**TIME-REVERSIBILITY
AND PARTICLE SEDIMENTATION**

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§0. Introduction

Jayaweera, Mason & Slack [JMS] and Hocking [H] study the sedimentation of small clusters of small spheres in a highly viscous fluid experimentally and analytically. Experimentally [JMS] find that when the cluster consists of fewer than seven falling particles, the spheres arrange themselves in a regular polygon. When the cluster consists of seven or more spheres, they find that the regular polygon is unstable. When the cluster size is three, they also find evidence for periodic motion where the spheres arrange themselves at the vertices of (a time varying) isosceles triangle, at least on a short to moderate time scale. This periodic motion takes place in a frame moving with the center of gravity of the cluster.

To explain the transition in observed phenomena when the cluster size reaches seven, Hocking [H] studies an ODE model called the Stokeslet. In the Stokeslet model, one replaces spheres by point particles with gravity acting as a delta function, assumes Stokes flow for individual particles (an infinite viscosity limit), and assumes an interaction between particles (caused by the flow fields of the individual particles). In this model, the regular polygon is an equilibrium. By direct calculation of the Jacobian matrix at the regular polygon, [H] shows that this equilibrium is elliptic (all eigenvalues on the imaginary axis) when the cluster size is less than seven and hyperbolic for cluster sizes between seven and twelve, with implication that the equilibria are hyperbolic for all cluster

sizes greater than seven. Stability is equated with ellipticity.

Caflisch, Lim, Luke & Sangani [CLLS] continue the study of the Stokeslet model, showing that when the cluster size is three there is indeed a periodic solution where the particles arrange themselves for all time as isosceles triangles. They also study numerically a more complicated model which assumes spheres of finite size rather than point particles. The [CLLS] proof of existence of the isosceles triangle periodic solution proceeds by explicitly reducing the Stokeslet system of ODE to a lower-dimensional phase space in which the equations are Hamiltonian, and then using the Hamiltonian structure to determine the periodic solutions. One of the main interests of [CLLS] in finding this periodic solution is to use its existence as a test for numerical codes developed to solve the more complicated finite sphere model.

In this paper we continue the study of periodic and quasi-periodic solutions the Stokeslet model for all cluster sizes. Our point of view is distinct from those of the previous papers in that our entire discussion is based on the existence of both time-reversal and spatial symmetries in the Stokeslet model. The time-reversal symmetry is due to the infinite viscosity limit and the spatial symmetries are due to the assumption of identical particles with symmetric coupling. Indeed, in general, the Stokeslet model does not have a Hamiltonian structure, but it does have a time-reversal symmetry, which turns out to be just as good as a Hamiltonian structure for finding periodic solutions.

Using symmetry arguments, we will show that, when the cluster size is three, there are two families of periodic solu-

tions to the Stokeslet model and one family of two-frequency motions. The periodic solutions form isosceles triangles; in one family the vertices of the side of unequal length move synchronously and in the other family these vertices move with a half-period phase shift. The family of two-frequency motions consists of equilateral triangles whose vertices move with a third of a period phase lag while rotating slowly on a circle (ponies on a merry-go-round). In addition, our methods enable us to prove the existence of solutions corresponding to these for the n particle cluster. Our theory shows, moreover, that only two calculations are needed using the Stokeslet model to prove the existence of these solutions: one is the computation of eigenvalues done in [H], and the other is the verification of a transversality condition showing that ponies actually rotate on the merry-go-round. This calculation is completed only when $n = 3$.

The mathematical theory we develop is one of time-reversible systems with spatial symmetries. We begin in Section 1 by describing a Liapunov-center theorem for reversible systems (following Vanderbauwhede [V]). This theorem assumes simple imaginary eigenvalues and nonresonance in the Jacobian matrix, which is not valid for the Stokeslet model. In Sections 2 and 3 we generalize this theorem to include zero eigenvalues in the Jacobian and 1:1 resonances caused by symmetry. Our main result is summarized in Theorem 3.1. Our method follows ideas in [GSt] for finding periodic solutions in symmetric systems undergoing Hopf bifurcation and ideas in [MRS] for finding periodic solutions in symmetric Hamiltonian systems.

In Section 4 we describe in detail the symmetries of the Stokeslet model (whose derivation is described briefly in an Appendix) and in Section 5 we describe how these symmetries affect the linear (Jacobian) analysis. The basic group theory of the Stokeslet is presented in these sections. In Section 6 we describe how to use Theorem 3.1 and symmetries to find periodic solutions for the Stokeslet. We emphasize here how the symmetries of the model determine the nature of the periodic solutions we find. The periodic solutions for $n \leq 6$ are described in Section 7.

In Section 8 we show how the two-frequency ponies on a merry-go-round are formed. Here we use results of Krupa [K] on bifurcation from group orbits of equilibria to organize the discussion. The remainder of this paper is devoted to the three particle Stokeslet model. The calculations mentioned previously verifying that the ponies actually rotate when $n = 3$ are also given in this section. Genericity arguments tell us that this result is expected; indeed, we expect the corresponding statement concerning ponies in the n particle case to be valid.

Finally, in Section 9 we comment on the linearized stability of the three families of solutions for $n = 3$. We prove that generically one of the two families of isosceles triangles is elliptic and the other hyperbolic and that the family of ponies is always elliptic. These remarks are based on theoretical normal form calculations (which introduce a temporal phase shift symmetry), spatial symmetries and time-reversal symmetry. Our linear stability results are similar to those of [MRS].

§1. Liapunov Center Theorem for Time-Reversible Systems

Vanderbauwhede [V] proves a Liapunov Center Theorem for time-reversible systems. We sketch his proof here so that we may generalize it to system with spatial symmetries.

Consider the system of ODE

$$\frac{dx}{dt} = f(x), \quad x \in V \quad (1.1)$$

where $f: V \rightarrow V$ is C^∞ and $V \cong \mathbb{R}^n$ for some n . A time-reversal symmetry of (1.1) is a reflection $R: V \rightarrow V$ satisfying

$$f(Rx) = -Rf(x). \quad (1.2)$$

Thus, when $x(t)$ is a solution to (1.1) so is

$$(T_R x)(t) \equiv Rx(-t). \quad (1.3)$$

We assume that x_0 is an equilibrium of (1.1) that is invariant under the time-reversal symmetry, i.e.,

$$f(x_0) = 0 \quad \text{and} \quad Rx_0 = x_0. \quad (1.4)$$

Observe that (1.4) coupled with differentiation of (1.2) leads to the antisymmetry

$$(df)_{x_0} R = -R(df)_{x_0}. \quad (1.5)$$

Hence, if λ is an eigenvalue of $(df)_{x_0}$ with eigenvector v , then $-\lambda$ is an eigenvalue with eigenvector Rv . Note that if $n = 2$, then λ must be either real or purely imaginary (depending on the sign of $\det(df)_{x_0}$).

This eigenvalue dichotomy occurs frequently in higher

dimensions, especially when spatial symmetries are present. In this section we assume:

$$\pm\omega_0 i \text{ are simple eigenvalues of } (df)_{x_0}, \text{ and} \quad (1.6)$$

$$k\omega_0 i \text{ is not an eigenvalue of } (df)_{x_0} \text{ (} k = 0, 2, 3, \dots \text{)}.$$

The Liapunov Center Theorem is:

Theorem 1.1: Assuming (1.2), (1.4) and (1.6) there exists a smooth one-parameter family of periodic solutions $x_a(t)$ of (1.1) with periods near $2\pi/\omega_0$ where $x_0(t) \equiv x_0$.

Proof: The proof of this theorem is an adaptation of the Liapunov-Schmidt reduction proof of Hopf bifurcation where the role of a bifurcation parameter is replaced by time-reversibility. We follow the exposition in [V,7.5.3] and [GS,VIII,§2].

Begin by rescaling time in (1.1) so that $\pm i$ are eigenvalues of $(df)_{x_0}$. Define $F: \mathcal{E}_{2\pi}^1 \times \mathbb{R} \rightarrow \mathcal{E}_{2\pi}$ by

$$F(x, \tau) = (1 + \tau) \frac{dx}{dt} - f(x) \quad (1.7)$$

where $\mathcal{E}_{2\pi}$ is the Banach space of V -valued, 2π -periodic mappings and $\mathcal{E}_{2\pi}^1$ consists of those $x(t) \in \mathcal{E}_{2\pi}$ that are continuously differentiable. Observe that solutions to $F(x, \tau) = 0$ correspond to $2\pi/(1+\tau)$ -periodic solutions of (1.1).

We solve $F = 0$ by Liapunov-Schmidt reduction. Hypothesis (1.6) implies that $\ker(df)_{x_0, 0}$ is two-dimensional, being spanned by $\operatorname{Re}(e^{it}v)$ and $\operatorname{Im}(e^{it}v)$ where $(df)_{x_0} v = iv$. Indeed, we can

identify $\ker(dF)_{x_0,0}$ with $\mathbb{C}$ by $z \rightarrow \operatorname{Re}(ze^{it}v)$. Liapunov-Schmidt reduction proves the existence of a smooth mapping

$$\phi: \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C}$$

whose zeros are in 1:1 correspondence with the (near 2π)-periodic solutions of (1.1).

An essential ingredient in the proof of both Hopf bifurcation and of this theorem is the existence of S^1 phase shift symmetry in the reduced equations ϕ . Observe that $\theta \in S^1$ acts on $x(t) \in \mathbb{C}_{2\pi}$ by

$$\theta \cdot x(t) = x(t - \theta), \quad (1.8)$$

and that F is S^1 -equivariant with respect to this action. Hence, when the Liapunov-Schmidt reduction is done correctly, the reduced function ϕ is also S^1 -equivariant. The action induced on $\ker(dF)_{x_0,0}$ by (1.8) is just the natural action

$$\theta \cdot z = e^{i\theta} z. \quad (1.9)$$

It follows from (1.9) and S^1 -equivariance that

$$\phi(z, \tau) = p(z\bar{z}, \tau)z + q(z\bar{z}, \tau)iz \quad (1.10)$$

where p and q are smooth, real-valued functions. See [GS, VIII, Proposition 2.3]. The details of Liapunov-Schmidt reduction show that

$$p(0,0) = 0, \quad q(0,0) = 0 \text{ and } q_\tau(0,0) \neq 0. \quad (1.11)$$

In Hopf bifurcation f and hence F, ϕ, p and q depend smoothly on a bifurcation parameter μ . Solving $\phi = 0$ for

nontrivial solutions ($z \neq 0$) is equivalent by (1.10) to solving $p = q = 0$. By (1.11) and the Implicit Function Theorem the equation $q = 0$ can be solved locally near the origin for $\tau = \tau(z\bar{z}, \mu)$, leading to the single equation $p(z\bar{z}, \tau(z\bar{z}, \mu), \mu) = 0$. Finally, in Hopf bifurcation, one shows that $p_\mu(0, 0, 0) \neq 0$ if complex conjugate eigenvalues of dF cross the imaginary axis with nonzero speed as μ is varied. Thus, the Implicit Function Theorem can be applied once again to solve $p = 0$ locally for $\mu = \mu(z\bar{z})$, thus giving the desired branch of periodic solutions.

