**SUPERLINEAR CONSERVATION LAW
WITH VISCOSITY**

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IMA Preprint Series # 558

August 1989

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Abstract. This paper gives some global existence results of the superlinear conservation law with viscosity.

§1. Introduction. There have been a lot of work about the reaction-diffusion equation

$$(1) \quad u_t = \Delta u + |u|^{\gamma-1}u, \quad \gamma > 1$$

with initial condition

$$(2) \quad u(x, 0) = \phi(x).$$

For global existence problem in L_p -setting the following theorem of F.B. Weissler [1] is typical.

THEOREM. Suppose $\frac{n}{2}(\gamma - 1) > 1$. If $\phi \geq 0$ and $\|\phi\|_{n(\gamma-1)/2}$ is sufficiently small, then there exists a non-negative continuous curve $u : [0, \infty) \rightarrow L^{n(\gamma-1)/2}$ which is a global solution to (1) with initial value ϕ .

Now we are going to consider the superlinear conservation law with viscosity

$$u_t + \sum_{i=1}^n \frac{\partial f_i(u)}{\partial x_i} + f_0(u) = \varepsilon \Delta u,$$

where

$$(3) \quad \begin{aligned} f_i(u) &\in C^1, \quad i = 0, 1, \dots, n, \\ |f_i(u)| &\leq C_i |u|^\beta, \quad |f'_i(u)| \leq C_i |u|^{\beta-1}, \quad i = 1, \dots, n \\ |f_0(u)| &\leq C_0 |u|^\gamma, \quad |f'_0(u)| \leq C_0 |u|^{\gamma-1}. \end{aligned}$$

We have an important example $f_i(u) = C_i |u|^{\beta-1}u$, $f_0(u) = -|u|^{\gamma-1}u$ which is typical in the following consideration. Without loss of generality we may suppose $\varepsilon = 1$. So we have

$$(4) \quad u_t + \sum_{i=1}^n \frac{\partial f_i(u)}{\partial x_i} + f_0(u) = \Delta u$$

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In [1] [2], Weissler considered the problem in L_p -setting. But now we consider the problem in weighted L_p -setting with power weights. This kind of consideration leads to new results even in the reaction-diffusion case.

We define the weighted L_p class $L_{p,l}$ as follows:

$$L_{p,l} \equiv \{u; u \in L_p, |x|^l u \in L_p\}.$$

The norm of u in $L_{p,l}$ is defined as

$$\|u\|_{p,l} = \|u\|_p + \| |x|^l u \|_p.$$

The solution of the problem (4),(2) reduces to the solution of the integral-differential equation

$$u = T(t)\phi - \int_0^t T(t-s)f_0(u)ds - \sum_{i=1}^n \int_0^t T(t-s) \frac{\partial}{\partial \xi_i} f_i(u)ds$$

or the integral equation

$$(5) \quad u = T(t)\phi - \int_0^t T(t-s)f_0(u)ds - \sum_{i=1}^n \int_0^t \left[\frac{\partial}{\partial x_i} T(t-s) \right] f_i(u)ds$$

where

$$T(t)\phi = \frac{1}{(4nt)^{n/2}} \int_{E_n} e^{-\frac{(x-\xi)^2}{4t}} \phi(\xi) d\xi.$$

We call any solution of (5) the generalized solution of (4), (2). We have

THEOREM 1. *Let $\gamma > \frac{2+\sqrt{n^2+4}}{n}$, β near $\frac{\gamma+1}{2}$, for sufficiently small $\alpha > 0$, $\|\phi\|_{q,k\alpha}$ be also sufficiently small, then there exists global solution of the problem (4), (2) in L_q , $\|u\|_q \leq C$, where C is a constant independent of t ,*

$$q = \frac{n(\gamma-1)}{2-\lambda\alpha}, \text{ for some } \lambda, k \text{ satisfying } 0 \leq \lambda \leq \gamma - k, 1 \leq k \leq \beta.$$

If $f_i(u) \equiv 0$, $i = 1, \dots, n$, (4) becomes

$$(4') \quad u_t + f_0(u) = \Delta u$$

we have

THEOREM 2. Let $\frac{n(\gamma-1)}{2} > 1, \alpha(> 0)$ be sufficiently small, $\|\phi\|_{q_1, k\alpha}$ also be sufficiently small, then there exists global solution of the problem (5),(2) in $L_{q_1}, \|u\|_{q_1} < C$, where C is independent of $t, q_1 = \frac{n(\gamma-1)}{2-\gamma\lambda\alpha}, 0 \leq k \leq \gamma, 0 \leq \lambda \leq k-1$.

