*(\phi)$ is the unique real number such that $\phi_b \in \mathcal{N}$, is not a solution of (4.12). Then $\|xu(t)\|_2^2$ does *not* converge to 0 as $t \to T^*(\phi)$.

Proof. By Proposition 4.3, it suffices to show that $E(\phi_b) \neq 0$. Suppose to the contrary that $E(\phi_b) = 0$. We repeat part of the proof of Theorem B in [30]; and we refer the reader to that proof to fill in the details of our argument. Since $\|\phi_b\|_2 = G$, and since $E(\psi) \geq 0$ for all $\psi \in H^1$ with $\|\psi\|_2 = G$, it follows that ϕ_b is a minimizer for the energy functional E on $\{\psi \in H^1: \|\psi\|_2 = G\}$. As a minimizer, it satifies the Euler-Lagrange equation, which is precisely (4.12). This contradicts the assumption that ϕ_b does not satisfy (4.12).

In two recent papers, Merle and Tsutsumi [23] and Tsutsumi [28] show that for radially symmetric non-global H^1 solutions of (1.1), there is a certain amount of L^2 concentration at blow up, i.e. for all $R > 0$, $\lim\inf_{t\to T^*(\phi)} \|u(t)\|_{L^2(|x|\leq R)} \geq G$. It follows that if ψ is a weak L^2 limit point of u(t) as $t \to T^*(\phi)$, then $\|\psi\|_2^2 \leq \|\phi\|_2^2 - G^2$. The next proposition generalizes this last result to arbitrary L^2 non-global solutions.

Proposition 4.9. There exists $\rho > 0$ such that $\|\psi\|_2^2 \leq \|\phi\|_2^2 - \rho^2$ whenever $\phi \in L^2$, $T^*(\phi) < \infty$, and ψ is a weak limit point of the resulting solution u(t) of (1.2) as $t \to T^*(\phi)$.

Proof. By Proposition 1 in [5], there exists $\delta > 0$ such that if $\|S(\cdot)\xi\|_{L^{\gamma+2}(0,\tau;L^{\gamma+2})} < \delta$, then $T^*(\xi) > \tau$. Applying this to an arbitrary point in the trajectory u(t), we see that for all $t \in [0,T^*(\phi))$,

$$\|S(\cdot)u(t)\|_{L^{\gamma+2}(0,T^*(\phi)-t,L^{\gamma+2})} \geq \delta.$$

Also, it is well known [9,12,5] that $\|S(\cdot)\xi\|_{L^{\gamma+2}(0,\infty;L^{\gamma+2})} \leq K\|\xi\|_2$. Thus, for *any* $\psi \in L^2$,

$$\begin{aligned} \delta &\leq \|S(\cdot)(u(t) - \psi)\|_{L^{\gamma+2}(0,T^*(\phi)-t,L^{\gamma+2})} + \|S(\cdot)\psi\|_{L^{\gamma+2}(0,T^*(\phi)-t,L^{\gamma+2})} \\ &\leq K\|u(t) - \psi\|_2 + \|S(\cdot)\psi\|_{L^{\gamma+2}(0,T^*(\phi)-t,L^{\gamma+2})}; \end{aligned}$$

and so $\lim\inf_{t\to T^*(\phi)} \|u(t) - \psi\|_2^2 \geq (\delta/K)^2$. Now suppose that $u(t_k)$ converges weakly to ψ in L^2, where $t_k \to T^*(\phi)$. The result follows easily by multiplying out $\|u(t_k) - \psi\|_2^2$ and passing to the limit.

