**MULTIVARIABLE BIORTHOGONAL
CONTINUOUS-DISCRETE
WILSON AND RACA POLYNOMIALS**

By

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**Multivariable biorthogonal continuous–discrete
Wilson and Racah polynomials**

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Abstract

Several families of multivariable, biorthogonal, partly continuous and partly discrete, Wilson polynomials are presented. These yield limit cases which are purely continuous in some of the variables and purely discrete in the others, or purely discrete in all the variables. The latter are referred to as the multivariable biorthogonal Racah polynomials. Interesting further limit cases include the multivariable biorthogonal Hahn and dual Hahn polynomials.

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I. Introduction

The Wilson polynomials^{1,2} are a very general family that include as special or limiting cases all the classical orthogonal polynomials and many related families. They can be expressed as the following ${}_4F_3$ hypergeometric series³

$$P_n(x) = (a+b)_n(a+c)_n(a+d)_n {}_4F_3 \left(\begin{matrix} -n, n+a+b+c+d-1, a-ix, a+ix \\ a+b, a+c, a+d \end{matrix} ; 1 \right), \quad (1.1)$$

where a, b, c, d are complex parameters, $(\alpha)_n \equiv \Gamma(n+\alpha)/\Gamma(\alpha)$ denotes the usual Pochhammer symbol, and n is a nonnegative integer. These are polynomials in x of degree $2n$ which one can show¹ are symmetric in all four of the parameters a, b, c, d . They are associated with the following weight function

$$w(x) = \frac{\Gamma(a+ix)\Gamma(a-ix)\Gamma(b+ix)\Gamma(b-ix)\Gamma(c+ix)\Gamma(c-ix)\Gamma(d+ix)\Gamma(d-ix)}{\Gamma(2ix)\Gamma(-2ix)}, \quad (1.2)$$

and satisfy a complex orthogonality relation

$$\int_C dx P_n(x) P_m(x) w(x) = \lambda_n \delta_{nm}, \quad (1.3)$$

where the normalization constant λ_n is given by

$$\lambda_n = 4\pi n! (n+a+b+c+d-1)_n \times \frac{\Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)\Gamma(n+c+d)}{\Gamma(2n+a+b+c+d)}, \quad (1.4)$$

and the contour C is deformed from the real axis so that it separates the increasing sequences of poles of the weight function from the decreasing sequences.

When the real parts of the parameters a, b, c, d are positive one can choose C to be the real axis. If in addition the parameters are real or occur in complex conjugate pairs then the polynomials and weight function are real and the latter is positive. In this case the Wilson polynomials satisfy a continuous orthogonality relation with respect to a positive measure on the real line. If the real part of one parameter is less than zero, let us say $\text{Re}(a) < 0$, $\text{Re}(b, c, d) > 0$ ($2a, a+b, a+c, a+d \neq 0, -1, -2, \dots$), then C must be deformed from the real axis to pass over the decreasing sequence of poles given by $x = -ia - ij$, $j =$

$0, 1, 2, \dots$ and under the increasing sequence found at $x = ia + ij, j = 0, 1, 2, \dots$. This contour can then be deformed back to the real axis plus closed loops about a finite number of poles. If the closed loops are then evaluated by the method of residues the inner product in (1.3) can be written as an integral over the real axis plus a finite discrete sum. In this case the Wilson polynomials satisfy a partly continuous and partly discrete orthogonality relation

$$\int_{-\infty}^{\infty} dx w(x) P_n(x) P_m(x) + \sum_{j=0}^{\text{Re}(a)+j<0} \tilde{w}(j) P_n(ia + ij) P_m(ia + ij) = \lambda_n \delta_{nm} , \quad (1.5)$$

where the discrete part of the weight function $\tilde{w}(j)$ is given by

$$\begin{aligned} \tilde{w}(j) = (4\pi) \Gamma(a+b) \Gamma(a+c) \Gamma(a+d) \frac{\Gamma(b-a) \Gamma(c-a) \Gamma(d-a)}{\Gamma(-2a)} \\ \times \frac{(2a)_j (a+1)_j (a+b)_j (a+c)_j (a+d)_j}{(a)_j (a-b+1)_j (a-c+1)_j (a-d+1)_j j!} . \end{aligned} \quad (1.6)$$

Formula (1.5) also yields a purely discrete orthogonality relation. Take $a+b = -\Delta + \epsilon$, where Δ is a nonnegative integer, divide (1.5) by $\Gamma(a+b) = \Gamma(-\Delta + \epsilon)$, and then take the limit $\epsilon \rightarrow 0$. The continuous term vanishes because $1/\Gamma(-\Delta + \epsilon) \rightarrow 0$ but the discrete part survives leaving

$$\sum_{j=0}^{\Delta} \rho(j) P_n(ia + ij) P_m(ia + ij) = \lambda'_n \delta_{nm} , \quad 0 \leq n, m \leq \Delta , \quad (1.7)$$

where the weight function and normalization constant are given by

$$\rho(j) = \frac{(2a)_j (a+1)_j (-\Delta)_j (a+c)_j (a+d)_j}{(a)_j (2a+\Delta+1)_j (a-c+1)_j (a-d+1)_j j!} , \quad (1.8)$$

$$\begin{aligned} \lambda'_n = n! (n - \Delta + c + d - 1)_n (2a+1)_{\Delta} (1-c-d)_{\Delta} \\ \times \frac{(-\Delta)_n (a+c)_n (a+d)_n (c-a-\Delta)_n (d-a-\Delta)_n (c+d)_n}{(-\Delta+c+d)_{2n} (a-c+1)_{\Delta} (a-d+1)_{\Delta}} . \end{aligned}$$

The orthogonality relation (1.7) is equivalent to Racah's orthogonality for what are called Racah coefficients or $6j$ symbols. Accordingly the polynomials in (1.7) are referred to in the literature as the Racah polynomials. It is customary to redefine the parameters and write these polynomials as

$$r_n(x) = (\alpha+1)_n(\gamma+1)_n(\beta+\delta+1)_n {}_4F_3\left(\begin{matrix} -n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{matrix}; 1\right), \quad (1.9)$$

and then the orthogonality relation becomes

$$\sum_{x=0}^{\Delta} \frac{(\gamma+\delta+1)_x(\gamma/2+\delta/2+3/2)_x(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x}{(\gamma/2+\delta/2+1/2)_x(\gamma+\delta-\alpha+1)_x(\gamma-\beta+1)_x(\delta+1)_x x!} r_n(x)r_m(x) = \lambda_n \delta_{nm}, \quad (1.10)$$

$$\lambda_n = n! (\alpha+1)_n(\beta+1)_n(\gamma+1)_n(\alpha-\delta+1)_n(\alpha+\beta-\gamma+1)_n(\beta+\delta+1)_n \\ \times \frac{(n+\alpha+\beta+1)_n}{(\alpha+\beta+2)_{2n}} \frac{\Gamma(\gamma+\delta-\alpha+1)\Gamma(-\beta-\alpha-1)\Gamma(\gamma-\beta+1)\Gamma(\delta+1)}{\Gamma(\gamma+\delta+2)\Gamma(-\beta)\Gamma(\gamma-\beta-\alpha)\Gamma(\delta-\alpha)},$$

where $\alpha+1, \beta+\delta+1$, or $\gamma+1 = -\Delta$, and $0 \leq n, m \leq \Delta$.

Two interesting limit cases are the Hahn and dual Hahn polynomials. The limit $\delta \rightarrow \infty$ with $\gamma+1 = -\Delta$ gives the Hahn polynomial orthogonality

$$\sum_{x=0}^{\Delta} \frac{(\alpha+1)_x(-\Delta)_x}{(-\Delta-\beta)_x x!} h_n(x)h_m(x) = \lambda_n \delta_{nm}, \quad (1.11)$$

where

$$h_n(x) = {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -\Delta \end{matrix}; 1\right), \quad 0 \leq n \leq \Delta, \quad (1.12)$$

$$\lambda_n = \frac{(\Delta-n)!n!}{\Delta!} \frac{(n+\alpha+\beta+1)_{\Delta}(\beta+1)_n}{(\alpha+1)_n(\beta+1)_{\Delta}} \frac{(\Delta+n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)}.$$

Letting $\beta \rightarrow \infty$ in (1.10) with $\alpha+1 = -\Delta$ gives the dual Hahn orthogonality

$$\sum_{x=0}^{\Delta} \binom{\Delta}{x} \frac{(\gamma+\delta+1)_x(\gamma/2+\delta/2+3/2)_x(\gamma+1)_x}{(\gamma/2+\delta/2+1/2)_x(\delta+1)_x(\Delta+\gamma+\delta+2)_x} d_n(x)d_m(x) = \lambda_n \delta_{nm}, \quad (1.13)$$

with

$$d_n(x) = {}_3F_2\left(\begin{matrix} -n, -x, x+\gamma+\delta+1 \\ -\Delta, \gamma+1 \end{matrix}; 1\right), \quad 0 \leq n \leq \Delta, \quad (1.14)$$

$$\lambda_n = \frac{(\Delta-n)!n!}{\Delta!} \frac{(\gamma+\delta+2)_{\Delta}}{(\gamma+1)_n(\delta+1)_{\Delta-n}}.$$

A generalization of the Wilson polynomials to p variables $x_1, x_2, \dots, x_p$ is given by the following four families⁴

$$P \left(\begin{matrix} x_1, x_2, \dots, x_p \\ n_1, n_2, \dots, n_p \end{matrix} \middle| \begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_p \end{matrix} \middle| c, d \right) = \left[\prod_{k=1}^p \Gamma(n_k + a_k + b_k) / \Gamma(a_k + b_k) \right] (A + c)_N (A + d)_N \\ \times F_{2:1; \dots; 1}^{2:2; \dots; 2} \left(\begin{matrix} N + A + B + c + d - 1, A - iX : -n_1, a_1 + ix_1; \dots; -n_p, a_p + ix_p \\ A + c, A + d : a_1 + b_1; \dots; a_p + b_p \end{matrix} \right), \quad (1.15)$$

$$\overline{P} \left(\begin{matrix} x_1, x_2, \dots, x_p \\ n_1, n_2, \dots, n_p \end{matrix} \middle| \begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_p \end{matrix} \middle| c, d \right) = \left[\prod_{k=1}^p \Gamma(n_k + a_k + b_k) / \Gamma(a_k + b_k) \right] (B + c)_N (B + d)_N \\ \times F_{2:1; \dots; 1}^{2:2; \dots; 2} \left(\begin{matrix} N + A + B + c + d - 1, B + iX : -n_1, b_1 - ix_1; \dots; -n_p, b_p - ix_p \\ B + c, B + d : a_1 + b_1; \dots; a_p + b_p \end{matrix} \right), \quad (1.16)$$

$$Q \left(\begin{matrix} x_1, x_2, \dots, x_p \\ n_1, n_2, \dots, n_p \end{matrix} \middle| \begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_p \end{matrix} \middle| c, d \right) = \left[\prod_{k=1}^p \Gamma(n_k + a_k + b_k) / \Gamma(a_k + b_k) \right] (c - iX)_N (d - iX)_N \\ \times F_{2:1; \dots; 1}^{2:2; \dots; 2} \left(\begin{matrix} -N - c - d + 1, B + iX : -n_1, a_1 + ix_1; \dots; -n_p, a_p + ix_p \\ -N - c + iX + 1, -N - d + iX + 1 : a_1 + b_1; \dots; a_p + b_p \end{matrix} \right), \quad (1.17)$$

$$\overline{Q} \left(\begin{matrix} x_1, x_2, \dots, x_p \\ n_1, n_2, \dots, n_p \end{matrix} \middle| \begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_p \end{matrix} \middle| c, d \right) = \left[\prod_{k=1}^p \Gamma(n_k + a_k + b_k) / \Gamma(a_k + b_k) \right] (c + iX)_N (d + iX)_N \\ \times F_{2:1; \dots; 1}^{2:2; \dots; 2} \left(\begin{matrix} -N - c - d + 1, A - iX : -n_1, b_1 - ix_1; \dots; -n_p, b_p - ix_p \\ -N - c - iX + 1, -N - d - iX + 1 : a_1 + b_1; \dots; a_p + b_p \end{matrix} \right), \quad (1.18)$$

where $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_p, c, d$ are complex parameters and $F_{r:v_1; \dots; v_p}^{q:\ell_1; \dots; \ell_p}$ is the generalized multivariable Kampé de Fériet hypergeometric series⁵ defined as