In this theorem we use the existence of the time-reversible symmetry R to solve $\phi = 0$. Observe that F , as defined in (1.7), anticommutes with T_R , as defined in (1.3), that is,

$$F(T_R x, \tau) = -T_R F(x, \tau). \quad (1.12)$$

Differentiation of (1.12) leads to the anticommutativity

$$(dF)_{x_0, 0} T_R = -T_R (dF)_{x_0, 0}, \quad (1.13)$$

and hence that $T_R: \ker(dF)_{x_0, 0} \rightarrow \ker(dF)_{x_0, 0}$. Let $R' = T_R|_{\ker(dF)_{x_0, 0}}$.

We assert that R' and S^1 do not commute on $\ker(dF)_{x_0, 0}$, and hence that $R' \notin S^1$, since S^1 is commutative. To prove this assertion we let $x(t)$ be in $\mathcal{E}_{2\pi}$ and compute

$$(\theta \cdot T_R \cdot x)(t) = (T_R \cdot x)(t - \theta) = Rx(-t + \theta)$$

$$(T_R \cdot \theta \cdot x)(t) = R(\theta \cdot x)(-t) = Rx(-t - \theta).$$

Thus, as long as $x(t) \neq 0$ and $\theta \neq 0$ or π , $\theta \cdot T_R \cdot x \neq T_R \cdot \theta \cdot x$.

Since R' and S^1 act orthogonally on $\mathbb{C}$, they generate $O(2)$. It follows that there exists $\theta \in S^1$ such that $R'' = \theta \cdot R'$ acts as complex conjugation on $\mathbb{C}$. If we define $T' = \theta \cdot T_R$ acting on $\mathbb{C}_{2\pi}$, then T' is also an antisymmetry of F .

With these comments, it is now straightforward to check that Liapunov-Schmidt reduction respects time-reversible symmetries, that is,

$$\phi(\bar{z}, \tau) = -\overline{\phi(z, \tau)}. \quad (1.14)$$

Identity (1.14) applied to (1.10) implies that $p \equiv 0$, that is,

$$\phi(z, \tau) = q(z\bar{z}, \tau)iz. \quad (1.15)$$

It follows from (1.11) and (1.15) that $\phi = 0$ can always be solved, using the Implicit Function Theorem, for $\tau = \tau(z\bar{z})$, $\tau(0) = 0$. This solution translates to a one-parameter family of periodic solutions to (1.1) parameterized by the amplitude $z\bar{z}$.

§2. Liapunov-Center Theorem with Zero Eigenvalues

As noted in Section 1, eigenvalues of $(df)_{x_0}$ come in pairs $\pm\lambda$. When $Rv = v$, where v is an eigenvector with eigenvalue λ , λ must equal 0. For this reason, time-reversibility often leads to zero eigenvalues, but usually when $R|_{\ker(df)_{x_0}}$ is the identity map. In this section, we generalize Theorem 1.1 to include this possibility.

Define $V_0 \equiv \ker(df)_{x_0}$ and let $m = \dim V_0$.

Theorem 2.1: Assume that (1.1) satisfies (1.2), (1.4),

$$\begin{aligned} \pm\omega_0 i \text{ are simple eigenvalues of } (df)_{x_0} \text{ and} \\ k\omega_0 i \text{ is not an eigenvalue of } (df)_{x_0} \text{ (} k = 2, 3, \dots \text{),} \end{aligned} \tag{2.1}$$

and

$$R|_{V_0} = I_{V_0}. \tag{2.2}$$

Then there exists a unique $(m + 1)$ -parameter family of periodic solutions $x_a(t)$ of (1.1) with period near $2\pi/\omega_0$.

Remark: Hypothesis (2.2) was used by Scheurle [S] when developing a KAM theory for reversible systems.

Proof: The proof of this theorem is similar to the proof of Theorem 1.1; we indicate the changes.

Assumption (2.1) implies that

$$\ker(dF)_{x_0,0} = V_i \oplus V_0 \quad (2.3)$$

where V_i is the two-dimensional real eigenspace of $(df)_{x_0}$ associated with the eigenvalues $\pm i$ and, as in Section 1, can be identified with $\mathbb{C}$. Note that S^1 phase shift symmetry acts as in (1.9) on V_i and trivially on V_0 (since V_0 consists of constant functions).

Hence, the Liapunov-Schmidt reduction leads to a smooth S^1 -equivariant mapping

$$\phi: \mathbb{C} \times V_0 \times \mathbb{R} \rightarrow \mathbb{C} \times V_0 \quad (2.4)$$

whose zeroes are in 1:1 correspondence with the desired periodic solutions to (1.1). Writing $\phi = (\phi_i, \phi_0)$ in coordinates and using S^1 -equivariance leads to:

$$\begin{aligned} (a) \quad \phi_i(e^{i\theta}z, w, \tau) &= e^{i\theta}\phi_i(z, w, \tau) \\ (b) \quad \phi_0(e^{i\theta}z, w, \tau) &= \phi_0(z, w, \tau). \end{aligned} \quad (2.5)$$

From (2.6) we see that the w -coordinates may be treated as parameters. Hence ϕ has the form

$$\begin{aligned} (a) \quad \phi_i(z, w, \tau) &= p(z\bar{z}, w, \tau)z + q(z\bar{z}, w, \tau)iz \\ (b) \quad \phi_0(z, w, \tau) &= r(z\bar{z}, w, \tau) \end{aligned} \quad (2.7)$$

where p, q and r are smooth, real-valued functions.

Next, we consider the effects of the time-reversal symmetry R . As in Section 1, ϕ anticommutes with the action of T_R on $\ker(dF)_{x_0,0}$, which after a possible phase shift, we may write as:

$$R''(z, w) = (\bar{z}, w). \quad (2.8)$$

Hence

$$\begin{aligned} (a) \quad \phi_i(\bar{z}, w, \tau) &= -\overline{\phi_i(z, w, \tau)} \\ (b) \quad \phi_0(\bar{z}, w, \tau) &= -\phi_0(z, w, \tau). \end{aligned} \quad (2.9)$$

It follows from (2.7) and (2.9) that

$$\begin{aligned} (a) \quad \phi_i(z, w, \tau) &= q(z\bar{z}, w, \tau)iz \\ (b) \quad \phi_0(z, w, \tau) &= 0. \end{aligned} \quad (2.10)$$

Thus $q(0) = 0$ and $q_\tau(0) \neq 0$ together with the Implicit Function Theorem allow us to solve $\phi = 0$ for $\tau = \tau(z\bar{z}, w)$. This yields the desired $(m+1)$ -parameter family of solutions.

§3. Liapunov Center Theorem with Spatial Symmetry

In this section we generalize Theorem 2.1 to the situation where f in (1.1) commutes with a compact group of spatial symmetries Γ . That is,

$$f(\gamma x) = \gamma f(x) \quad \forall \gamma \in \Gamma. \quad (3.1)$$

We also assume

$$\gamma x_0 = x_0 \quad \forall \gamma \in \Gamma. \quad (3.2)$$

The existence of spatial symmetries often causes the eigenvalues of $(df)_{x_0}$ to be multiple. This arises from the commutativity relation

$$(df)_{x_0} \gamma = \gamma (df)_{x_0} \quad \forall \gamma \in \Gamma \quad (3.3)$$

which is obtained from (3.1) by differentiation.

The prototypical situation occurs when

$$V = W \oplus W$$

and Γ acts absolutely irreducibly on W . (Absolute irreducibility means that the only linear maps on W commuting with Γ are scalar multiples of the identity.) Since $(df)_{x_0}$ commutes with Γ , absolute irreducibility implies:

$$(df)_{x_0} = \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}.$$

Hence the eigenvalues of $(df)_{x_0}$ are the eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, each with multiplicity equal to $\dim W$.

Suppose now that R commutes with Γ . Then the ± 1 eigenspaces of R are invariant under Γ . Assuming that both eigenspaces are nontrivial allows us, after a linear change of coordinates, to write $R(w_1, w_2) = (w_1, -w_2)$. Anticommutativity of $(df)_{x_0}$ with R implies $a = d = 0$. Hence the eigenvalues of $(df)_{x_0}$ are either real or purely imaginary. This is the generalization of the eigenvalue dichotomy referred to in Section 1.

In our generalization of Theorem 2.1, we assume

$$V_i = W \oplus W \tag{3.4}$$

where Γ acts absolutely irreducibly on W . See [GSS, XVI, §1]. The idea behind this assumption is the generalization of Hopf bifurcation to systems with symmetry obtained by detecting solutions by their symmetries [GSS, Theorem XVI, 4.1]. Their symmetries, however, contain both a spatial (Γ) component and a temporal (S^1) component. More precisely, $(\gamma, \theta) \in \Gamma \times S^1$ acts on $x(t)$ by

$$(\gamma, \theta) \cdot x(t) = \gamma x(t - \theta), \tag{3.5}$$

and the group of symmetries of a periodic mapping is a subgroup $\Sigma \subset \Gamma \times S^1$.

Observe that $V_i \oplus V_0$ is $\Gamma \times S^1$ -invariant. The S^1 -invariance follows from the discussion in Section 2, while the Γ -invariance from the fact that (3.3) implies Γ -invariance of the eigenspaces of $(df)_{x_0}$. Let $\Sigma \subset \Gamma \times S^1$ be a subgroup. Define

$$\begin{aligned}
(a) \quad \text{Fix}(\Sigma) &= \{v \in V_i \oplus V_0 : \sigma v = v \quad \forall \sigma \in \Sigma\} \\
(b) \quad \text{Fix}_i(\Sigma) &= \text{Fix}(\Sigma) \cap V_i \\
(c) \quad \text{Fix}_0(\Sigma) &= \text{Fix}(\Sigma) \cap V_0
\end{aligned} \tag{3.6}$$

Then $\text{Fix}(\Sigma) = \text{Fix}_i(\Sigma) \oplus \text{Fix}_0(\Sigma)$. Let $m = \dim \text{Fix}_0(\Sigma)$.

We now prove the generalization of Theorem 2.1 to Γ -equivariant systems that we shall use in later sections. For completeness we restate all hypotheses here.

Theorem 3.1: Assume that the Γ -equivariant system of ODE

$$dx/dt = f(x) \tag{1.1}$$

has a Γ -equivariant equilibrium x_0 . Assume

$$\begin{aligned}
\pm \omega_0 i \text{ are nonzero eigenvalues of } (df)_{x_0}, \text{ and} \\
k\omega_0 i \text{ is not an eigenvalue of } (df)_{x_0} \text{ (} k = 2, 3, \dots \text{)}.
\end{aligned} \tag{3.7}$$

Assume that the generalized eigenspace V_i corresponding to the eigenvalues $\pm \omega_0 i$ has the form

$$V_i = W \oplus W \tag{3.4}$$

where Γ acts absolutely irreducible on W .

Assume that (1.1) has a time-reversal symmetry R that fixes x_0 and let $\Sigma \subset \Gamma \times S^1$ be a subgroup satisfying:

$$\begin{aligned}
(a) \quad \dim \text{Fix}_i(\Sigma) &= 2, \\
(b) \quad R(\text{Fix}_i(\Sigma)) &= \text{Fix}_i(\Sigma), \\
(c) \quad R|_{\text{Fix}_0(\Sigma)} &= I
\end{aligned} \tag{3.8}$$

Then there exists an $(m+1)$ -parameter family of periodic solutions to (1.1) with period $2\pi/\omega_0$ and symmetry Σ .