Finally, we indulge in some speculation. The proof of the existence of non-global solutions to (1.2) formally suggests that $\|xu(t)\|_2^2$ converges to zero at the blow-up time. This is clearly not true in general: it suffices to translate spatially any non-global solution, so at least either the original or the translated solution will fail to satisfy this condition. Moreover, Merle [22] has recently given an example of a non-global solution, none of whose translates verifies $\|xu(t)\|_2^2 \to 0$ as t approaches blow-up time. However, it seems to be an open question as to whether or not every radially symmetric non-global solution has this property. (See [14,16,17,18,21,24] for additional recent work on blow up for equation (1.1).) Note that Proposition 4.8 shows that any radially symmetric solution in X satisfying the hypotheses of Theorem B, other than the example discussed above, fails to have this property. However, it is not clear if such a solution exists. Thus, we are led to the following conjecture.

Conjecture I : If $\phi \in X$ is radially symmetric and $T^*(\phi) < \infty$, then $\|xu(t)\|_2^2 \to 0$ as $t \to T^*(\phi)$. In other words, $T^*(\phi) < \infty$ if and only if $P[\phi](t)$ has a positive zero, and $T^*(\phi)$ is the smallest positive zero of $P[\phi](t)$.

Although we are not yet able to prove (or disprove) this conjecture, it is amusing to look at some of its immediate consequences. First we establish some notation. X_{rad} denotes the set of radially symmetric functions in X; $\mathcal{R}_{rad} = \mathcal{R} \cap X_{rad}$, $\mathcal{N}_{rad} = \mathcal{N} \cap X_{rad}$, and $\mathcal{B}_{rad} = \mathcal{B} \cap X_{rad}$. Recall that by Proposition 2.3, $\mathcal{B}_{rad} = \cup_{b<0}[M_b(\mathcal{N}_{rad})]$.

Proposition 4.10. Suppose Conjecture I is correct.

(a) $\mathcal{N}_{rad} = \{\phi \in X_{rad} : \phi$ is not identically zero, $E(\phi) = 0$ and $F(\phi) \geq 0\}$.

(b) $\mathcal{N}_{rad} \cap \overline{\mathcal{N}}_{rad} = \{\phi \in X_{rad} : \phi$ is not identically zero, $E(\phi) = 0$ and $F(\phi) = 0\}$.

(c) $\mathcal{R}_{rad} = \{\phi \in X_{rad} : E(\phi) > 0$ and $P[\phi](t) > 0$ for all $t \geq 0\} \cup \{\phi \equiv 0\}$.

(d) $\mathcal{R}_{rad} \cap \overline{\mathcal{R}}_{rad} = \{\phi \in X_{rad} : E(\phi) > 0$ and $P[\phi](t) > 0$ for all $t \in \mathbf{R}\} \cup \{\phi \equiv 0\} = \{\phi \in X_{rad} : \phi_b \in \mathcal{R}$ for all $b \in \mathbf{R}\}$. This set is unbounded in X_{rad}. In particular, the scattering operator is defined on an unbounded set.

(e) $T^*: \cup_{b \in \mathbf{R}} [M_b(\mathcal{N}_{rad})] \to (0,\infty]$ is continuous; in particular $T^*: \mathcal{B}_{rad} \cup \mathcal{N}_{rad} \to (0,\infty]$ is continuous. (Note that $\mathcal{B}_{rad} \cup \mathcal{N}_{rad}$ is an X_{rad}-closed set since its complement $\mathcal{R}_{rad}$ is open.) Moreover, the rate of blow up is continuous on $\mathcal{B}_{rad}$ in the sense described in Theorem 4.6.

(f) $T^*: X_{rad} \to (0,\infty]$ is discontinuous precisely on $\{\phi_b : \phi$ is not identically zero, $E(\phi) = F(\phi) = 0$, and $b < 0\} = \{\phi_b : \phi \in \mathcal{N}_{rad} \cap \overline{\mathcal{N}}_{rad}$, and $b < 0\}$. Moreover, $T^*(\psi) = \infty$ for some $\psi \in X_{rad}$ arbitrarily close to such a ϕ_b.

