**ELECTRO-OPTIC MODULATION IN
AN ARBITRARY CROSS-SECTION WAVEGUIDE**

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Abstract

The electromagnetic field of an optical wave of frequency ω traveling through a long, electro-optic waveguide of arbitrary cross section and composition in the presence of an arbitrary external field of strength E^{ext} and radio frequency Ω , is derived by a perturbative argument, assuming $\Omega/\omega \ll 1$ and $|r_{ijk}E^{ext}| \ll 1$, where r_{ijk} are the linear electro-optic coefficients. An idealized model is solved exactly in the context of rigorous perturbation theory, and the solution is shown to be valid whenever $(r_{ijk}E^{ext})^2\omega/\Omega \ll 1$, without restriction on $r_{ijk}E^{ext}\omega/\Omega$. The arbitrary cross-section formula, when evaluated in the model case, agrees exactly with the rigorous result, so it is argued that the arbitrary cross-section formula should also be valid whenever $(r_{ijk}E^{ext})^2\omega/\Omega \ll 1$.

I. INTRODUCTION

The electro-optic effect can be exploited to effect a modulation of an optical wave passing through a nonlinear-optical waveguide by a radio-frequency external field. In general, the effect of the external field is to produce a phase shift $\varphi(x,t)$ of the optical wave, where x is the distance along the axial direction of the waveguide [1]. The phase shift depends on the electro-optic coefficients r_{ijk} of the medium, the strength E^{ext} and frequency Ω of the external field and the frequency ω of the optical wave. The phase modulation thus obtained is converted to amplitude modulation by including the waveguide in a larger optical system--for example, in a Mach-Zehnder interferometer--where it can be observed and utilized in various applications [2]. The quantity of interest in the analysis of such a system is $\varphi(L,t)$, where L is the length of the waveguide.

Waveguides used as modulators are often multi-layer dielectric stacks or optical fibers with at least one electro-optically active layer. Calculations of $\varphi(x,t)$ in such waveguides generally take into account only heuristically their geometry and composition. For example, formulas strictly valid in homogeneous media of infinite extent are applied with very little modification to waveguides only a few microns thick [3]. The region of validity of these formulas, in the parameter space of r_{ijk} , E^{ext} , Ω and ω , is unknown.

It is the purpose of this paper to present a general derivation of $\varphi(x,t)$ for an arbitrary cross-section waveguide and to discover the region of validity of the resultant formula.

The plan of the work is as follows. First, in section II, we make use of a perturbative argument to derive a general formula for $\varphi(x,t)$ in the arbitrary cross-section case. Since there is an important step in this derivation whose validity is difficult to assess, and since the perturbation theory we use is not necessarily systematic, we turn, in section III, to the rigorous solution of a model waveguide, one consisting of a nonlinear optical planar slab sandwiched between two perfectly-conducting planes. For this system, a rigorous perturbation theory can be constructed, and we show that the solution for $\varphi(x,t)$ so obtained is valid whenever $(r_{ijk}E^{ext})^2\omega/\Omega \ll 1$ (we assume throughout, in addition, that $\Omega/\omega \ll 1$ and $|r_{ijk}E^{ext}| \ll 1$.) Finally, in section IV, we demonstrate that the general formula of section II, when evaluated in the model case, gives precisely the same $\varphi(x,t)$ as that derived by the rigorous theory. Arguing that the region of validity of the general formula should not be geometry dependent, we arrive at an informal proof that the general formula is also valid whenever $(r_{ijk}E^{ext})^2\omega/\Omega \ll 1$.

II. GENERAL WAVEGUIDE MODULATOR

In this section we consider the arbitrary-cross-section waveguide pictured in Fig. 1. The axial direction is x , and we designate the two perpendicular directions by a vector $\vec{r}_\perp$. The waveguide is composed of material of (linear) dielectric tensor $\vec{\epsilon}(\vec{r}_\perp)$ at frequency ω . In general, $\vec{\epsilon}(\vec{r}_\perp)$ is taken to be complex in order to include the effects of absorption. We assume that the composition of the waveguide is x -independent [i.e., cross sections taken perpendicular to the axis at $x = x_1$ and $x = x_2$ are identical.] At least one of the dielectric regions is assumed to be a material possessing a non-zero second-order susceptibility tensor $\chi^{(2)}(\omega; \omega - \omega', \omega') \equiv \chi(\omega; \omega - \omega', \omega')$. We apply an arbitrary external electric field $\vec{E}^{ext}(\vec{r}_\perp; x, t)$, where we separate the x and $\vec{r}_\perp$ dependence explicitly, for reasons that will become clear presently. We assume that the external field oscillates at frequency Ω , the modulation frequency.

We introduce into the waveguide at $x = 0$ a plane wave of frequency $\omega \gg \Omega$ and couple the wave out again at $x = L$. It is the purpose of this section to calculate the electric field $\vec{E}(\vec{r}, t)$ of the wave at an arbitrary point x inside the waveguide, where $\vec{r} = (\vec{r}_\perp, x)$.

vec- To begin, we define [4] the electric displacement $\vec{D}(\vec{r}, t)$ tor:

$$\vec{D}(\vec{r}, t) = \vec{\epsilon}(\vec{r}_\perp) \cdot \vec{E}(\vec{r}, t) + 4\pi \vec{P}'(\vec{r}, t), \quad (2.1a)$$

where $\vec{P}'(\vec{r}, t)$ is the nonlinear polarization, given by

$$\vec{P}'(\vec{r}, t) = 2 \vec{\chi}(\vec{r}_\perp) : \vec{E}^{e \times t}(\vec{r}_\perp; x, t) \vec{E}(\vec{r}, t) , \quad (2.1b)$$

and $\vec{\chi}(\vec{r}_\perp)$ is the second-order susceptibility tensor at frequency ω .

Maxwell's equations are

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= - \frac{1}{c} \frac{\partial}{\partial t} \vec{H} , \\ \vec{\nabla} \times \vec{H} &= \frac{1}{c} \frac{\partial}{\partial t} \vec{D} . \end{aligned} \quad (2.2)$$

In addition to $\vec{E}$ and $\vec{H}$, we define the fields $\vec{e}$ and $\vec{h}$. The latter are the limits of former as $\vec{E}^{e \times t} \rightarrow 0$. Thus, the equations satisfied by $\vec{e}(\vec{r}, t)$ and $\vec{h}(\vec{r}, t)$ are

$$\begin{aligned} \vec{\nabla} \times \vec{e} &= - \frac{1}{c} \frac{\partial}{\partial t} \vec{h} , \\ \vec{\nabla} \times \vec{h} &= \frac{1}{c} \frac{\partial}{\partial t} \vec{\epsilon} \cdot \vec{e} . \end{aligned} \quad (2.3)$$

We now form the identity [5]

$$\vec{\nabla} \cdot (\vec{E} \times \vec{h}^* + \vec{e}^* \times \vec{H}) + \frac{1}{c} \frac{\partial}{\partial t} (\vec{h}^* \cdot \vec{H} + \vec{e}^* \cdot \vec{\epsilon} \cdot \vec{E}) = - \frac{4\pi}{c} \vec{e}^* \cdot \frac{\partial \vec{P}'}{\partial t} , \quad (2.4)$$

which follows from (2.1)-(2.3).