$$F_{r:v_1; \dots; v_p}^{q:\ell_1; \dots; \ell_p} \left(\begin{matrix} \alpha_1, \dots, \alpha_q : \beta_1^{(1)}, \dots, \beta_{\ell_1}^{(1)}; \dots; \beta_1^{(p)}, \dots, \beta_{\ell_p}^{(p)}; \\ \gamma_1, \dots, \gamma_r : \xi_1^{(1)}, \dots, \xi_{v_1}^{(1)}; \dots; \xi_1^{(p)}, \dots, \xi_{v_p}^{(p)}; \end{matrix} \middle| z_1, z_2, \dots, z_p \right) = \\ = \sum_{\{j_k\}} \frac{\prod_{i=1}^q (\alpha_i)_J \prod_{i=1}^{\ell_1} (\beta_i^{(1)})_{j_1} \dots \prod_{i=1}^{\ell_p} (\beta_i^{(p)})_{j_p}}{\prod_{i=1}^r (\gamma_i)_J \prod_{i=1}^{v_1} (\xi_i^{(1)})_{j_1} \dots \prod_{i=1}^{v_p} (\xi_i^{(p)})_{j_p}} \frac{z_1^{j_1}}{j_1!} \frac{z_2^{j_2}}{j_2!} \dots \frac{z_p^{j_p}}{j_p!}, \quad (1.19)$$

where $\{j_k\}$ denotes summation indices $j_1, j_2 \dots j_p$ which run over all nonnegative integers and we have introduced the following shorthand notation

$$X \equiv \sum_{k=1}^p x_k, \quad N \equiv \sum_{k=1}^p n_k, \quad J \equiv \sum_{k=1}^p j_k, \quad A \equiv \sum_{k=1}^p a_k, \quad B \equiv \sum_{k=1}^p b_k, \quad (1.20)$$

and in the absence of specifying the arguments $z_1, z_2 \dots z_p$ unity is to be understood. The overbars in (1.16) and (1.18) denote distinct families of polynomials and should not be confused with complex conjugation. The p -tuple of nonnegative integers $n_1, n_2, \dots n_p$ labels the different polynomials whose degrees are given by $2N$ where N is defined in (1.20). These polynomials are associated with the following multivariable weight function

$$w\left(x_1, x_2, \dots x_p \left| \begin{matrix} a_1, a_2, \dots a_p \\ b_1, b_2, \dots b_p \end{matrix} \right| c, d\right) = \left[\prod_{k=1}^p \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right] \times \frac{\Gamma(A - iX) \Gamma(B + iX) \Gamma(c + iX) \Gamma(c - iX) \Gamma(d + iX) \Gamma(d - iX)}{\Gamma(2iX) \Gamma(-2iX)}, \quad (1.21)$$

which notice is symmetric under the interchange of c and d as are all four families of polynomials. When no ambiguity arises we simply write $P_n(x)$, $\overline{P}_n(x)$, $Q_n(x)$, $\overline{Q}_n(x)$, and $w(x)$ for the polynomials and weight function, respectively.

These satisfy the following biorthogonality relations⁴

$$P_n \cdot Q_m = \overline{P}_n \cdot \overline{Q}_m = \lambda_n \prod_{k=1}^p \delta_{n_k m_k}, \quad (1.22)$$

$$P_n \cdot \overline{P}_m = Q_n \cdot \overline{Q}_m = 0 \quad \text{if } N \neq M,$$

where the normalization constant is given by

$$\lambda_n = 2(2\pi)^p \left[\prod_{k=1}^p \Gamma(n_k + a_k + b_k) n_k! \right] \times \frac{\Gamma(N + A + c) \Gamma(N + A + d) \Gamma(N + B + c) \Gamma(N + B + d) \Gamma(N + c + d)}{(2N + A + B + c + d - 1) \Gamma(N + A + B + c + d - 1)}, \quad (1.23)$$

and for positive real parts of the parameters the inner product is defined

$$P_n \cdot Q_m \equiv \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_1 w(x_1 \dots x_p) P_n(x_1 \dots x_p) Q_m(x_1 \dots x_p), \quad (1.24)$$

$$\text{Re}(a_1, a_2 \dots a_p, b_1, b_2 \dots b_p, c, d) > 0,$$

where the integration contours are simply the real axes.

In Section II we extend these purely continuous multivariable biorthogonal Wilson polynomials to several “mixed” cases where the inner product is partly continuous and partly discrete. In Section III we discuss the purely discrete family which are the multivariable biorthogonal Racah polynomials. Taking appropriate limits then yields multivariable biorthogonal Hahn and dual Hahn polynomials.

II. Multivariable mixed type inner products

The biorthogonality relations (1.22) are still valid for negative real parts of the parameters provided each of the contours are suitably deformed to separate the increasing sequences of poles of the weight function from the decreasing sequences, assuming these two sets are disjoint. However, due to the multiple integrals involved it is not always clear where the poles lie in each of the variables and what the appropriate contours are.

We consider several specific cases the first being the following parameter domain

$$\operatorname{Re}(a_1, a_2 \dots a_p, b_1, b_2 \dots b_p, d) > 0, \quad \operatorname{Re}(c) < 0, \quad (2.1)$$

$$\operatorname{Re}(a_1 + c), \operatorname{Re}(b_1 + c) > 0, \quad \operatorname{Re}(c), 2c, c + d \neq 0, -1, -2, \dots,$$

for which the x_1 contour C_1 is deformed from the real axis to pass above the decreasing sequence of poles given by $x_1 = -X_2^p - ic - ij$, $j = 0, 1, 2, \dots$ and underneath the increasing sequence found at $x_1 = -X_2^p + ic + ij$, $j = 0, 1, 2, \dots$, and also above and below the remaining decreasing and increasing sequences, respectively. We have introduced the following shorthand notation to denote partial sums

$$X_j^\ell \equiv \sum_{k=j}^{\ell} x_k, \quad A_j^\ell \equiv \sum_{k=j}^{\ell} a_k, \quad B_j^\ell \equiv \sum_{k=j}^{\ell} b_k. \quad (2.2)$$

If we furthermore choose C_1 sufficiently close to (and under) $-X_2^p + ic$ and also sufficiently close to (and over) $-X_2^p - ic$ then the remaining contours $C_2 \dots C_p$ can be chosen on the real axes.

Let us first demonstrate that this choice of contours leads to the norm of the weight function as given by (1.23) with $N = 0$. Making a change of variables from $x_1, x_2 \dots x_p$ to

$X, x_2 \dots x_p$ and reversing the order of the integrations gives

$$\begin{aligned}
& \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_2 \int_{C_1} dx_1 w(x_1 \dots x_p) = \\
& = \int_C dX \frac{\Gamma(A - iX) \Gamma(B + iX) \Gamma(c + iX) \Gamma(c - iX) \Gamma(d + iX) \Gamma(d - iX)}{\Gamma(2iX) \Gamma(-2iX)} \\
& \times \int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_p \Gamma(a_1 + iX - iX_2^p) \Gamma(b_1 - iX + iX_2^p) \left[\prod_{k=2}^p \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right],
\end{aligned} \tag{2.3}$$

where the contour C passes over the decreasing sequence of poles $X = -ic - ij$, $j = 0, 1, 2, \dots$ and under the increasing sequence $X = ic + ij$, $j = 0, 1, 2, \dots$, and also above and below the remaining decreasing and increasing sequences, respectively. To evaluate the $x_2, x_3 \dots x_p$ integrations we introduce the following single variable integral formula³

$$\begin{aligned}
& \int_{C'} dx \Gamma(\alpha + ix) \Gamma(\beta + ix) \Gamma(\gamma - ix) \Gamma(\delta - ix) = \\
& = (2\pi) \frac{\Gamma(\alpha + \gamma) \Gamma(\alpha + \delta) \Gamma(\beta + \gamma) \Gamma(\beta + \delta)}{\Gamma(\alpha + \beta + \gamma + \delta)}, \\
& \alpha + \gamma, \alpha + \delta, \beta + \gamma, \beta + \delta \neq 0, -1, -2, \dots,
\end{aligned} \tag{2.4}$$

where the contour C' separates the increasing sequences of poles of the integrand from the decreasing sequences; in the special case when $\text{Re}(\alpha, \beta, \gamma, \delta) > 0$, C' can be chosen on the real axis. Returning to (2.3) and recalling that $\text{Re}(a_1 + c), \text{Re}(b_1 + c) > 0$, the contour C is assumed to pass sufficiently close to (and over) $-ic$ so that $\text{Re}(a_1 + iX) > 0$ and also sufficiently close to (and under) $+ic$ so that $\text{Re}(b_1 - iX) > 0$. In this case formula (2.4) with C' on the real axis and induction can be used to evaluate the $x_2, x_3 \dots x_p$ integrations leading to

$$\begin{aligned}
& \int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_p \Gamma(a_1 + iX - iX_2^p) \Gamma(b_1 - iX + iX_2^p) \left[\prod_{k=2}^p \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right] = \\
& = (2\pi)^{p-1} \left[\prod_{k=1}^p \Gamma(a_k + b_k) \right] \Gamma(A + iX) \Gamma(B - iX) / \Gamma(A + B),
\end{aligned} \tag{2.5}$$

and if this is substituted into (2.3) the norm of the weight function becomes

$$\begin{aligned}
& \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_2 \int_{C_1} dx_1 w(x_1 \dots x_p) = \\
& = (2\pi)^{p-1} \left[\prod_{k=1}^p \Gamma(a_k + b_k) \right] [\Gamma(A + B)]^{-1} \int_C dX \Gamma(A + iX) \Gamma(A - iX) \\
& \quad \times \frac{\Gamma(B + iX) \Gamma(B - iX) \Gamma(c + iX) \Gamma(c - iX) \Gamma(d + iX) \Gamma(d - iX)}{\Gamma(2iX) \Gamma(-2iX)},
\end{aligned} \tag{2.6}$$

which is simply proportional to the single variable integral given by (1.2)–(1.4) with $n = m = 0$. Using this result in (2.6) then yields the multivariable norm as defined in (1.23) with $N = 0$.

Next we express the multiple integral in the left of (2.6) as a finite discrete sum and integrations over the real axes. This is achieved by deforming C_1 to the real axis plus closed loops about a finite number of poles and then evaluating the latter by the method of residues. This leads to two additional terms involving discrete sums which by using (2.5) one can show are equal. The norm then becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_2 \int_{C_1} dx_1 w(x_1 \dots x_p) &= \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_1 w(x_1 \dots x_p) \\
&+ \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_2 \sum_{j=0}^{\text{Re}(c)+j<0} \tilde{w}(j, x_2 \dots x_p),
\end{aligned} \tag{2.7}$$

where all the integrals on the right are over the real axes and the “mixed” weight function in the second term is given by

$$\begin{aligned}
\tilde{w}(j, x_2 \dots x_p) &= (4\pi) \Gamma(a_1 - c - j - iX_2^p) \Gamma(b_1 + c + j + iX_2^p) \left[\prod_{k=2}^p \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right] \\
&\times \frac{\Gamma(A + c + j) \Gamma(B - c - j) \Gamma(2c + j) \Gamma(d - c - j) \Gamma(d + c + j) (-1)^j}{\Gamma(2c + 2j) \Gamma(-2c - 2j) j!}.
\end{aligned} \tag{2.8}$$

The first purely continuous term in the right of (2.7) is that which arises for positive real parts of the parameters. The second mixed term represents the contribution arising from the negative real part of the parameter c .

The inner product of the multivariable Wilson polynomials is defined

$$P_n \cdot Q_m \equiv \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_2 \int_{C_1} dx_1 w(x_1 \dots x_p) P_n(x_1 \dots x_p) Q_m(x_1 \dots x_p), \quad (2.9)$$

which in analogy with (2.7) can be written as

$$\begin{aligned} P_n \cdot Q_m &= \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_1 w(x_1 \dots x_p) P_n(x_1 \dots x_p) Q_m(x_1 \dots x_p) \\ &+ \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_2 \sum_{j=0}^{\operatorname{Re}(c)+j<0} \tilde{w}(j, x_2 \dots x_p) P_n(-X_2^p + ic + ij, x_2 \dots x_p) Q_m(-X_2^p + ic + ij, x_2 \dots x_p) \end{aligned} \quad (2.10)$$

and similarly for $\overline{P}_n \cdot \overline{Q}_m$, $P_n \cdot \overline{P}_m$, and $Q_n \cdot \overline{Q}_m$. Having verified the norm of the weight function it then follows by the same proof as for the purely continuous family⁴ that these inner products satisfy biorthogonality relations (1.22).