Proof. (a) Suppose $\phi \in \mathcal{N}_{rad}$. It follows from Proposition 4.3 and Conjecture I that $E(\phi) = 0$. Also, we clearly must have $F(\phi) \geq 0$, since if $F(\phi) < 0$, then $P[\phi](t)$ will be negative for large t, and so $T^*(\phi) < \infty$. On the other hand, if $\phi \in X_{rad}$ is not identically zero, and $E(\phi) = 0$ and $F(\phi) \geq 0$, then by Proposition 4.1, $\phi \notin \mathcal{R}$. Moreover, $P[\phi](t) > 0$ for all $t \geq 0$. Therefore, if $T^*(\phi) < \infty$, then $\|xu(t)\|_2^2$ cannot tend to zero at blow-up time, which violates Conjecture I. Thus, $\phi \in \mathcal{N}_{rad}$.

(b) This is an immediate consequence of statement (a), applied to both forward and backward solutions.

(c) Clearly $\mathcal{R}_{rad} \subset \{\phi \in X_{rad} : E(\phi) > 0 \text{ and } P[\phi](t) > 0 \text{ for all } t \geq 0\} \cup \{\phi \equiv 0\}$. On the other hand, if $\phi \in X_{rad}$ and $P[\phi](t) > 0$ for all $t \geq 0$, then by Conjecture I, $T^*(\phi) = \infty$. Furthermore, if $E(\phi) > 0$, then by statement (a), $\phi \notin \mathcal{N}_{rad}$; and so $\phi \in \mathcal{R}_{rad}$.

(d) The first equality is a consequence of statement (c) applied to both forward and backward solutions. For the second equality, we have already seen in Propositions 4.1 and 4.2 that if $\phi \in X_{rad}$ is not identically zero and $\phi_b \in \mathcal{R}$ for all $b \in \mathbf{R}$, then $E(\phi) > 0$ and $P[\phi](t) > 0$ for all $t \in \mathbf{R}$. Suppose instead that $\phi \in X_{rad}$, $E(\phi) > 0$, and $P[\phi](t) > 0$ for all $t \in \mathbf{R}$. It follows from (4.6) and (4.7) that $P[\phi_b](t) > 0$ for all real t and all real b. Moreover, by (4.4) $E(\phi_b) > 0$ for all real b. It now follows from statement (c) that $\phi_b \in \mathcal{R}$ for all $b \in \mathbf{R}$. That this set is unbounded is easy to see since it includes $\{\phi \in X_{rad} : E(\phi) > 0 \text{ and } F(\phi) = 0\}$. See Remark 3.3 to see how this applies to the scattering operator.

(e) This is an immediate consequence of part (a) above and Theorem 4.6, part (b), applied just to radially symmetric functions.

(f) We recall first that T^* is certainly continuous on the open set $\mathcal{R}_{rad}$, since it is identically ∞ there. Furthermore, if $\phi \in X_{rad}$ is not identically zero, $E(\phi) = 0$, $F(\phi) > 0$, (and $b \leq 0$), we claim that ϕ_b is in the interior of $\cup_{b \in \mathbf{R}} [M_b(\mathcal{N}_{rad})]$, and so T^* is continuous there. To prove the claim it suffices to consider $b = 0$. We must show that if ψ is sufficiently close to ϕ, then $\psi \in \cup_{b \in \mathbf{R}} [M_b(\mathcal{N}_{rad})]$. Since $E(\phi) = 0$ and $F(\phi) > 0$, it follows that $P[\phi](t) < 0$ for large negative t. If ψ is close to ϕ in X, the same must be true for $P[\psi](t)$. By Propositions 4.2 and 2.3, $\psi_b \in \mathcal{N}_{rad}$ for some real b. This proves the claim.