In (2.4), $\vec{e}^*$ is computed as follows. The vector function $\vec{e}$ is a functional of $\vec{\epsilon}$ because of Eqs. (2.3). Let us, for the moment, make this dependence explicit by writing $\vec{e}[\vec{\epsilon}]$. Let z_c

denote the complex conjugate of a complex number z . Then, we define $\vec{e}^*[\vec{\epsilon}] \equiv \vec{e}_c[\vec{\epsilon}]$, where $\vec{e}_c$ is the functional of $\vec{\epsilon}$ obtained by replacing explicit occurrences of i by $-i$ (while treating ϵ as a variable independent of i). By contrast, the complex conjugate of the vector $\vec{e}[\vec{\epsilon}]$ is given by $(\vec{e}[\vec{\epsilon}])_c = \vec{e}_c[\vec{\epsilon}_c]$; that is, both the explicit dependence of $\vec{e}$ on i and the implicit dependence of $\vec{e}$ on i through the dependence of $\vec{\epsilon}$ on i are affected. Thus, we may view the "*" operation as being the same as complex conjugation, except that $\vec{\epsilon}$ is treated as if it were a real quantity. In the absence of absorption, when $\vec{\epsilon}$ is real, the "*" operation is identical to complex conjugation.

Next, we integrate (2.4) over the cross-sectional variables $\vec{r}_\perp$. It is expected that the fields fall off sufficiently rapidly for large $|\vec{r}_\perp|$ that $\int d^2 r_\perp \vec{\nabla}_\perp \cdot \vec{V} = 0$, where $\vec{\nabla}_\perp$ is the transverse gradient. Thus, we obtain

$$\begin{aligned} \frac{\partial}{\partial x} \int d^2 r_\perp (\vec{E} \times \vec{h}^* + \vec{e}^* \times \vec{H})_x + \frac{1}{c} \frac{\partial}{\partial t} \int d^2 r_\perp (\vec{h}^* \cdot \vec{H} + \vec{e}^* \cdot \vec{\epsilon} \cdot \vec{E}) \\ = - \frac{4\pi}{c} \int d^2 r_\perp \vec{e}^* \cdot \frac{\partial \vec{P}'}{\partial t}, \end{aligned} \quad (2.5)$$

where the "x" subscript in the first integral on the left-hand side indicates the x component of the vector in parentheses.

Let us now make an Ansatz regarding the x and t dependence of the fields. We assume that $\vec{e}$ and $\vec{h}$ are the fields of a normal mode of the waveguide in the absence of $\vec{E}^{e \times t}$, so they have the simple x and t dependence $\exp[i(\beta x - \omega t)]$, where β is the propagation constant characteristic of the particular mode. As for the

fields $\vec{E}$ and $\vec{H}$, we shall assume that they differ from $\vec{e}$ and $\vec{h}$ by a (possibly complex) phase factor [6] $\exp(i\varphi(x,t))$, where $\varphi(x,t)$ is a function to be determined by satisfying Eq. (2.5). Thus, we let

$$\begin{aligned}\vec{E}(\vec{r},t) &= \exp(i\varphi(x,t)) \vec{e}(\vec{r},t) , \\ \vec{H}(\vec{r},t) &= \exp(i\varphi(x,t)) \vec{h}(\vec{r},t) .\end{aligned}\tag{2.6}$$

We assume that the function $\varphi(x,t)$ is at least first order in $\vec{\chi} \cdot \vec{E}^{ex t}$.

Putting (2.6) into (2.5), dropping terms of higher than first order in $\vec{\chi} \cdot \vec{E}^{ex t}$, and recalling [7] that $\Omega \ll \omega$, we get a first-order partial differential equation for $\varphi(x,t)$:

$$\left[\frac{\partial}{\partial t} + v_g \frac{\partial}{\partial x} \right] \varphi(x,t) = I(x,t) ,\tag{2.7a}$$

where the "source" $I(x,t)$ is given by

$$I(x,t) = \frac{\omega}{2U} \int d^2 r_{\perp} \vec{e}^*(\vec{r}_{\perp}) \cdot \vec{\chi}(\vec{r}_{\perp}) : \vec{e}(\vec{r}_{\perp}) \vec{E}^{ex t}(\vec{r}_{\perp}; x, t) ,\tag{2.7b}$$

and we define the group velocity

$$v_g = \frac{P}{U} ,\tag{2.7c}$$

with P , the time-averaged power flow, and U , the energy per unit length, defined, as usual [8], by

$$P = \frac{c}{8\pi} \operatorname{Re} \int d^2 r_{\perp} (\vec{e}(\vec{r}_{\perp}) \times \vec{h}^*(\vec{r}_{\perp}))_x, \quad (2.7d)$$

$$U = \frac{1}{16\pi} \int d^2 r_{\perp} (\vec{e}^*(\vec{r}_{\perp}) \cdot \vec{e}(\vec{r}_{\perp}) + \vec{h}^*(\vec{r}_{\perp}) \cdot \vec{h}(\vec{r}_{\perp})).$$

Here, we have separated the $\vec{r}_{\perp}$ dependence from the x and t dependence of the modal fields $\vec{e}$ and $\vec{h}$. That is, we write $\vec{e}(\vec{r}, t) \equiv \exp(i[\beta x - \omega t]) \vec{e}(\vec{r}_{\perp})$, and a similar equation for $\vec{h}(\vec{r}, t)$.

We may solve the differential equation (2.7a) as follows. We define two new variables

$$s_{\pm} = t \pm x/v_g, \quad (2.8a)$$

corresponding derivatives

$$\frac{\partial}{\partial s_{\pm}} = \frac{1}{2} \left[\frac{\partial}{\partial t} \pm v_g \frac{\partial}{\partial x} \right], \quad (2.8b)$$

and phase and source functions

$$\Phi(s_+, s_-) = \varphi(\frac{1}{2}v_g[s_+ - s_-], \frac{1}{2}[s_+ + s_-]), \quad (2.8c)$$

$$J(s_+, s_-) = \frac{1}{2}I(\frac{1}{2}v_g[s_+ - s_-], \frac{1}{2}[s_+ + s_-]).$$

