COMPLETE INTEGRABILITY IN STATISTICAL MECHANICS
AND THE YANG-BAXTER EQUATIONS

BY

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IMA Preprint Series #57

February 1984

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Complete Integrability in Statistical Mechanics
and the Yang-Baxter Equations^{*}

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^{*} This work was supported in part by the National Science Foundation under Grant MCS-8301261

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ABSTRACT

In this paper we give a differential formulation of the Yang-Baxter equations. This formulation leads to the introduction of the Yang-Baxter ideal $\mathfrak{g}_{\text{YB}}$, the basic geometric object in this formulation. These ideas are illustrated in the context of the Baxter model and the general eight-vertex model.

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1.0. Vertex Models and Transfer Matrices

We consider a square lattice Λ of M rows and N columns with periodic boundary conditions. A configuration v is specified by assigning to each bond in Λ an element of the state space $\mathcal{A}$. In this paper $\mathcal{A}$ will always be a finite set and we let $q \equiv |\mathcal{A}|$. The phase space $\mathcal{N}$ is the set of all such configurations. To a given vertex at site $j \in \Lambda$ (see Figure 1)

Figure 1

is assigned an energy $\epsilon^{\lambda\lambda'}(\alpha, \alpha') \in \mathbb{R} \cup \{+\infty\}$ for all choices of $\lambda, \lambda', \alpha, \alpha' \in \mathcal{A}$. In the formulation here $\epsilon^{\lambda\lambda'}(\alpha, \alpha')$ does not depend upon the site j . The energy E_Λ of configuration $v \in \mathcal{N}$ is the sum of the energies of the individual vertices, and the probability of configuration v is given by the Gibbs measure

$$P_\Lambda(v) = \exp(-\beta E(v)) / Z_\Lambda(\beta)$$

where $Z_\Lambda(\beta)$ is the normalizing factor $\sum_{v \in \mathcal{N}} \exp(-\beta E_\Lambda(v))$ (called the partition function) and $0 < \beta < \infty$ is proportional to the inverse temperature. The thermodynamics of this model is obtained from the free energy per lattice site, f , given by

$$-\beta f = \lim_{\Lambda \rightarrow \mathbb{Z}^2} \frac{1}{|\Lambda|} \log Z_\Lambda(\beta).$$

The phrase "exactly solved model" is used in the sense that f can be given in closed form.

To facilitate the evaluation of Z_Λ a transfer matrix is introduced (see [3]) and it is straightforward to show that

$$Z_\Lambda = \text{Tr}(T^M) \quad (1.1)$$

where the trace is with respect to the space

$$\mathcal{H}_N = \mathcal{C}_1^q \otimes \mathcal{C}_2^q \otimes \dots \otimes \mathcal{C}_N^q$$

with $\mathcal{C}_j^q \cong \mathbb{C}^q$ (recall q is the number of elements in the set $\mathcal{J}$), and

$T: \mathcal{H}_N \rightarrow \mathcal{H}_N$ is the transfer matrix. In the basis $\{f_\alpha\}$ where

$f_\alpha = e_{\alpha_1} \otimes \dots \otimes e_{\alpha_N}$, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathcal{J}^N$, and $\{e_{\alpha_i}\}$ is the standard basis for $\mathcal{C}_i^q$, T is given by

$$T_{\alpha, \alpha'} = \sum_{\lambda \in \mathcal{J}^N} T^{\lambda_1 \lambda_2}_{(\alpha_1, \alpha'_1)} T^{\lambda_2 \lambda_3}_{(\alpha_2, \alpha'_2)} \dots T^{\lambda_N \lambda_1}_{(\alpha_N, \alpha'_N)}$$

where $\lambda = (\lambda_1, \dots, \lambda_N)$, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathcal{J}^N$ and

$T^{\lambda \lambda'}_{(\alpha, \alpha')} \equiv \exp(-\beta \epsilon^{\lambda \lambda'}_{(\alpha, \alpha')})$ is the Boltzmann factor corresponding to the vertex configuration given by $\lambda, \lambda', \alpha$, and α' (see Figure 1). Define $T^{\lambda \lambda'}: \mathcal{C}^q \rightarrow \mathcal{C}^q$, $\lambda, \lambda' \in \mathcal{J}$ by

$$(T^{\lambda \lambda'})_{\alpha \alpha'} = T^{\lambda \lambda'}_{(\alpha, \alpha')}, \quad \alpha, \alpha' \in \mathcal{J}$$

and

$$T_j^{\lambda \lambda'}: \mathcal{H}_N \rightarrow \mathcal{H}_N, \quad 1 \leq j \leq N$$

by

$$T_j^{\lambda\lambda'} : I \otimes \dots \otimes \underset{\substack{\downarrow \\ \text{jth slot}}}{T^{\lambda\lambda'}} \otimes \dots \otimes I .$$

Finally, if we denote by $\mathcal{H}_{N,q}$ the tensor product of $\mathcal{H}_N$ with itself q -times and by $L_j : \mathcal{H}_{N,q} \rightarrow \mathcal{H}_{N,q}$, $1 \leq j \leq N$, the $q \times q$ matrix whose entries are

$$L_j(\lambda, \lambda') = T_j^{\lambda\lambda'}, \quad \lambda, \lambda' \in \lambda \quad (1.2)$$

we have the following compact expression for the transfer matrix T :

$$T = \text{Tr}_q(L_1 L_2 \dots L_N) \quad (1.3)$$

where $\text{Tr}_q(\dots)$ indicates the trace is only over the $q \times q$ matrix structure of the L_j 's.