Formula (2.10) also yields a simpler and even more interesting mixed type inner product. Take $c + d = -\Delta_1 + \epsilon$, where Δ_1 is a nonnegative integer, divide the biorthogonality relations (1.22) by $\Gamma(c + d) = \Gamma(-\Delta_1 + \epsilon)$, and then take the limit $\epsilon \rightarrow 0$. The first purely continuous term in (2.10) vanishes because $1/\Gamma(-\Delta_1 + \epsilon) \rightarrow 0$ but the second mixed term survives leaving (writing x_1 in place of j)

$$P_n^{(1)} \cdot Q_m^{(1)} \equiv \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_2 \sum_{x_1=0}^{\Delta_1} w^{(1)}(x_1, x_2 \dots x_p) P_n^{(1)}(x_1, x_2 \dots x_p) Q_m^{(1)}(x_1, x_2 \dots x_p), \quad (2.11)$$

$$P_n^{(1)} \cdot Q_m^{(1)} = \overline{P}_n^{(1)} \cdot \overline{Q}_m^{(1)} = \lambda_n^{(1)} \prod_{k=1}^p \delta_{n_k m_k}, \quad P_n^{(1)} \cdot \overline{P}_m^{(1)} = Q_n^{(1)} \cdot \overline{Q}_m^{(1)} = 0 \quad \text{if } N \neq M,$$

where the mixed weight function and normalization constant are given by

$$\begin{aligned} w^{(1)}(x_1 \dots x_p) &= \Gamma(a_1 + d + \Delta_1 - x_1 - iX_2^p) \Gamma(b_1 + c + x_1 + iX_2^p) \left[\prod_{k=2}^p \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right] \\ &\times \frac{\Gamma(A + c + x_1)}{\Gamma(A + c)} \frac{\Gamma(B + d + \Delta_1 - x_1)}{\Gamma(B + d)} \frac{\Gamma(1 + 2d + 2\Delta_1 - 2x_1)}{\Gamma(1 + 2d + 2\Delta_1 - x_1)} \frac{\Gamma(2d + \Delta_1 - x_1)}{\Gamma(2d + 2\Delta_1 - 2x_1)} \binom{\Delta_1}{x_1} (-1)^{x_1}, \end{aligned} \quad (2.12)$$

$$\lambda_n^{(1)} = (2\pi)^{p-1} \left[\prod_{k=1}^p \Gamma(n_k + a_k + b_k) n_k! \right] \frac{\Delta_1!}{(\Delta_1 - N)!} (-1)^N (A + c)_N (B + d)_N \quad (2.13)$$

$$\times \frac{\Gamma(N + A + d) \Gamma(N + B + c)}{(2N + A + B + c + d - 1) \Gamma(N + A + B + c + d - 1)} ,$$

and the polynomials are defined

$$\begin{aligned} P_n^{(1)}(x_1 \dots x_p) &\equiv P_n(-X_2^p + ic + ix_1, x_2 \dots x_p) , & N = 0, 1, 2, \dots \infty , \\ \overline{P}_n^{(1)}(x_1 \dots x_p) &\equiv \overline{P}_n(-X_2^p + ic + ix_1, x_2 \dots x_p) , & N = 0, 1, 2, \dots \infty , \\ Q_n^{(1)}(x_1 \dots x_p) &\equiv Q_n(-X_2^p + ic + ix_1, x_2 \dots x_p) , & 0 \leq N \leq \Delta_1 , \\ \overline{Q}_n^{(1)}(x_1 \dots x_p) &\equiv \overline{Q}_n(-X_2^p + ic + ix_1, x_2 \dots x_p) , & 0 \leq N \leq \Delta_1 , \end{aligned} \quad (2.14)$$

with $c + d = -\Delta_1$. These biorthogonality relations can be verified independently of the limiting procedure. To calculate the norm of the weight function one uses (2.4) and induction to perform the $x_2, x_3, \dots, x_p$ integrations and then the following summation theorem³

$$\begin{aligned} {}_5F_4 \left(\begin{matrix} 2\alpha, \alpha + 1, \alpha + \beta, \alpha + \gamma, \alpha + \delta \\ \alpha, \alpha - \beta + 1, \alpha - \gamma + 1, \alpha - \delta + 1 \end{matrix} ; 1 \right) &= \\ &= \frac{\Gamma(\alpha - \beta + 1) \Gamma(\alpha - \gamma + 1) \Gamma(\alpha - \delta + 1) \Gamma(-\alpha - \beta - \gamma - \delta + 1)}{\Gamma(2\alpha + 1) \Gamma(-\beta - \gamma + 1) \Gamma(-\beta - \delta + 1) \Gamma(-\gamma - \delta + 1)} , \end{aligned} \quad (2.15)$$

$$\operatorname{Re}(\alpha + \beta + \gamma + \delta) < 1 ,$$

to evaluate the x_1 sum. Having calculated the norm the biorthogonality relations can then be verified in the same manner as was proved for the purely continuous family.⁴ Also, some of the original restrictions in (2.1) can be removed from (2.11) leaving only

$$\operatorname{Re}(a_1, a_2 \dots a_p, b_1, b_2 \dots b_p, d) > 0 , \quad \operatorname{Re}(b_1 + c) > 0 , \quad c + d = -\Delta_1 , \quad (2.16)$$

where recall Δ_1 is a nonnegative integer.

We consider a further generalization of (2.11)–(2.14) by also allowing some or all of the a -parameters to have negative real parts. The parameter domain in this case is defined by

$$\begin{aligned}
\operatorname{Re}(a_1, a_2 \dots a_r) &< 0, & \operatorname{Re}(a_{r+1} \dots a_p, b_1 \dots b_p, d) &> 0, \\
\operatorname{Re}(b_1 + c) &> 0, & \operatorname{Re}(A_1^r + d) &> 0, & c + d &= -\Delta_1, \\
\operatorname{Re}(a_2 \dots a_r), & a_2 + b_2 \dots a_r + b_r &\neq 0, -1, -2 \dots, & r = 1, 2, \dots p,
\end{aligned} \tag{2.17}$$

for which the contours $C_2 \dots C_r$, that in (2.11) were on the real axes, are here deformed to pass underneath the increasing sequences of poles $x_k = ia_k + ij_k$, $j_k = 0, 1, 2, \dots$, $k = 2, 3 \dots r$, while still passing above the decreasing sequences. We furthermore choose these contours sufficiently close to (and under) ia_k , $k = 2, 3 \dots r$, so that $\operatorname{Re}(a_1 + d - iX_2^r) > 0$, which is always possible in light of the restriction $\operatorname{Re}(A_1^r + d) > 0$. In this case the remaining contours $C_{r+1} \dots C_p$ can be chosen to lie on the real axes. The norm of the weight function is calculated in the same manner as was done for (2.11). That is, one uses (2.4) and induction to perform the $x_2 \dots x_p$ integrations and then the ${}_5F_4$ summation theorem (2.15) to evaluate the x_1 sum resulting in (2.13) with $N = 0$. Having verified the norm one can then prove the biorthogonality relations (2.11) in the same manner as was done for the purely continuous family,⁴ but with the more general inner product defined above or as re-expressed in (2.18).

The contour integrals can be transformed to discrete sums and real integrations by deforming $C_2 \dots C_r$ to the real axes plus closed loops about a finite number of poles and then evaluating the latter by the method of residues. The inner product of the polynomials can then be schematically written as

$$\begin{aligned}
P_n^{(1)} \cdot Q_m^{(1)} &\equiv \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_{r+1} \left(\prod_{k=2}^r \left\{ \int_{-\infty}^{\infty} dx_k + (2\pi i) \sum_{j_k=0}^{\operatorname{Re}(a_k) + j_k < 0} \operatorname{res}(x_k = ia_k + ij_k) \right\} \right) \\
&\times \sum_{x_1=0}^{\Delta_1} w^{(1)}(x_1 \dots x_p) P_n^{(1)}(x_1 \dots x_p) Q_m^{(1)}(x_1 \dots x_p),
\end{aligned} \tag{2.18}$$

and similarly for $\overline{P}_n^{(1)} \cdot \overline{Q}_m^{(1)}$, $P_n^{(1)} \cdot \overline{P}_m^{(1)}$, and $Q_n^{(1)} \cdot \overline{Q}_m^{(1)}$, where $\operatorname{res}(x_k)$ denotes the residue at x_k . The right side of (2.18) represents a multitude of mixed type terms involving integrations over the real axes and finite discrete sums.

This example which is a generalization of a limit case of (2.10) has itself an interesting limit case. Set $a_k + b_k = -\Delta_k + \epsilon$, $k = 2, 3 \dots r$, where Δ_k are nonnegative integers, divide the biorthogonality relations by $\prod_{k=2}^r \Gamma(a_k + b_k) = \prod_{k=2}^r \Gamma(-\Delta_k + \epsilon)$, and then take the limit $\epsilon \rightarrow 0$. Since $1/\Gamma(-\Delta_k + \epsilon) \rightarrow 0$ the only term in (2.18) that survives is the one with r discrete sums leaving (writing $x_2 \dots x_r$ in place of $j_2 \dots j_r$ and transforming $x_1 \rightarrow \Delta_1 - x_1$)

$$P_n^{(2)} \cdot Q_m^{(2)} \equiv \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_{r+1} \sum_{x_r=0}^{\Delta_r} \dots \sum_{x_2=0}^{\Delta_2} \sum_{x_1=0}^{\Delta_1} w^{(2)}(x_1 \dots x_p) P_n^{(2)}(x_1 \dots x_p) Q_m^{(2)}(x_1 \dots x_p) , \quad (2.19)$$

$$P_n^{(2)} \cdot Q_m^{(2)} = \overline{P}_n^{(2)} \cdot \overline{Q}_m^{(2)} = \lambda_n^{(2)} \prod_{k=1}^p \delta_{n_k m_k} , \quad P_n^{(2)} \cdot \overline{P}_m^{(2)} = Q_n^{(2)} \cdot \overline{Q}_m^{(2)} = 0, \quad \text{if } N \neq M ,$$

where the weight function, normalization constant, and polynomials are given by

$$w^{(2)}(x_1 \dots x_p) = \left[\prod_{k=1}^r \binom{\Delta_k}{x_k} \right] \left[\prod_{k=r+1}^p \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right] \Gamma(A_1^r + d + X_1^r - iX_{r+1}^p) \\ \times \Gamma(B_1^r + c + \Delta_1^r - X_1^r + iX_{r+1}^p) \frac{\Gamma(A + c + \Delta_1 - x_1)}{\Gamma(A + c)} \frac{\Gamma(B + d + x_1)}{\Gamma(B + d)} \\ \times \frac{\Gamma(1 + 2d + 2x_1)}{\Gamma(1 + 2d + \Delta_1 + x_1)} \frac{\Gamma(2d + x_1)}{\Gamma(2d + 2x_1)} (-1)^{\Delta_1 - x_1} , \quad (2.20)$$

$$\lambda_n^{(2)} = (2\pi)^{p-r} \Gamma(n_1 + a_1 + b_1) n_1! \left[\prod_{k=2}^r \frac{\Delta_k!}{(\Delta_k - n_k)!} n_k! (-1)^{n_k} \right] \\ \times \left[\prod_{k=r+1}^p \Gamma(n_k + a_k + b_k) n_k! \right] \frac{\Delta_1!}{(\Delta_1 - N)!} (-1)^N (A + c)_N (B + d)_N \\ \times \frac{\Gamma(N + A + d) \Gamma(N + B + c)}{(2N + A + B + c + d - 1) \Gamma(N + A + B + c + d - 1)} , \quad (2.21)$$

$$P_n^{(2)}(x_1 \dots x_p) \equiv P_n(-iA_2^r - iX_1^r - X_{r+1}^p - id, ia_2 + ix_2 \dots ia_r + ix_r, x_{r+1} \dots x_p) ,$$

$$\overline{P}_n^{(2)}(x_1 \dots x_p) \equiv \overline{P}_n(-iA_2^r - iX_1^r - X_{r+1}^p - id, ia_2 + ix_2 \dots ia_r + ix_r, x_{r+1} \dots x_p) ,$$

$$0 \leq n_k \leq \Delta_k, \quad k = 2, 3 \dots r, \quad n_1, n_{r+1} \dots n_p = 0, 1, 2 \dots \infty , \quad (2.22)$$

$$Q_n^{(2)}(x_1 \dots x_p) \equiv Q_n(-iA_2^r - iX_1^r - X_{r+1}^p - id, ia_2 + ix_2 \dots a_r + ix_r, x_{r+1} \dots x_p) ,$$

$$\overline{Q}_n^{(2)}(x_1 \dots x_p) \equiv \overline{Q}_n(-iA_2^r - iX_1^r - X_{r+1}^p - id, ia_2 + ix_2 \dots a_r + ix_r, x_{r+1} \dots x_p) ,$$

$$0 \leq n_k \leq \Delta_k, k = 2, 3 \dots r, \quad 0 \leq N \leq \Delta_1 ,$$

with $c + d = -\Delta_1$ and $a_k + b_k = -\Delta_k, k = 2, 3 \dots r$. As an independent verification of the norm of the weight function one uses (2.4) and induction to evaluate the $x_{r+1} \dots x_p$ integrations, the following summation theorem⁶ to perform the $x_2 \dots x_r$ sums,

$$F_{1:0;\dots;0}^{1:1;\dots;1} \left(\begin{matrix} \alpha : \beta^{(1)}; \dots; \beta^{(p)}; \\ \gamma : -; \dots; -; \end{matrix} ; 1, 1 \dots 1 \right) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta^{(1)} - \dots - \beta^{(p)})}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta^{(1)} - \dots - \beta^{(p)})} , \quad (2.23)$$

and then theorem (2.15) to evaluate the remaining x_1 sum, resulting in (2.21) with $N = 0$. The biorthogonality relations (2.19) can then be independently verified in the manner described for the purely continuous family.⁴ Also, the restriction $\text{Re}(a_2 \dots a_r) \neq 0, -1, -2 \dots$, is removable from (2.19)–(2.22).