Equation (2.7a) can thus be rewritten in the form

$$\frac{\partial \Phi(s_+, s_-)}{\partial s_+} = J(s_+, s_-). \quad (2.9)$$

The general solution of (2.9) is

$$\Phi(s_+, s_-) = \int_0^{s_+} d\zeta J(\zeta, s_-) + A(s_-), \quad (2.10)$$

where $A(s_-)$ is an arbitrary function of s_- . In order to solve uniquely for the function $\Phi(s_+, s_-)$ appropriate to our problem, we need to specify a boundary condition. This can be derived as follows. Before the wave arrives at the leading end of the waveguide ($x = 0$), it is certainly independent of $\vec{E}^{e \times t}$. By continuity of the wave at $x = 0$, then, we must have $\varphi(x=0, t) = 0$, for all t . The point $x = 0$ corresponds to $s_+ = s_- = t$, so, in terms of $\Phi(s_+, s_-)$, this condition reads

$$\Phi(t, t) = 0. \quad (2.11)$$

Using the boundary condition (2.11) in the general solution (2.10), we can solve for $A(s_-)$. The unique solution of (2.9) satisfying (2.11) is thus

$$\Phi(s_+, s_-) = \int_{s_-}^{s_+} d\zeta J(\zeta, s_-), \quad (2.12)$$

Returning to the original variables, we can rewrite (2.12) as

$$\varphi(x, t) = \frac{4\pi\omega}{c} \int_0^x dx' \frac{\int d^2 r_{\perp} \vec{e}^*(\vec{r}_{\perp}) \cdot \vec{\chi}(\vec{r}_{\perp}) : \vec{e}(\vec{r}_{\perp}) \vec{E}^{e \times t}(\vec{r}_{\perp}; x', t - [x - x']/v_g)}{\text{Re} \int d^2 r_{\perp} (\vec{e}(\vec{r}_{\perp}) \times \vec{h}^*(\vec{r}_{\perp}))_x}, \quad (2.13)$$

where we have used Eqs. (2.7) and (2.8) and have defined a new

integration variable $x' = \frac{1}{2}x + \frac{1}{2}v_g(\zeta - t)$. It is more customary to write the field-induced phase shift $\varphi(x, t)$ in terms of the electro-optic coefficients [1], r_{ijk} . These coefficients are given by

$$r_{ijk} = -8\pi\eta_{il}\eta_{mj}\chi_{lmk} , \quad (2.14)$$

where $\vec{\eta} = (\vec{\epsilon})^{-1}$, and a sum over repeated indices is implied. In terms of $\vec{r}$, we have the formula

$$\varphi(x, t) = - \frac{\omega}{2c} \int_0^x dx' \frac{\int d^2 r_{\perp} \vec{d}^*(\vec{r}_{\perp}) \cdot \vec{r}(\vec{r}_{\perp}) : \vec{d}(\vec{r}_{\perp}) \vec{E}^{e \times t}(\vec{r}_{\perp}; x', t - [x - x']/v_g)}{\text{Re} \int d^2 r_{\perp} (\vec{e}(\vec{r}_{\perp}) \times \vec{h}^*(\vec{r}_{\perp}))_x} , \quad (2.15)$$

where $\vec{d}(\vec{r}_{\perp}) = \vec{\epsilon}(\vec{r}_{\perp}) \cdot \vec{e}(\vec{r}_{\perp})$, and we assume $(\epsilon^*)_{ij} = \epsilon_{ji}$.

Equation (2.15) is the phase shift induced by an arbitrary position- and time-dependent external field on an optical wave propagating in a waveguide of arbitrary cross section, to first order in $\vec{r} \cdot \vec{E}^{e \times t}$. As an example of the use of this formula, let us compute the phase shift in the case of a traveling-wave external field of the form

$$\vec{E}^{e \times t}(\vec{r}_{\perp}; x, t) = \vec{e}^{e \times t}(\vec{r}_{\perp}) \sin \Omega(t - n_m x/c) , \quad (2.16)$$

where $\vec{e}^{e \times t}(\vec{r}_{\perp})$ is an arbitrary function, and n_m is a constant, called the "modulation index." By (2.15), we need to evaluate the integral

$$\int_0^x dx' \sin \Omega \left[t - \frac{x-x'}{v_g} - n_m \frac{x'}{c} \right] = \frac{2v_{\text{eff}}}{\Omega} \sin(\% \Omega x / v_{\text{eff}}) \sin[\Omega t - \phi(x)] , \quad (2.17)$$

where v_{eff} is an effective group velocity, given by

$$\frac{1}{v_{\text{eff}}} = \frac{1}{v_g} - \frac{n_m}{c} , \quad (2.18a)$$

with v_g as defined by Eqs. (2.7c-d), and

$$\phi(x) = \Omega \frac{x}{2} \left[\frac{1}{v_g} + \frac{n_m}{c} \right] . \quad (2.18b)$$

Using (2.15) with (2.17), then, we deduce for this case

$$\begin{aligned} \varphi(x, t) = & - \frac{\omega v_{\text{eff}}}{\Omega c} \frac{\int d^2 r_{\perp} \vec{d}^*(\vec{r}_{\perp}) \cdot \vec{r}(\vec{r}_{\perp}) : \vec{d}(\vec{r}_{\perp}) \vec{e}^{e \times t}(\vec{r}_{\perp})}{\text{Re} \int d^2 r_{\perp} (\vec{e}(\vec{r}_{\perp}) \times \vec{h}^*(\vec{r}_{\perp}))_x} \\ & \times \sin(\% \Omega x / v_{\text{eff}}) \sin[\Omega t - \phi(x)] . \end{aligned} \quad (2.19)$$

It is interesting to compare (2.19) with the result one obtains in a homogeneous, nonlinear medium of infinite extent [9].

We may rewrite (2.19) in the form

$$\varphi(x, t) = - \frac{\pi n^3 r e^{e \times t} x}{\lambda} \Gamma \frac{\sin(\% \Omega x / v_{\text{eff}})}{(\% \Omega x / v_{\text{eff}})} \sin[\Omega t - \phi(x)] , \quad (2.20a)$$

where n , r and $e^{e \times t}$ are effective parameters, an index, an electro-optic coefficient and a field strength, respectively, which characterize the infinite-homogeneous-medium case, λ is the vacuum wavelength of the optical wave and Γ is an "overlap integral," defined by

$$\Gamma = \frac{\int d^2 r_{\perp} \vec{d}^*(\vec{r}_{\perp}) \cdot \vec{r}(\vec{r}_{\perp}) : \vec{d}(\vec{r}_{\perp}) \vec{e}^{e \times t}(\vec{r}_{\perp})}{n^3 r e^{e \times t} \text{Re} \int d^2 r_{\perp} (\vec{e}(\vec{r}_{\perp}) \times \vec{h}^*(\vec{r}_{\perp}))_x} . \quad (2.20b)$$

In the limit of an infinite, homogeneous nonlinear medium, the factor $\Gamma \rightarrow 1$, which can be seen as follows. For such a medium, $\vec{e}$, $\vec{h}$ and $\vec{d}$ are all constant vectors. The wave propagates in the x direction with propagation constant $\beta = n\omega/c$, where n is the effective index, and we assume that $\vec{e} \cdot \hat{x} = 0$. Thus, $\vec{h} = n(\hat{x} \times \vec{e})$, and $\vec{d} = n^2 \vec{e}$. We define the product $re^{e \times t} \equiv \vec{e} \cdot \vec{r} : \vec{e} \vec{e}^{e \times t} / |\vec{e}|^2$. Then, the numerator in (2.20b) is simply $n^4 re^{e \times t} A |\vec{e}|^2$, where A is the cross-sectional area. The integral in the denominator has the value $nA |\vec{e}|^2$. Combining these factors as in (2.20b), we see that $\Gamma = 1$ in this limit. Equation (2.20a) with $\Gamma = 1$ is identical to the standard result obtained for an infinite, homogeneous medium [9]. When $\Gamma \neq 1$, Eqs. (2.20) agree in essential details with the result of Alferness [3].