For weighted Lebesgue class $L_{p,l}$, we have the inequality (PWI1) (see [5]):

$$\|f\|_q \leq C \|f\|_p^\lambda \| |x|^l f \|_p^{1-\lambda}, \quad (0 < \lambda < 1)$$

where $p \geq q > q^* = \frac{np}{n+lp}$ and C is best possible. If we apply this inequality to the class $L_{q,k\alpha}$ in Theorem 2, we get

$$q^* = \frac{nq_1}{n+lq_1} = \frac{n(\gamma-1)}{2+((k-\lambda)\gamma-k)\alpha} \leq \frac{n(\gamma-1)}{2},$$

and only the case $\lambda = k-1, k = \gamma$ gives an equality. So in other cases $q^* < \frac{n(\gamma-1)}{2}, L_{q,k\alpha} \hookrightarrow L_{n(\gamma-1)/2}$, and the result is contained in Weissler's theorem stated above. But in the case $\lambda = k-1, k = \gamma, q^* = \frac{n(\lambda-1)}{2}, L_{q_1,k\alpha}$ is not surely imbedded into $L_{n(\lambda-1)/2}$. In fact we have functions $u \in L_{q_1,k\alpha}$ with $\|u\|_{n(\lambda-1)/2} = \infty$. So Theorem 2 gives new results.

§2. Generalized Young's Inequality. Suppose

$$f(x) = \int_{E_n} G(x-y)g(y)dy$$

is a convolution of G, g . We have the classical Young Inequality

$$(6) \quad \|f\|_{q'} \leq \|G\|_r \|g\|_p, \quad \frac{1}{r} = \frac{1}{p'} + \frac{1}{q'}, \quad (p, r \geq 1).$$

Now if $G(x)$ is radial, and $G(|x|) = |x|^\mu k(|x|)$, where $k(|x|)$ is non-increasing, then we have

Generalized Young's Inequality. If $l < n(1 - \frac{1}{p}), h < n(1 - \frac{1}{q}), h+k \geq 0, \frac{1}{r} = \frac{1}{p'} + \frac{1}{q'}$ then

$$(7) \quad \| |x|^{-h} f \|_{q'} \leq C \| |x|^l g \|_p \| |x|^{-h-l} G \|_r, \quad (p, r \geq 1).$$

The proof of this inequality is based on a Hardy-Littlewood proof of the inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)dx dy}{|x|^h |y|^k |x-y|^{\lambda-h-k}} \leq C(p, q, h, k, \lambda) \|f\|_p \|g\|_q.$$

But there appears some mistake in Hardy-Littlewood's proof as we pointed out before (see [5]), and we adopt the correct proof we have made in [5]. Firstly we may suppose $\mu = 0$. In the case $\mu \neq 0$, the proof is purely similar.

Proof. Put $|y|^l g(y) = g_1(y) \in L_p$. For any $\psi(x) \in L_q$, we have

$$\int_{E_n} |x|^{-h} f(x) \psi(x) dx = \int_{E_n} \int_{E_n} \frac{G(x-y) g_1(y) \psi(x)}{|x|^h |y|^l} dx dy.$$

We decompose the integral domain E_{2n} into three parts:

$$\begin{aligned} \int_{E_n} \int_{E_n} &= \iint_{\frac{|x|}{2^{n(1-b)/2b}} \leq |y| \leq 2^{n(1-a)/2a} |x|} + \iint_{|y| \geq 2^{n(1-a)/2a} |x|} \\ &+ \iint_{|y| \leq 2} = I_1 + I_2 + I_3, \quad (0 < a, b < 1). \end{aligned}$$

In the above equality, if $h \leq 0$, then I_2 does not appear; if $l \leq 0$, then I_3 does not appear. In the integral I_1 , it is easy to see $|x|^h |y|^l \geq C_2 |x-y|^{h+l}$, ($h+l \geq 0$). So

$$\begin{aligned} I_1 &\leq C_2^{-1} \iint_{\frac{|x|}{2^{n(1-b)/2b}} \leq |y| \leq 2^{\frac{n(1-a)}{2a}} |x|} \frac{G(|x-y|) |\psi(x)| |g_1(y)|}{|x-y|^{h+l}} dy dx \\ &\leq C \|\psi\|_q \left\| \int_{E_n} \frac{G(x-y)}{|x-y|^{h+l}} |g_1(y)| dy \right\|_{q'} \quad (\text{H\"older's inequality}) \\ &\leq C \|\psi\|_q \|g_1\|_p \left\| \frac{G(|x|)}{|x|^{h+l}} \right\|_r \quad (\text{Young's inequality}). \end{aligned}$$