Returning to the purely continuous family (1.22)–(1.24) we consider another mixed type generalization arising from the following parameter domain

$$\begin{aligned} \text{Re}(a_1, a_2 \dots a_r) < 0 , \quad \text{Re}(a_{r+1}, a_{r+2} \dots a_p, b_1, b_2 \dots b_p, c, d) > 0 , \\ \text{Re}(A_1^r + A) , \text{Re}(A_1^r + c) , \text{Re}(A_1^r + d) > 0 , \quad r = 1, 2 \dots p-1 , \\ \text{Re}(a_1, a_2 \dots a_r) , \quad a_1 + b_1, a_2 + b_2 \dots a_r + b_r \neq 0, -1, -2, \dots , \end{aligned} \quad (2.24)$$

for which the first r contours $C_1, C_2 \dots C_r$ are deformed below the real axes to pass underneath the increasing sequences of poles $x_k = ia_k + ij_k, k = 1, 2 \dots r, j_k = 0, 1, 2, \dots$, while still passing above the decreasing sequences. If these are chosen sufficiently close to (and under) $ia_k, k = 1, 2 \dots r$ then the remaining contours $C_{r+1} \dots C_p$ can be chosen to lie on the real axes.

To show that these contours give (1.23) with $N = 0$ for the norm of the weight function we begin with a change of variables from $x_1, x_2 \dots x_p$ to $x_1, x_2 \dots x_r, X_{r+1}^p, x_{r+2} \dots x_p$ yielding

$$\begin{aligned}
& \int_{C_1} dx_1 \dots \int_{C_r} dx_r \int_{-\infty}^{\infty} dx_{r+1} \dots \int_{-\infty}^{\infty} dx_p w(x_1 \dots x_p) = \int_{C_1} dx_1 \dots \int_{C_r} dx_r \left[\prod_{k=1}^r \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right] \\
& \times \int_{-\infty}^{\infty} dX_{r+1}^p \frac{\Gamma(A - iX) \Gamma(B + iX) \Gamma(c + iX) \Gamma(c - iX) \Gamma(d + iX) \Gamma(d - iX)}{\Gamma(2iX) \Gamma(-2iX)} \\
& \times \int_{-\infty}^{\infty} dx_{r+2} \dots \int_{-\infty}^{\infty} dx_p \Gamma(a_{r+1} + iX_{r+1}^p - iX_{r+2}^p) \Gamma(b_{r+1} - iX_{r+1}^p + iX_{r+2}^p) \\
& \times \left[\prod_{k=r+2}^p \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right], \tag{2.25}
\end{aligned}$$

and then the $x_{r+2} \dots x_p$ integrations are performed by using (2.4) and induction leading to

$$\begin{aligned}
& \int_{C_1} dx_1 \dots \int_{C_r} dx_r \int_{-\infty}^{\infty} dx_{r+1} \dots \int_{-\infty}^{\infty} dx_p w(x_1 \dots x_p) = (2\pi)^{p-r-1} \left[\prod_{k=r+1}^p \Gamma(a_k + b_k) \right] \\
& \times [\Gamma(A_{r+1}^p + B_{r+1}^p)]^{-1} \int_{C_1} dx_1 \dots \int_{C_r} dx_r \left[\prod_{k=1}^r \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right] \\
& \times \int_{-\infty}^{\infty} dX_{r+1}^p \Gamma(A_{r+1}^p + iX_{r+1}^p) \Gamma(B_{r+1}^p - iX_{r+1}^p) \Gamma(A - iX) \Gamma(B + iX) \\
& \times \frac{\Gamma(c + iX) \Gamma(c - iX) \Gamma(d + iX) \Gamma(d - iX)}{\Gamma(2iX) \Gamma(-2iX)}. \tag{2.26}
\end{aligned}$$

Another change of variables from $x_1 \dots x_r, X_{r+1}^p$ to $X, x_1 \dots x_r$ then transforms this into

$$\begin{aligned}
& \int_{C_1} dx_1 \dots \int_{C_r} dx_r \int_{-\infty}^{\infty} dx_{r+1} \dots \int_{-\infty}^{\infty} dx_p w(x_1 \dots x_p) = \frac{(2\pi)^{p-r-1}}{\Gamma(A_{r+1}^p + B_{r+1}^p)} \left[\prod_{k=r+1}^p \Gamma(a_k + b_k) \right] \\
& \times \int_C dX \frac{\Gamma(A - iX) \Gamma(B + iX) \Gamma(c + iX) \Gamma(c - iX) \Gamma(d + iX) \Gamma(d - iX)}{\Gamma(2iX) \Gamma(-2iX)} \\
& \times \int_{C_1} dx_1 \dots \int_{C_r} dx_r \Gamma(A_{r+1}^p + iX - iX_1^r) \Gamma(B_{r+1}^p - iX + iX_1^r) \left[\prod_{k=1}^r \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right], \tag{2.27}
\end{aligned}$$

where C passes underneath iA_1^r . The contours $C_1 \dots C_r$, which recall pass underneath the increasing sequences $x_k = ia_k + ij_k$, $k = 1, 2, \dots, r$, $j_k = 0, 1, 2, \dots$, are assumed to pass

sufficiently close to (and under) $ia_k, k = 1, 2 \dots r$ so that $\text{Re}(A_{r+1}^p - iX_1^r) > 0$, which is possible on account of $\text{Re}(A_1^r + A) > 0$. Also $\text{Re}(B_{r+1}^p + iX_1^r) > 0$ since $\text{Re}(b_1 \dots b_p) > 0$ and X_1^r has negative or zero imaginary part. In this case the sequences of poles in the variable X do not cross the real axis and so C can be deformed to this axis. With X real the $x_1 \dots x_r$ integrations can then be performed by (2.4) and induction giving

$$\begin{aligned} \int_{C_1} dx_1 \dots \int_{C_r} dx_r \int_{-\infty}^{\infty} dx_{r+1} \dots \int_{-\infty}^{\infty} dx_p w(x_1 \dots x_p) &= (2\pi)^{p-1} \left[\prod_{k=1}^p \Gamma(a_k + b_k) \right] [\Gamma(A + B)]^{-1} \\ &\times \int_{-\infty}^{\infty} dX \Gamma(A + iX) \Gamma(A - iX) \Gamma(B + iX) \Gamma(B - iX) \\ &\times \frac{\Gamma(c + iX) \Gamma(c - iX) \Gamma(d + iX) \Gamma(d - iX)}{\Gamma(2iX) \Gamma(-2iX)}, \end{aligned} \quad (2.28)$$

which is simply proportional to the single variable integral given by (1.2)–(1.4) with $n = m = 0$. Using this result in (2.28) then yields the multivariable norm (1.23) with $N = 0$. Having verified the norm the biorthogonality relations (1.22) then follow by the same proof as for the purely continuous family,⁴ but with the more general inner product defined above or as re-expressed in (2.30).

As before the contour integrals in the left of (2.28) can be expressed as multiple finite sums and real integrations by deforming $C_1, C_2 \dots C_r$ to the real axes plus closed loops about a finite number of poles and then evaluating the latter by the method of residues. The norm can then be schematically written as

$$\begin{aligned} \int_{C_1} dx_1 \dots \int_{C_r} dx_r \int_{-\infty}^{\infty} dx_{r+1} \dots \int_{-\infty}^{\infty} dx_p w(x_1 \dots x_p) &= \\ = \left(\prod_{k=1}^r \left\{ \int_{-\infty}^{\infty} dx_k + (2\pi i) \sum_{j_k=0}^{\text{Re}(a_k) + j_k < 0} \text{res}(x_k = ia_k + ij_k) \right\} \right) &\int_{-\infty}^{\infty} dx_{r+1} \dots \int_{-\infty}^{\infty} dx_p w(x_1 \dots x_p), \end{aligned} \quad (2.29)$$

representing a multitude of mixed type terms involving real integrations and finite discrete sums. Accordingly the inner product of the polynomials becomes

$$\begin{aligned} P_n \cdot Q_m &\equiv \left(\prod_{k=1}^r \left\{ \int_{-\infty}^{\infty} dx_k + (2\pi i) \sum_{j_k=0}^{\text{Re}(a_k) + j_k < 0} \text{res}(x_k = ia_k + ij_k) \right\} \right) \\ &\times \int_{-\infty}^{\infty} dx_{r+1} \dots \int_{-\infty}^{\infty} dx_p w(x_1 \dots x_p) P_n(x_1 \dots x_p) Q_m(x_1 \dots x_p), \end{aligned} \quad (2.30)$$

and similarly for $\overline{P}_n \cdot \overline{Q}_m$, $P_n \cdot \overline{P}_m$, and $Q_n \cdot \overline{Q}_m$.

Formula (2.30) also yields a much simpler mixed type inner product. Set $a_k + b_k = -\Delta_k + \epsilon$, $k = 1, 2, \dots, r$, where Δ_k are nonnegative integers, divide the biorthogonality relations (1.22) by $\prod_{k=1}^r \Gamma(a_k + b_k) = \prod_{k=1}^r \Gamma(-\Delta_k + \epsilon)$, and then take the limit $\epsilon \rightarrow 0$. The only term in (2.30) that survives is the one with r discrete sums leaving (writing $x_1 \dots x_r$ in place of $j_1 \dots j_r$)

$$P_n^{(3)} \cdot Q_m^{(3)} \equiv \sum_{x_1=0}^{\Delta_1} \sum_{x_2=0}^{\Delta_2} \dots \sum_{x_r=0}^{\Delta_r} \int_{-\infty}^{\infty} dx_{r+1} \dots \int_{-\infty}^{\infty} dx_p w^{(3)}(x_1 \dots x_p) P_n^{(3)}(x_1 \dots x_p) Q_m^{(3)}(x_1 \dots x_p) \quad (2.31)$$

$$P_n^{(3)} \cdot Q_m^{(3)} = \overline{P}_n^{(3)} \cdot \overline{Q}_m^{(3)} = \lambda_n^{(3)} \prod_{k=1}^p \delta_{n_k m_k}, \quad P_n^{(3)} \cdot \overline{P}_m^{(3)} = Q_n^{(3)} \cdot \overline{Q}_m^{(3)} = 0 \quad \text{if } N \neq M,$$

where the mixed weight function and normalization constant are given by ($\Delta_1^r \equiv \sum_{k=1}^r \Delta_k$)

$$w^{(3)}(x_1 \dots x_p) = \left[\prod_{k=1}^r \binom{\Delta_k}{x_k} \right] \left[\prod_{k=r+1}^p \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right] \quad (2.32)$$

$$\begin{aligned} & \times \Gamma(A + A_1^r + X_1^r - iX_{r+1}^p) \Gamma(B + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p) \Gamma(c + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p) \\ & \times \Gamma(c + A_1^r + X_1^r - iX_{r+1}^p) \frac{\Gamma(d + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p) \Gamma(d + A_1^r + X_1^r - iX_{r+1}^p)}{\Gamma(2B_1^r + 2\Delta_1^r - 2X_1^r + 2iX_{r+1}^p) \Gamma(2A_1^r + 2X_1^r - 2iX_{r+1}^p)}, \end{aligned}$$

$$\lambda_n^{(3)} = 2(2\pi)^{p-r} \left[\prod_{k=1}^r \frac{\Delta_k!}{(\Delta_k - n_k)!} n_k! (-1)^{n_k} \right] \left[\prod_{k=r+1}^p \Gamma(n_k + a_k + b_k) n_k! \right] \quad (2.33)$$

$$\times \frac{\Gamma(N + A + c) \Gamma(N + A + d) \Gamma(N + B + c) \Gamma(N + B + d) \Gamma(N + c + d)}{(2N + A + B + c + d - 1) \Gamma(N + A + B + c + d - 1)},$$

and the polynomials are defined

$$\begin{aligned} P_n^{(3)}(x_1 \dots x_p) & \equiv P_n(ia_1 + ix_1 \dots ia_r + ix_r, x_{r+1} \dots x_p), \\ \overline{P}_n^{(3)}(x_1 \dots x_p) & \equiv \overline{P}_n(ia_1 + ix_1 \dots ia_r + ix_r, x_{r+1} \dots x_p), \quad 0 \leq n_k \leq \Delta_k, \quad k = 1, 2, \dots, r, \\ Q_n^{(3)}(x_1 \dots x_p) & \equiv Q_n(ia_1 + ix_1 \dots ia_r + ix_r, x_{r+1} \dots x_p), \quad n_{r+1} \dots n_p = 0, 1, 2, \dots, \infty, \\ \overline{Q}_n^{(3)}(x_1 \dots x_p) & \equiv \overline{Q}_n(ia_1 + ix_1 \dots ia_r + ix_r, x_{r+1} \dots x_p), \end{aligned} \quad (2.34)$$

where the indicated range of the indices applies to all four families.