There are two major reasons that Γ differs from unity in a waveguide. First, the vector $\vec{d}$ has more non-zero components in a waveguide than in an infinite medium. Thus, a different com-

bination of components of the tensor $\vec{\epsilon}$ appears in the numerator of (2.20b) than in the denominator (we shall see an explicit example of this phenomenon at the end of the next section, when (2.19) is applied to a simple planar waveguide structure.) Secondly, the modal fields are generally non-zero in linear optical regions (the "inactive" layers) as well as in the nonlinear optical region (the "active" layer.) The integration in the numerator is over the active layer only, while that in the denominator is over all space. If a substantial fraction of the optical energy travels through the inactive layers, the phase shift is thus reduced from its infinite medium value roughly by the ratio of the energy in the active layer to the total energy.

We should point out some reservations we have concerning the general result (2.15) and its derivation in this section. First, we do not know for certain that the Ansatz (2.6) is sufficiently general. The mathematical simplicity and consistency of the steps leading thence to (2.15) tend to suggest that it is, but we have no proof. Secondly, the procedure used in this section is not a rigorous perturbation theory. That is, (2.15) is not necessarily the first term in a systematic expansion for $\varphi(x,t)$ in the quantity $\vec{\epsilon} \cdot \vec{E}^{ext}$; furthermore, we have no proof that, could such an expansion be constructed by a generalization of the methods of this section, that expansion would converge. Thus, we do not know a priori the range of validity of the result in the parameter space of $\vec{\epsilon}$, $\vec{E}^{ext}$, ω and Ω .

For these reasons, we now turn to a rigorous derivation of the phase shift $\varphi(x,t)$ in an idealized model waveguide system. We shall derive an expression for the phase shift in the model case which agrees precisely with (2.15) when the latter is applied to the same geometry; moreover, since the alternative derivation is a rigorous perturbation theory, it furnishes us with a region of validity for the phase shift in the model system. Finally, since, based on physical considerations, the region of validity of the result should not depend on the details of the geometry of the waveguide or of the external field, we may argue that (2.15) can be expected to hold in the same region of the parameter space as the model result. This region is described by

$$\begin{aligned} \Omega/\omega &\ll 1, \\ |r_{ijk} E^{ext}| &\ll 1, \\ (\omega/\Omega) (r_{ijk} E^{ext})^2 &\ll 1. \end{aligned} \tag{2.21}$$

III. RIGOROUS SOLUTION OF A MODEL WAVEGUIDE MODULATOR

We consider the idealized model waveguide system shown in Fig. 2, in which a planar, birefringent dielectric slab of thickness d and ordinary and extraordinary refractive indices n_o and n_e , respectively (z is the extraordinary axis, normal to the slab), occupies the region $0 < z < d$. The slab is enclosed (at $z = 0$ and $z = d$) by perfectly conducting planes. We take the slab to be infinite in the y direction and of length L in the x direction (occupying $0 < x < L$).

The dielectric slab is assumed to have linear electro-optic coefficients

$$r_{ijk} = r_{13}\delta_{ij}\hat{z}_k + r_{51}(\delta_{ik}\hat{z}_j + \delta_{jk}\hat{z}_i) + r_{33}\hat{z}_i\hat{z}_j\hat{z}_k, \quad (3.1)$$

where i, j and k are vector indices in three-space, and we use the standard correspondence $r_{ijk} \sim r_{ak}$, where $1 \leq a \leq 6$ [1]. Here, $\hat{z} = (0, 0, 1)$, and $\delta_{ij} = \delta_{ij} - \hat{z}_i\hat{z}_j$.

A time-dependent (but position-independent) external electric field

$$\vec{E}^{ext}(t) = \hat{z}E^{ext}\sin\Omega t, \quad (3.2)$$

is applied in the slab. By definition, the inverse dielectric tensor in the presence of this external field is given by

$$\vec{\eta}(t) = \begin{bmatrix} (a_o(t)/n_o)^2 & 0 & 0 \\ 0 & (a_o(t)/n_o)^2 & 0 \\ 0 & 0 & (a_e(t)/n_e)^2 \end{bmatrix}, \quad (3.3a)$$

where

$$a_\alpha(t) = \sqrt{1 + \epsilon_\alpha \sin \Omega t}, \quad (\alpha = o, e), \quad (3.3b)$$

with

$$\begin{aligned} \epsilon_o &= n_o^2 r_{13} E^{ext}, \\ \epsilon_e &= n_e^2 r_{33} E^{ext}. \end{aligned} \quad (3.3c)$$

We shall construct the modes of the waveguide in the presence of the modulating field. The y direction is trivial in this problem, so the fields are independent of y. The TE modes have $E_x = E_z = H_y = 0$. It is convenient to define a function $F_1(x, z, t)$ [here, the "1" subscript is for "TE," while "2" is for "TM"] such that

$$E_y = \frac{1}{c} \frac{\partial F_1}{\partial t}, \quad H_x = \frac{\partial F_1}{\partial z}, \quad H_z = -\frac{\partial F_1}{\partial x}. \quad (3.5a)$$

Noting that $\vec{E} = \vec{\eta} \cdot \vec{D}$, with $\vec{\eta}$ given by Eqs. (3.3), Maxwell's equations yield a wave equation for F_1 of the form

$$\left\{ \nabla^2 - \frac{n_o^2}{c^2} \frac{\partial}{\partial t} \frac{1}{[a_o(t)]^2} \frac{\partial}{\partial t} \right\} F_1(x, z, t) = 0. \quad (3.5b)$$

Because there are conducting planes at $z = 0$ and $z = d$, the function $F_1(x, z, t)$ must satisfy

$$F_1(x, 0, t) = F_1(x, d, t) = 0. \quad (3.5c)$$

For the TM case, $E_y = H_x = H_z = 0$, and we define a function $F_2(x, z, t)$ such that

$$H_y = \frac{1}{c} \frac{\partial F_2}{\partial t}, \quad D_x = - \frac{\partial F_2}{\partial z}, \quad D_z = \frac{\partial F_2}{\partial x}. \quad (3.6a)$$