In the integral I_2 , if $h > 0$, put $h = n(1-a)/q'$, $l_1 = n(1-a)/p'$,

$$\bar{x} = |x|^a \left(\frac{x_1}{|x|}, \dots, \frac{x_n}{|x|} \right), \quad \bar{y} = |y|^a \left(\frac{y_1}{|y|}, \dots, \frac{y_n}{|y|} \right).$$

Then

$$\begin{aligned} I_2 &\leq C \int \dots \int_{|y| \geq 2^{n(1-a)/2a} |x|} \frac{|G(x-y)| |\psi(x)| |g_1(y)|}{|x|^h |y|^l} dx dy \\ &\leq C \int \dots \int_{|\bar{y}| \geq 2^{n(1-a)/2a} |\bar{x}|} \frac{|G(|x-y|)| |\psi(x)| |g_1(y)| |x|^{n(1-a)} |y|^{n(1-a)}}{|x|^{n(1-a)/q'} |y|^{n(1-a)/p'} |y|^{l-l_1}} d\bar{x} d\bar{y} \\ &\leq C \|\psi(x)\|_q \left\| \int_{E_n} \frac{|G(|x-y|)| |g_1(y)| |y|^{n(1-a)/p} d\bar{y}}{|x-y|^{l-l_1}} \right\|_{L_{q'}} \\ &\leq C \|\psi(x)\|_q \|g_1(y)\|_p \left\| \frac{|G(|\bar{x}|^{\frac{1}{a}})|}{|\bar{x}|^{\frac{l-l_1}{a}}} \right\|_{L_r}. \end{aligned}$$

Substituting l_1 into $l - l_1 + n(1-a)$, we get

$$\left\| \frac{G(|\bar{x}|^{\frac{1}{a}})}{|\bar{x}|^{\frac{l-l_1}{a}}} \right\|_{L_r} = \left\| \frac{G(x)}{|x|^{h+l}} \right\|_r.$$

The norms in $L_{q'}, L_r$ above involve integrals with respect to $\bar{x}$. So

$$I_2 \leq C \|\psi(x)\|_q \|g_1(y)\|_p \left\| \frac{G}{|x|^{h+l}} \right\|_r.$$

We can get the estimate in I_3 as well as in I_2 ,

$$I_3 \leq C \|\psi(x)\|_q \|g_1(y)\|_p \left\| \frac{G}{|x|^{h+l}} \right\|_r.$$

So lastly we have

$$\int |x|^{-h} f(x) \psi(x) dx \leq C \|\psi(x)\|_q \|x|^l g\|_p \left\| \frac{G}{|x|^{h+l}} \right\|_r.$$

Because $\psi(x)$ is arbitrary in L_q , so we have

$$\| |x|^{-h} f(x) \|_{q'} \leq C \| |x|^l g \|_p \| |x|^{-h-l} G(x) \|_r.$$

So the Generalized Young Inequality has been proved.

Put

$$P(x, t) = \int_0^t T(t-s) \frac{g(\xi, s)}{|\xi|^l} ds = \int_0^t \frac{ds}{(\sqrt{4\pi(t-s)})^n} \int_{E_n} e^{-\frac{(x-\xi)^2}{4(t-s)}} \frac{g(\xi, s)}{|\xi|^l} d\xi.$$

From Generalized Young's Inequality we get

COROLLARY 1. If $h < n(1 - \frac{1}{q}), l < n(1 - \frac{1}{p}), h + l \geq 0$, then

$$(8) \quad \| |x|^{-h} P(x, t) \|_{q'} \leq C \int_0^t (t-s)^{b_1} \|g(\xi, s)\|_p ds$$

where

$$(9) \quad b_1 = \frac{n}{2} \left(\frac{1}{r} - 1 \right) - \frac{h+l}{2}, \quad \frac{1}{r} = \frac{1}{p'} + \frac{1}{q'}.$$

Proof. From Generalized Young's Inequality, we get immediately

$$\| |x|^{-h} P(x, t) \|_{q'} \leq C \int_0^t \frac{1}{[4\pi(t-s)]^{n/2}} \|g(\xi, s)\|_p \left\| \frac{e^{-\frac{\xi^2}{4(t-s)}}}{|\xi|^{h+l}} \right\|_r ds.$$