Remark 1.1.

The introduction of a transfer matrix (1.1) is classical, but the form (1.3) for vertex models is due to Baxter [2].

Remark 1.2.

The operator $L = L_1 L_2 \dots L_N$ has close analogies with inverse scattering methods as was first emphasized by Faddeev and co-workers (see [10]).

2.0. Yang-Baxter Equations

Suppose there are n distinct Boltzmann weights $T^{\lambda\lambda'}(\alpha, \alpha')$, $\lambda, \lambda', \alpha, \alpha' \in \mathcal{A}$. By relabeling we can introduce a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ of Boltzmann weights. Then to indicate the dependence upon x , we write the transfer matrix T and the operators L_j of Section 1 as $T(x)$ and $L_j(x)$, respectively. The next step is the key question first posed by Baxter [2]:

Is there a deformation of the Boltzmann vector x to, say, x' and hence a deformation of $T(x)$ to $T(x')$ such that $T(x)$ and $T(x')$ commute?

From the representation (1.3) of the transfer matrix $T(x)$ as a trace and from the commutativity of L_j and L_k it follows that

$$T(x)T(x') = \text{Tr}_{q^2} \left[(L_1 \otimes L'_1) \cdots (L_N \otimes L'_N) \right]$$

where $L_j = L_j(x)$ and $L'_j = L_j(x')$, and the subscript q^2 on the trace denotes the trace is only over the $q^2 \times q^2$ matrix structure of $L_j \otimes L'_j$. To compute $T(x')T(x)$ one simply switches the primed and unprimed labels. From the invariance of the trace under similarity transformations, a sufficient condition for the commutativity of $T(x)$ and $T(x')$ is the existence of nonsingular matrix $R(x, x'): \mathbb{C}^{q^2} \rightarrow \mathbb{C}^{q^2}$ such that

$$\left(L_j(x') \otimes L_j(x) \right) R(x, x') = R(x, x') \left(L_j(x) \otimes L_j(x') \right) \quad (2.1)$$

for all j . This equation is the Yang-Baxter equation.

Remark 2.1.

Yang [11] first derived similar relations as the factorizability condition for a 3-particle S-matrix into products of 2-particle S-matrices. For this reason, these relations have independent interest in quantum field theories of 1 space - 1 time. Baxter [2,3] calls these relations (and their dual version) the generalized star-triangle relations since they generalize the star-triangle relation discovered by Onsager in the two-dimensional Ising model. I believe it was Zamolodchikov [12] who first emphasized the identical structure of these equations in these two areas.

Remark 2.2.

In setting up the YB equations, it was assumed that the energy of a vertex, $\epsilon^{\lambda\lambda'}(\alpha, \alpha')$, did not depend upon the lattice site j . This was assumed for simplicity of presentation. In fact, the staggered vertex models are of considerable theoretical interest [3].

Remark 2.3.

The YB equations "usually" admit only the trivial solution $x' = \rho x$, $R(x, x') = \rho I$ corresponding to the deformation $T(x) \rightarrow \rho^N T(x)$ of transfer matrices.

Remark 2.4.

To understand why a nontrivial solution to the YB equation results in the computation of the free energy per lattice site would require further explanation. This has been explained by a number of authors and the reader is

referred to Baxter [2,3] for the original Q-matrix method or to [9] for the more recent functional equation methods of Stroganov and later workers.

Remark 2.5.

The reader is referred to Perk and Shultz [8] for a review of the literature on the YB equations.

3.0. Differential YB Equations and the YB Ideal

In the YB equation (2.1) it is natural to impose the normalization

$$R(x, x) = I . \quad (3.1)$$

If the YB equation is differentiated with respect to the x' parameters, evaluated at $x' = x$, and (3.1) is imposed, then the differential YB equation is obtained

$$\Omega_{YB} \equiv dL \otimes L - L \otimes dL + (L \otimes L) dR - dR(L \otimes L) = 0 \quad (3.2)$$

where "d" denotes the exterior derivative in $\mathbb{R}^n$, the space of independent Boltzmann weights. Suppose that R has m independent entries (typically, $m=n$). Then a slightly more general setting for (3.2) is to interpret the "d" as exterior differentiation with respect to the independent Boltzmann weights $x = (x_1, x_2, \dots, x_n)$ and the independent r -variables $r = (r_1, r_2, \dots, r_m)$.

We denote by $\{\Omega_\alpha\}$ the collection of nonzero elements of Ω_{YB} . In terms of coordinates $(x, r) \in \mathbb{R}^{n+m}$, each Ω_α has the expansion

$$\Omega_\alpha = \sum_{\beta=1}^n e_{\alpha\beta}(x) dx_\beta + \sum_{\beta=1}^m f_{\alpha\beta}(x) dr_\beta \quad (3.3a)$$

or in the obvious matrix notation

$$\Omega = E dx + F dr . \quad (3.3b)$$

The functions $e_{\alpha\beta}$ and $f_{\alpha\beta}$ depend only upon $x \in \mathbb{R}^n$ since only dR and not R appears in (3.2). In many applications, $e_{\alpha\beta}$ ($f_{\alpha\beta}$) are homogeneous polynomials of degree one (two) in the variables $x_1, \dots, x_n$. To "solve" (3.2)

means to find a manifold $\mathfrak{M}$ (at least locally) such that Ω_α restricted to $\mathfrak{M}$ is identically zero for all α .