Proof: The proof of this theorem is similar to that of Theorem 2.1; we indicate the changes.

The Liapunov-Schmidt reduction leads to a smooth $\Gamma \times S^1$ -equivariant mapping

$$\phi: V_i \times V_0 \times \mathbb{R} \rightarrow V_i \times V_0,$$

the Γ -equivariance following as in the proof of [GSS,XVI, Theorem 4.1]. This $\Gamma \times S^1$ -equivariance implies that

$$\phi: \text{Fix}(\Sigma) \times \mathbb{R} \rightarrow \text{Fix}(\Sigma). \quad (3.10)$$

Writing $\text{Fix}(\Sigma) = \text{Fix}_i(\Sigma) \oplus \text{Fix}_0(\Sigma)$ we see that the hypotheses of Theorem 2.1 are satisfied if we identify $\text{Fix}_i(\Sigma)$ with $\mathbb{C}$ in Theorem 2.1 and $\text{Fix}_0(\Sigma)$ with V_0 . $\square$

Remarks 3.2: (a) Let H be the projection of $\Sigma \subset \Gamma \times S^1$ into Γ . Since S^1 acts trivially on V_0 , we see that

$$\text{Fix}_0(\Sigma) = \text{Fix}_0(H) \equiv \{v \in V_0 : hv = v \quad \forall h \in H\}.$$

Thus, we have an alternative way of computing m in Theorem 3.1.

(b) The hypotheses of Theorem 3.1 involve both Σ and R . Therefore, it is possible to find an appropriate Σ but to have the conditions (3.8b,c) on R fail. In these cases, R can sometimes be replaced by a time-reversal symmetry $R' = \gamma R$ for some $\gamma \in \Gamma$ where R' satisfies (3.8b,c).

§4. Abstract Structure of the Stokeslet Model

Let $x_1, \dots, x_n \in \mathbb{R}^3$ denote the consecutive edges of an n -sided polygon in $\mathbb{R}^3$. Thus, the $3n$ -vector $(x_1, \dots, x_n)$ lies in the space

$$V = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^{3n}: x_1 + \dots + x_n = 0 \right\}. \quad (4.1)$$

The Stokeslet model, whose derivation is sketched in the Appendix, is

$$\dot{x}_j = \sum_{k=1}^{n-2} \left[U(x_{j+1} + \dots + x_{j+k}) - U(x_{j-1} + \dots + x_{j-k}) \right] \quad (4.2)$$

for $j = 1, \dots, n$ where the indices are taken mod n and where

$$U(x) = \frac{e_1}{|x|} + (e_1, x) \frac{x}{|x|^3}, \quad x \in \mathbb{R}^3. \quad (4.3)$$

We make three observations concerning this model.

- (i) $D_n \times O(2)$ is a group of symmetries of (4.2).
- (ii) Regular horizontal n -gons are equilibria that are fixed by a subgroup $\Gamma \subset D_n \times O(2)$ isomorphic to D_n .
- (iii) There is a time-reversal symmetry fixing regular n -gons.

We discuss these points in order.

An action of $O(2)$ on $\mathbb{R}^3$ is generated by:

$$(a) \quad R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \quad \forall \theta \in SO(2)$$

(4.4)

$$(b) \quad \kappa = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This action extends to V by:

$$(a) \quad \theta \cdot (x_1, \dots, x_n) = (R_\theta x_1, \dots, R_\theta x_n)$$

(4.5)

$$(b) \quad \kappa \cdot (x_1, \dots, x_n) = (\kappa x_1, \dots, \kappa x_n).$$

The dihedral group D_n , the symmetries of a regular n -gon, has $2n$ elements and is generated by an element of order n and a reflection of order 2. An action of D_n on V is defined by:

$$(x_1, \dots, x_n) \rightarrow (x_2, \dots, x_n, x_1)$$

(4.6)

$$(x_1, \dots, x_n) \rightarrow -(x_n, x_{n-1}, \dots, x_1).$$

The Stokeslet model (4.2) is equivariant with respect to these actions of $O(2)$ and D_n . This is clear for the permutation symmetries (4.6). $O(2)$ -equivariance is verified by first observing that

$$U(\gamma x) = \gamma U(x) \quad \forall \gamma \in O(2).$$

(4.7)

Using this group action it is simple to write down the regular n -gon. Let

$$\theta_n = 2\pi/n,$$

(4.8)

fix x_1 in the horizontal e_2, e_3 plane, and set

$$x_{j+1} = R_{\theta_n} x_j \quad (j = 1, \dots, n-1). \quad (4.9)$$

Using (4.7) it is now a simple matter to show that the regular n -gon (4.9) is an equilibrium for (4.2).

The precise subgroup of $D_n \times O(2)$ fixing (4.9) depends on the choice of x_1 , though it is always isomorphic to D_n . Henceforth, we assume

$$x_1 = R_{\theta_n/2} e_3$$

and call the symmetries of this regular n -gon Γ . Γ is generated by:

$$(a) \quad \mathcal{E}(x_1, \dots, x_n) = (-\theta_n) \cdot (x_2, \dots, x_n, x_1) \quad (4.10)$$

$$(b) \quad \mathcal{T}(x_1, \dots, x_n) = -x \cdot (x_n, x_{n-1}, \dots, x_1).$$

Finally $R = \begin{pmatrix} -1 & 1 \\ & 1 \end{pmatrix}$ extends to a time-reversal symmetry of (4.2) by

$$R \cdot (x_1, \dots, x_n) = (Rx_1, \dots, Rx_n). \quad (4.11)$$

Note that R fixes regular horizontal n -gons.

§5. Group Theoretic Restrictions on $(df)_{x_0}$

Spatial and time-reversal symmetries greatly restrict the form of the matrix

$$L = (df)_{x_0}. \quad (5.1)$$

In this section we use the specific group structure described in Section 4 to analyze L . This analysis is based on the commutativity properties:

$$\begin{aligned} (a) \quad Ly &= \gamma L & \forall \gamma \in \Gamma \\ (b) \quad LR &= -RL. \end{aligned} \quad (5.2)$$

Recall from [GSS,XII,§2] that an isotypic component of V corresponding to an irreducible representation W of Γ is the sum of all irreducible subspaces of V that are Γ -isomorphic to W . Standard theorems show that L can be block diagonalized with respect to the isotypic decomposition of V into isotypic components. To describe these isotypic components we begin by listing (up to isomorphism) all of the irreducible representations of D_n . See Miller [M] for a proof that what follows is a complete list.

The irreducible representations of $\Gamma \cong D_n$ are either one or two-dimensional. The actual number and type of these representations depends on the parity of n . When n is even there are four nonisomorphic one-dimensional irreducible representations of Γ , determined by whether $\mathcal{E}$ and $\mathcal{T}$, see (4.10), act by $+1$ or -1 . We denote these representations by $W_{\epsilon\delta}$

where $\mathcal{E}$ acts by ϵI and $\mathcal{F}$ acts by δI . (Of course, $\delta, \epsilon = \pm 1$.) W_{++} is the trivial representation.

When n is odd there are only two nonisomorphic one-dimensional irreducible representations of Γ , namely, W_{++} and W_{+-} . This follows since $\mathcal{E}$ is a group element of order n and hence $\mathcal{E}^n = I$. Thus, if $\mathcal{E}$ acts by $-I$, then $(-1)^n$ must be 1, that is, n must be even.

There are $[(n-1)/2]$ distinct two-dimensional irreducible representations of Γ . We denote these by $W_k \cong \mathbb{C}$ for $k = 1, \dots, [(n-1)/2]$. On W_k

$$\begin{aligned} (a) \quad \mathcal{E}z &= e^{ki\theta} z \\ (b) \quad \mathcal{F}z &= \bar{z}. \end{aligned} \tag{5.3}$$

Note that Γ acts absolutely irreducibly on each W_k , and that W_{-k} , using the obvious extension to negative k in (5.3), is Γ -isomorphic to W_k .

The isotypic decomposition of V is given by:

Theorem 5.1: (a) When n is odd

$$V \cong W_{++} \oplus W_{+-} \oplus W_1^2 \oplus W_2^3 \oplus \dots \oplus W_{(n-1)/2}^3.$$

(b) When n is even

$$V \cong W_{++} \oplus W_{+-} \oplus W_{--} \oplus W_{-+}^2 \oplus W_1^2 \oplus W_2^3 \oplus \dots \oplus W_{(n-2)/2}^3$$

(c) On the repeated components W^s , the time-reversal symmetry R acts by $-I$ on the first copy of W and $+I$ on the remaining copies. R acts as $+I$ on the other (one-dimensional) representations.

Before proving Theorem 5.1 we describe explicitly the consequences of Theorem 5.1 for $L = (df)_{x_0}$.

Theorem 5.2: (a) $L = 0$ on W_{++} , W_{+-} and W_{--} .

(b) $L|W_{-+}^2$ has a pair of simple eigenvalues $\pm\lambda$, where λ is either real or purely imaginary.

(c) $L|W_k^s$ has a pair of eigenvalues $\pm\lambda$, each of multiplicity two, where λ is either real or purely imaginary. When $s = 3$, L has a pair of zero eigenvalues and $R|(\ker L \cap W_k^s) = +I$.

Remark: It follows that L has $[n/2] + 1$ zero eigenvalues forced by symmetry and $[n/2]$ independent pairs of eigenvalues with each pair either real or purely imaginary. In the generic case, when all of these independent eigenvalue pairs are nonzero, $R|\ker L = +I$.

Proof: Since L commutes with Γ , L leaves isotypic components invariant. See [GSS, Theorem XII, 3.5].

(a) As observed earlier (5.2b) implies that L has eigenvalues $\pm\lambda$. On a one-dimensional space λ must equal zero.

(b,c) Suppose that γ acts absolutely irreducible on W and diagonally on the isotypic component W^s . Then $L|W^s$ has the form

$$\begin{pmatrix} a_{11}I & \dots & a_{1s}I \\ \vdots & & \vdots \\ a_{s1}I & \dots & a_{ss}I \end{pmatrix}$$

where each $a_{ij} \in \mathbb{R}$. It follows that the eigenvalues of $L|W^s$ are just the eigenvalue of the $s \times s$ matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \cdots & a_{ss} \end{pmatrix}.$$

each with multiplicity $\dim W$. Theorem 5.1(c) states that $R(w_1, w_2, \dots, w_s) = (-w_1, w_2, \dots, w_s)$. Thus (5.2b) implies that

$$A = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1s} \\ a_{21} & & & \\ \vdots & & 0 & \\ a_{s1} & & & \end{pmatrix}$$

Any vector in $(a_{12}, \dots, a_{1s})^\perp \in \mathbb{R}^{s-1}$ yields two null-vectors for $L|W^s$. Moreover, these null-vectors lie in the second through s -th components of W^s and hence $R = +I$ on these null-vectors.