Next we consider $\phi \in X_{rad}$, not identically zero, such that $E(\phi) = F(\phi) = 0$; and so $T^*(\phi) = \infty$. If $\phi_k \to \phi$ in X_{rad}, then each ϕ_k is either in $\mathcal{R}_{rad}$, in which case $T^*(\phi_k) = \infty$, or in $\mathcal{B}_{rad} \cup \mathcal{N}_{rad}$, in which case $T^*(\phi_k) \to \infty$ by statement (f). Finally, if ϕ is not identically zero, $E(\phi) = 0$, $F(\phi) = 0$, and $b < 0$, then $T^*(\phi_b) = -1/b$. However, there exists ψ arbitrarily close to ϕ in X_{rad} such that $E(\psi) > 0$ and $F(\psi) = 0$. (Indeed, let $\psi = (1-\epsilon)\phi$.) By statement (d), $\psi_b \in \mathcal{R}$ for all real b, and in particular all $b < 0$. Since ψ_b is arbitrarily close to ϕ_b, and since $T^*(\psi_b) = \infty$, it follows that T^* is not continuous at ϕ_b.

In particular, the above proposition shows (assuming Conjecture I) that the initial values in X_{rad} which lead to 'stable' blow up, i.e. such that nearby initial values also lead to blow up and at approximately the same time, are precisely those functions of the form ϕ_b, where ϕ is not identically zero, $E(\phi) = 0$, $F(\phi) > 0$, and $b < 0$. Moreover, (still assuming Conjecture I) the non-global solutions described in Proposition 4.4 are precisely those which exhibit *unstable* blow up, as described in statement (f) of the proposition. Thus, any properties of such unstable blow up might be hard to observe with numerical calculations. In particular, if we further conjecture that the blow-up rate given by (4.9) is unique to unstable blow up, this explains why such a blow-up rate has not been numerically observed. (See Section 6 in [15].)

There is another conjecture which has a bearing on these questions.

Conjecture II : $\mathcal{F}(\mathcal{N}_X) \subset \overline{\mathcal{N}}_X$, or less generally $\mathcal{F}(\mathcal{N}_{rad}) \subset \overline{\mathcal{N}}_{rad}$.

(We use the Fourier transform as defined by (3.6).) One asks immediately why this is a reasonable conjecture, and, if true, of what interest it would be. The second question is easier to answer. Indeed, Conjecture II implies that Conjecture I is false. To see this (or rather its contrapositive), it suffices to exhibit $\phi \in X_{rad}$ such that $E(\phi) = 0$, $F(\phi) \geq 0$, and $E(\mathcal{F}\phi) \neq 0$. Then $\phi \in \mathcal{N}_{rad}$ by statement (a) of Proposition 4.10, but $\mathcal{F}\phi$ cannot be in $\overline{\mathcal{N}}_{rad}$, also by statement (a) of Proposition 4.10. (We will exhibit such a ϕ: see property (P_b) below.) Moreover, if Conjecture II is true, it seems likely that also $\mathcal{F}(\mathcal{R}_X) \subset \overline{\mathcal{R}}_X$. This would provide new sufficient conditions for blow up: for example, if $E(\mathcal{F}\phi) < 0$, then $T^*(\phi) < \infty$.

As to whether or not this conjecture is plausible, we explain first that we came to it via the considerations in Sections 2 and 3 of this paper. For example, $\mathcal{N}$ (as well as $\mathcal{R}$) is invariant under *both* spatial translations and multiplication by $\exp(ix \cdot x_0)$, which certainly is suggestive of the Fourier transform. Moreover, the Fourier transform enters into the scattering theory for (1.1) in a somewhat surprising way; and Proposition 3.4 (whose negation was our first version of Conjecture II) and its proof hint at something else.