Let M_Ω denote the matrix $(E|F)$ so that (3.3b) can be written as

$$\Omega = M_\Omega \begin{pmatrix} dx \\ dr \end{pmatrix}. \quad (3.4)$$

For each $\xi = (x, r) \in R^{n+m}$, we denote by r_ξ the rank of M_Ω evaluated at ξ . A point ξ is said to be a regular point (or sometimes generic point) if $r_\xi \geq r_\eta$ for all $\eta \in R^{n+m}$. We have the following well known result (see [5], for example).

Proposition 3.1.

The set of regular points of M_Ω is an open set in R^{n+m} .

The proof follows from the relation of the rank of a matrix to determinants and the fact that the determinants is a continuous function of the Boltzmann variables $x \in R^n$.

Let ξ be a regular point and B_ξ an open ball centered at ξ such that $r_\eta = r_\xi$ for all $\eta \in B_\xi$. We can assume that in B_ξ a linearly independent set $\{\Omega_\alpha\}_{\alpha=1}^{r_\xi}$ has been chosen. This pfaffian system $\{\Omega_\alpha\}_{\alpha=1}^{r_\xi}$ is said to be completely integrable in B_ξ if there exists r_ξ independent, differentiable functions $f^\alpha: R^{n+m} \rightarrow R$ such that

$$\Omega_\alpha = \sum_{\beta=1}^{r_\xi} a_{\alpha\beta}(x, r) df^\beta \quad (3.5)$$

for all $\eta = (x, r) \in B_\xi$ with $(a_{\alpha\beta}(x, r))_{\alpha, \beta=1}^{r_\xi}$ invertible. The functions f^α are called first integrals, the coefficients $a_{\alpha\beta}$ integrating factors, and the manifold $\mathfrak{M}$ is defined by (at least locally in B_ξ)

$$f^\alpha(x, r) = c^\alpha, \quad c^\alpha \in \mathbb{R}, \quad \alpha = 1, \dots, r_\xi.$$

The following result of Frobenius is fundamental (see [1,5]).

Theorem 3.2.

Let $\xi \in \mathbb{R}^{n+m}$ be a regular point. A necessary and sufficient condition for the pfaffian system $\left\{ \Omega_\alpha \right\}_{\alpha=1}^{r_\xi}$ to be completely integrable in B_ξ is

$$d\Omega_\alpha \wedge V = 0, \quad \alpha = 1, \dots, r_\xi \quad (3.6)$$

where $V = \Omega_1 \wedge \dots \wedge \Omega_{r_\xi}$.

Given a regular point $\xi \in \mathbb{R}^{n+m}$ and $\left\{ \Omega_\alpha \right\}_{\alpha=1}^{r_\xi}$ a pfaffian system, we write this system in the E and F notation of (3.3b). Thus E is a $r_\xi \times n$ matrix, F a $r_\xi \times m$ matrix, and M_Ω a $r_\xi \times (n+m)$ matrix of rank r_ξ at all points $\eta \in B_\xi$. Since the rank of M_Ω is constant in B_ξ , it follows that the $\text{rank}(F) \equiv r_F$ is also constant throughout B_ξ . Partition F (by relabeling if necessary) into F_1 and F_2 where F_1 is an $r_F \times m$ matrix, $\text{rank}(F_1) = r_F$, $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$. This induces a partition for E , $E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$. We now assume the existence of the manifold $\mathfrak{M}$ (that is, we assume that in B_ξ $\left\{ \Omega_\alpha \right\}_{\alpha=1}^{r_\xi}$ satisfies the Frobenius integrability condition (3.6)). On $\mathfrak{M}$ we have

$$E_i dx + F_i dr = 0, \quad i = 1, 2. \quad (3.7)$$

If $\{f_1, \dots, f_{m-r_F}\}$ is a basis for the null space of F_1 , it follows from (3.7) for $i=1$ that there exists a matrix G and constants c_i such that

$$dx = G dx + \sum_{i=1}^{m-r_F} c_i f_i. \quad (3.8)$$

Substituting this into (3.7) gives

$$\omega^\alpha \equiv \sum_{\beta=1}^n (M_\omega)_{\alpha\beta} dx_\beta = 0, \quad \alpha = 1, \dots, (r_\xi - r_F) \quad (3.9)$$

where $M_\omega = E_2 + F_2 G$ is a $(r_\xi - r_F) \times n$ matrix, $\text{rank}(M_\omega) = r_\xi - r_F \equiv n-d$. We denote by ω_{YB} this collection of 1-forms. If the entries of E and F are homogeneous polynomials, then so too are the entries of M_ω . Let $\pi = \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ be the projection $\pi(x, r) = x$.

Theorem 3.3.

If Ω_{YB} is completely integrable in a neighborhood of a regular point $\xi_0 = (x_0, r_0) \in \mathbb{R}^{n+m}$, then $\left\{ \omega^\alpha \right\}_{\alpha=1}^{n-d}$ is completely integrable in a neighborhood of $\pi(\xi_0)$.

Proof.