Thus there is only one set of nonzero independent eigenvalues in A and since $\text{tr } A = 0$, these eigenvalues must be $\pm\lambda$ where λ is either real or purely imaginary. $\square$

The remainder of this section is devoted to:

Proof of Theorem 5.1: We begin by decomposing $V = V_1 \oplus V_2$ where

$$\begin{aligned} (a) \quad V_1 &= \left\{ (x_1, \dots, x_n) \in V : x_j = a_j e_1 \right\} \\ (b) \quad V_2 &= \left\{ (x_1, \dots, x_n) \in V : x_j \cdot e_1 = 0 \right\} \end{aligned} \tag{5.4}$$

are Γ -invariant subspaces. Observe that $R = -I$ on V_1 and $+I$ on V_2 .

We divide the proof of Theorem 5.1 into two lemmas.

Lemma 5.3: (a) When n is odd

$$V_2 \cong W_{++} \oplus W_{+-} \oplus W_1 \oplus W_2^2 \oplus \dots \oplus W_{(n-1)/2}^2.$$

(b) When n is even

$$V_2 \cong W_{++} \oplus W_{+-} \oplus W_{-+} \oplus W_{--} \oplus W_1 \oplus W_2^2 \oplus \dots \oplus W_{(n-2)/2}^2.$$

Lemma 5.4: (a) When n is odd

$$V_1 = W_1 \oplus \dots \oplus W_{(n-1)/2}.$$

(b) When n is even

$$V_1 = W_{-+} \oplus W_1 \oplus \dots \oplus W_{(n-2)/2}.$$

We prove both of these lemmas by explicitly producing the irreducible subspaces making up V_1 and V_2 and counting the number of isomorphic representations.

For $1 \leq \ell \leq n-1$ define

$$V_{2,\ell} = \left\{ (x_1, \dots, x_n) \in V_2 : x_{j+1} = R_{\ell\theta_n} x_j \right\}. \quad (5.5)$$

Observe that

$$V_2 = V_{2,1} \oplus \dots \oplus V_{2,\ell} \quad (5.6)$$

and that each $V_{2,\ell}$ is Γ -invariant. Indeed, if we identify a vector $x \in \mathbb{R}^2$ with $x_1 = x$ in $V_{2,\ell}$, we may explicitly compute the action of $\mathcal{E}$ and $\mathcal{T}$ on $V_{2,\ell}$. We find:

$$\begin{aligned}
(a) \quad \mathcal{E}x &= R_{(\ell-1)\theta_n} x \\
(b) \quad \mathcal{T}x &= -x R_{-\ell\theta_n} x.
\end{aligned}
\tag{5.7}$$

It follows from (5.7) that $\mathcal{E}$ acts trivially on $V_{2,1}$ and that $V_{2,1}$ decomposes into two one-dimensional irreducible representations (the eigenvalues of $\mathcal{T}$) isomorphic to W_{++} and W_{+-} . When n is even, $\mathcal{E}$ acts as $-I$ on $V_{2,n/2}$ and this space also decomposes into two one-dimensional irreducible representations isomorphic to W_{-+} and W_{--} .

Γ acts irreducibly on the remaining two-dimensional invariant subspaces in (5.6). Now (5.7a) shows that $V_{2,\ell}$ is isomorphic to $W_{\ell-1}$ for $\ell = 2, \dots, [(n-1)/2]+1$; this yields one copy each of the distinct W_j 's in V_2 . For $\ell > [(n-1)/2]+1$, the representation of Γ on $V_{2,\ell}$ is isomorphic to $W_{n-\ell-1}$. (Here we use $W_{-k} \cong W_k$.) Since $\ell \leq n-1$ and $n-\ell+1 \geq 2$ it follows that $W_2, \dots, W_{[(n-1)/2]}$ each make a second appearance as irreducibles in V_2 . This proves Lemma 5.3.

To prove Lemma 5.4 we define

$$\begin{aligned}
(a) \quad V_{1,k} &= \mathbb{R}\{v_k^r, v_k^i\}, \text{ where} \\
(b) \quad v_k^r &= (1, \cos(k\theta), \dots, \cos((n-1)k\theta)) \\
(c) \quad v_k^i &= (0, \sin(k\theta), \dots, \sin((n-1)k\theta)).
\end{aligned}
\tag{5.8}$$

Here we use $\theta = \theta_n = 2\pi/n$. We assert that

$$V_1 = V_{1,1} \oplus \dots \oplus V_{1,[n/2]}. \tag{5.9}$$

To verify (5.9) we first show that each $V_{1,j} \subset V_1$. Let

$\phi_k = e^{ik\theta}$ for $1 \leq k \leq n-1$. Then

$$v_k^r = \operatorname{Re} \phi_k \quad \text{and} \quad v_k^i = \operatorname{Im} \phi_k$$

where

$$\phi_k = (1, \phi_k, \dots, \phi_k^{n-1}) .$$

Since $\phi_k \neq 1$ is an n -th root of unity

$$1 + \phi_k + \dots + \phi_k^{n-1} = 0 .$$

It follows that $v_k^r, v_k^i \in V_1$.

Note that each $V_{1,k}$ is two-dimensional with the single exception that when n is even $V_{1,n/2}$ is one-dimensional (since $v_{n/2}^i = 0$).

Next we show that each subspace $V_{1,k}$ is Γ -invariant and Γ -irreducible. Since all of the irreducibles are distinct representations of Γ , they must be linearly independent. Hence (5.9) will follow by a simple dimension count.

We compute:

$$\begin{aligned} (a) \quad \mathcal{E}v_k^r &= \cos(k\theta) v_k^r - \sin(k\theta) v_k^i \\ (b) \quad \mathcal{E}v_k^i &= \sin(k\theta) v_k^r + \cos(k\theta) v_k^i \\ (c) \quad \mathcal{F}v_k^r &= -\cos(k\theta) v_k^r + \sin(k\theta) v_k^i \\ (d) \quad \mathcal{F}v_k^i &= \sin(k\theta) v_k^r + \cos(k\theta) v_k^i . \end{aligned} \tag{5.10}$$

It follows that each $V_{1,k}$ is Γ -invariant. Moreover, in the

basis $\{v_k^r, v_k^i\}$, $\mathcal{E}$ has the matrix form

$$\begin{pmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{pmatrix}$$

and hence the action of Γ on $V_{1,k}$ is isomorphic to W_k . $\square$

Remark 5.5: Let K be that part of $\ker(df)_{x_0}$ that is forced to exist due to spatial and time-reversal symmetry. The space K consists of $W_{++} \oplus W_{+-}$ plus W_{--} if n is even plus the subspaces of W_j^3 specified in the proof of Theorem 5.2(c). Observe that

$$\text{Fix}_K(\mathcal{E}) = W_{++} \oplus W_{+-} \tag{5.11}$$

since $\mathcal{E}$ acts nontrivially on W_{--} and by a nontrivial rotation on the relevant W_j^3 's.

§6. Two-Dimensional Fixed-Point Subspaces

Theorem 5.2 states that in the Stokeslet model the eigenvalues of $(df)_{x_0}$ are either real or purely imaginary. We expect, since it is true generically, that the real eigenspace V_i associated with a purely imaginary eigenvalue of $(df)_{x_0}$ will have the form (3.4), that is, $V_i = W \oplus W$ where the group $\Gamma \cong D_n$ acts absolutely irreducibly on W . Moreover, in the Stokeslet model, the possible choices for W are any of the two-dimensional irreducibles W_k ($1 \leq k \leq [(n-1)/2]$) and, when n is even, the one-dimensional irreducible W_{-+} .

In this section we recall the classification - up to conjugacy - of all (isotropy) subgroups $\Sigma \subset D_n \times S^1$ that have two-dimensional fixed-point subspaces. This classification is derived in [GSt], see also [GSS, XVIII, §1], and is implicit in [vGV]. We also describe briefly the implications of having symmetry Σ for each family of periodic solutions.

When n is even and $W = W_{-+}$ the eigenspace V_i is automatically two-dimensional. The symmetry of periodic solutions consists of those elements in $D_n \times S^1$ that act trivially on W_{-+}^2 . By definition $\mathcal{C}$ acts by -1 and $\mathcal{T}$ by $+1$ on W_{-+} . Hence the group Σ generated by $(\mathcal{C}, \pi)$ and $\mathcal{T}$ acts trivially on W_{-+} and Σ is the group of symmetries of the associated periodic solutions. Note that Σ is isomorphic to D_n .

These symmetries force solutions $(x_1(t), \dots, x_n(t))$ to have the form

$$(x, -R_{2\theta}xx, R_{2\theta}x, -R_{4\theta}xx, R_{4\theta}x, \dots, -xx). \quad (6.1)$$

 

for all time t where $\theta = 2\pi/n$. To verify (6.1) observe that if $(\mathcal{E}, \pi)$ fixes a 2π -periodic solution, then

$$R_{-\theta} x_{k+1}(t + \pi) = x_k(t)$$

and solutions have the form

$$(x(t), R_{\theta} x(t+\pi), R_{2\theta} x(t), \dots, R_{(n-1)\theta} x(t+\pi)).$$

Invariance under $\mathcal{T}$ implies $-x R_{(n-1)\theta} x(t+\pi) = x(t)$. Thus

$$x(t + \pi) = -R_{\theta} x(t).$$

Substitution yields (6.1).

Note that when $n = 4$ and $x \cdot e_1 = 0$ (6.1) reduced to the rhombus

$$(x, \chi x, -x, -\chi x).$$

In general, for $n = 4$, the horizontal projection of this periodic solution is a (time-dependent) rhombus and whose opposite vertices move synchronously in the vertical direction e_1 . For general even n , these types of solutions project onto n -gons in the horizontal plane with sides of equal lengths, and alternate vertices move synchronously in the vertical direction e_1 . The actual the horizontal n -gon, however, may be somewhat complicated.

Next we consider solution types that may be associated with two-dimensional irreducible representations of D_n . To simplify the discussion we first consider the case $n = 3$ and then indicate some of the differences when $n > 3$. The subgroups of $D_3 \times S^1$ that have two-dimensional fixed point subspaces are classified in [GSS, XVIII, Table 1.1] and are:

 

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PLEASE PRINT NAME AND ADDRESS OF THE PERSON TO WHOM THE ORDER IS TO BE PAID:

There are two types of information that may be associated with a given information type:

THESE RESULTS ARE IN ACCORD WITH THE CONCLUSIONS OF OTHER STUDIES THAT THE EFFECT OF A SINGLE DOSE OF 100 mg OF DILTIAZEM ON THE PHARMACOKINETICS OF THEO-100 IS NOT SIGNIFICANTLY DIFFERENT FROM THAT OF A SINGLE DOSE OF 100 mg OF THEO-100.