Perhaps more convincingly, however, Conjecture II is in fact true for a significant class of functions in $\mathcal{N}_X$. First of all, if $\phi \in \mathcal{N}_X$ is a (real) eigenfunction of $\mathcal{F}$, then certainly $\mathcal{F}\phi \in \overline{\mathcal{N}}_X$. In addition, if $\phi = k \exp(-c\pi|x|^2)$, where Re $c > 0$, then $\mathcal{F}\phi = (\bar{c}/|c|)^{n/2} D_{1/|c|}(\bar{\phi})$. Since the dilation operators D_β and multiplication by a constant of modulus 1 both preserve $\overline{\mathcal{N}}_X$, it follows that if $\phi \in \mathcal{N}_X$, then $\mathcal{F}\phi \in \overline{\mathcal{N}}_X$. More generally, functions $\phi \in X$ such that $\mathcal{F}\phi = \alpha D_\beta \bar{\phi}$ for some $\alpha \in \mathbf{C}$ of modulus 1 and some $\beta > 0$ clearly satisfy Conjecture II, i.e. if such a ϕ is in $\mathcal{N}_X$, then $\mathcal{F}\phi \in \overline{\mathcal{N}}_X$. This class of functions contains certain complex polynomials multiplied by complex exponentials, but it is not clear if its intersection with $\mathcal{N}_{rad}$ is in fact dense in $\mathcal{N}_{rad}$.

If we take the point of view that Conjecture I is false, Conjecture II suggests a good place to look for a counter example: we look for an element ϕ of $\mathcal{N}_{rad}$ which does *not* satisfy $\mathcal{F}\phi = \alpha D_\beta \bar{\phi}$ as described above, and in particular such that $E(\phi) = 0$, $F(\phi) \geq 0$, and $E(\mathcal{F}\phi) \neq 0$. However, since if Conjecture I is false, it is not completely clear when such a function is in $\mathcal{N}_{rad}$, and since $\mathcal{N}_{rad}$ is an unstable set, we need to be a little more clever. We are looking here for an initial value ϕ for a numerical experiment which would provide strong evidence for or against Conjecture I (and II).

Let $\phi \in X_{rad}$ be such that $F(\phi) = 0$, (for example if ϕ is real valued). If $k > 0$, then by (4.4)

$$E(k\phi_b) = k^2\{(1/2)\|\nabla\phi\|_2^2 + (b^2/8)\|x\phi\|_2^2\} - \{k^{\gamma+2}/(\gamma+2)\|\phi\|_{\gamma+2}^{\gamma+2}\}. \quad (4.14)$$

Thus, the condition $E(k\phi_b) = 0$ determines a one-to-one correspondence between $k > 0$ and $b > 0$. Suppose for all $b > 0$ that ϕ has the following property:

(P_b) If $E(k\phi_b) = 0$, then $E(k\mathcal{F}(\phi_b)) \neq 0$.

We consider the solutions to (1.1) starting from two specific initial values. For the first of these two initial values we take $k\phi$, where k is chosen sufficiently large that $E(k\phi) < 0$. Such a k is now fixed, and we describe the second inital value as follows. Let $b > 0$ be the unique positive number such that $E(k\phi_b) = 0$. (Note that $F(k\phi_b) > 0$.) Let $\psi = \alpha D_\beta \overline{\mathcal{F}(\phi_b)}$, where α is a complex number of modulus 1 and $\beta > 0$. (We are free to choose such α and β as we like.) Property (P_b) implies that $E(k\psi) \neq 0$. We take $k\psi_{-b}$ as the second inital value. Let u(t) and v(t) denote the solutions to (1.1) with these respective initial values.

Suppose first that Conjecture I is correct. Then $k\phi_b \in \mathcal{N}_{rad}$ and $\|xu(t)\|_2^2 \to 0$ at blow up. Also, if $T^*(k\psi_{-b}) < 0$, then $\|xv(t)\|_2^2 \to 0$ at blow up.

Suppose next that Conjecture II is correct. If $\|xu(t)\|_2^2 \to 0$ at blow up, then $k\phi_b \in \mathcal{N}_{rad}$, and so $\overline{\mathcal{F}(k\phi_b)} \in \mathcal{N}_{rad}$. Therefore $k\psi \in \mathcal{N}_{rad}$, and $T^*(k\psi_{-b}) < \infty$. Since $E(k\psi) \neq 0$, it follows that $\|xv(t)\|_2^2$ does *not* converge to 0 at blow up.