Let $D = m+n-r_\xi$ and $V^1, \dots, V^D$ independent vector fields such that $\Omega^\alpha(V^i) = 0$, $\alpha = 1, \dots, r_\xi$, $i = 1, \dots, D$. In terms of coordinates

$$V^i = \sum_{j=1}^n A_j^i \frac{\partial}{\partial x_j} + \sum_{j=1}^m B_j^i \frac{\partial}{\partial r_j} \quad (3.10)$$

where A_j^i and B_j^i are functions of $x \in \mathbb{R}^n$ defined in some neighborhood of $\pi(\xi_0)$. The complete integrability of $\left\{ \Omega^\alpha \right\}_{\alpha=1}^{r_\xi}$ implies there exist $C_k^{ij}: \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ such that in a neighborhood of ξ_0

$$[v^i, v^j] = \sum_{k=1}^D c_k^{ij} v^k \quad (3.11)$$

where $[,]$ is the Lie bracket. In fact, c_k^{ij} depend only upon $\pi(\xi)$ since A_j^i and B_j^i in (3.10) are independent of the r -variable. Define the vector fields

$$X^i = \sum_{j=1}^n A_j^i \frac{\partial}{\partial x_j} . \quad (3.12)$$

Then it is a consequence of (3.7)-(3.9) that $\omega^\alpha(X^i) = 0$. There will be $d = n - (r_{\xi_0} - r_F)$ such linearly independent vector fields. Let $\{X^i\}_{i=1}^d$ be such a choice. To show that $\{\omega^\alpha\}_{\alpha=1}^{n-d}$ is completely integrable, it suffices to show (Frobenius theorem for vector fields [1,5])

$$[X^i, X^j] = \sum_{k=1}^d c_k^{ij} X^k . \quad (3.13)$$

But (3.13) follows from (3.11) and the fact that A_j^i and B_j^i appearing in (3.10) are independent of r .

Definition 3.3.

The manifold $\mathcal{M}_{YB} \subset \mathbb{R}^n$ defined by a completely integrable system $\{\omega^\alpha\}_{\alpha=1}^{n-d}$ in some neighborhood of a regular point $x = \pi(\xi)$ will be called the Yang-Baxter manifold for the associated vertex model.

Definition 3.4.

A vertex model is completely integrable if the differential YB equation is completely integrable in the Frobenius sense and $d = \dim \mathcal{M}_{YB} \geq 2$.

We require $d \geq 2$ since $d = 1$ corresponds to the trivial deformation

$$x \mapsto \rho x, \quad x \in \mathbb{R}^n, \quad \rho \in \mathbb{R}.$$

If we denote by $\Lambda(\mathbb{R}^n)$ the Grassman algebra of differential forms on $\mathbb{R}^n$, then the pfaffian system ω_{YB} generates an ideal $\mathcal{I}_{\text{YB}}$, called the Yang-Baxter ideal. An equivalent condition for the existence of $\mathfrak{M}_{\text{YB}}$ is that $\mathcal{I}_{\text{YB}}$ is closed under exterior differentiation, i.e., $d\mathcal{I}_{\text{YB}} \subset \mathcal{I}_{\text{YB}}$. The ideal $\mathcal{I}_{\text{YB}}$ occupies a central geometric role in Cartan's theory of pfaffian systems. In particular, in the context of nonlinear evolution equations, Estabrook and Wahlquist [4] were the first to emphasize the role the corresponding ideal plays in soliton theory. It will be interesting to see if the techniques developed in this context have applications to the differential Yang-Baxter equations.

In a typical application of this formalism, the differential YB equation (3.2) at a generic point ξ will be completely integrable but $\dim \mathfrak{M}_{\text{YB}} = 1$. To proceed further, one must look for conditions amongst the Boltzmann weights that allow the rank of M_Ω (or M_ω) to decrease. Such conditions can be found by computing determinants. For example, at generic points $\xi \in \mathbb{R}^{n+m}$, there exists at least one $r_\xi \times r_\xi$ block of M_Ω such that the determinant of this block is nonzero. In general, this determinant will be a homogeneous polynomial in $x_1, \dots, x_n$. If we require all such determinants to vanish, then this defines an algebraic variety in $\mathbb{R}^n$. We then restrict Ω_{YB} to this variety, and then proceed with a similar analysis as outlined above except that now we are working on an algebraic variety rather than $\mathbb{R}^{n+m}$.

We amplify a case which commonly arises in applications. Suppose $\omega_{\text{YB}} = \left\{ \omega^\alpha \right\}_{\alpha=1}^{n-2}$; that is, we seek a two-dimensional manifold $\mathfrak{M}_{\text{YB}} \subset \mathbb{R}^n$. In terms of vector fields, we first find vector fields X and Y such that $\omega^\alpha(X) = 0$ and $\omega^\alpha(Y) = 0$ for $\alpha=1, \dots, n-2$. Now the dilation vector field

$$D = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \quad (3.14)$$

satisfies $\omega^\alpha(D) = 0$, $\alpha=1, \dots, n-2$, since the deformation

$(x_1, \dots, x_n) \mapsto \rho(x_1, \dots, x_n)$ is the trivial solution to the YB equation. We write a vector field Y as

$$Y = \sum_{i=1}^n c_i x_i \frac{\partial}{\partial x_i} \quad (3.15)$$

and use the equations

$$\omega^\alpha(Y) = 0, \quad \alpha=1, \dots, n-2 \quad (3.16)$$

to determine, say, $c_3, \dots, c_n$ in terms of c_1 and c_2 . This is possible since the ω^α are linearly independent. If we write

$$c_i = c_1 + (c_2 - c_1)a_i, \quad i=2, \dots, n$$

then

$$Y = c_1 D + (c_2 - c_1) \sum_{i=2}^n a_i V_i \quad (3.17)$$

where $V_i = x_i \frac{\partial}{\partial x_i}$. The coefficients $a_i(c_2 - c_1)$ are homogeneous of degree, say, k . Thus an easy calculation gives

$$[D, Y] = kY - kc_1 D \quad (3.18)$$

which shows that the system is completely integrable. This should be compared with the equivalent, but algebraically messy, condition $d\omega^\alpha \wedge \Omega = 0$, $\Omega = \omega^1 \wedge \dots \wedge \omega^{n-2}$.