Maxwell's equations provide

$$\left\{ \frac{[a_e(t)]^2}{n_e^2} \frac{\partial^2}{\partial x^2} + \frac{[a_o(t)]^2}{n_o^2} \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right\} F_2(x, z, t) = 0, \quad (3.6b)$$

with boundary conditions

$$\frac{\partial F_2(x, 0, t)}{\partial z} = \frac{\partial F_2(x, d, t)}{\partial z} = 0. \quad (3.6c)$$

We may separate the z dependence of the F_μ from the x and t dependence, as follows. Using the boundary conditions (3.5c) and (3.6c), we define functions $R_{\mu, m}(x, t)$ by

$$F_1(x, z, t) = \sqrt{\frac{2}{d}} \sum_{m=1}^{\infty} \sin\left[\frac{m\pi z}{d}\right] R_{1, m}(x, t), \quad (3.7)$$

$$F_2(x, z, t) = \sqrt{\frac{2}{d}} \sum_{m=1}^{\infty} \cos\left[\frac{m\pi z}{d}\right] R_{2, m}(x, t).$$

Let us focus attention in what follows directly on the component functions $R_{\mu,m}(x,t)$. For simplicity of notation, we suppress the "m" index in the sequel. The functions $R_{\mu}(x,t)$ satisfy

$$\left\{ \frac{\partial^2}{\partial x^2} - \left[\frac{m\pi}{d} \right]^2 - \frac{n_o^2}{c^2} \frac{\partial}{\partial t} \frac{1}{[a_o(t)]^2} \frac{\partial}{\partial t} \right\} R_1(x,t) = 0, \quad (3.8a)$$

$$\left\{ \frac{[a_e(t)]^2}{n_e^2} \frac{\partial^2}{\partial x^2} - \frac{[a_o(t)]^2}{n_o^2} \left[\frac{m\pi}{d} \right]^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right\} R_2(x,t) = 0. \quad (3.8b)$$

We can determine some qualitative features of the time dependence of $R_{\mu}(x,t)$ by a simple physical picture. Before the wave arrives at $x = 0$, it oscillates in time at frequency ω . While inside the waveguide, the wave interacts with the external field, producing wavelets which oscillate at frequencies $\omega_k \equiv \omega + k\Omega$, with $k = 0, \pm 1, \dots$. If we set $R_{\mu}(x,t) = \exp(-i\omega t)\Phi_{\mu}(x,t)$, then the function $\Phi_{\mu}(x,t)$ is composed of frequencies $k\Omega$ only: that is, $\Phi_{\mu}(x,t)$ is periodic in time with period $2\pi/\Omega$. That a solution to the problem exists in this form, with ω equal to some parameter, follows from the periodicity of the functions $a_{\alpha}(t)$ and the Floquet-Bloch theorem [10]. That the parameter ω must be identically equal to the incident frequency follows from our physical considerations.

Appropriate boundary conditions in x for the $R_{\mu}(x,t)$ may now be derived as follows. At $x = 0$, before the wave has interacted with the external field, we demand that only frequency ω be present. We may normalize the fields so that

$$R_{\mu}(0,t) = \exp(-i\omega t). \quad (3.9a)$$

Since the waveguide is very long compared to the wavelength of the beam, we do not expect any substantial reflection from $x = L$ back into the waveguide. This condition may be expressed as

$$k_x > 0, \quad (3.9b)$$

where k_x is the x component of the wavevector. With the boundary conditions (3.9), the differential equations (3.8) now possess a unique solution.

First, we note that the differential equations (3.8) are separable in x and t . We can write the general solution in the form

$$R_{\mu}(x,t) = \sum_k C_{k\mu} T_{k\mu}(t) \exp(i\beta_{k\mu} x), \quad (3.10)$$

where k is an index whose precise nature will be specified shortly, the $C_{k\mu}$ are constants, and $\beta_{k\mu} > 0$, in accordance with (3.9b). The eigenfunctions $T_{k\mu}(t)$ satisfy

$$\left\{ \frac{d}{dt} \frac{1}{(a_{k1}(t))^2} \frac{d}{dt} + \lambda_{k1} \right\} T_{k1}(t) = 0, \quad (3.11a)$$

$$\left\{ \frac{1}{(a_{k2}(t))^2} \frac{d^2}{dt^2} + \lambda_{k2} \right\} T_{k2}(t) = 0, \quad (3.11b)$$

where we define the eigenvalues

$$\lambda_{k1} = \frac{c^2}{n_o^2} \left[\beta_{k1}^2 + \left[\frac{m\pi}{d} \right]^2 \right], \quad (3.11c)$$

$$\lambda_{k2} = c^2 \left[\left[\frac{\beta_{k2}}{n_e} \right]^2 + \left[\frac{m\pi}{n_o d} \right]^2 \right],$$

and the new functions

$$a_{k1}(t) = a_o(t), \quad (3.11d)$$

$$a_{k2}(t) = \sqrt{\frac{(a_o(t))^2 (\beta_{k2}/n_e)^2 + (a_o(t))^2 (m\pi/n_o d)^2}{(\beta_{k2}/n_e)^2 + (m\pi/n_o d)^2}}.$$

(We note that, although $a_{k2}(t)$ depends on the separation constant β_{k2} , it possesses an expansion of the form $a_{k2}(t) = 1 + \delta a_{k2}(t)$, where $|\delta a_{k2}(t)| \ll 1$, for all β_{k2} , as long as $\epsilon_e, \epsilon_o \ll 1$.) It is convenient to make a change of time and frequency variables in Eqs. (3.11). We define

$$\tau_{k\mu}(t) = \int_0^t dt' a_{k\mu}(t'), \quad (3.12a)$$

$$\Omega_{k\mu} = \frac{2\pi}{\tau_{k\mu}(2\pi/\Omega)}. \quad (3.12b)$$

Alternatively, one may solve Eq. (3.12a) for t in terms of $\tau_{k\mu}$ and Eq. (3.12b) for Ω in terms of $\Omega_{k\mu}$, using the explicit formulas Eqs. (3.11d) and (3.3b) for $a_{k\mu}(t)$. The variables $\tau_{k\mu}(t)$ and $\Omega_{k\mu}$ satisfy the important identity

$$\tau_{k\mu}(t + 2\pi/\Omega) = \tau_{k\mu}(t) + 2\pi/\Omega_{k\mu}, \quad (3.13a)$$

which follows simply from the definitions (3.12). Alternatively, this identity can be written

$$t(\tau_{k\mu} + 2\pi/\Omega_{k\mu}) = t(\tau_{k\mu}) + 2\pi/\Omega, \quad (3.13b)$$

where $t(\tau_{k\mu})$ is the function obtained by inverting (3.12a).