These results can also be verified independently of the limit. To calculate the norm of the weight function begin with a change of variables from $x_{r+1} \dots x_p$ to $X_{r+1}^p, x_{r+2} \dots x_p$ and use (2.4) and induction to evaluate the $x_{r+2} \dots x_p$ integrations. This gives

$$\begin{aligned}
& \sum_{x_1=0}^{\Delta_1} \dots \sum_{x_r=0}^{\Delta_r} \int_{-\infty}^{\infty} dx_{r+1} \dots \int_{-\infty}^{\infty} dx_p w^{(3)}(x_1 \dots x_p) = \frac{(2\pi)^{p-r-1}}{\Gamma(A_{r+1}^p + B_{r+1}^p)} \left[\prod_{k=r+1}^p \Gamma(a_k + b_k) \right] \\
& \times \sum_{x_1=0}^{\Delta_1} \dots \sum_{x_r=0}^{\Delta_r} \left[\prod_{k=1}^r \binom{\Delta_k}{x_k} \right] \int_{-\infty}^{\infty} dX_{r+1}^p \Gamma(A_{r+1}^p + iX_{r+1}^p) \Gamma(B_{r+1}^p - iX_{r+1}^p) \quad (2.35) \\
& \times \Gamma(A + A_1^r + X_1^r - iX_{r+1}^p) \Gamma(B + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p) \Gamma(c + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p) \\
& \times \Gamma(c + A_1^r + X_1^r - iX_{r+1}^p) \frac{\Gamma(d + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p)}{\Gamma(2B_1^r + 2\Delta_1^r - 2X_1^r + 2iX_{r+1}^p)} \frac{\Gamma(d + A_1^r + X_1^r - iX_{r+1}^p)}{\Gamma(2A_1^r + 2X_1^r - 2iX_{r+1}^p)},
\end{aligned}$$

which by a further change of variable from X_{r+1}^p to $Z \equiv X_{r+1}^p + i(A_1^r + X_1^r)$ becomes

$$\begin{aligned}
& \sum_{x_1=0}^{\Delta_1} \dots \sum_{x_r=0}^{\Delta_r} \int_{-\infty}^{\infty} dx_{r+1} \dots \int_{-\infty}^{\infty} dx_p w^{(3)}(x_1 \dots x_p) = \frac{(2\pi)^{p-r-1}}{\Gamma(A_{r+1}^p + B_{r+1}^p)} \left[\prod_{k=r+1}^p \Gamma(a_k + b_k) \right] \\
& \times \sum_{x_1=0}^{\Delta_1} \dots \sum_{x_r=0}^{\Delta_r} \left[\prod_{k=1}^r \binom{\Delta_k}{x_k} \right] \int_{C(X_1^r)} dZ \Gamma(A + X_1^r + iZ) \Gamma(B + \Delta_1^r - X_1^r - iZ) \quad (2.36) \\
& \times \frac{\Gamma(A - iZ) \Gamma(B + iZ) \Gamma(c + iZ) \Gamma(c - iZ) \Gamma(d + iZ) \Gamma(d - iZ)}{\Gamma(2iZ) \Gamma(-2iZ)},
\end{aligned}$$

where the contours $C(X_1^r)$, $X_1^r = 0, 1, 2 \dots \Delta_1^r$ run parallel to the real axis with imaginary part $i(A_1^r + X_1^r)$. Taking into account (2.24) one finds that no poles of the integrand lie in the region bounded by and including $C(X_1^r)$ and the real axis. In this case each of the contours $C(X_1^r)$ can be deformed to the real axis and the integral can be brought outside of the multiple sums. The latter are then evaluated by theorem (2.23) resulting in

$$\begin{aligned}
& \sum_{x_1=0}^{\Delta_1} \dots \sum_{x_r=0}^{\Delta_r} \int_{-\infty}^{\infty} dx_{r+1} \dots \int_{-\infty}^{\infty} dx_p w^{(3)}(x_1 \dots x_p) = \frac{(2\pi)^{p-r-1}}{\Gamma(A + B)} \left[\prod_{k=r+1}^p \Gamma(a_k + b_k) \right] \quad (2.37) \\
& \times \int_{-\infty}^{\infty} dZ \frac{\Gamma(A + iZ) \Gamma(A - iZ) \Gamma(B + iZ) \Gamma(B - iZ) \Gamma(c + iZ) \Gamma(c - iZ) \Gamma(d + iZ) \Gamma(d - iZ)}{\Gamma(2iZ) \Gamma(-2iZ)},
\end{aligned}$$

which is proportional to the single variable integral given by (1.2)–(1.4) with $n = m = 0$. Using this result in (2.37) then confirms the norm as given by (2.33) with $N = 0$. The biorthogonality relations (2.31) are then verified in the same manner as for the purely continuous family.⁴ Also, the restriction $\text{Re}(a_1, a_2 \dots a_r) \neq 0, -1, -2, \dots$ is removable from (2.31)–(2.34).

III. Multivariable biorthogonal Racah polynomials

The mixed type family (2.31), which is a limit case of (2.30), requires that at least one of the a -parameters have a positive real part. We consider a further generalization where all of the a -parameters have negative real parts. Choosing $r = p - 1$ in (2.31) we then define the parameter domain as

$$\begin{aligned} \text{Re}(a_1, a_2 \dots a_p) < 0, \quad \text{Re}(b_1, b_2 \dots b_p, c, d), \quad \text{Re}(A_1^{p-1} + c), \quad \text{Re}(A_1^{p-1} + d) > 0, \\ 2A, \quad A + B, \quad A + c, \quad A + d, \quad \text{Re}(A + A_1^{p-1}), \quad \text{Re}(a_p), \quad a_p + b_p \neq 0, -1, -2 \dots, \\ a_k + b_k = -\Delta_k, \quad k = 1, 2 \dots p - 1, \end{aligned} \quad (3.1)$$

for which the x_p contour C_p , which in (2.31) was on the real axis, is here deformed to pass underneath the increasing sequence $x_p = ia_p + ij_p, j_p = 0, 1, 2 \dots$ and above the decreasing sequence $x_p = -iA - iA_1^{p-1} - ij, j = 0, 1, 2 \dots$ while still passing above and below the remaining decreasing and increasing sequences, respectively.

To evaluate the norm of the weight function one begins with a change of variable from x_p to $Z \equiv x_p + i(A_1^{p-1} + X_1^{p-1})$ giving

$$\begin{aligned} \sum_{x_1=0}^{\Delta_1} \dots \sum_{x_{p-1}=0}^{\Delta_{p-1}} \int_{C_p} dx_p w^{(3)}(x_1 \dots x_p) &= \sum_{x_1=0}^{\Delta_1} \dots \sum_{x_{p-1}=0}^{\Delta_{p-1}} \left[\prod_{k=1}^{p-1} \binom{\Delta_k}{x_k} \right] \int_{C(X_1^{p-1})} dZ \Gamma(A + X_1^{p-1} + iZ) \\ &\times \Gamma(B + \Delta_1^{p-1} - X_1^{p-1} - iZ) \frac{\Gamma(A - iZ) \Gamma(B + iZ) \Gamma(c + iZ) \Gamma(c - iZ) \Gamma(d + iZ) \Gamma(d - iZ)}{\Gamma(2iZ) \Gamma(-2iZ)}, \end{aligned} \quad (3.2)$$

and then the contours $C(X_1^{p-1}), X_1^{p-1} = 0, 1, 2 \dots \Delta_1^{p-1}$ are deformed back to the real axes but with indentations to pass underneath the increasing sequence $Z = iA + ij, j = 0, 1, 2 \dots$ and above the decreasing sequence $Z = -iA - ij, j = 0, 1, 2 \dots$ while still passing above

and below the remaining decreasing and increasing sequences, respectively. The multiple sums can then be brought inside the integral and evaluated by theorem (2.23) resulting in

$$\sum_{x_1=0}^{\Delta_1} \dots \sum_{x_{p-1}=0}^{\Delta_{p-1}} \int_{C_p} dx_p w^{(3)}(x_1 \dots x_p) = \frac{\Gamma(a_p + b_p)}{\Gamma(A + B)} \int_C dZ \Gamma(A + iZ) \Gamma(A - iZ) \quad (3.3)$$

$$\times \frac{\Gamma(B + iZ) \Gamma(B - iZ) \Gamma(c + iZ) \Gamma(c - iZ) \Gamma(d + iZ) \Gamma(d - iZ)}{\Gamma(2iZ) \Gamma(-2iZ)},$$

which is proportional to the single variable integral given by (1.2)–(1.4) with $n = m = 0$. Substituting this result into (3.3) then yields the norm defined by (2.33) with $r = p - 1$ and $N = 0$. The biorthogonality relations (2.31) then once again follow in the same manner as was proved for the purely continuous family⁴ but with the inner product defined above or as expressed in (3.5).

Deforming C_p to the real axis plus closed loops about a finite number of poles and then evaluating the latter by the method of residues allows us to write schematically

$$\int_{C_p} dx_p = \int_{-\infty}^{\infty} dx_p + (2\pi i) \sum_{j_p=0}^{\text{Re}(a_p) + j_p < 0} \text{res}(x_p = ia_p + ij_p) - (2\pi i) \sum_{j=0}^{\text{Re}(A + A_1^{p-1}) + j < 0} \text{res}(x_p = -iA - iA_1^{p-1} - ij), \quad (3.4)$$

and then the inner product of the polynomials can be expressed as

$$P_n^{(3)} \cdot Q_m^{(3)} \equiv \sum_{x_1=0}^{\Delta_1} \dots \sum_{x_{p-1}=0}^{\Delta_{p-1}} \left\{ \int_{-\infty}^{\infty} dx_p + (2\pi i) \sum_{j_p=0}^{\text{Re}(a_p) + j_p < 0} \text{res}(x_p = ia_p + ij_p) \right. \quad (3.5)$$

$$\left. - (2\pi i) \sum_{j=0}^{\text{Re}(A + A_1^{p-1}) + j < 0} \text{res}(x_p = -iA - iA_1^{p-1} - ij) \right\} w^{(3)}(x_1 \dots x_p) P_n^{(3)}(x_1 \dots x_p) Q_m^{(3)}(x_1 \dots x_p),$$

and similarly for $P_n^{(3)} \cdot \overline{P}_m^{(3)}$, $Q_n^{(3)} \cdot \overline{Q}_m^{(3)}$, and $\overline{P}_n^{(3)} \cdot \overline{Q}_m^{(3)}$.

Formula (3.5) has a limit to a purely discrete inner product. Take $a_p + b_p = -\Delta_p + \epsilon$, where Δ_p is a nonnegative integer, divide by $\Gamma(a_p + b_p) = \Gamma(-\Delta_p + \epsilon)$ and then take the limit $\epsilon \rightarrow 0$. The two purely discrete terms survive giving (writing x_p in place of j_p)

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} P_n^{(3)} \cdot Q_m^{(3)} / \Gamma(-\Delta_k + \epsilon) &= (2\pi) \sum_{x_1=0}^{\Delta_1} \dots \sum_{x_p=0}^{\Delta_p} \left[\prod_{k=1}^p \binom{\Delta_k}{x_k} \right] \Gamma(2A + X) \Gamma(B - A - X) \\
&\times \frac{\Gamma(c + A + X) \Gamma(c - A - X) \Gamma(d + A + X) \Gamma(d - A - X)}{\Gamma(2A + 2X) \Gamma(-2A - 2X)} \\
&\times P_n^{(3)}(x_1 \dots x_{p-1}, ia_p + ix_p) Q_m^{(3)}(x_1 \dots x_{p-1}, ia_p + ix_p) \\
&+ (2\pi) \sum_{x_1=0}^{\Delta_1} \dots \sum_{x_{p-1}=0}^{\Delta_{p-1}} \sum_{j=0}^{\Delta + X_1^{p-1}} \left[\prod_{k=1}^{p-1} \binom{\Delta_k}{x_k} \right] \frac{\Delta_p!}{(\Delta + X_1^{p-1} - j)! (j - X_1^{p-1})!} (-1)^{\Delta + \Delta_p} \\
&\times \Gamma(2A + j) \Gamma(B - A + \Delta_1^{p-1} - j) \Gamma(c + A - X_1^{p-1} + j) \Gamma(c - A + X_1^{p-1} - j) \\
&\times \frac{\Gamma(d + A - X_1^{p-1} + j) \Gamma(d - A + X_1^{p-1} - j)}{\Gamma(2A - 2X_1^{p-1} + 2j) \Gamma(-2A + 2X_1^{p-1} - 2j)} \\
&\times P_n^{(3)}(x_1 \dots x_{p-1}, -iA - iA_1^{p-1} - ij) Q_m^{(3)}(x_1 \dots x_{p-1}, -iA - iA_1^{p-1} - ij) ,
\end{aligned} \tag{3.6}$$

and in turn one can show that these two terms are equal. To demonstrate this make a change of summation index in the second term, $j \rightarrow j + X_1^{p-1}$, substitute representations (1.15) and (1.17) for $P_n(x)$ and $Q_m(x)$, and then use theorem (2.23) to evaluate the $x_1, x_2 \dots x_{p-1}$ sums. This leaves only the j sum but if (2.23) is used again with a different choice of parameters this can be re-expressed as a multiple sum that is equal to the first term in (3.6). The inner product can then be taken as twice the first term