1981-1982 1983-1984 1985-1986 1987-1988 1989-1990 1991-1992 1993-1994 1995-1996 1997-1998 1999-2000 2001-2002 2003-2004 2005-2006 2007-2008 2009-2010 2011-2012 2013-2014 2015-2016 2017-2018 2019-2020 2021-2022 2023-2024 2025-2026 2027-2028 2029-2030 2031-2032 2033-2034 2035-2036 2037-2038 2039-2040 2041-2042 2043-2044 2045-2046 2047-2048 2049-2050 2051-2052 2053-2054 2055-2056 2057-2058 2059-2060 2061-2062 2063-2064 2065-2066 2067-2068 2069-2070 2071-2072 2073-2074 2075-2076 2077-2078 2079-2080 2081-2082 2083-2084 2085-2086 2087-2088 2089-2090 2091-2092 2093-2094 2095-2096 2097-2098 2099-2100 2101-2102 2103-2104 2105-2106 2107-2108 2109-2110 2111-2112 2113-2114 2115-2116 2117-2118 2119-2120 2121-2122 2123-2124 2125-2126 2127-2128 2129-2130 2131-2132 2133-2134 2135-2136 2137-2138 2139-2140 2141-2142 2143-2144 2145-2146 2147-2148 2149-2150 2151-2152 2153-2154 2155-2156 2157-2158 2159-2160 2161-2162 2163-2164 2165-2166 2167-2168 2169-2170 2171-2172 2173-2174 2175-2176 2177-2178 2179-2180 2181-2182 2183-2184 2185-2186 2187-2188 2189-2190 2191-2192 2193-2194 2195-2196 2197-2198 2199-2200 2201-2202 2203-2204 2205-2206 2207-2208 2209-2210 2211-2212 2213-2214 2215-2216 2217-2218 2219-2220 2221-2222 2223-2224 2225-2226 2227-2228 2229-2230 2231-2232 2233-2234 2235-2236 2237-2238 2239-2240 2241-2242 2243-2244 2245-2246 2247-2248 2249-2250 2251-2252 2253-2254 2255-2256 2257-2258 2259-2260 2261-2262 2263-2264 2265-2266 2267-2268 2269-2270 2271-2272 2273-2274 2275-2276 2277-2278 2279-2280 2281-2282 2283-2284 2285-2286 2287-2288 2289-2290 2291-2292 2293-2294 2295-2296 2297-2298 2299-2300 2301-2302 2303-2304 2305-2306 2307-2308 2309-2310 2311-2312 2313-2314 2315-2316 2317-2318 2319-2320 2321-2322 2323-2324 2325-2326 2327-2328 2329-2330 2331-2332 2333-2334 2335-2336 2337-2338 2339-2340 2341-2342 2343-2344 2345-2346 2347-2348 2349-2350 2351-2352 2353-2354 2355-2356 2357-2358 2359-2360 2361-2362 2363-2364 2365-2366 2367-2368 2369-2370 2371-2372 2373-2374 2375-2376 2377-2378 2379-2380 2381-2382 2383-2384 2385-2386 2387-2388 2389-2390 2391-2392 2393-2394 2395-2396 2397-2398 2399-2400 2401-2402 2403-2404 2405-2406 2407-2408 2409-2410 2411-2412 2413-2414 2415-2416 2417-2418 2419-2420 2421-2422 2423-2424 2425-2426 2427-2428 2429-2430 2431-2432 2433-2434 2435-2436 2437-2438 2439-2440 2441-2442 2443-2444 2445-2446 2447-2448 2449-2450 2451-2452 2453-2454 2455-2456 2457-2458 2459-2460 2461-2462 2463-2464 2465-2466 2467-2468 2469-2470 2471-2472 2473-2474 2475-2476 2477-2478 2479-2480 2481-2482 2483-2484 2485-2486 2487-2488 2489-2490 2491-2492 2493-2494 2495-2496 2497-2498 2499-2500 2501-2502 2503-2504 2505-2506 2507-2508 2509-2510 2511-2512 2513-2514 2515-2516 2517-2518 2519-2520 2521-2522 2523-2524 2525-2526 2527-2528 2529-2530 2531-2532 2533-2534 2535-2536 2537-2538 2539-2540 2541-2542 2543-2544 2545-2546 2547-2548 2549-2550 2551-2552 2553-2554 2555-2556 2557-2558 2559-2560 2561-2562 2563-2564 2565-2566 2567-2568 2569-2570 2571-2572 2573-2574 2575-2576 2577-2578 2579-2580 2581-2582 2583-2584 2585-2586 2587-2588 2589-2590 2591-2592 2593-2594 2595-2596 2597-2598 2599-2600 2601-2602 2603-2604 2605-2606 2607-2608 2609-2610 2611-2612 2613-2614 2615-2616 2617-2618 2619-2620 2621-2622 2623-2624 2625-2626 2627-2628 2629-2630 2631-2632 2633-2634 2635-2636 2637-2638 2639-2640 2641-2642 2643-2644 2645-2646 2647-2648 2649-2650 2651-2652 2653-2654 2655-2656 2657-2658 2659-2660 2661-2662 2663-2664 2665-2666 2667-2668 2669-2670 2671-2672 2673-2674 2675-2676 2677-2678 2679-2680 2681-2682 2683-2684 2685-2686 2687-2688 2689-2690 2691-2692 2693-2694 2695-2696 2697-2698 2699-2700 2701-2702 2703-2704 2705-2706 2707-2708 2709-2710 2711-2712 2713-2714 2715-2716 2717-2718 2719-2720 2721-2722 2723-2724 2725-2726 2727-2728 2729-2730 2731-2732 2733-2734 2735-2736 2737-2738 2739-2740 2741-2742 2743-2744 2745-2746 2747-2748 2749-2750 2751-2752 2753-2754 2755-2756 2757-2758 2759-2760 2761-2762 2763-2764 2765-2766 2767-2768 2769-2770 2771-2772 2773-2774 2775-2776 2777-2778 2779-2780 2781-2782 2783-2784 2785-2786 2787-2788 2789-2790 2791-2792 2793-2794 2795-2796 2797-2798 2799

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$$\mathbb{Z}_2(\mathcal{T}), \mathbb{Z}_2(\mathcal{T}, \pi) \text{ and } \mathbb{Z}_3(\mathcal{E}, 2\pi/3). \quad (6.2)$$

Solutions (x_1, x_2, x_3) having symmetry $\mathbb{Z}_2(\mathcal{T})$ satisfy

$$x_1 = -x_3 \quad \text{and} \quad x_2 = -x_2. \quad (6.3)$$

It follows that such solutions form isosceles triangles where the vertices of the side with unequal length move synchronously in the vertical direction. These are the type of solution found in the Stokeslet model by [CLLS].

Solutions having symmetry $\mathbb{Z}_2(\mathcal{T}, \pi)$ satisfy

$$x_1(t) = -x_3(t + \pi) \quad \text{and} \quad x_2(t) = -x_2(t + \pi). \quad (6.4)$$

Since $|x_1| = |x_3|$ these solutions also form isosceles triangles. In this case, however, the end points of the side with unequal length move in the vertical direction with a precise half-period phase lag.

Solutions having symmetry $\mathbb{Z}_3(\mathcal{E}, 2\pi/3)$ satisfy

$$R_\theta x_{j+1}(t + 2\pi/3) = x_j(t) \quad j = 1, 2. \quad (6.5)$$

Such solutions form equilateral triangles whose vertices move with precise one-third period phase lags. These solutions resemble three ponies on a (stationary) merry-go-round.

Two complications arise when generalizing to the case $n > 3$. First, for the standard irreducible representation the isotropy subgroups differ slightly depending on whether n is odd or congruent to zero or two mod four. Second, there are $[(n-1)/2]$ different two-dimensional irreducible representation, each leading to solutions with slightly different geometric form. We consider

these complications in order.

Let $W = W_1$ be the standard two-dimensional irreducible representation of D_n . For each $n \geq 3$ there are three (nonconjugate) isotropy subgroups of $D_n \times S^1$ acting on W^2 which have two-dimensional fixed point subspaces and those are analogous to the $n = 3$ case. First, for all n there is a subgroup $\mathbb{Z}_n(\mathbb{C}, 2\pi/n)$ whose solutions (as in the case $n = 3$) correspond to n ponies on a stationary merry-go-round.

When n is odd there are solutions with $\mathbb{Z}_2(\mathcal{T})$ symmetry. They have $(n-1)/2$ pairs of sides with equal lengths which move synchronously, the pairing of sides being given by $\mathcal{T}$. We call these solutions *isosceles n-gons*. The unpaired side has vertices that move synchronously. When n is even, the corresponding solution has symmetry $\mathbb{Z}_2(\mathcal{T}) \oplus \mathbb{Z}_2(\mathbb{C}^{n/2}, \pi)$. So when n is even solutions are paired across a line connecting two opposite vertices and there is no unpaired side. Opposite sides (those with $(n/2)-1$ sides between them) move with a half period phase lag.

Finally, when n is odd there are solutions with $\mathbb{Z}_2(\mathcal{T}, \pi)$ symmetry. Again sides are paired, as in the $\mathbb{Z}_2(\mathcal{T})$ case, but for these solutions paired sides move with a half period phase lag. When $n \equiv 2 \pmod{4}$ solutions of this type persist but with symmetry $\mathbb{Z}_2(\mathcal{T}, \pi) \oplus \mathbb{Z}_2(\mathbb{C}^{n/2}, \pi)$. Again sides are paired across a line connecting opposite vertices and there is no unpaired side when n is even. Opposite sides move with a half-period phase lag. The two symmetries together force all but two (opposite) sides to come in quartets with two sides moving synchronously and two with the phase lag. When $n \equiv 0 \pmod{4}$ isotropy of the

third solution type is $\mathbb{Z}_2(\mathcal{R}_{2\pi/n}) \oplus \mathbb{Z}_2(\mathcal{C}^{n/2}, \pi)$. Here solutions are paired across a line connecting the midpoints of two opposite sides. These pairs move synchronously while opposite sides move with a phase lag of one-half period.

Periodic solutions corresponding to $V_i = W_k^2$, where $k > 1$ differ from those with $k = 1$ in two ways. First when k divides n , $\mathcal{C}^{n/k}$ acts trivially on W_k . Thus these periodic solutions all divide the n -gon into n/k groups of evenly spaced sides. All of the sides in each group move synchronously. The D_n symmetry reduces to $D_{n/k}$ acting on these groups of sides - rather than individual sides.

Second when n and k are relatively prime, solution behave similarly to the $k = 1$ case with two exceptions. The ponies on a merry-go-round move to a different tune. You have to move k ponies around to get to the one which is $2\pi/n$ out of phase. Similarly when n is even, the sides which move with half-period time lag are not opposite but rather are separated by $\ell-1$ sides where $k\ell \equiv n/2 \pmod{n}$.

§7. Periodic Solutions in the Stokeslet Model

In this section we apply Theorem 3.1 to the Stokeslet model. It follows from Theorem 5.1 and 5.2 that there are $[n/2]$ pairs of eigenvalues, $\pm\lambda$, of the Jacobian at an equilibrium regular n -gon which can be either real or purely imaginary. These eigenvalues have been computed by [H] for $n \leq 12$. He finds that when $n \geq 7$ there is at least one pair which is real and nonzero. In such a case the equilibrium regular polygons are saddles and are asymptotically unstable; this observation is consistent with experiments where the regular n -gons, for $n \geq 7$, seem to break apart. For this reason, we focus here on the case $3 \leq n \leq 6$.

To apply Theorem 3.1 with the isotropy subgroups described in Section 6 we need two additional pieces of information. First, the various distinct, purely imaginary eigenvalues must be nonresonant. This point is trivially true for $n = 3$ and may be checked for the cases $n = 4, 5, 6$. It follows immediately that the simple eigenvalues corresponding to W_{+-} yield a family of periodic solutions when n equals 4 or 6.