One might ask why integrability of ω_{YB} is so easily proven in this case. The reason is simply that since all the coefficients of ω^α are

homogeneous, one is really working in P^{n-1} . In P^{n-1} one is finding a curve, not a surface, and hence integrability is then automatic.

4.0. Baxter Model

The Baxter model is defined by choosing the state space λ to be a two-element set and assigning the following Boltzmann weights

$$T^{RR} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \quad T^{RL} = \begin{pmatrix} 0 & x_4 \\ x_3 & 0 \end{pmatrix}, \quad T^{LR} = \begin{pmatrix} 0 & x_3 \\ x_4 & 0 \end{pmatrix}$$

and $T^{LL} = \begin{pmatrix} x_2 & 0 \\ 0 & x_1 \end{pmatrix}$ where "R" refers to "right arrow" and "L" to "left arrow" (see Figure 2). In Baxter's notation $(x_1, x_2, x_3, x_4) = (a, b, c, d)$.

Figure 2: Boltzmann weights for the 8 nonzero vertex configurations in the Baxter model

The matrix R is taken to have the form

$$R = \begin{pmatrix} r_1 & 0 & 0 & r_4 \\ 0 & r_2 & r_3 & 0 \\ 0 & r_3 & r_2 & 0 \\ r_4 & 0 & 0 & r_1 \end{pmatrix} \quad (4.1)$$

Then Ω_{YB} consists of six nonzero differential forms

$$\begin{aligned} \Omega^1 &= x_3 dx_4 - x_4 dx_3 + (x_1^2 - x_2^2) dr_4 \\ \Omega^2 &= x_1 dx_2 - x_2 dx_1 + (x_3^2 - x_4^2) dr_3 \\ \Omega^3 &= x_2 dx_4 - x_4 dx_2 - x_2 x_4 dr_1 + x_2 x_4 dr_2 + x_1 x_4 dr_3 - x_2 x_3 dr_4 \\ \Omega^4 &= x_1 dx_3 - x_3 dx_1 - x_1 x_3 dr_1 + x_1 x_3 dr_2 + x_2 x_3 dr_3 - x_1 x_4 dr_4 \end{aligned} \quad (4.2)$$

$$\Omega^5 = x_4 dx_1 - x_1 dx_4 - x_1 x_4 dr_1 + x_1 x_4 dr_2 + x_2 x_4 dr_3 - x_1 x_3 dr_4$$

$$\Omega^6 = x_3 dx_2 - x_2 dx_3 - x_2 x_3 dr_1 + x_2 x_3 dr_2 + x_1 x_3 dr_3 - x_2 x_4 dr_4.$$

These forms are not linearly independent and at a regular point ξ ,

$r_\xi = \text{rank}(M_\Omega) = 5$. It can be further checked that the first five forms are

linearly independent and the Frobenius condition (3.6) is satisfied. I

emphasize that such computations can be done using symbolic computer algebra

programs (MACSYMA, for example). The F matrix has $r_F = \text{rank}(F) = 3$ so

that, upon elimination of the dr_i from (4.2), there results $\omega_{YB} = \left\{ \omega^\alpha \right\}_{\alpha=1}^2$

where

$$\omega^1 = \frac{dx_1}{x_1} + \frac{dx_2}{x_2} - \frac{dx_3}{x_3} - \frac{dx_4}{x_4} \quad (4.3)$$

$$\omega^2 = (x_1^2 - x_2^2) \left(\frac{dx_1}{x_1} - \frac{dx_2}{x_2} \right) - (x_3^2 - x_4^2) \left(\frac{dx_3}{x_3} - \frac{dx_4}{x_4} \right).$$

We now rephrase the problem in terms of vector fields. We must find two linearly independent vector fields X and Y so that $\omega^\alpha(X) = 0$ and $\omega^\alpha(Y) = 0$. We choose $Y = D$ and a calculation shows that X must be of the form

$$X = \left(A + B \frac{x_3^2 - x_4^2}{x_1^2 - x_2^2} \right) x_1 \partial_1 + \left(A - B \frac{x_3^2 - x_4^2}{x_1^2 - x_2^2} \right) x_2 \partial_2 \quad (4.4)$$

$$+ (A+B)x_3 \partial_3 + (A-B)x_4 \partial_4.$$

Choosing $A = 0$ and $B = (x_1^2 - x_2^2)$, we get the symmetrical choice

$$X = (x_3^2 - x_4^2)(x_1 \partial_1 - x_2 \partial_2) + (x_1^2 - x_2^2)(x_3 \partial_3 - x_4 \partial_4). \quad (4.5)$$