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} P_n^{(3)} \cdot Q_m^{(3)} / \Gamma(-\Delta_k + \epsilon) &= (4\pi) \sum_{x_1=0}^{\Delta_1} \dots \sum_{x_p=0}^{\Delta_p} \left[\prod_{k=1}^p \binom{\Delta_k}{x_k} \right] \Gamma(2A + X) \Gamma(B - A - X) \\
&\times \frac{\Gamma(c + A + X) \Gamma(c - A - X) \Gamma(d + A + X) \Gamma(d - A - X)}{\Gamma(2A + 2X) \Gamma(-2A - 2X)} \\
&\times P_n^{(3)}(x_1 \dots x_{p-1}, ia_p + ix_p) Q_m^{(3)}(x_1 \dots x_{p-1}, ia_p + ix_p) ,
\end{aligned} \tag{3.7}$$

and similarly for $\overline{P}_n^{(3)} \cdot \overline{Q}_m^{(3)}$, $P_n^{(3)} \cdot \overline{P}_m^{(3)}$, and $Q_n^{(3)} \cdot \overline{Q}_m^{(3)}$. The biorthogonality relations (2.31) then yield in this limit, with a change in notation,

$$R_n \cdot W_m \equiv \sum_{\{x_k\}} \rho(x_1 \dots x_p) R_n(x_1 \dots x_p) W_m(x_1 \dots x_p), \quad (3.8)$$

$$R_n \cdot W_m = \bar{R}_n \cdot \bar{W}_m = \lambda_n \prod_{k=1}^p \delta_{n_k m_k}, \quad R_n \cdot \bar{R}_m = W_n \cdot \bar{W}_m = 0, \quad \text{if } N \neq M,$$

where $R_n(x)$, $\bar{R}_n(x)$, $W_n(x)$, and $\bar{W}_n(x)$ are the multivariable biorthogonal Racah polynomials and $\{x_k\}$ denotes the p discrete variables $x_1, x_2, \dots, x_p$. After a customary redefinition of the parameters to $\alpha_1, \alpha_2, \dots, \alpha_p, \beta, \delta, \gamma$ ($\alpha \equiv \sum_{k=1}^p \alpha_k$) this family is given by

$$\rho(x_1 \dots x_p) = \left[\prod_{k=1}^p \frac{\Gamma(x_k + \alpha_k + 1)}{\Gamma(\alpha_k + 1) x_k!} \right] \frac{(\gamma + \delta + 1)_X (\gamma/2 + \delta/2 + 3/2)_X}{(\gamma/2 + \delta/2 + 1/2)_X (\gamma + \delta - \alpha - p + 2)_X} \times \frac{(\beta + \delta + 1)_X (\gamma + 1)_X}{(\gamma - \beta + 1)_X (\delta + 1)_X}, \quad (3.9)$$

$$\lambda_n = \left[\prod_{k=1}^p n_k! (\alpha_k + 1)_{n_k} \right] (\beta + 1)_N (\gamma + 1)_N (\alpha - \delta + p)_N (\alpha + \beta - \gamma + p)_N (\beta + \delta + 1)_N \times \frac{(N + \alpha + \beta + p)_N \Gamma(\gamma + \delta - \alpha - p + 2) \Gamma(-\beta - \alpha - p) \Gamma(\gamma - \beta + 1) \Gamma(\delta + 1)}{(\alpha + \beta + p + 1)_{2N} \Gamma(\gamma + \delta + 2) \Gamma(-\beta) \Gamma(\gamma - \beta - \alpha - p + 1) \Gamma(\delta - \alpha - p + 1)}, \quad (3.10)$$

$$R_n(x_1 \dots x_p) = \left[\prod_{k=1}^p (\alpha_k + 1)_{n_k} \right] (\beta + \delta + 1)_N (\gamma + 1)_N \times F_{2:1; \dots; 1}^{2:2; \dots; 2} \left(\begin{matrix} N + \alpha + \beta + p, X + \gamma + \delta + 1 : -n_1, -x_1; \dots; -n_p, -x_p \\ \beta + \delta + 1, \gamma + 1 : \alpha_1 + 1; \dots; \alpha_p + 1 \end{matrix} \right), \quad (3.11)$$

$$\bar{R}_n(x_1 \dots x_p) = \left[\prod_{k=1}^p (\alpha_k + 1)_{n_k} \right] (\alpha + \beta - \gamma + p)_N (\alpha - \delta + p)_N \times F_{2:1; \dots; 1}^{2:2; \dots; 2} \left(\begin{matrix} N + \alpha + \beta + p, -X + \alpha - \gamma - \delta + p - 1 : -n_1, x_1 + \alpha_1 + 1; \dots; -n_p, x_p + \alpha_p + 1 \\ \alpha + \beta - \gamma + p, \alpha - \delta + p : \alpha_1 + 1; \dots; \alpha_p + 1 \end{matrix} \right), \quad (3.12)$$

$$W_n(x_1 \dots x_p) = \left[\prod_{k=1}^p (\alpha_k + 1)_{n_k} \right] (X + \beta + \delta + 1)_N (X + \gamma + 1)_N \times F_{2:1; \dots; 1}^{2:2; \dots; 2} \left(\begin{matrix} -N - \beta, -X + \alpha - \gamma - \delta + p - 1 : -n_1, -x_1; \dots; -n_p, -x_p \\ -N - X - \beta - \delta, -N - X - \gamma : \alpha_1 + 1; \dots; \alpha_p + 1 \end{matrix} \right), \quad (3.13)$$

$$\begin{aligned} \overline{W}_n(x_1 \dots x_p) &= \left[\prod_{k=1}^p (\alpha_k + 1)_{n_k} \right] (-X + \beta - \gamma)_N (-X - \delta)_N \\ &\times F_{2:1;\dots;1}^{2:2;\dots;2} \left(\begin{matrix} -N - \beta, X + \gamma + \delta + 1 : -n_1, x_1 + \alpha_1 + 1; \dots; -n_p, x_p + \alpha_p + 1 \\ -N + X + \gamma - \beta + 1, -N + X + \delta + 1 : \alpha_1 + 1; \dots; \alpha_p + 1 \end{matrix} \right), \end{aligned} \quad (3.14)$$

which according to the present derivation satisfy relations (3.8) when $\alpha_k + 1 = -\Delta_k$, $k = 1, 2 \dots p$, and the $\{x_k\}$ sum is over the region $0 \leq x_k \leq \Delta_k$, $k = 1, 2 \dots p$. As an independent evaluation of the norm of the weight function one uses theorem (2.23) to perform the $x_1 \dots x_{p-1}$ summations at constant X and then (2.15) to evaluate the X sum resulting in (3.10) with $N = 0$. The biorthogonality relations (3.8) can then be independently verified in the same manner as for the purely continuous family⁴ apart from a redefinition of the parameters. In this way one finds that (3.8) are also valid for $\beta + \delta + 1$ or $\gamma + 1 = -\Delta_{p+1}$, where Δ_{p+1} is another nonnegative integer, and in this case the $\{x_k\}$ sum is over the region $0 \leq X \leq \Delta_{p+1}$. Another possibility is to have a combination of $\beta + \delta + 1$ or $\gamma + 1 = -\Delta_{p+1}$ and only a subset of the α -parameters satisfying $\alpha_k + 1 = -\Delta_k$. These different possibilities and the corresponding regions of the $\{x_k\}$ sums are summarized below

$$\begin{aligned} \alpha_k + 1 &= -\Delta_k, \quad k = 1, 2 \dots p, & 0 \leq x_k \leq \Delta_k, \\ \beta + \delta + 1 \text{ or } \gamma + 1 &= -\Delta_{p+1}, & 0 \leq X \leq \Delta_{p+1}, \\ \beta + \delta + 1 \text{ or } \gamma + 1 &= -\Delta_{p+1} \text{ and } \alpha_k + 1 = -\Delta_k, \quad k \in S \subseteq (1, 2 \dots p), \\ & & (0 \leq x_k \leq \Delta_k) \cap (0 \leq X \leq \Delta_{p+1}). \end{aligned} \quad (3.15)$$

In the special case of a single variable all four families of polynomials (3.11)–(3.14) reduce, through a transformation formula satisfied by the ${}_4F_3(1)$ hypergeometric function, to (1.9) and the biorthogonality relations (3.8) reduce to the single orthogonality relation (1.10).

Alternatively one could have chosen the inner product to be twice the second term in (3.6) which leads to a different but within a change of variables equivalent family.

Another inequivalent multivariable generalization of the Racah polynomials has been studied by Gustafson.⁷ These are closely related to the so called $U(n)$ multivariable hypergeometric series introduced by Holman, Biedenharn, and Louck⁸ and Holman,⁹ and

which have been q -extended by Milne.^{10–15} Gustafson's polynomials are associated with a different weight function than (3.9) and are orthogonal as opposed to biorthogonal. The difference in these two families is a reflection of the distinct hypergeometric series to which they are related, the Kampé de Fériet series (1.19) in the present case and the $U(n)$ series in Gustafson's case.

IV. Multivariable biorthogonal Hahn and dual Hahn polynomials

In analogy with the single variable case there is an interesting limit to a family of multivariable Hahn polynomials. These are obtained upon dividing the Racah polynomials by δ^N and taking the limit $\delta \rightarrow \infty$. The first two families $R_n(x)$ and $\bar{R}_n(x)$ limit to the same Hahn polynomials $H_n(x)$ while $W_n(x)$ and $\bar{W}_n(x)$ limit to the same biorthogonal counterparts $\bar{H}_n(x)$,

$$\begin{aligned} \lim_{\delta \rightarrow \infty} \delta^{-N} R_n(x) &= \lim_{\delta \rightarrow \infty} \delta^{-N} \bar{R}_n(x) = \left[\prod_{k=1}^p (\alpha_k + 1)_{n_k} \right] (\gamma + 1)_N H_n(x) , \\ \lim_{\delta \rightarrow \infty} \delta^{-N} W_n(x) &= \lim_{\delta \rightarrow \infty} \delta^{-N} \bar{W}_n(x) = \left[\prod_{k=1}^p (\alpha_k + 1)_{n_k} \right] (\gamma + 1)_N \bar{H}_n(x) , \\ \lim_{\delta \rightarrow \infty} \rho(x) &= \left[\prod_{k=1}^p \frac{\Gamma(x_k + \alpha_k + 1)}{\Gamma(\alpha_k + 1) x_k!} \right] \frac{(\gamma + 1)_X}{(\gamma - \beta + 1)_X} , \end{aligned} \quad (4.1)$$

where $H_n(x)$ and $\bar{H}_n(x)$ are given by

$$\begin{aligned} H_n(x) &= F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} N + \alpha + \beta + p : -n_1, -x_1; \dots; -n_p, -x_p \\ \gamma + 1 : \alpha_1 + 1; \dots; \alpha_p + 1 \end{matrix} \right) , \\ &= (-1)^N \frac{(\alpha + \beta - \gamma + p)_N}{(\gamma + 1)_N} F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} N + \alpha + \beta + p : -n_1, x_1 + \alpha_1 + 1; \dots; -n_p, x_p + \alpha_p + 1 \\ \alpha + \beta - \gamma + p : \alpha_1 + 1; \dots; \alpha_p + 1 \end{matrix} \right) , \\ \bar{H}_n(x) &= \frac{(X + \gamma + 1)_N}{(\gamma + 1)_N} F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} -N - \beta : -n_1, -x_1; \dots; -n_p, -x_p \\ -N - X - \gamma : \alpha_1 + 1; \dots; \alpha_p + 1 \end{matrix} \right) , \\ &= (-1)^N \frac{(-X + \beta - \gamma)_N}{(\gamma + 1)_N} F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} -N - \beta : -n_1, x_1 + \alpha_1 + 1; \dots; -n_p, x_p + \alpha_p + 1 \\ -N + X + \gamma - \beta + 1 : \alpha_1 + 1; \dots; \alpha_p + 1 \end{matrix} \right) . \end{aligned} \quad (4.2)$$

These satisfy the biorthogonality relations

$$\begin{aligned}
H_n \cdot \bar{H}_m &\equiv \sum_{\{x_k\}} \left[\prod_{k=1}^p \frac{\Gamma(x_k + \alpha_k + 1)}{\Gamma(\alpha_k + 1) x_k!} \right] \frac{(\gamma + 1)_X}{(\gamma - \beta + 1)_X} H_n(x) \bar{H}_m(x) , \\
H_n \cdot \bar{H}_m &= \lambda_n \prod_{k=1}^p \delta_{n_k m_k} , \quad H_n \cdot H_m = \bar{H}_n \cdot \bar{H}_m = 0 \quad \text{if } N \neq M , \\
\lambda_n &= (-1)^N \left[\prod_{k=1}^p n_k! / (\alpha_k + 1)_{n_k} \right] (\alpha + \beta - \gamma + p)_N \frac{(\beta + 1)_N (N + \alpha + \beta + p)_N}{(\gamma + 1)_N (\alpha + \beta + p + 1)_{2N}} \\
&\quad \times \frac{\Gamma(-\beta - \alpha - p) \Gamma(\gamma - \beta + 1)}{\Gamma(-\beta) \Gamma(\gamma - \beta - \alpha - p + 1)} ,
\end{aligned} \tag{4.3}$$

where the $\{x_k\}$ sum is over one of the regions

$$\begin{aligned}
\alpha_k + 1 &= -\Delta_k, \quad k = 1, 2 \dots p, & 0 \leq x_k \leq \Delta_k , \\
\gamma + 1 &= -\Delta_{p+1} , & 0 \leq X \leq \Delta_{p+1} , \\
\gamma + 1 &= -\Delta_{p+1} \text{ and } \alpha_k + 1 = -\Delta_k, \quad k \in S \subseteq (1, 2 \dots p) , \\
&& (0 \leq x_k \leq \Delta_k) \cap (0 \leq X \leq \Delta_{p+1}) .
\end{aligned} \tag{4.4}$$

These polynomials have already been discussed in detail¹⁶ for the specific case $\gamma + 1 = -\Delta_{p+1}$ and the equivalence of each pair of representations in (4.2) has been demonstrated. Among other interesting properties these polynomials possess discrete Rodrigues formulas.