Second, in the case of the two-dimensional irreducibles W_k , we need to observe whether the time-reversal symmetry R preserves $\text{Fix}'_i(\Sigma)$. It turns out that R does preserve $\text{Fix}_i(\Sigma)$ for either Σ that is generated by reflections, but does not for $\Sigma \cong \mathbb{Z}_n$. Consequently, using Theorem 3.1, we will prove the existence of two families of periodic solutions (generalizing the synchronous and asynchronous isosceles triangles) for $3 \leq n \leq 6$. We will show in the next section, that there are two-frequency, quasiperiodic solutions corresponding to $\mathbb{Z}_n$, but additional

remarks are necessary to verify this point. These quasiperiodic solutions may be thought of as ponies on a (rotating) merry-go-round.

The proof that the time-reversal symmetry R preserves $\text{Fix}_i(\Sigma)$ when $\Sigma = \mathbb{Z}_2(\mathcal{T})$ or $\mathbb{Z}_2(\mathcal{T}, \pi)$ is straightforward since R commutes with all spatial symmetries, including $\mathcal{T}$, and with the phase shifts 0 and π . This commutativity insures that $\text{Fix}_i(\Sigma)$ is an invariant subspace for R . Note that R does not commute with $(\mathcal{C}, 2\pi/n)$ and it is for this reason that $\text{Fix}_i(\mathbb{Z}_n)$ is not invariant under R .

In Table 7.1 we list all of those families of periodic solutions whose existence are proved by combining the calculations of [H] with Theorems 3.1 and 5.1.

n	W	Solution type	isotropy
3	W_1	Synchronous isosceles triangles	$\mathbb{Z}_2(\mathcal{T})$
		Asynchronous isosceles triangle	$\mathbb{Z}_2(\mathcal{T}, \pi)$
4	W_{-+}	Synchronous rhombus	$D_4((\mathcal{E}, \pi), \mathcal{T})$
	W_1	rectangles	$\mathbb{Z}_2(\mathcal{R}_{\pi/2})$
		isosceles quadrilateral	$\mathbb{Z}_2(\mathcal{T})$
5	W_1	synchronous isosceles pentagon 1	$\mathbb{Z}_2(\mathcal{T})$
		asynchronous isosceles pentagon 1	$\mathbb{Z}_2(\mathcal{T}, \pi)$
	W_2	synchronous isosceles pentagon 2	$\mathbb{Z}_2(\mathcal{T})$
		asynchronous isosceles pentagon 2	$\mathbb{Z}_2(\mathcal{T}, \pi)$
	W_{-+}	synchronous equilateral hexagon	$D_6((\mathcal{E}, \pi), \mathcal{T})$
		synchronous isosceles hexagon 1	$\mathbb{Z}_2(\mathcal{T})$
6	W_1	asynchronous isosceles hexagon 1	$\mathbb{Z}_2(\mathcal{T}, \pi)$
		synchronous isosceles hexagon 2	$\mathbb{Z}_2(\mathcal{T}) \oplus \mathbb{Z}_2(\mathcal{E}^3)$
	W_2	asynchronous isosceles hexagon 2	$\mathbb{Z}_2(\mathcal{T}, \pi) \oplus \mathbb{Z}_2(\mathcal{E}^3)$
		synchronous isosceles hexagon 2	$\mathbb{Z}_2(\mathcal{T}, \pi) \oplus \mathbb{Z}_2(\mathcal{E}^3)$

Table 7.1: Periodic solutions determined for n particles in the Stokeslet model with $3 \leq n \leq 6$. (See Section 8 for quasiperiodic solutions).

§8. Quasiperiodic Solutions in the Stokeslet Model

Krupa [K] considers bifurcation from group orbits of equilibria and, indeed, in the Stokeslet model the basic equilibrium, the regular n -gon, lies on an $O(2)$ -orbit of such equilibria. Until now, we have neglected these symmetries having needed only to concentrate on the discrete D_n symmetry to obtain our results. We will use Krupa's results to reduce the dimension of the Stokeslet model by one - essentially we quotient by the space W_{+-} in the isotypic decomposition of Theorem 5.1. On this reduced space, we use Theorem 3.1 to prove the existence of the periodic ponies on a stationary merry-go-round solutions. Finally, we show that the periodic ponies correspond to two-frequency trajectories in the full phase space. These solutions may be interpreted as ponies on a rotating merry-go-round.

We begin by describing Krupa's results along with their adaptation to the time-reversible situation. Let Δ be a compact Lie group acting linearly on $\mathbb{R}^n$. Let $f(x)$ be a smooth Δ -equivariant vector field with equilibrium x_0 . Let $X = \Delta x_0$ be the group orbit of equilibria containing x_0 and let Γ be the isotropy subgroup of x_0 in Δ .

Theorem 8.1: There exists a Δ -invariant tubular neighborhood N of X and a decomposition on N

$$f = f_T + f_N \tag{8.1}$$

where

- (i) f_T and f_N are smooth and Δ -equivariant
- (ii) f_T is tangent to group orbits of Δ
- (iii) f_N is normal to X .

Remark: By 'normal to X ' we mean the following. Let $\pi:N \rightarrow X$ be the projection associated with the tubular neighborhood and let N_x be the fiber $\pi^{-1}(x)$ for all $x \in X$. (We can certainly choose N so that N_x is normal to X in $\mathbb{R}^n$.) Then " f_N is normal to X " means that $f_N(y) \in N_{\pi(y)}$ for each $y \in N$.

The second result concerns the relationship between solutions of the differential equations $dx/dt = f(x)$ and $dy/dt = f_N(y)$.

Theorem 8.2: Let $x(t)$ be the solution to $dx/dt = f(x)$ with initial condition $x(0) = y_0 \in N_{x_0}$. Let $y(t)$ be the solution in N_{x_0} to $dy/dt = f_N(y)$ with the same initial condition $y(0) = y_0$. Then there exists a smooth curve $\delta(t) \in \Delta$ with $\delta(0) = 1$ satisfying

$$x(t) = \delta(t)y(t).$$

Remarks 8.3: (a) We can think of $\delta(t)$ as representing a drift in the solution $x(t)$ along the group orbit through $y(t)$. In the Stokeslet model, group orbits of $\Delta = O(2) \times D_n$ are finite unions of circles. So if $y(t)$ is periodic and $\delta(t)$ is nonzero, we expect $x(t)$ to be a flow on a 2-torus.

(b) In general, $\delta(t)$ can be chosen to be a one parameter group in Δ . Hence, the closure of the image of δ is abelian and hence a torus.

(c) Whether drifting actually occurs ($\delta(t) \neq 1$) depends on whether f_T is nonzero on $y(t)$. Generically it can be determined from group theoretic calculations alone whether drifting is possible. For any given f , however, such as the Stokeslet model,

one needs to check that explicitly that f_T is nonzero. Not surprising, for the Stokeslet model, these algebraic conditions show that drifting is not possible (that is, $f_T(y(t)) \equiv 0$) for the periodic solutions constructed in Section 7. We show below that $f_T \neq 0$ for the ponies on a merry-go-round solution.

We now show how the decomposition (8.1) respects time-reversibility. We assume

$$f(Rx) = -Rf(x)$$

for some orthogonal matrix R , that commutes with Δ . We also assume $Rx_0 = x_0$ for the equilibrium x_0 of f . We assert:

$$\begin{aligned} (a) \quad R(T_x \Delta x) &= T_{Rx} \Delta(Rx) \quad \text{and} \quad RN_x = N_x \\ (b) \quad f_N(Rx) &= -Rf_N(x) \\ (c) \quad f_T(Rx) &= -Rf_T(x). \end{aligned} \tag{8.2}$$

We actually prove (8.2) only when the dimension of group orbits in N is constant, though this restriction is not necessary. To verify (8.2a) let $z(t) = \delta(t)x$ be a curve in the group orbit of Δ through x . The typical element of $T_x \Delta x$ is $v = dz/dt(0)$. Observe that

$$Rv = \frac{d}{dt} R\delta(t)x = \frac{d}{dt} \delta(t)Rx \in T_{Rx} \Delta Rx,$$

since R commutes with Δ . Note that $Rx = x$ since $Rx_0 = x_0$ and R commutes with Δ . Hence $R(T_x X) = T_x X$ and since R is orthogonal $R(N_x) = N_x$.

To verify (8.2b) observe that

$$0 = Rf(x) + f(Rx) = \left[Rf_N(x) + f_N(Rx) \right] + \left[Rf_T(x) + f_T(Rx) \right],$$

where the first sum lies in $N_{\pi(x)}$ and the second sum lies in $T_{Rx}\Delta(Rx)$. By assumption these spaces are transverse; hence each sum is zero.

Remarks 8.4: (a) In the Stokeslet model f_T is tangent to $SO(2)$ group orbits. Tangent vectors to such orbits are horizontal, that is, perpendicular to e_1 . R acts trivially on f_T . Thus (8.2c) is: $f_T(Rx) = -f_T(x)$.

(b) Indeed, if we define

$$J \equiv \frac{dR_\theta}{d\theta} \Big|_{\theta=0} \quad (8.3)$$

then tangent vector fields all have the form $a(x)Jx$ where

$$a(x) = \frac{f(x) \cdot Jx}{|Jx|^2}. \quad (8.4)$$

Note that by abuse of notation

$$Jx = (Jx_1, \dots, Jx_n) \quad \text{where} \quad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

In the Stokeslet model, we fixed an equilibrium regular n -gon x_0 by (4.9) and (4.10). For this equilibrium the isotropy subgroup $\Gamma \cong D_n$ is defined by (4.11). It is easy to check that the $\Delta = D_n \times O(2)$ group orbit through x_0 is one-dimensional and the tangent space to this group orbit is W_{+-} . (To verify this assertion one shows that $\mathcal{E}$ acts trivially on the tangent

space while $\mathcal{T}$ acts as $-I$.) Thus N_{x_0} is the sum of all isotypic components of V under Γ as listed in Theorem 5.1 except for W_{+-} . We now show, using Theorem 3.1, that there is a family of periodic solutions for $f_N|_{N_{x_0}}$ corresponding to isotopy $\Sigma = \mathbb{Z}_n(\mathcal{C}, 2\pi/n)$. These are the ponies on a stationary merry-go-round.

As indicated in the last section, we cannot use the time-reversal symmetry R to prove directly the existence of the periodic solutions. The obstacle is that R does not leave the subspace $\text{Fix}_1(\Sigma)$ invariant. To see this, observe

$$R \circ (\mathcal{C}, 2\pi/n) = (\mathcal{C}, -2\pi/n)R. \quad (8.5)$$

Hence, if $v \in \text{Fix}_1(\Sigma)$, then Rv is fixed by $(\mathcal{C}, -2\pi/n)$. As shown in [GSS, XVIII] the subgroups Σ and $\Sigma' = \mathbb{Z}_n(\mathcal{C}, -2\pi/n)$ are conjugate (indeed $\mathcal{T}: \text{Fix}(\Sigma) \rightarrow \text{Fix}(\Sigma')$) and $\text{Fix}(\Sigma) \perp \text{Fix}(\Sigma')$.