But

$$\begin{aligned} \left\| \frac{e^{-\frac{\xi^2}{4(t-s)}}}{|\xi|^{h+l}} \right\|_r &= \left\{ \int_{E_n} \frac{e^{-\frac{r\xi^2}{4(t-s)}}}{|\xi|^{r(h+l)}} d\xi \right\}^{\frac{1}{r}} \\ &= C \left[\int_0^\infty \frac{e^{-\frac{rR^2}{4(t-s)}}}{R^{r(h+l)}} R^{n-1} dR \right]^{\frac{1}{r}}. \end{aligned}$$

Put $\bar{R} = \sqrt{\frac{r}{4(t-s)}}R$, noticing $h < n(1 - \frac{1}{q}), l < n(1 - \frac{1}{p})$, we know

$$\begin{aligned} \left\| \frac{e^{-\frac{\xi^2}{4(t-s)}}}{|\xi|^{h+l}} \right\|_r &\leq C(t-s)^{\frac{n}{2r} - \frac{h+l}{2}} \left[\int_0^\infty \frac{e^{-\bar{R}^2} d\bar{R}}{\bar{R}^{r(h+l)+n-1}} \right]^{\frac{1}{r}} \\ &\leq C(t-s)^{\frac{1}{2r} - \frac{h+l}{2}}. \end{aligned}$$

So

$$\begin{aligned} |||x|^{-h}P(x,t)||_{q'} &\leq C \int_0^t (t-s)^{\frac{n}{2}(\frac{1}{r}-1) - \frac{h+l}{2}} ||g(\xi,s)||_p ds \\ &= C \int_0^t (t-s)^{b_1} ||g||_p ds. \end{aligned}$$

Put

$$\begin{aligned} Q(x,t) &= \int_0^t \left[\frac{\partial}{\partial x_i} T(t-s) \right] \cdot \frac{g(\xi,s)}{|\xi|^l} ds \\ &= -\frac{1}{2} \int_0^t \frac{ds}{[4\pi(t-s)]^{n/2}} \int_{E_n} \frac{x_i - \xi_i}{t-s} \frac{g(\xi,s)}{|\xi|^l} d\xi. \end{aligned}$$

From Generalized Young's Inequality we get

COROLLARY 2. If $h < n(1 - \frac{1}{q}), k < n(1 - \frac{1}{p}), \frac{1}{r} = \frac{1}{p'} + \frac{1}{q'}$, then

$$(10) \quad |||x|^{-h}Q(x,t)||_{q'} \leq C \int_0^t (t-s)^{b_2} ||g||_p ds,$$

where

$$(11) \quad b_2 = \frac{n}{2}(\frac{1}{r} - 1) - \frac{h+l+1}{2}.$$

Proof. From Generalized Young's Inequality we get

$$\begin{aligned} |||x|^{-h}Q(x, t)||_{q'} &\leq C \int_0^t (t-s)^{n/2} \left\| \int_{E_n} \frac{|x_i - \xi_i|}{(t-s)} e^{-\frac{(x-\xi)^2}{4(t-s)}} \frac{|g(\xi, s)|}{|\xi|^l} d\xi \right\|_{q'} \\ &\leq C \int_0^t (t-s)^{-\frac{n}{2}-1} ||g(\xi, s)||_p \left\| \frac{e^{-\frac{|x|^2}{4(t-s)}}}{|x|^{h+l-1}} \right\|_r ds. \end{aligned}$$

By direct calculation, we have

$$\left\| \frac{e^{-\frac{x^2}{4(t-s)}}}{|x|^{h+l-1}} \right\|_r \leq C(t-s)^{\frac{n}{2r} - \frac{h+l-1}{2}}.$$

So

$$\begin{aligned} |||x|^{-h}Q(x, t)||_{q'} &\leq C \int_0^t (t-s)^{-\frac{n}{2}-1+\frac{1}{2r}-\frac{(h+l)-1}{2}} ||g||_p ds \\ &= C \int_0^t (t-s)^{b_2} ||g||_p ds. \end{aligned}$$

Corollary 2 is proved now.