We now look for the first integrals of ω_{YB} . A first integral f of X satisfies $\mathcal{L}_X f = 0$, where $\mathcal{L}_X$ is the Lie derivative with respect to the vector field X . We want to find f^1 and f^2 such that $\mathcal{L}_D f^\alpha = 0$ and $\mathcal{L}_X f^\alpha = 0$ (X given by (4.5)). Now $\mathcal{L}_D f^\alpha = 0$ simply says the f^α are homogeneous of degree zero. Thus we write

$$f^\alpha = \frac{g^\alpha}{h^\alpha} \quad (4.6)$$

where g^α and h^α are homogeneous polynomials in $x_1, \dots, x_4$ of the same degree. Since $f \neq 0$ (dropping the index α), we have

$$\frac{1}{g} \mathcal{L}_X g - \frac{1}{h} \mathcal{L}_X h = 0. \quad (4.7)$$

Thus a sufficient condition for $\mathcal{L}_X f = 0$ is

$$\mathcal{L}_X g = 0 \quad \text{and} \quad \mathcal{L}_X h = 0. \quad (4.8)$$

It is easy to check that polynomials of degree 1 cannot satisfy (4.7). The next simplest guess is a polynomial of degree 2. Thus we make the ansatz for g (and similarly for h)

$$g(x_1, x_2, x_3, x_4) = \sum_{i \leq j} a_{ij} x_i x_j. \quad (4.9)$$

Now $\mathcal{L}_X g$ is a polynomial of degree four in $x_1, \dots, x_4$ and linear in the a_{ij} . Requiring this to be the zero polynomial, we obtain equations for the a_{ij} . Solving these we find that the most general g of the form (4.9) to satisfy $\mathcal{L}_X g = 0$ is

$$g(x_1, x_2, x_3, x_4) = a_{12} x_1 x_2 + a_{34} x_3 x_4 + a_{11} (x_1^2 + x_2^2 - x_3^2 - x_4^2). \quad (4.10)$$

Of course, the same result holds for h and we conclude that we can choose

$$f^1(x) = \frac{x_3 x_4}{x_1 x_2} \quad (4.11)$$

and

$$f^2(x) = \frac{x_1^2 + x_2^2 - x_3^2 - x_4^2}{2x_1 x_2}.$$

In this way we have arrived at the fundamental result of Baxter [2].

Theorem 4.1 (Baxter).

The Baxter transfer matrices $T(x)$ and $T(x')$ commute if x and x' both lie on the manifold $\mathfrak{m}_B$ defined by

$$f^1(x) = \frac{x_3 x_4}{x_1 x_2} = c_1 \quad (4.12)$$

and

$$f^2(x) = \frac{x_1^2 + x_2^2 - x_3^2 - x_4^2}{2x_1 x_2} = c_2$$

for constants c_1 and c_2 .

If we complexify the coordinates x^i , then $x = (x_1, x_2, x_3, x_4)$ can be viewed as the homogeneous coordinates of an element of $P^3(\mathbb{C})$. The manifold $\mathfrak{m}_B \subset P^3(\mathbb{C})$ is the intersection of the two algebraic surfaces defined by f^α . It can be shown that, for generic c_1 and c_2 , this embedded curve is an elliptic curve.

Defining $w_1 = \frac{1}{2}(x_1 + x_2)$, $w_2 = \frac{1}{2}(x_1 - x_2)$, $w_3 = \frac{1}{2}(x_3 + x_4)$, and $w_4 = \frac{1}{2}(x_3 - x_4)$, we see that the differential equations $\dot{x}_i = X_i(x)$ where X is the vector field (4.5) imply $\dot{w}_1 = 4w_2 w_3 w_4$, $\dot{w}_2 = 4w_1 w_3 w_4$, $\dot{w}_3 = 4w_1 w_2 w_4$,

and $\dot{w}_4 = 4w_1w_2w_3$. Introducing the ratios $\Omega_1 = w_1/w_3$, $\Omega_2 = w_2/w_3$, and $\Omega_3 = w_4/w_3$ we obtain the differential equations

$$\begin{aligned}\dot{\Omega}_1 &= (w_3^2 - w_1^2)\Omega_2\Omega_3 \\ \dot{\Omega}_2 &= (w_3^2 - w_2^2)\Omega_3\Omega_1 \\ \dot{\Omega}_3 &= (w_3^2 - w_4^2)\Omega_1\Omega_2\end{aligned}\tag{4.14}$$

The functions $(w_3^2 - w_j^2)$ are in fact first integrals of the vector field X as is the function $\frac{1}{2}x_1x_2$. Since the vector field X determines the differential equations only up to a parametrization, we change the parametrization in (4.14) so that

$$\begin{aligned}\dot{\Omega}_1 &= \alpha_1\Omega_2\Omega_3 \\ \dot{\Omega}_2 &= \alpha_2\Omega_3\Omega_1 \\ \dot{\Omega}_3 &= \alpha_3\Omega_1\Omega_2\end{aligned}\tag{4.15}$$

where $\alpha_1 = \frac{2(w_3^2 - w_1^2)}{x_1x_2} = c_1 - c_2 - 1$, $\alpha_2 = \frac{2(w_3^2 - w_2^2)}{x_1x_2} = c_1 - c_2 + 1$, and

$\alpha_3 = \frac{2(w_3^2 - w_4^2)}{x_1x_2} = 2c_2$ with c_1 and c_2 the constants from Theorem 4.1. The advantage of (4.15) is that the α_i are also first integrals of the dilation vector field D . These equations are essentially the Euler equations for a rigid body. As shown by Mishchenko [7], the Euler equations can be written as a Lax pair [6]

$$\dot{M} = [M, \Omega]\tag{4.16}$$

where $M = \Omega J + J\Omega$,

$$\Omega = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \quad (4.17)$$

$$J = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix}$$

where now the constants appearing in (4.15) are $\frac{J_3-J_2}{J_3+J_2}$, $\frac{J_1-J_3}{J_1+J_3}$, and $\frac{J_2-J_1}{J_2+J_1}$, respectively.