We also define scaled eigenfunctions

$$\psi_{k\mu}(\tau_{k\mu}) = F_{k\mu}(t(\tau_{k\mu})) T_{k\mu}(t(\tau_{k\mu})), \quad (3.14a)$$

where

$$F_{k1}(t) = \frac{1}{\sqrt{a_{k1}(t)}}, \quad (3.14b)$$

$$F_{k2}(t) = \sqrt{a_{k2}(t)}.$$

The functions $\psi_{k\mu}(\tau_{k\mu})$ satisfy

$$\left\{ \frac{d^2}{d\tau_{k\mu}^2} + \lambda_{k\mu} - V_{k\mu}(\tau_{k\mu}) \right\} \psi_{k\mu}(\tau_{k\mu}) = 0, \quad (3.15a)$$

where the "potentials" $V_{k\mu}(\tau_{k\mu})$ are defined as

$$V_{k\mu}(\tau_{k\mu}) = \frac{F_{k\mu}''(t(\tau_{k\mu})) \left[\frac{dt}{d\tau_{k\mu}} \right]^2 + F_{k\mu}'(t(\tau_{k\mu})) \frac{d^2 t}{d\tau_{k\mu}^2}}{F_{k\mu}(t(\tau_{k\mu}))}. \quad (3.15b)$$

Here, primes denote differentiation with respect to t : $F'(t) \equiv dF(t)/dt$. The precise functional dependence on $\tau_{k\mu}$ of $V_{k\mu}$ could be computed using (3.15b), (3.14b), (3.12a), (3.11d) and (3.3b), but, fortunately, we do not require the explicit formulas. Only two properties of $V_{k\mu}$ are of interest to us here: First, $V_{k\mu}(\tau_{k\mu})$ is periodic in $\tau_{k\mu}$ with period $2\pi/\Omega_{k\mu}$. Secondly,

$$V_{k\mu}(\tau_{k\mu}) = \Omega^2 O(\epsilon) , \quad (3.16)$$

where ϵ is ϵ_e or ϵ_o . (As with $a_{k2}(t)$ above, the potential $V_{k2}(\tau_{k2})$ depends on β_{k2} , but the property (3.16) is independent of β_{k2} .) Both properties are easy to establish from the definitions (3.15b), (3.14b) and (3.12a).

Since $V_{k\mu}$ is periodic in $\tau_{k\mu}$, by another application of the Floquet-Bloch theorem [10], this time in the variable $\tau_{k\mu}$, the solution of (3.15a) may be written in the form

$$\psi_{k\mu}(\tau_{k\mu}) = \exp(-i\omega_{k\mu}\tau_{k\mu}) \Psi_{k\mu}(\tau_{k\mu}) , \quad (3.17a)$$

where we choose

$$\omega_{k\mu} = \omega \Omega_{k\mu}/\Omega . \quad (3.17b)$$

The significance of this particular choice of $\omega_{k\mu}$ will soon emerge. Here, the function $\Psi_{k\mu}(\tau_{k\mu})$ is periodic in $\tau_{k\mu}$ with period $2\pi/\Omega_{k\mu}$.

The boundary condition at $x = 0$ [Eq. (3.9a)] can be written in terms of the new variables as

$$\sum_k C_{k\mu} \frac{\exp(i[\omega t - \omega_{k\mu} \tau_{k\mu}(t)])}{F_{k\mu}(t)} \Psi_{k\mu}(\tau_{k\mu}(t)) = 1. \quad (3.18)$$

This condition must be used to determine the coefficients $C_{k\mu}$. We do so as follows. We first show that the coefficient of $C_{k\mu}$ in the summand of (3.18) is a periodic function of t with period $2\pi/\Omega$. Then, we consider a perturbative solution of (3.15), valid under the conditions (2.19). Finally, we expand all quantities in (3.18) to the lowest non-trivial order in ϵ and solve the resulting equation for $C_{k\mu}$.

The proof that the summand in (3.18) is a periodic function of t with period $2\pi/\Omega$ runs as follows. First, by hypothesis, $\Psi_{k\mu}$ and $F_{k\mu}$ are periodic. In the remaining exponential, let $t \rightarrow t + 2\pi/\Omega$. Using Eqs. (3.13), we have

$$\begin{aligned} \omega(t+2\pi/\Omega) - \omega_{k\mu} \tau_{k\mu}(t+2\pi/\Omega) &= \omega t - \omega_{k\mu} \tau_{k\mu}(t) + 2\pi(\omega/\Omega - \omega_{k\mu}/\Omega_{k\mu}) \\ &= \omega t - \omega_{k\mu} \tau_{k\mu}(t), \end{aligned} \quad (3.19)$$

where the equality follows from (3.17b). (Alternatively, we could have left $\omega_{k\mu}$ initially as an unspecified parameter which would then have been determined by requiring that (3.18) be periodic.)

It is always true in practice that $\epsilon \ll 1$ and $\Omega/\omega \ll 1$. We wish to study the system (3.15) when both these conditions are true, but $\epsilon\omega/\Omega$ is not necessarily small. We shall, however, that $\epsilon^2\omega/\Omega \ll 1$.

Under these conditions, Eqs. (3.15) may be solved using perturbation theory. When $\epsilon \equiv 0$, $\Psi_{k\mu} \sim \exp(i[\omega_{k\mu} - (\lambda_{k\mu})^{1/2}]\tau_{k\mu})$. The suitable set of eigenfunctions in this limit is furnished by $\Psi_{k\mu} \sim \exp(ik\Omega_{k\mu}\tau_{k\mu})$, where k runs through the integers. Using standard perturbation theory techniques, then, we obtain the eigenfunctions and eigenvalues in the form

$$\begin{aligned}\Psi_{k\mu}(\tau_{k\mu}) &= \exp(ik\Omega_{k\mu}\tau_{k\mu}) + O(\epsilon), \\ \lambda_{k\mu} &= (\omega_{k\mu} - k\Omega_{k\mu})^2 + O(\epsilon^2), \quad k \in \mathbb{Z}.\end{aligned}\tag{3.20}$$

The $O(\epsilon)$ and $O(\epsilon^2)$ corrections in (3.20) can be ignored when $\epsilon^2\omega/\Omega \ll 1$. If this condition is not met, then such corrections must be taken into account, and the treatment becomes correspondingly more complicated.