§3. Proof of Theorem 1. We shall solve the integral equation (5) by iteration

$$u_n = T(t)\phi - \int_0^t T(t-s)f_0(u_{n-1}) - \sum_{i=1}^n \int_0^t \left[\frac{\partial}{\partial x_i} T(t-s) \right] f_i(u_{n-1}) ds.$$

Put

$$u_0 = T(t)\phi, \quad Fu = T(t)\phi - \int_0^t T(t-s)f_0(u) - \sum_{i=1}^n \int_0^t \left[\frac{\partial}{\partial x_i} T(t-s) \right] f_i(u) ds.$$

We take the fundamental space as

$$(12) \quad X = \left\{ u \in C([0, \infty), L_{q, k\alpha}), t^b u \in C([0, \infty), L_{p\gamma, \alpha}), \sup_{t>0} t^b ||u||_{p\gamma, \alpha} \leq \varepsilon \right\},$$

and define

$$(13) \quad ||u||_X = \sup_{t>0} ||u||_{q, k\alpha} + \sup_{t>0} t^b ||u||_{p\gamma, \alpha},$$

where

$$q = \frac{n(\gamma-1)}{2-\lambda\alpha}, \quad b = \frac{1}{\gamma-1} - \frac{n}{2p\gamma} - \frac{\lambda\alpha}{2(\gamma-1)}, \quad (0 < b < b\gamma < 1)$$

and some λ, k satisfying $0 \leq \lambda \leq \gamma - k, 1 \leq k \leq \beta$.

We are going to prove F is a contraction operator over X .

For any $u, v \in X$,

$$\begin{aligned}
 (14) \quad \|Fu - Fv\|_{q, k\alpha} &\leq \int_0^t \|T(t-s)(f_0(u) - f_0(v))\|_{q, k\alpha} ds \\
 &+ \sum_{i=1}^n \int_0^t \left\| \frac{\partial}{\partial x_i} T(t-s)(f_i(u) - f_i(v)) \right\|_{q, k\alpha} ds \\
 &= J_1 + J_2.
 \end{aligned}$$

$$\begin{aligned}
 (15) \quad J_1 &\leq \int_0^t \|T(t-s)(f_0(u) - f_0(v))\|_q ds + \int_0^t \|T(t-s)(f_0(u) - f_0(v))|x|^{k\alpha}\|_q ds \\
 &= J_{11} + J_{12}.
 \end{aligned}$$

$$\begin{aligned}
 (16) \quad J_{11} &= \int_0^t \|T(t-s)(f'_0(w)(u-v))\|_q ds \quad w \in (u, v) \\
 &\leq C_0 \int_0^t \left\| T(t-s) \frac{||\xi|^\alpha w|^\lambda |w|^{\gamma-\lambda-1} |u-v|}{|\xi|^{\lambda\alpha}} \right\|_q ds \\
 &\leq C_0 \int_0^t (t-s)^{\frac{n}{2q} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} |||\xi|^\alpha w|^\lambda |w|^{\gamma-\lambda-1} |u-v|||_p ds \\
 &\quad \left(\text{By Coro. 1, } \lambda\alpha < n(1 - \frac{1}{p}) \right) \\
 &\leq C_0 \int_0^t (t-s)^{\frac{n}{2q} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} |||\xi|^\alpha w|^\lambda_{p\gamma} ||w|^{\gamma-\lambda-1}|_{p\gamma} ||u-v||_{p\gamma} ds \\
 &\leq C_0 \int_0^t (t-s)^{\frac{n}{2q} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} ||w|^{\gamma-1}|_{p\gamma, \alpha} ||u-v||_{p\gamma} ds \\
 &= C_0 \int_0^t (t-s)^{\frac{n}{2q} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} s^{-b\gamma} (s^b ||w||_{p\gamma, \alpha})^{\gamma-1} (s^b ||u-v||_{p\gamma}) ds \\
 &\leq C(2\varepsilon)^{\gamma-1} t^{\frac{n}{2q} - \frac{n}{2p} - \frac{\lambda\alpha}{2} + 1 - b\gamma} \int_0^1 (1-s)^{\frac{n}{2q} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} s^{-b\gamma} ds \sup_{t>0} t^b ||u-v||_{p\gamma} \\
 &\leq C(2\varepsilon)^{\gamma-1} \sup_{t>0} t^b ||u-v||_{p\gamma},
 \end{aligned}$$

where we suppose

$$(17) \quad \frac{n}{2q} - \frac{n}{2p} - \frac{\lambda\alpha}{2} + 1 - b\gamma = 0.$$