We now apply Theorem 3.1 to the time-reversal symmetry $R' = \mathcal{T}R$. This time-reversal symmetry does leave $\text{Fix}(\Sigma)$ invariant, but now we must check that (3.8c) is valid for R' . Namely, we must verify that

$$R'|_{\text{Fix}_0(\Sigma)} = I. \quad (8.6)$$

Remark 3.2 implies that $\text{Fix}_0(\Sigma) = \text{Fix}_0(\mathcal{C})$ and Remark 5.5 states that $\text{Fix}_0(\mathcal{C}) = W_{++} \oplus W_{+-}$. Clearly $R'|_{W_{++}} = I$ since both R and $\mathcal{T}$ act trivially on W_{++} . The same is not true for W_{+-} since $\mathcal{T}$ acts nontrivially on W_{+-} ; that is why we are obliged to use the normal vector field f_N and Theorem 8.1. Indeed, W_{+-} is not in the domain of f_N and (8.4) is satisfied for f_N . Thus, ponies on a stationary merry-go-round exist for f_N .

In the remainder of this section we verify that these ponies drift around the $SO(2)$ group orbits, as described in Theorem 8.2. To do this we must first give a precise definition of drift. Write the Stokeslet model abstractly as

$$dx/dt = F(x), \quad (8.6)$$

where $F = F_N + F_T$, as in Theorem 8.1. Let $y(t)$ denote a pony on a merry-go-round solution for F_N which is T -periodic, and let $x(t)$ denote the corresponding trajectory for F . It follows from Theorem 8.2 that there exists a curve $\delta(t)$ in $SO(2)$ such that $x(t) = \delta(t)y(t)$.

Definition 8.5. Drift occurs on $y(t)$ if $\delta(t) \neq 1$. The net drift of $y(t)$ is $\delta(T)$.

If the net drift is unequal to 1, then the solution $x(t)$ will drift around a 2-torus. If $\delta(T)$ and T are rationally independent, then $x(t)$ will be a true two-frequency motion and its trajectory will be dense on a 2-torus.

So to determine whether $x(t)$ has two-frequencies, we must compute $\delta(T)$. It follows from (8.6) that:

$$d\delta/dt = \delta F_T(y). \quad (8.7)$$

Using Remark 8.4(b), we write $F_T(y) = a(y)Jy$ for some smooth function $a(y)$. We can also rewrite $\delta(t) = R_{\phi(t)}$ with $\phi(0) = 0$. This gives δ explicitly as a curve in the action of $SO(2)$. Direct computation using the definition of R_{θ} leads to:

$$\frac{dR}{dt}\phi = \dot{\phi} R_{\phi} J. \quad (8.8)$$

Substitution of (8.8) into (8.7) yields

$$\dot{\phi} = -a(y(t)). \quad (8.9)$$

Integration of (8.9) yields a formula for the net drift:

$$\phi(T) = - \int_0^T a(y(t)) dt \quad (8.10)$$

Thus, to compute $\phi(T)$ we need formulas for $a(y)$ and $y(t)$. The formula for a in terms of F is given by (8.4); the more complicated part of this calculation involves finding an approximate formula for $y(t)$. To simplify the exposition we restrict our attention to the three particle Stokeslet model.

For $n = 3$, (4.2) states:

$$F(y) = (U(y_2) - U(y_3), U(y_3) - U(y_1), U(y_1) - U(y_2)).$$

Using (4.3) and (8.4) we find that

$$a(y) = \frac{1}{|Jy|^2} \left[\frac{e_1 \cdot y_1}{|y_1|^3} y_1 \cdot J(y_3 - y_2) + \frac{e_1 \cdot y_2}{|y_2|^3} y_2 \cdot J(y_1 - y_3) + \frac{e_1 \cdot y_3}{|y_3|^3} y_3 \cdot J(y_2 - y_1) \right] \quad (8.11)$$

Recall from (6.5) that ponies satisfy:

$$y(t) = (p(t), R_{\theta} p(t+T/3), R_{2\theta} p(t+2T/3))$$

where $\theta = 2\pi/3$. It follows that integrating each of the summands in (8.11) yields the same value. Hence

$$\phi(T) = -3 \int_0^T \frac{1}{|Jy|^2} \frac{1}{|y|^3} e_1 \cdot y_1 y_1 \cdot J(y_3 - y_2) dt. \quad (8.12)$$

Recall that Theorem 3.1 guarantees that F_N has a continuous family of T_ϵ -periodic ponies $y_\epsilon(t)$ with $y_0(t) \equiv x_0$. Hence

$$y_\epsilon(t) = x_0 + \epsilon z(\omega_\epsilon t) + O(\epsilon^2) \quad (8.13)$$

where $z(t)$ is 2π -periodic, $\omega_\epsilon = T_\epsilon/2\pi$, and from (4.9)

$$x_0 = (R_{\theta/2}e_3, R_{3\theta/2}e_3, R_{5\theta/2}e_3) \equiv (x_1^0, x_2^0, x_3^0). \quad (8.14)$$

Indeed, if we rescale time in the Stokeslet model so that the imaginary eigenvalues of $(dF)_{x_0}$ are $\pm i$, then the proof of

Theorem 3.1 shows that $T_\epsilon = 2\pi + O(\epsilon^2)$.

Theorem 8.6: $\phi(T_\epsilon) = K\epsilon^2 + O(\epsilon^3)$ where $K \neq 0$.

Remark: Theorem 8.6 implies that $\phi(T_\epsilon)$ is nonzero for small values of ϵ and hence net drift is nonzero. Since both ϕ and T vary continuously with ϵ , it follows that for most values of ϵ near zero $\phi(T_\epsilon)/T_\epsilon$ is irrational. Hence, the corresponding trajectories $x_\epsilon(t)$ for the full Stokeslet are dense in 2-tori.

The proof of Theorem 8.6 is based on the explicit calculation of $z(t) = (z_1(t), z_2(t), z_3(t))$, which we summarize by:

Lemma 8.7: There exists a horizontal vector h and a 2π -periodic function $c(t)$ such that

$$\begin{aligned}
z_1(t) &= c(\omega_\epsilon t) e_1 + R_{\omega_\epsilon t} h \\
z_2(t) &= c(\omega_\epsilon t + \theta) e_1 + R_{\omega_\epsilon t + 2\theta} h \\
z_3(t) &= c(\omega_\epsilon t + 2\theta) e_1 + R_{\omega_\epsilon t + \theta} h.
\end{aligned} \tag{8.15}$$

Proof of Theorem 8.6: Since $e_1 \cdot x_1^0 = 0$, $\phi(T_\epsilon)$ is at least order ϵ . Indeed, $\phi(T_\epsilon)$ is of order at least ϵ^2 since from (8.15a) the order ϵ term is:

$$C \int_0^T c(\omega_\epsilon t) dt = 0,$$

where $C \equiv x_1^0 \cdot J(x_3^0 - x_2^0) \neq 0$.

Up to nonzero constant multiples, there are four terms contributing to the order ϵ^2 term in $\phi(T_\epsilon)$:

$$\begin{aligned}
\text{(i)} \quad & \int_0^T e_1 \cdot (\text{order } \epsilon^2 \text{ term in } y_1) dt \\
\text{(ii)} \quad & \int_0^T c(\omega_\epsilon t) \cdot (\text{order } \epsilon \text{ term in } y_1 \cdot J(y_3 - y_2)) dt \\
\text{(iii)} \quad & \int_0^T c(\omega_\epsilon t) \cdot (\text{order } \epsilon \text{ term in } |Jy|^{-2}) dt \\
\text{(iv)} \quad & \int_0^T c(\omega_\epsilon t) \cdot (\text{order } \epsilon \text{ term in } |y_1|^{-3}) dt.
\end{aligned}$$

We show that the first three of these are zero and that the fourth is nonzero.

To verify that (i) is zero, recall that y is in V and hence that $y_1 + y_2 + y_3 = 0$. Thus, from (8.11), the sum of the contributions of the three integrands is zero.

To compute (ii), observe that the order ϵ term in $y_1 \cdot J(y_3 - y_2)$ is $z_1 \cdot J(x_3^0 - x_2^0) + x_1^0 \cdot J(z_3 - z_2)$ which after substitution of (8.14) and (8.15), is identically zero.

Contribution (iii) is shown to be zero by verifying that

$$|Jy|^2 = 3 + O(\epsilon^2).$$

Finally, we compute (iv). By direct calculation

$$|y_1|^{-3} = 1 - 3\epsilon c(\omega_\epsilon t) + O(\epsilon^2).$$

Since $\int_0^T c(\omega_\epsilon t)^2 dt \neq 0$, we have proved the theorem. $\square$

Proof of Lemma 8.7: The basic idea behind the Liapunov-Schmidt reduction proof of the existence of periodic solutions in Theorem 3.1 is to restrict the reduced operator to the invariant subspace $\text{Fix}(\Sigma)$ and to find solutions on that subspace. The difficulty in actually finding these periodic solutions in the phase space V occurs when Σ is a subgroup of $\Gamma \times S^1$ having elements with a nontrivial temporal (S^1) part. Since S^1 does not act directly on V (it does act by phase shift on $\mathbb{C}_{2\pi}$), it is somewhat difficult to find $\text{Fix}(\Sigma)$ in V . Indeed, for the periodic solutions $\Sigma = \mathbb{Z}_3(\mathbb{C}, \pm 2\pi/3)$, with either choice of sign possible.

In the the Liapunov-Schmidt reduction, however, we find periodic solutions by implicitly solving certain equations for higher order terms that are functions of periodic solutions of the linearized equations. More precisely, let $\mathcal{W}$ be the space of 2π -periodic solutions to $du/dt = (dF)_{x_0} u$. That is, $\mathcal{W}$ is spanned by $e^{it} v$ where v is an eigenvector of $(dF)_{x_0}$ with eigenvalue i . On the space $\mathcal{W}$, S^1 acts naturally by phase shifts and we can find $\text{Fix}(\Sigma)$. Moreover, the periodic solutions in $\mathcal{W}$ are, after rescaling the period by ω_ϵ , the first order approximations

to the actual periodic solutions of the Stokeslet model. Hence, we can prove the lemma if we can identify $\text{Fix}(\Sigma)$ inside $\mathcal{W}$.

Recall from Theorem 5.1 that $(dF)_{x_0}$ has an invariant subspace isomorphic to W_1^2 , which is $V_1 \oplus V_{2,2}$ in coordinates. See (5.8, 5.6). According to Theorem 5.2, the eigenvalues of $L \equiv (dF)_{x_0}|_{W_1^2}$ are either real or purely imaginary, and the remaining eigenvalues are zero. Indeed, [H] shows that the nonzero eigenvalues are purely imaginary, and we have rescaled time so that they are $\pm i$. In addition, Theorem 5.2 implies that

$$L = \begin{pmatrix} 0 & aA \\ bA^{-1} & 0 \end{pmatrix}$$

where $A: V_{2,2} \rightarrow V_1$ is a fixed Γ -equivariant isomorphism and $ab = -1$ since the imaginary eigenvalues of L are $\pm i$.

Replacing aA by $-A$, we see that

$$L = \begin{pmatrix} 0 & -A \\ A^{-1} & 0 \end{pmatrix}.$$

The periodic solutions in $\mathcal{W}$ are generated by:

$$\exp(tL) = \begin{pmatrix} \cos(t)I & -\sin(t)A \\ \sin(t)A^{-1} & \cos(t)I \end{pmatrix}. \quad (8.16)$$

Note that (8.16) determines the action of S^1 explicitly.