We can now compute the summand of (3.18) consistent with (2.21). Using the explicit formulas (3.3), (3.11) and (3.12),

$$\begin{aligned}\tau_{k\mu}(t) &= t + (\epsilon_\mu/\Omega)\sin^2\frac{1}{2}\Omega t + O(\epsilon_\mu^2/\Omega), \\ \Omega_{k\mu} &= \Omega + O(\Omega\epsilon_\mu^2),\end{aligned}\tag{3.21}$$

where

$$\begin{aligned}\epsilon_1 &= \epsilon_0, \\ \epsilon_2 &= \frac{\epsilon_0(\beta_2/n_0)^2 + \epsilon_0(m\pi/n_0d)^2}{(\omega/c)^2}.\end{aligned}\tag{3.22}$$

Here,

$$\beta_{\mu} = n_{\mu} \sqrt{(\omega/c)^2 - (m\pi/n_o d)^2}, \quad (3.23a)$$

with

$$n_{\mu} = \begin{cases} n_o, & (\mu = 1), \\ n_e, & (\mu = 2). \end{cases} \quad (3.23b)$$

Using (3.11c), (3.20) and (2.21), we can also write

$$\beta_{k\mu} \approx \beta_{\mu} - k\omega\Omega n_{\mu}^2/c^2 \beta_{\mu}. \quad (3.24)$$

Thus, to this order, Ω and $\Omega_{k\mu}$ are interchangeable. The only places where the distinction between t and $\tau_{k\mu}$ on the one hand and β_{μ} and $\beta_{k\mu}$ on the other is important are, respectively, in the exponent of (3.18), where t and $\tau_{k\mu}$ are multiplied by ω , and in the exponent of (3.10), where β_{μ} and $\beta_{k\mu}$ are multiplied by x . Rewriting (3.18) using (3.20) and (3.21), we have

$$\sum_k C_{k\mu} \exp(ik\Omega_{k\mu} \tau_{k\mu}) \exp(-i\epsilon_{\mu}(\omega/\Omega) \sin^2 \frac{1}{2} \Omega t) = 1. \quad (3.25)$$

But, to this order, we can replace $\tau_{k\mu}$ in (3.25) by t , and $\Omega_{k\mu}$ by Ω , so we obtain

$$\sum_k C_{k\mu} \exp(ik\Omega t) = \exp(i\epsilon_{\mu}(\omega/\Omega) \sin^2 \frac{1}{2} \Omega t). \quad (3.26)$$

Now, the quantity on the right-hand side of (3.26) can be expanded in a Fourier series by using the formula [11]

$$\exp(-ix\cos\theta) = \sum_{k=-\infty}^{\infty} (-i)^k J_k(x) \exp(ik\theta), \quad (3.27)$$

where $J_k(x)$ is a Bessel function. Thus, we obtain

$$C_{k\mu} = (-i)^k J_k(\frac{1}{2}\epsilon_{\mu}\omega/\Omega) \exp(\frac{1}{2}i\epsilon_{\mu}\omega/\Omega). \quad (3.28)$$

Returning to $R_{\mu}(x,t)$, using (3.10), (3.14), (3.17), (3.20), (3.21) and (3.28), we may write at once

$$R_{\mu}(x,t) = \exp(-i\omega t)\Phi_{\mu}(x,t) + \Delta R_{\mu}(x,t), \quad (3.29a)$$

where

$$\begin{aligned} \Phi_{\mu}(x,t) = & \exp(\frac{1}{2}i\epsilon_{\mu}(\omega/\Omega)\cos\Omega t) \\ & \times \sum_{k=-\infty}^{\infty} (-i)^k J_k(\frac{1}{2}\epsilon_{\mu}\omega/\Omega) \exp(i\beta_{k\mu}x + ik\Omega t), \end{aligned} \quad (3.29b)$$

and

$$\Delta R_{\mu}(x,t) = O(\epsilon_{\mu} + \epsilon_{\mu}^2\omega/\Omega). \quad (3.29c)$$

Using (3.24), the sum in (3.29) can be performed explicitly with the result

$$R_{\mu}(x,t) \approx \exp(-i(\omega t - \beta_{\mu}x)) \exp(i\varphi_{\mu}(x,t)), \quad (3.30a)$$

where

$$\varphi_{\mu}(x,t) = -\frac{\epsilon_{\mu}\omega}{\Omega} \sin\left[\frac{\Omega x}{2v_{\mu}}\right] \sin\left[\Omega\left[t - \frac{x}{2v_{\mu}}\right]\right], \quad (3.30b)$$

and we recognize the group velocity in polarization state μ :

$$v_{\mu} \equiv (\partial\beta_{\mu}/\partial\omega)^{-1} = c^2\beta_{\mu}/\omega n_{\mu}^2. \quad (3.30c)$$

Equation (3.30b) is the rigorous, perturbative solution for the field-induced phase shift $\varphi_\mu(x,t)$ in the idealized model. Because of Eq. (3.29c), this solution is expected to be valid whenever $\epsilon_\mu^2 \omega / \Omega \ll 1$. We note that Eqs. (3.30) become equivalent to Eqs. (2.20) when we let $d \rightarrow \infty$ in the former and $n_m \rightarrow 0$ in the latter. Thus, the infinite-extent, homogeneous result [9] is correctly obtained. We may now proceed to compare Eqs. (3.30) with what one obtains by applying the arbitrary cross-section formula, Eq. (2.19), to the model case.

IV. COMPARISON OF THE TWO APPROACHES IN THE IDEALIZED CASE

In this section, we compare the rigorous result (3.30) with the general calculation of section II by applying the traveling-wave formula (2.19) to the case of the model waveguide, with an external field of the form (3.2). We may rewrite Eq. (2.19) for this case (i.e., for $n_m = 0$) as

$$\varphi(x,t) = - \frac{\omega v_g}{\Omega c} \frac{\int dz \vec{d}^*(z) \cdot \vec{r} : \vec{d}(z) \vec{E}^{e \times t}}{\text{Re} \int dz (\vec{e}(z) \times \vec{h}^*(z))_x} \sin(\frac{1}{2}\Omega x/v_g) \sin\Omega(t-x/2v_g), \quad (4.1)$$

where $\vec{e}(z)$ and $\vec{h}(z)$ are the modal electric and magnetic fields, respectively (for $\vec{E}^{e \times t} = 0$) derived in section III, $\vec{d}(z) = \vec{e} \cdot \vec{e}(z)$, and the z integrations run from 0 to d . We note that Eq. (4.1) is already in the same form as Eq. (3.30b). Indeed, since [12] $v_g = v_\mu$, the arguments of the two sine functions in (4.1) and (3.30b) match exactly. The only remaining equality to establish, then, is $\delta = \epsilon_\mu \omega / \Omega$, where

$$\delta = - \frac{\omega v_g}{\Omega c} \frac{\int dz \vec{d}^*(z) \cdot \vec{r} : \vec{d}(z) \vec{E}^{e \times t}}{\text{Re} \int dz (\vec{e}(z) \times \vec{h}^*(z))_x}. \quad (4.2)$$

For the idealized case, with $\vec{E}^{e \times t}$ proportional to $\hat{z}$, the relevant non-zero components of $\vec{r}$ are $r_{113} = r_{223} \equiv r_{13}$ and $r_{333} \equiv r_{33}$.