Another important limit not yet studied are the multivariable biorthogonal dual Hahn polynomials. These result upon dividing the Racah polynomials by β^N and taking the limit $\beta \rightarrow \infty$. In this case $R_n(x)$ and $\bar{W}_n(x)$ limit to the same dual Hahn family $D_n(x)$ while $\bar{R}_n(x)$ and $W_n(x)$ limit to the same biorthogonal counterparts $\bar{D}_n(x)$,

$$\begin{aligned}
\lim_{\beta \rightarrow \infty} \beta^{-N} R_n(x) &= \lim_{\beta \rightarrow \infty} \beta^{-N} \bar{W}_n(x) = \left[\prod_{k=1}^p (\alpha_k + 1)_{n_k} \right] (\gamma + 1)_N D_n(x) , \\
\lim_{\beta \rightarrow \infty} \beta^{-N} \bar{R}_n(x) &= \lim_{\beta \rightarrow \infty} \beta^{-N} W_n(x) = \left[\prod_{k=1}^p (\alpha_k + 1)_{n_k} \right] (\gamma + 1)_N \bar{D}_n(x) , \\
\lim_{\beta \rightarrow \infty} \rho(x) &= \left[\prod_{k=1}^p \frac{\Gamma(x_k + \alpha_k + 1)}{\Gamma(\alpha_k + 1) x_k!} \right] \frac{(\gamma + \delta + 1)_X (\gamma/2 + \delta/2 + 3/2)_X (\gamma + 1)_X}{(\gamma/2 + \delta/2 + 1/2)_X (\gamma + \delta - \alpha - p + 2)_X (\delta + 1)_X} (-1)^X ,
\end{aligned} \tag{4.5}$$

where $D_n(x)$ and $\overline{D}_n(x)$ are given by

$$\begin{aligned}
D_n(x) &= F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} X + \gamma + \delta + 1 : -n_1, -x_1; \dots; -n_p, -x_p \\ \gamma + 1 : \alpha_1 + 1; \dots; \alpha_p + 1 \end{matrix} \right) \\
&= \frac{(-X - \delta)_N}{(\gamma + 1)_N} F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} X + \gamma + \delta + 1 : -n_1, x_1 + \alpha_1 + 1; \dots; -n_p, x_p + \alpha_p + 1 \\ -N + X + \delta + 1 : \alpha_1 + 1; \dots; \alpha_p + 1 \end{matrix} \right), \\
\overline{D}_n(x) &= \frac{(\alpha - \delta + p)_N}{(\gamma + 1)_N} F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} -X + \alpha - \gamma - \delta + p - 1 : -n_1, x_1 + \alpha_1 + 1; \dots; -n_p, x_p + \alpha_p + 1 \\ \alpha - \delta + p : \alpha_1 + 1; \dots; \alpha_p + 1 \end{matrix} \right) \\
&= \frac{(X + \gamma + 1)_N}{(\gamma + 1)_N} F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} -X + \alpha - \gamma - \delta + p - 1 : -n_1, -x_1; \dots; -n_p, -x_p \\ -N - X - \gamma : \alpha_1 + 1; \dots; \alpha_p + 1 \end{matrix} \right).
\end{aligned} \tag{4.6}$$

The four biorthogonality relations satisfied by the Racah polynomials, in this limit, imply the single relation

$$\begin{aligned}
&\sum_{\{x_k\}} \left[\prod_{k=1}^p \frac{\Gamma(x_k + \alpha_k + 1)}{\Gamma(\alpha_k + 1) x_k!} \right] \frac{(\gamma + \delta + 1)_X (\gamma/2 + \delta/2 + 3/2)_X (\gamma + 1)_X (-1)^X}{(\gamma/2 + \delta/2 + 1/2)_X (\gamma + \delta - \alpha - p + 2)_X (\delta + 1)_X} D_n(x) \overline{D}_m(x) \\
&= \left[\prod_{k=1}^p n_k! / (\alpha_k + 1)_{n_k} \right] \frac{(\alpha - \delta + p)_N}{(\gamma + 1)_N} \frac{\Gamma(\gamma + \delta - \alpha - p + 2) \Gamma(\delta + 1)}{\Gamma(\gamma + \delta + 2) \Gamma(\delta - \alpha - p + 1)} \prod_{k=1}^p \delta_{n_k m_k},
\end{aligned} \tag{4.7}$$

where the $\{x_k\}$ sum is again over one of the regions given in (4.4).

To demonstrate the equivalence of the two representations of $D_n(x)$ we first use (2.23) do deduce the identity

$$\frac{\Gamma(X + \delta + 1)}{\Gamma(X + \delta + 1 - N + J)} = \sum_{\{\ell_k\}} \left[\prod_{k=1}^p \binom{n_k - j_k}{\ell_k} \right] \frac{\Gamma(N - L + X + \delta + \gamma + 1)}{\Gamma(J + X + \delta + \gamma + 1)} \frac{\Gamma(L - N - \gamma)}{\Gamma(-N - \gamma)}, \tag{4.8}$$

where j_k denotes the summation indices in the Kampé de Fériet hypergeometric series (1.19). This is substituted into the second expression for $D_n(x)$, the $\{j_k\}$ and $\{\ell_k\}$ sums are interchanged, and (2.23) is used again to evaluate the former. Reversing the remaining $\{\ell_k\}$ sums by transforming $\ell_k \rightarrow n_k - \ell_k$ then yields the first expression for $D_n(x)$. For the $\overline{D}_n(x)$ representations one begins instead with the following identity also deduced from (2.23)

$$\frac{\Gamma(N + \alpha - \delta + p)}{\Gamma(J + \alpha - \delta + p)} = \sum_{\{\ell_k\}} \left[\prod_{k=1}^p \binom{n_k - j_k}{\ell_k} \right] \frac{\Gamma(N - L + \alpha + p - X - \delta - \gamma - 1)}{\Gamma(J + \alpha + p - X - \delta - \gamma - 1)} \quad (4.9)$$

$$\times \frac{\Gamma(L + X + \gamma + 1)}{\Gamma(X + \gamma + 1)}.$$

Substituting this into the first expression for $\overline{D}_n(x)$ and proceeding as before then yields the second representation.

There are also mixed type counterparts to these Hahn and dual Hahn polynomials which are obtained as limit cases of the mixed type Wilson polynomials. In (2.19)–(2.22) put

$$\begin{aligned} a_k &= a'_k + \tfrac{1}{2}i w_k, \quad b_k = b'_k - \tfrac{1}{2}i w_k, \quad k = 1, 2 \dots p, \\ x_k &= x'_k \quad k = 1, 2 \dots r, \quad x_k = x'_k - \tfrac{1}{2}w_k, \quad k = r + 1 \dots p, \\ c &= c' + \tfrac{1}{2}i W, \quad d = d' - \tfrac{1}{2}i W, \quad W \equiv \sum_{k=1}^p w_k, \end{aligned} \quad (4.10)$$

divide the polynomials by $(iW)^N$ and then take the limit $W \rightarrow \infty$ (and drop the primes). The mixed Wilson families $P_n^{(2)}(x)$ and $\overline{P}_n^{(2)}(x)$ both limit to the same mixed Hahn polynomials while $Q_n^{(2)}(x)$ and $\overline{Q}_n^{(2)}(x)$ limit to the same biorthogonal counterparts

$$\begin{aligned} \lim_{W \rightarrow \infty} (iW)^{-N} P_n^{(2)}(x) &= \lim_{W \rightarrow \infty} (iW)^{-N} \overline{P}_n^{(2)}(x) = H_n^{(2)}(x), \\ \lim_{W \rightarrow \infty} (iW)^{-N} Q_n^{(2)}(x) &= \lim_{W \rightarrow \infty} (iW)^{-N} \overline{Q}_n^{(2)}(x) = \overline{H}_n^{(2)}(x), \end{aligned} \quad (4.11)$$

$$\begin{aligned} \lim_{W \rightarrow \infty} w^{(2)}(x) &= \left[\prod_{k=1}^r \binom{\Delta_k}{x_k} \right] \left[\prod_{k=r+1}^p \Gamma(a_k + i x_k) \Gamma(b_k - i x_k) \right] \\ &\quad \times \Gamma(A_1^r + d + X_1^r - i X_{r+1}^p) \Gamma(B_1^r + c + \Delta_1^r - X_1^r + i X_{r+1}^p), \end{aligned}$$

where $H_n^{(2)}(x)$ and $\overline{H}_n^{(2)}(x)$ are given by

$$\begin{aligned} H_n^{(2)}(x) &= \left[\prod_{k=1}^p (a_k + b_k)_{n_k} \right] (A + d)_N F_{1:1; \dots; 1}^{1:2; \dots; 2} \left(\begin{matrix} N + A + B + c + d - 1 : -n_1, A_1^r + d \\ A + d : a_1 + b_1; \end{matrix} \right. \\ &\quad \left. + X_1^r - i X_{r+1}^p; -n_2, -x_2; \dots; -n_r, -x_r; -n_{r+1}, a_{r+1} + i x_{r+1}; \dots; -n_p, a_p + i x_p \right. \\ &\quad \left. - \Delta_2; \dots; -\Delta_r; a_{r+1} + b_{r+1}; \dots; a_p + b_p \right) \end{aligned}$$

$$\begin{aligned}
&= (-1)^N \left[\prod_{k=1}^p (a_k + b_k)_{n_k} \right] (B + c)_N F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} N + A + B + c + d - 1 : -n_1, B_1^r + c + \Delta_1^r \\ B + c : a_1 + b_1; \end{matrix} \right. \\
&\quad \left. -X_1^r + iX_{r+1}^p; -n_2, -\Delta_2 + x_2; \dots; -n_r, -\Delta_r + x_r; -n_{r+1}, b_{r+1} - ix_{r+1}; \dots; -n_p, b_p - ix_p \right. \\
&\quad \left. -\Delta_2; \dots; -\Delta_r; a_{r+1} + b_{r+1}; \dots; a_p + b_p \right), \tag{4.12} \\
\bar{H}_n^{(2)}(x) &= (-1)^N \left[\prod_{k=1}^p (a_k + b_k)_{n_k} \right] (-\Delta_1 + x_1)_N F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} \Delta_1 - N + 1 : -n_1, B_1^r + c + \Delta_1^r \\ \Delta_1 - N - x_1 + 1 : a_1 + b_1; \end{matrix} \right. \\
&\quad \left. -X_1^r + iX_{r+1}^p; -n_2, -\Delta_2 + x_2; \dots; -n_r, -\Delta_r + x_r; -n_{r+1}, b_{r+1} - ix_{r+1}; \dots; -n_p, b_p - ix_p \right. \\
&\quad \left. -\Delta_2; \dots; -\Delta_r; a_{r+1} + b_{r+1}; \dots; a_p + b_p \right) \\
&= \left[\prod_{k=1}^p (a_k + b_k)_{n_k} \right] (-x_1)_N F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} \Delta_1 - N + 1 : -n_1, A_1^r + d + X_1^r - iX_{r+1}^p; \\ -N + x_1 + 1 : a_1 + b_1; \end{matrix} \right. \\
&\quad \left. -n_2, -x_2; \dots; -n_r, -x_r; -n_{r+1}, a_{r+1} + ix_{r+1}; \dots; -n_p, a_p + ix_p \right. \\
&\quad \left. -\Delta_2; \dots; -\Delta_r; a_{r+1} + b_{r+1}; \dots; a_p + b_p \right),
\end{aligned}$$

with $c + d = -\Delta_1$ and $a_k + b_k = -\Delta_k$, $k = 2, 3 \dots r$. These satisfy

$$\begin{aligned}
H_n^{(2)} \cdot \bar{H}_m^{(2)} &\equiv \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_{r+1} \sum_{x_r=0}^{\Delta_r} \dots \sum_{x_1=0}^{\Delta_1} \left[\prod_{k=1}^r \binom{\Delta_k}{x_k} \right] \left[\prod_{k=r+1}^p \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right] \\
&\quad \times \Gamma(A_1^r + d + X_1^r - iX_{r+1}^p) \Gamma(B_1^r + c + \Delta_1^r - X_1^r + iX_{r+1}^p) H_n^{(2)}(x) \bar{H}_m^{(2)}(x), \\
H_n^{(2)} \cdot \bar{H}_m^{(2)} &= \lambda_n^{(2)} \prod_{k=1}^p \delta_{n_k m_k}, \quad H_n^{(2)} \cdot H_m^{(2)} = \bar{H}_n^{(2)} \cdot \bar{H}_m^{(2)} = 0 \quad \text{if } N \neq M, \tag{4.13} \\
\lambda_n^{(2)} &= (2\pi)^{p-r} \Gamma(n_1 + a_1 + b_1) n_1! \left[\prod_{k=2}^r (-\Delta_k)_{n_k} n_k! \right] \left[\prod_{k=r+1}^p \Gamma(n_k + a_k + b_k) n_k! \right] \\
&\quad \times \frac{\Delta_1!}{(\Delta_1 - N)!} \frac{\Gamma(N + A + d) \Gamma(N + B + c)}{(2N + A + B + c + d - 1) \Gamma(N + A + B + c + d - 1)},
\end{aligned}$$

and the equivalence of each pair of representations in (4.12) follows from (4.2) upon a redefinition of the parameters and a change of variables.