To identify the α_j in (4.15) with these expressions, we are lead to a system of homogeneous equations for the J_i which in general have only the trivial solution. To avoid this we again change the "time" parameter $t \rightarrow \alpha t$ where α satisfies

$$\alpha^2 \alpha_1 \alpha_2 \alpha_3 + \alpha_1 + \alpha_2 + \alpha_3 = 0 \quad (4.18)$$

which is the determinant condition from the homogeneous equations. In this manner we can solve for the ratios J_1/J_3 and J_2/J_3 . The results are

$$\frac{J_1}{J_3} = - \frac{(\alpha_2 \alpha_3 - \alpha_1 \alpha_3 - 2\lambda + \alpha_2^2 + \alpha_1 \alpha_2)}{(\alpha_1 + \alpha_2)(\alpha_2 + \alpha_3)}, \quad (4.19)$$

$$\frac{J_2}{J_3} = \frac{(\alpha_2 \alpha_3 - \alpha_1 \alpha_3 - 2\lambda - \alpha_1^2 - \alpha_1 \alpha_2)}{(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_3)},$$

and $\lambda = [-\alpha_1 \alpha_2 \alpha_3 (\alpha_1 + \alpha_2 + \alpha_3)]^{1/2}$.

Thus we have shown

Theorem 4.2.

The nonlinear differential equations that determine the Baxter curve m_B in $P^3(\mathbb{C})$ can be written as a Lax pair. This Lax pair is equivalent to the Euler equations for a rigid body.

The reduction of (4.15) to the Lax form assumes that $\alpha_i \neq 0$, $i=1,2$ and 3. In the 6-vertex case ($x_4=0 \rightarrow w_3=w_4$), (4.15) becomes

$$\begin{aligned}\dot{\Omega}_1 &= \lambda_1 \Omega_2 \\ \dot{\Omega}_2 &= \lambda_2 \Omega_1\end{aligned}$$

where $\lambda_1 = (1+c_2)\Omega_3$, $\lambda_2 = (1-c_2)\Omega_3$, and $\Omega_3 = \text{constant}$.

Remark 4.3.

That there exists a connection between asymmetric tops and the Baxter parametrization of the w_i was conjectured by R. Hermann in a private communication.

5.0. General 8-Vertex

The 8-vertex model with periodic boundary conditions is equivalent to choosing six independent Boltzmann weights as shown in Figure 3 [3].

Figure 3: Boltzmann weights for 8-vertex model

The matrices $T^{\lambda\lambda'}$ are $T^{RR} = \begin{pmatrix} x_1 & 0 \\ 0 & x_3 \end{pmatrix}$, $T^{RL} = \begin{pmatrix} 0 & x_6 \\ x_5 & 0 \end{pmatrix}$,

$$T^{LR} = \begin{pmatrix} 0 & x_5 \\ x_6 & 0 \end{pmatrix}, \text{ and } T^{LL} = \begin{pmatrix} x_4 & 0 \\ 0 & x_2 \end{pmatrix}.$$

We make the ansatz

$$R = \begin{pmatrix} r_1 & 0 & 0 & r_4 \\ 0 & r_2 & r_3 & 0 \\ 0 & r_3 & r_5 & 0 \\ r_4 & 0 & 0 & r_6 \end{pmatrix}$$

A more general matrix R leads to the same conclusions. Assuming x_5 and x_6 are nonzero, Ω_{YB} consists of 13 nonzero forms. These forms are given in Appendix A. At a generic point, $\text{rank}(M_\Omega) = 10$ and $\text{rank}(F) = 5$. Thus upon elimination of the dr_i 's, the set $\omega_{\text{YB}} = \left\{ \omega^\alpha \right\}_{\alpha=1}^5$ has five linearly independent forms. Each ω^α satisfies $\omega^\alpha(D) = 0$ where D is the dilation vector field in $\mathbb{R}^6$.

Thus we conclude

Theorem 5.1.

The general 8-vertex model admits only the trivial deformation at a generic point $x \in \mathbb{R}^6$. Thus, in the sense of Definition 3.4, the general 8-vertex model is not integrable.

To find conditions on the Boltzmann weights so that the 8-vertex model is integrable, we must determine the singular points of M_Ω . Unfortunately, the author has no concise way to do this and must resort to long calculations to determine the singular points. Recall that in this case, a point $x \in \mathbb{R}^6$ is singular if $\text{rank}(M_\Omega) < 10$. In principle we look at all 10×10 blocks of M_Ω obtained by deleting three rows and two columns from M_Ω . The determinants of these blocks define a family $\{\sigma_\alpha\}$ of homogeneous polynomials. The zero set of $\{\sigma_\alpha\}$ then defines an algebraic variety on which M_Ω is singular.

At a generic point $x \in \mathbb{R}^6$ we choose a basis for $\{\Omega^\alpha\}$, say $\{\Omega^\alpha\}_{\alpha=4}^{13}$ (see Appendix A). For this basis we form the 10×12 coefficient matrix $M_\Omega^{(B)}$ (that is, the M_Ω matrix associated with $\{\Omega^\alpha\}_{\alpha=4}^{13}$). If we define

$$d(i,j) = \det(M_\Omega^{(B)}(i,j))$$

where $M_\Omega^{(B)}(i,j)$ is the 10×10 matrix obtained from $M_\Omega^{(B)}$ by deleting columns i and j , then a calculation (using MACSYMA) shows that either $d(i,j)$ is identically zero or $d(i,j) = (-1)^{i+j} x_i x_j \theta(x)$ ($i=1,2,\dots,6$; $j=7,8,11,12$) where

$$\begin{aligned}
\sigma(x) = & 4x_2^3x_5^2(x_6^2+x_5^2-x_3x_4-x_1x_2)(x_1x_4x_6^2-x_2x_3x_6^2 \\
& - x_1x_4x_5^2 + x_2x_3x_5^2 - x_2x_4^3 + 2x_1x_3x_4^2 - x_2x_3^2x_4 \\
& + x_2^3x_4 - x_1^2x_2x_4) .
\end{aligned} \tag{5.1}$$

A necessary condition for the rank of M_Ω to be less than 10 is $\sigma \equiv 0$. In the Baxter case ($x_1 = x_2$ and $x_3 = x_4$), the quartic factor in (5.1) is identically zero. The condition

$$x_1x_2 + x_3x_4 - x_5^2 - x_6^2 = 0 \tag{5.2}$$

on the Boltzmann weights which also implies $\sigma = 0$ is called the "free fermion condition" and it is known that the free fermion model is exactly solvable.