The integrations in (4.1) are trivial because of the simple form of the unmodulated fields in the idealized case, so we find $\delta = \delta_\mu$, where

$$\delta_\mu = -r_\mu E^{e \times t} (\omega^2 n_\mu^2 v_\mu / \Omega c^2 \beta_\mu), \quad (4.3a)$$

with

$$r_1 = r_{13} n_o^2, \quad (4.3b)$$

$$r_2 = (c/\omega)^2 [(\beta_2)^2 r_{33} + (m\pi/d)^2 r_{13}].$$

Finally, using Eqs. (3.3c), (3.22), (3.30c) and (4.3), it is a matter of simple algebra to show that

$$\delta_\mu = -\epsilon_\mu (\omega/\Omega). \quad (4.4)$$

Thus, we see that Eq. (3.30b) are equivalent to (2.19) when the latter are applied to the idealized model waveguide structure.

This equivalence has its primary significance in establishing, because of (3.29c), a region of validity [in the parameter space of r_{ijk} , $E^{e \times t}$, Ω and ω] for the general formula (2.15), [viz., the region described by conditions (2.21)], even though (3.29c) was proved rigorously only for the idealized case. It is, after all, reasonable that the region of validity of these formulas should not depend on waveguide geometry, and since we have established the validity of (2.15) in one such geometry, it

should also hold in the general case. We may regard the work of this section, then, as an informal proof of this statement. In other words, we expect that Eqs. (2.15) will be valid whenever

$$\begin{aligned} |r_{ijk} E^{ext}| &\ll 1, \\ \Omega/\omega &\ll 1, \\ (r_{ijk} E^{ext})^2 \omega/\Omega &\ll 1. \end{aligned} \tag{4.5}$$

REFERENCES

- [1] See, for example, A. Yariv and P. Yeh, Optical Waves in Crystals, (Wiley-Interscience, New York, 1984), chapter 7.
- [2] See, for example, H.A. Haus, "Nonlinear Optical Waveguide Devices," in Nonlinear Optics: Materials and Devices, Proc. 1985 Int. School of Mat. Sci. and Tech., Erice, eds. C. Flytzanis and J.L. Oudar, (Springer, New York, 1986), pp. 2-30.
- [3] R. C. Alferness, "Waveguide Electrooptic Modulators," IEEE Trans. Microwave Theory Tech., MTT-23, 1121-1137, (1982). The homogeneous result is simply multiplied by an "overlap integral" Γ , in Eqs. (6)-(8) of the paper, to account for the reduced energy in the active region.
- [4] We use Gaussian units. The relationship between $\vec{P}'(\vec{r}, t)$ and $\vec{E}(\vec{r}, t)$ is local in time, since $\Omega \ll \omega$ and $|\vec{\chi} \cdot \vec{E}^{ex t}| \ll 1$.
- [5] Here, we apply to our two-dimensional case a perturbative method which is discussed by J.A. Kong in connection with three-dimensional dielectric perturbations in an electromagnetic cavity in Electromagnetic Wave Theory, (Wiley, New York, 1986), section 3.7d. (See also problem P3.48.)
- [6] We are only fixing the component of $\vec{E}(\vec{r}, t)$ parallel to $\vec{e}(\vec{r}, t)$ and the component of $\vec{H}(\vec{r}, t)$ parallel to $\vec{h}(\vec{r}, t)$ by this Ansatz.

We could choose a more general Ansatz of the form

$$\begin{aligned}\vec{E}(\vec{r}, t) &= \exp(i\varphi(x, t))\vec{e}(\vec{r}, t) + \vec{E}_\perp(\vec{r}, t), \\ \vec{H}(\vec{r}, t) &= \exp(i\varphi(x, t))\vec{h}(\vec{r}, t) + \vec{H}_\perp(\vec{r}, t),\end{aligned}$$

where $\vec{E}_\perp(\vec{r}, t)$ and $\vec{H}_\perp(\vec{r}, t)$ satisfy

$$\begin{aligned}\int d^2 r_\perp (\vec{e}^*(\vec{r}, t) \cdot \vec{e}(r_\perp) \cdot \vec{E}_\perp(\vec{r}, t)) &= \int d^2 r_\perp (\vec{h}^*(\vec{r}, t) \cdot \vec{H}_\perp(\vec{r}, t)) = 0, \\ \int d^2 r_\perp (\vec{E}_\perp(\vec{r}, t) \times \vec{h}^*(\vec{r}, t))_x &= \int d^2 r_\perp (\vec{e}^*(\vec{r}, t) \times \vec{H}_\perp(\vec{r}, t))_x = 0.\end{aligned}$$

The fields $\vec{E}_\perp(\vec{r}, t)$ and $\vec{H}_\perp(\vec{r}, t)$ do not contribute to Eq. (2.5).

[7] As a result of $\omega \gg \Omega$, we can take $\partial \vec{P}' / \partial t = -i\omega \vec{P}'$. Corrections to this formula are $O(\Omega/\omega)$. It would be inconsistent to keep these corrections at this point, because such corrections were already ignored in writing Eqs. (2.1).

[8] See, in the context of waveguide theory, A. Yariv and P. Yeh, op. cit., chapter 11. Here, "Re" is the real part with respect to the "*" operation. U is real with respect to "*" because $\vec{E}^{e \times t}(\vec{r}_\perp; x, t)$ is real.

[9] A. Yariv and P. Yeh, op. cit., Section 7.4. In particular, section 7.4.1 contains a derivation of the phase modulation result given here.

[10] This theorem was formulated and proved by G. Floquet, "Sur les Équations Différentielles Linéaires à Coefficients Périodiques," Ann. École Norm. Sup. 12, 47-88 (1883). In physics, it is more often known as "Bloch's theorem," because of its use in solid state physics by F. Bloch, "Über die Quantenmechanik der Elektronen in Kristallgittern," Z. Physik 57, 555-600 (1929).

[11] See, for example, M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, (National Bureau of Standards, Washington, D.C., 1964), chapter 9.

[12] The proof that $v_g \equiv (\partial\beta/\partial\omega)^{-1} = P/U$ is contained in most books on waveguide theory. See, for example, A. Yariv and P. Yeh, op. cit., chapter 11.

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FIGURE CAPTIONS

Fig. 1. General N-layer dielectric waveguide of length L with its axis along x and perpendicular directions $\vec{r}_\perp$: (a) full view; (b) cross-sectional cut at an arbitrary point x . At least one of the layers is electro-optically active.

Fig. 2. Idealized model waveguide modulator: (a) a nonlinear dielectric slab of thickness d , infinitely wide in the y direction, of length L in the x direction, is surrounded by perfectly conducting plates; (b) a time-dependent potential difference $v^{ext}(t)$ is applied between the two conductors, and an optical wave is introduced into the waveguide at $x = 0$.

Fig. 1

(a)

(b)

Fig. 2

(a)

(b)

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