In other words, if one of $\|xu(t)\|_2^2$ and $\|xv(t)\|_2^2$ does not converge to 0 at blow up, then certainly Conjecture I is false, and there is some evidence for the truth of Conjecture II. If both $\|xu(t)\|_2^2$ and $\|xv(t)\|_2^2$ converge to 0 at blow up, then certainly Conjecture II is false, and there is some evidence (rather strong we feel) for the truth of Conjecture I.

More specifically now, let $\phi(x) = |x|^2\exp(-a\pi|x|^2)$, where $a > 0$ is fixed. Then $\phi_{4\pi b}(x) = |x|^2\exp(-c\pi|x|^2)$ and

$$[\mathcal{F}\phi_{4\pi b}](x) = -c^{-n/2}[(|x|^2/c^2) - (n/2\pi c)]\exp(-\pi|x|^2/c),$$

where $c = a - ib$. First taking the complex conjugate, then applying the dilation operator $D_{|c|}$, and finally multiplying by $-(\overline{c}/|c|)^{(n/2)+2}$, we arrive at

$$\psi(x) = [|x|^2 - (n/2\pi c)]\exp(-c\pi|x|^2),$$
$$\psi_{-4\pi b}(x) = [|x|^2 - (n/2\pi c)]\exp(-a\pi|x|^2).$$

We note in passing that if $k > 0$, and thus also $b > 0$, are large, then $\psi_{-4\pi b}$ is close in X to ϕ; and so $E(k\phi) < 0$ will imply $E(k\psi_{-4\pi b}) < 0$. Thus, even in the case where Conjecture I is correct, v(t) will necessarily blow up in finite time.

It remains to investigate property (P_b) for this choice of ϕ. Since $E(k\mathcal{F}\phi_{4\pi b}) = 0$ if and only if $E(k\psi) = 0$, we see that $(P_{4\pi b})$ holds precisely when

$$\frac{\|\psi\|_{\gamma+2}^{\gamma+2}}{\|\phi\|_{\gamma+2}^{\gamma+2}} \neq \frac{\|\nabla\psi\|_2^2}{4\pi^2 b^2\|x\phi\|_2^2 + \|\nabla\phi\|_2^2}, \tag{4.15}$$

where we have used (4.14). Since

$$\nabla\psi(x) = (-2\pi c|x|^2 + n + 2)x\cdot\exp(-c\pi|x|^2),$$
$$\|\nabla\psi\|_2^2 = 4\pi^2 b^2\|x\phi\|_2^2 + \|\nabla\phi\|_2^2 - 4\pi n a\|\phi\|_2^2 + n^2\|\phi/|x|\,\|_2^2,$$

the right side of (4.15) is given by

$$1 + \frac{n^2\|\phi/(|x|)\|_2^2 - 4\pi n a\|\phi\|_2^2}{4\pi^2 b^2\|x\phi\|_2^2 + \|\nabla\phi\|_2^2}.$$

Moreover,

$$\left|\,|x|^2 - (n/2\pi c)\right|^{\gamma+2} = \left[|x|^4 + \frac{n^2 - 4\pi a n|x|^2}{4\pi^2(a^2+b^2)}\right]^{1+(\gamma/2)}.$$

In particular, if γ is an even integer (i.e. if n = 1 or 2), then the left side of (4.15) is a polynomial in $(a^2+b^2)^{-1}$ of degree at least two. Therefore, (P_b) holds except at perhaps a finite number of values of b. For higher dimensions, the left side of (4.15) always converges to 1 as $b \to \infty$. However, subtracting 1 from both sides of (4.15), multiplying by b^2, and letting $b \to \infty$, we obtain different limits for the two sides of (4.15). Therefore, (P_b) holds for b sufficiently large.

Thus, at least if n = 1 or 2, the initial values $k|x|^2\exp(-a\pi|x|^2)$ and $k[|x|^2 - (n/2\pi c)]\exp(-a\pi|x|^2)$, where k is large, b is determined by the condition $E(k\phi_b) = 0$, and $c = a - ib$, provide a good test of Conjectures I and II. (The possible values where (P_b) might fail will be avoided with probability one.)

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