An additional complication arises. At a generic point $x \in \mathbb{R}^6$ we can write

$$\Omega^\alpha = \sum_{\beta=4}^{13} C_{\alpha\beta} \Omega^\beta, \quad \alpha=1,2,3 \tag{5.3}$$

where the coefficients $C_{\alpha\beta}$ depend upon x . At first one might say that restricting (5.3) to the algebraic variety $\sigma(x) \equiv 0$ shows that Ω^α , $\alpha=1,2,3$ are linearly dependent upon Ω^β , $\beta=4, \dots, 13$. This would, in fact, be the case if the coefficients $C_{\alpha\beta}$ in (5.3) are defined on the variety $\sigma = 0$.

However, this is not at all clear since it is precisely this condition that makes the (maximal) determinants vanish. At this point we have two choices.

Either we consider all 10×10 blocks of M_Ω and compute the family $\{\sigma_\alpha\}$ or we restrict $\{\Omega^\alpha\}_{\alpha=1}^{13}$ to $\sigma = 0$ and calculate a new M_Ω for the restricted $\{\Omega^\alpha\}$ and repeat the above process.

Of course there is a third possibility and that is to gain a better theoretical understanding and avoid these calculations altogether!

Acknowledgements

The author wishes to thank John Cremona, Robert Hermann, Blaise Morton, Tom Shemanske, and Ernst Snapper for useful conversations. I wish to particularly thank J.H.H. Perk for pointing out an error in an earlier version. Furthermore, the hospitality of Professors Weinberger and Sell at the Institute for Mathematics and Its Applications where this work was started is gratefully acknowledged. Some of the calculations described here were carried out at the Kiewit Computation Center at Dartmouth College on the MACSYMA symbolic manipulation system running on a VAX 11/750.

Appendix A

The differential YB equation for the 8-vertex model leads to the system

$$\left\{ \Omega^\alpha \right\}_{\alpha=1}^{13} \quad (\text{assuming } x_5 \text{ and } x_6 \text{ are nonzero}) \text{ where}$$

$$\begin{aligned} \Omega^1 &= x_6 dx_1 - x_1 dx_6 - x_1 x_6 dr_1 + x_1 x_6 dr_2 + x_3 x_6 dr_3 - x_2 x_5 dr_4 \\ \Omega^2 &= x_5 dx_3 - x_3 dx_5 - x_3 x_5 dr_1 + x_3 x_5 dr_2 + x_1 x_5 dr_3 - x_4 x_6 dr_4 \\ \Omega^3 &= -x_6 dx_3 + x_3 dx_6 - x_3 x_6 dr_1 + x_3 x_6 dr_2 + x_1 x_6 dr_3 - x_4 x_5 dr_4 \\ \Omega^4 &= -x_5 dx_1 + x_1 dx_5 - x_1 x_5 dr_1 + x_1 x_5 dr_2 + x_3 x_5 dr_3 - x_2 x_6 dr_4 \\ \Omega^5 &= -x_6 dx_5 + x_5 dx_6 - x_5 x_6 dr_1 + (x_1^2 - x_4^2) dr_4 + x_5 x_6 dr_6 \\ \Omega^6 &= x_6 dx_5 - x_5 dx_6 - x_5 x_6 dr_1 + (x_3^2 - x_2^2) dr_4 + x_5 x_6 dr_6 \\ \Omega^7 &= x_4 dx_1 - x_1 dx_4 + (x_6^2 - x_5^2) dr_3 \\ \Omega^8 &= -x_3 dx_2 + x_2 dx_3 + (x_5^2 - x_6^2) dr_3 \\ \Omega^9 &= dr_2 - dr_5 \\ \Omega^{10} &= -x_6 dx_2 + x_2 dx_6 - x_2 x_6 dr_2 - x_4 x_6 dr_3 + x_1 x_5 dr_4 + x_2 x_6 dr_6 \\ \Omega^{11} &= -x_5 dx_4 + x_4 dx_5 - x_4 x_5 dr_2 - x_2 x_5 dr_3 + x_3 x_6 dr_4 + x_4 x_5 dr_6 \\ \Omega^{12} &= x_6 dx_4 - x_4 dx_6 - x_4 x_6 dr_2 - x_2 x_6 dr_3 + x_3 x_5 dr_4 + x_4 x_6 dr_6 \\ \Omega^{13} &= x_5 dx_2 - x_2 dx_5 - x_2 x_5 dr_2 - x_4 x_5 dr_3 + x_1 x_6 dr_4 + x_2 x_5 dr_6 \end{aligned}$$

The set $\left\{ \Omega^\alpha \right\}_{\alpha=4}^{13}$ forms a linearly independent set at a generic point $x \in \mathbb{R}^6$.

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