We now use the symmetry of ponies solutions and (8.16) to determine the function $z(t)$ explicitly. Let $(p, q) \in V_{2,2} \oplus V_1$ and let

$$z(t) = e^{tL} (p, q) \quad (8.17)$$

be a periodic function in $\mathcal{W}$. Then $z(t)$ is a ponies periodic

function if $\mathcal{E}z(t+\theta) = z(t)$. It follows that

$$e^{-\theta L} (p, q) = (\mathcal{E}p, \mathcal{E}q). \quad (8.18)$$

Using (8.16), we can parametrize solutions to (8.18) by

$$p = -\frac{1}{\sqrt{3}} A(I+2\mathcal{E})q \equiv Aq^\perp \quad (8.19)$$

where $q \in V_{2,2}$ is arbitrary.

Since q is in $V_{2,2}$, it has the form $(h, R_\theta h, R_{2\theta} h)$ for some horizontal vector h . See (5.5). Using the action of $\mathcal{E}$ on $V_{2,2}$, see (5.7), it follows that $q^\perp = Jq$. We can now rewrite (8.19) as

$$p = AJq. \quad (8.20)$$

Next, we use (8.16) and (8.17) to show that

$$z(t) = \left(AJ(\cos(t)I + \sin(t)J)q, (\cos(t)I + \sin(t)J)q \right).$$

Finally, we observe that on $V_{2,2}$

$$R_t = \cos(t)I + \sin(t)J.$$

Hence,

$$z(t) = (AJR_t q, R_t q)$$

which proves the lemma since the first component of $AJR_t q$ is both a periodic function in t and a vertical vector (being in V_1), and the first component of $R_t q$ is $R_t h$ and h is a horizontal vector. □

§9. Remarks on Stability

In this section we discuss the stability of the families of periodic and quasiperiodic solutions that we found for the three particle Stokeslet model. Although many of our comments will be valid for the general n particle model, we consider here only the technically less complicated three particle case.

Our discussion will focus on linear stability, though we will remark briefly on the issue of nonlinear stability at the end of the section. Our results are similar to those of [MRS], who consider linear stability for periodic solutions of general Hamiltonian systems with spatial symmetry, and specifically consider D_3 symmetry in the Henon-Heiles system.

As in Hamiltonian systems, periodic solutions in time-reversible systems that are fixed by a time-reversal symmetry cannot be asymptotically stable, since if one trajectory approaches the cycle in forward time another must approach the cycle in backward time. Thus linear stability of such periodic solutions must correspond to Floquet exponents on the imaginary axis, the elliptic case, and instability to Floquet exponents with nonzero real part, the hyperbolic case.

When $n = 3$ the Stokeslet model is posed on the six-dimensional space $V = W_{++} \oplus W_{+-} \oplus W^2$ where $W \cong \mathbb{C}$. In fact, continuous spatial symmetries force stability questions onto the four-dimensional subspace W^2 . Besides the rotational symmetry $SO(2)$, whose tangent space at the equilibrium x_0 is W_{+-} , there is a scaling symmetry for the Stokeslet model (which until now has not been used explicitly):

$$F(rx) = \frac{1}{r} F(x). \quad (9.1)$$

See (4.3) to verify (9.1). The tangent space to the scaling symmetry is W_{++} , corresponding to equilibrium equilateral triangles of varying side length. Thus, when considering orbital stability, we need only view the stability of periodic solutions to the normal vector field F_N (defined on $W_{++} \oplus W^2$) restricted to W^2 .

Our discussion in Section 6 shows that there are three families of periodic solutions to F_N : ponies on a merry-go-round, $\mathbb{Z}_3(\mathbb{C}, 2\pi/3)$; synchronous isosceles triangles, $\mathbb{Z}_2(\mathcal{T})$; and asynchronous isosceles triangles, $\mathbb{Z}_2(\mathcal{T}, \pi)$. We prove the following:

Theorem 9.1: Generically, $\mathbb{Z}_3(\mathbb{C}, 2\pi/3)$ periodic solutions are elliptic, and one of the isosceles triangles solutions is elliptic while the other is hyperbolic.

Sketch of Proof: Our proof relies on many of the calculations presented in [GSS, XVIII], which we summarize here. The Poincaré-Birkhoff normal form of F_N on $W^2 \cong \mathbb{C}^2$ has $D_3 \times S^1$ equivariance - up to any finite order. Following [GSS, XVIII, Proposition 2.1] the normal form vector field has the form:

$$g(z_1, z_2) = A_1 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + A_2 \begin{bmatrix} z_1^2 \bar{z}_1 \\ z_2^2 \bar{z}_2 \end{bmatrix} + A_3 \begin{bmatrix} z_1^2 z_2^3 \\ z_1^3 \bar{z}_2^2 \end{bmatrix} + A_4 \begin{bmatrix} z_1^4 \bar{z}_2^3 \\ z_1^3 \bar{z}_2^4 \end{bmatrix} \quad (9.2)$$

where the A_j are complex-valued functions of

$$N = |z_1|^2 + |z_2|^2, \quad P = |z_1|^2 |z_2|^2, \quad S = (z_1 \bar{z}_2)^3 + (\bar{z}_1 z_2)^3 \quad (9.3)$$

$$T = i(|z_1|^2 - |z_2|^2)((z_1 \bar{z}_2)^3 - (\bar{z}_1 z_2)^3).$$

Normal form can be achieved respecting the time-reversal symmetry $R(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$. The anticommutativity of (9.2) with respect to R implies that

$$A_j = a_j T + i\alpha_j \quad (j = 1, \dots, 4) \quad (9.4)$$

where the a_j and α_j are real-valued functions of N, P, S since

$$T^2 = (4P - N^2)(S^2 - 4P^3).$$

See [GSS, XVIII, (2.2)].

For the normal form equations, the Floquet equations can be transformed to constant coefficient equations and solved explicitly [GSS, XVI, Proposition 6.4]. We denote the Floquet matrix for a periodic solution $u(t)$ by M_u .

By general theory, we know that both eigenvalues of M_u restricted to the two-dimensional subspace $\text{Fix}(\Sigma)$, where Σ is the isotropy subgroup corresponding to u , are zero. One zero eigenvalue corresponds to the eigenvector u' ; the other zero eigenvalue follows from the fact that $\text{Fix}(\Sigma)$ is foliated by periodic trajectories (Theorem 3.1). Thus, linear stability is determined in the two-dimensional Σ -invariant subspace V_1 transverse to $\text{Fix}(\Sigma)$.

For the Z_3 solutions, M_u is a rotation matrix on V_1 [GSS, Table XVIII, 3.4] and time-reversal symmetry implies that this rotation must be by $\pi/2$. Hence the eigenvalues of $M_u|_{V_1}$

lie on the imaginary axis. Assuming that these eigenvalues are nonzero, which is implied by a nondegeneracy condition on the third order terms in the normal form equations g , the ponies are elliptic - at least in normal form.

When normal form is broken at high order, the ponies still remain elliptic. The two zero eigenvalues of the Floquet equations are still forced by u' and the family of periodic solutions: by continuity the other two eigenvalues must be nonzero and close to purely imaginary. Time-reversal symmetry implies that these eigenvalues are $\pm\lambda$ even when the equations are not in normal form, and hence these eigenvalues must remain purely imaginary.

For the $Z_2(\mathcal{T})$, $Z_2(\mathcal{T}, \pi)$ solutions, time-reversal symmetry implies that the transverse eigenvalues are $\pm\lambda$, and hence either elliptic or hyperbolic. Which occurs depends on the sign of $\det(M_u|V_1)$. These determinants are computed for normal form equations in [GSS, Table XVIII, 3.2] and, at lowest order, have opposite signs. Indeed, the sign of this determinant is determined by the sign of $\alpha_2(0)\alpha_3(0)$, thus yielding a nondegeneracy condition involving terms of fifth order in the normal form equations. When normal form symmetry is broken at order greater than five, the argument given for the ponies solutions applies here also. Thus, generically, one of the isosceles triangles solutions will be elliptic and one hyperbolic. □

We end by discussing briefly nonlinear stability. As noted above, we know that these periodic solutions will never be asymptotically stable. Scheurle [S] develops a KAM theory for time-reversible systems indicating the existence of invariant 2-tori for the normal vector field F_N surrounding the periodic solutions. His results, however, do not apply to cases of 1:1 resonance in the eigenvalues of the Jacobian $(dF_N)_{x_0}$. Because of D_3 symmetry, this 1:1 resonance is forced.

Let us speculate that the source of the multiple eigenvalues, the spatial D_3 symmetry, restricts the form of the equations sufficiently so that Scheurle's proof can be adapted to this symmetry case. Then, typical trajectories of F_N would lie on invariant 2-tori. We also speculate that due to the drift, computed in Section 8, 2-tori near the ponies for F_N would lead to three-frequency motion in the full six-dimensional Stokeslet model.

Appendix: Derivation of the Stokeslet Model

Under the assumption of small Reynolds number based on sphere radius a , the Stokes equations:

$$\begin{aligned} (a) \quad \mu \Delta u - \nabla p &= 0 \\ (b) \quad \nabla \cdot u &= 0 \end{aligned} \tag{A.1}$$

with appropriate boundary conditions, describe the sedimentation of solid spheres. In (A.1) u is the fluid velocity, μ is the viscosity and p is the pressure. In the point particle approximation of the motion of spheres in the above limit, each particle applies a point load, $mg\mathbf{e}_1$ to the fluid position $\mathbf{x}$ of the particle. This delta function force appears on the right hand side of (A.1a). Let U be the fundamental solution (Green's function) of (A.1a), that is, U specifies the response of a Stokes fluid to a point load. U is explicitly defined in (4.3).

The resulting fluid velocity at $\mathbf{y}$ is given by the Stokeslet $\tau U(\mathbf{y}-\mathbf{x})$ $\tau = mg/8\pi\mu$. Faxen's first law (cf. [CR,HB]) states that the velocity $\mathbf{v}$ of a point particle in a flow $\mathbf{u}$ experiencing a force $\mathbf{F}$ is given by

$$\mathbf{v} = \mathbf{u} + \mathbf{v}_{st}$$

in which

$$\mathbf{v}_{st} = \mathbf{F}/6\pi\mu a$$

is the Stokes velocity due to the force $\mathbf{F}$. In order for the velocity $\mathbf{v}$ to make sense, the particles must be considered, at

this point in the derivation, to have radius a .

Consider a system of n identical point particles sedimenting in a fluid. Each particle experiences a flow u due to the Stokeslet generated by all of the remaining particles. For particles $z_i(t)$, $i = 1, \dots, n$, their motion are governed by the equations

$$dz_i/dt = v_{st} + \tau \sum_{i \neq j}^n U(z_i - z_j) \quad i = 1, \dots, n \quad (A.2)$$

which become singular whenever $z_i = z_j$, $i \neq j$. It is convenient to work with relative coordinates (the sides of the n -gon, rather than the vertices) and scaled time. Thus, if

$$x_j = z_{j+1} - z_j \quad j = 1, \dots, n \quad (A.3)$$

where the indices are taken mod n and

$$t = \tau t', \quad (A.4)$$

in which t' is the original time, the Stokeslet model equations are given by (4.2).

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