Another mixed type Hahn family is obtained from (2.31)–(2.34) using the same limiting procedure. That is, transform the parameters and variables as in (4.10) and then take the limit $W \rightarrow \infty$. This yields,

$$\begin{aligned}\lim_{W \rightarrow \infty} (iW)^{-N} P_n^{(3)}(x) &= \lim_{W \rightarrow \infty} (iW)^{-N} \overline{P}_n^{(3)}(x) = H_n^{(3)}(x), \\ \lim_{W \rightarrow \infty} (iW)^{-N} Q_n^{(3)}(x) &= \lim_{W \rightarrow \infty} (iW)^{-N} \overline{Q}_n^{(3)}(x) = \overline{H}_n^{(3)}(x),\end{aligned}\tag{4.14}$$

$$\begin{aligned}\lim_{W \rightarrow \infty} w^{(3)}(x) / \Gamma(A' + c' + iW) \Gamma(B' + d' - iW) &= \left[\prod_{k=1}^r \binom{\Delta_k}{x_k} \right] \left[\prod_{k=r+1}^p \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right] \\ &\times \Gamma(A_1^r + d + X_1^r - iX_{r+1}^p) \Gamma(B_1^r + c + \Delta_1^r - X_1^r + iX_{r+1}^p),\end{aligned}$$

where the polynomials are given by

$$\begin{aligned}H_n^{(3)}(x) &= \left[\prod_{k=1}^p (a_k + b_k)_{n_k} \right] (A + d)_N F_{1:1; \dots; 1}^{1:2; \dots; 2} \left(\begin{matrix} N + A + B + c + d - 1 : -n_1, -x_1; \\ A + d : -\Delta_1; \\ -n_2, -x_2; \dots; -n_r, -x_r; -n_{r+1}, a_{r+1} + ix_{r+1}; \dots; -n_p, a_p + ix_p \\ -\Delta_2; \dots; -\Delta_r; a_{r+1} + b_{r+1}; \dots; a_p + b_p \end{matrix} \right) \\ &= (-1)^N \left[\prod_{k=1}^p (a_k + b_k)_{n_k} \right] (B + c)_N F_{1:1; \dots; 1}^{1:2; \dots; 2} \left(\begin{matrix} N + A + B + c + d - 1 : -n_1, -\Delta_1 + x_1; \\ B + c : -\Delta_1; \\ -n_2, -\Delta_2 + x_2; \dots; -n_r, -\Delta_r + x_r; -n_{r+1}, b_{r+1} - ix_{r+1}; \dots; -n_p, b_p - ix_p \\ -\Delta_2; \dots; -\Delta_r; a_{r+1} + b_{r+1}; \dots; a_p + b_p \end{matrix} \right), \\ &\overline{H}_n^{(3)}(x) = \left[\prod_{k=1}^p (a_k + b_k)_{n_k} \right] (d + A_1^r + X_1^r - iX_{r+1}^p)_N F_{1:1; \dots; 1}^{1:2; \dots; 2} \left(\begin{matrix} -N - c - d + 1 : \\ -N - d - A_1^r - X_1^r \\ -n_1, -x_1; \dots; -n_r, -x_r; -n_{r+1}, a_{r+1} + ix_{r+1}; \dots; -n_p, a_p + ix_p \\ +iX_{r+1}^p + 1 : -\Delta_1; \dots; -\Delta_r; a_{r+1} + b_{r+1}; \dots; a_p + b_p \end{matrix} \right) \\ &= (-1)^N \left[\prod_{k=1}^p (a_k + b_k)_{n_k} \right] (c - A_1^r - X_1^r + iX_{r+1}^p)_N F_{1:1; \dots; 1}^{1:2; \dots; 2} \left(\begin{matrix} -N - c - d + 1 : \\ -N - c + A_1^r + X_1^r \\ -n_1, -\Delta_1 + x_1; \dots; -n_r, -\Delta_r + x_r; -n_{r+1}, b_{r+1} - ix_{r+1}; \dots; -n_p, b_p - ix_p \\ -iX_{r+1}^p + 1 : -\Delta_1; \dots; -\Delta_r; a_{r+1} + b_{r+1}; \dots; a_p + b_p \end{matrix} \right),\end{aligned}\tag{4.15}$$

with $a_k + b_k = -\Delta_k$, $k = 1, 2 \dots r$, and these satisfy

$$\begin{aligned}
H_n^{(3)} \cdot \overline{H}_m^{(3)} &\equiv \int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_{r+1} \sum_{x_r=0}^{\Delta_r} \dots \sum_{x_1=0}^{\Delta_1} \left[\prod_{k=1}^r \binom{\Delta_k}{x_k} \right] \left[\prod_{k=r+1}^p \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right] \\
&\quad \times \Gamma(A_1^r + d + X_1^r - iX_{r+1}^p) \Gamma(B_1^r + c + \Delta_1^r - X_1^r + iX_{r+1}^p) H_n^{(3)}(x) \overline{H}_m^{(3)}(x) , \\
H_n^{(3)} \cdot \overline{H}_m^{(3)} &= \lambda_n^{(3)} \prod_{k=1}^p \delta_{n_k m_k} , \quad H_n^{(3)} \cdot H_m^{(3)} = \overline{H}_n^{(3)} \cdot \overline{H}_m^{(3)} = 0 \quad \text{if } N \neq M , \quad (4.16) \\
\lambda_n^{(3)} &= (2\pi)^{p-r} \left[\prod_{k=1}^r (-\Delta_k)_{n_k} n_k! \right] \left[\prod_{k=r+1}^p \Gamma(n_k + a_k + b_k) n_k! \right] (-1)^N \\
&\quad \times \frac{\Gamma(N + A + d) \Gamma(N + B + c) \Gamma(N + c + d)}{(2N + A + B + c + d - 1) \Gamma(N + A + B + c + d - 1)} .
\end{aligned}$$

Notice that the weight function in (4.16) is equivalent to that of the first mixed type Hahn family (4.13); the two families of polynomials are however distinct. The equivalence of each pair of representations in (4.15) again follows from (4.2) upon a redefinition of the parameters and a change of variables.

The analogous mixed type dual Hahn family is obtained from (2.31)–(2.34) in the limit $d \rightarrow \infty$. In this case $P_n^{(3)}(x)$ and $\overline{Q}_n^{(3)}(x)$ limit to the same dual Hahn polynomials while $\overline{P}_n^{(3)}(x)$ and $Q_n^{(3)}(x)$ limit to the same biorthogonal counterparts

$$\begin{aligned}
\lim_{d \rightarrow \infty} d^{-N} P_n^{(3)}(x) &= \lim_{d \rightarrow \infty} d^{-N} \overline{Q}_n^{(3)}(x) = D_n^{(3)}(x) , \\
\lim_{d \rightarrow \infty} d^{-N} \overline{P}_n^{(3)}(x) &= \lim_{d \rightarrow \infty} d^{-N} Q_n^{(3)}(x) = \overline{D}_n^{(3)}(x) , \quad (4.17) \\
\lim_{d \rightarrow \infty} \frac{w^{(3)}(x)}{\Gamma^2(d)} &= \rho^{(3)}(x) ,
\end{aligned}$$

where

$$\begin{aligned}
\rho^{(3)}(x) &= \left[\prod_{k=1}^r \binom{\Delta_k}{x_k} \right] \left[\prod_{k=r+1}^p \Gamma(a_k + ix_k) \Gamma(b_k - ix_k) \right] \Gamma(A + A_1^r + X_1^r - iX_{r+1}^p) \\
&\quad \times \Gamma(B + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p) \frac{\Gamma(c + B_1^r + \Delta_1^r - X_1^r + iX_{r+1}^p) \Gamma(c + A_1^r + X_1^r - iX_{r+1}^p)}{\Gamma(2B_1^r + 2\Delta_1^r - 2X_1^r + 2iX_{r+1}^p) \Gamma(2A_1^r + 2X_1^r - 2iX_{r+1}^p)} ,
\end{aligned}$$

$$\begin{aligned}
D_n^{(3)}(x) &= \left[\prod_{k=1}^r (-\Delta_k)_{n_k} \right] \left[\prod_{k=r+1}^p (a_k + b_k)_{n_k} \right] (A + c)_N F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} A + A_1^r + X_1^r - iX_{r+1}^p : \\ A + c : \\ -n_1, -x_1; \dots; -n_r, -x_r; -n_{r+1}, a_{r+1} + ix_{r+1}; \dots; -n_p, a_p + ix_p \\ -\Delta_1; \dots; -\Delta_r; a_{r+1} + b_{r+1}; \dots; a_p + b_p \end{matrix} \right) \\
&= \left[\prod_{k=1}^r (-\Delta_k)_{n_k} \right] \left[\prod_{k=r+1}^p (a_k + b_k)_{n_k} \right] (c - A_1^r - X_1^r + iX_{r+1}^p)_N F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} A + A_1^r + X_1^r \\ -N - c + A_1^r \\ -iX_{r+1}^p : -n_1, -\Delta_1 + x_1; \dots; -n_r, -\Delta_r + x_r; -n_{r+1}, b_{r+1} - ix_{r+1}; \dots; -n_p, b_p - ix_p \\ +X_1^r - iX_{r+1}^p + 1 : -\Delta_1; \dots; -\Delta_r; a_{r+1} + b_{r+1}; \dots; a_p + b_p \end{matrix} \right), \tag{4.18}
\end{aligned}$$

$$\begin{aligned}
\overline{D}_n^{(3)}(x) &= \left[\prod_{k=1}^r (-\Delta_k)_{n_k} \right] \left[\prod_{k=r+1}^p (a_k + b_k)_{n_k} \right] (B + c)_N F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} B - A_1^r - X_1^r + iX_{r+1}^p : \\ B + c : \\ -n_1, -\Delta_1 + x_1; \dots; -n_r, -\Delta_r + x_r; -n_{r+1}, b_{r+1} - ix_{r+1}; \dots; -n_p, b_p - ix_p \\ -\Delta_1; \dots; -\Delta_r; a_{r+1} + b_{r+1}; \dots; a_p + b_p \end{matrix} \right) \\
&= \left[\prod_{k=1}^r (-\Delta_k)_{n_k} \right] \left[\prod_{k=r+1}^p (a_k + b_k)_{n_k} \right] (c + A_1^r + X_1^r - iX_{r+1}^p)_N F_{1:1;\dots;1}^{1:2;\dots;2} \left(\begin{matrix} B - A_1^r - X_1^r \\ -N - c - A_1^r \\ +iX_{r+1}^p : -n_1, -x_1; \dots; -n_r, -x_r; \dots; -n_{r+1}, a_{r+1} + ix_{r+1}; \dots; -n_p, a_p + ix_p \\ -X_1^r + iX_{r+1}^p + 1 : -\Delta_1; \dots; -\Delta_r; a_{r+1} + b_{r+1}; \dots; a_p + b_p \end{matrix} \right),
\end{aligned}$$

with $a_k + b_k = -\Delta_k$, $k = 1, 2 \dots r$. These satisfy the biorthogonality relation

$$\begin{aligned}
\int_{-\infty}^{\infty} dx_p \dots \int_{-\infty}^{\infty} dx_{r+1} \sum_{x_r=0}^{\Delta_r} \dots \sum_{x_1=0}^{\Delta_1} \rho^{(3)}(x) D_n^{(3)}(x) \overline{D}_m^{(3)}(x) &= 2(2\pi)^{p-r} \left[\prod_{k=1}^r (-\Delta_k)_{n_k} n_k! \right] \\
&\times \left[\prod_{k=r+1}^p \Gamma(n_k + a_k + b_k) n_k! \right] \Gamma(N + A + c) \Gamma(N + B + c) \prod_{k=1}^p \delta_{n_k m_k}, \tag{4.19}
\end{aligned}$$

and the equivalence of each pair of representations in (4.18) follows from (4.6) upon a redefinition of the parameters and a change of variables.

The purely continuous multivariable biorthogonal Hahn¹⁷ and dual Hahn⁴ polynomials are also known.

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