$$\begin{aligned}
J_{12} &\leq C_0 \int_0^t \left\| T(t-s) \frac{||\xi|^\alpha w|^{\lambda+k-1} |w|^{\gamma-\lambda-k} |\xi|^\alpha (u-v)|x|^{k\alpha}}{|\xi|^{(\lambda+k)\alpha}} \right\|_q ds \\
&\leq C \int_0^t (t-s)^{\frac{n}{2q}-\frac{n}{2p}-\frac{\lambda\alpha}{2}} |||\xi|^\alpha w|^{\lambda+k-1} |w|^{\gamma-\lambda-k} |\xi|^\alpha |u-v|||_p ds \\
(18) \quad &\quad \left((\lambda+k)\alpha < n(1-\frac{1}{p}), 0 \leq \lambda+k \leq \gamma \right) \\
&\leq C \int_0^t (t-s)^{\frac{n}{2q}-\frac{n}{2p}-\frac{\lambda\alpha}{2}} |||\xi|^\alpha w|^{\lambda+k-1} |w|^{\gamma-\lambda-k} ||\xi|^\alpha |u-v|||_{p\gamma} ds \\
&\leq C \int_0^t (t-s)^{\frac{n}{2q}-\frac{n}{2p}-\frac{\lambda\alpha}{2}} ||w|^{\gamma-1} ||\xi|^\alpha (u-v)||_{p\gamma} ds \\
&\leq C(2\varepsilon)^{\gamma-1} \sup t^b ||u-v||_{p\gamma, \alpha}.
\end{aligned}$$

$$\begin{aligned}
J_2 &= \sum_{i=1}^n \int_0^t \left\| \frac{\partial}{\partial x_i} T(t-s)(f_i(u) - f_i(v)) \right\|_{q, k\alpha} ds \\
(19) \quad &= \sum_{i=1}^n \int_0^t \left\| \frac{\partial}{\partial x_i} T(t-s)(f_i(u) - f_i(v)) \right\|_q ds \\
&+ \sum_{i=1}^n \int_0^t \left\| \frac{\partial}{\partial x_i} T(t-s)(f_i(u) - f_i(v)) |x|^{k\alpha} \right\|_q ds \\
&= J_{21} + J_{22}.
\end{aligned}$$

$$\begin{aligned}
J_{21} &= \sum_{i=1}^n \int_0^t \left\| \frac{\partial}{\partial x_i} T(t-s) f'_i(w)(u-v) \right\|_q ds \quad w \in (u, v) \\
&\leq \sum_{i=1}^n C_i \int_0^t (t-s)^{-\frac{n}{2}-1} ds \left\| \int_{E_n} |x_i - \xi_i| e^{-\frac{(x-\xi)^2}{4(t-s)}} \frac{||\xi|^\alpha w|^\mu |w|^{\beta-\mu-1}}{|\xi|^{\mu\alpha}} |u-v| \right\|_q \\
(20) \quad &\leq \sum_{i=1}^n C_i \int_0^t (t-s)^{-\frac{n}{2}-1} ds \int_{E_n} \frac{|\xi_i|}{|\xi|^{\mu\alpha}} e^{-\frac{\xi^2}{4(t-s)}} d\xi ||\xi|^\alpha w|^\mu_{\beta q} ||w|^{\beta-\mu-1}|u-v||_{q\beta} \\
&\leq C \int_0^t (t-s)^{-\frac{1}{2}-\frac{\mu\alpha}{2}} s^{-b\beta} ds (s^b ||w||_{p\gamma, \alpha})^{\beta-1} (s^b ||u-v||_{p\gamma}) \\
&\leq C(2\varepsilon)^{\beta-1} t^{\frac{1}{2}-\frac{\mu\alpha}{2}-b\beta} \sup_{t>0} t^b ||u-v||_{p\gamma} \\
&= C(2\varepsilon)^{\beta-1} \sup_{t>0} t^b ||u-v||_{p\gamma},
\end{aligned}$$

where we take

$$(21) \quad \frac{1}{2} - \frac{\mu\alpha}{2} - b\beta = 0, \quad \beta q = p\gamma.$$

$$\begin{aligned}
J_{22} &\leq \sum \int_0^t \left\| \frac{\partial}{\partial x_i} T(t-s) f'_i(w) (u-v) |x|^{k\alpha} \right\|_q ds \\
&= \sum \int_0^t \left\| \left[\frac{\partial}{\partial x_i} T(t-s) \right] \frac{[|\xi|^{(k+\mu)\alpha-\alpha} f'(w)] |\xi|^\alpha (u-v)}{|\xi|^{(k+\mu)\alpha}} \right\|_q ds \\
&\leq C \int_0^t (t-s)^{-\frac{1}{2}-\frac{\mu\alpha}{2}} ||| |\xi|^\alpha w|^{\mu+k-1} |w|^{\beta-(\mu+k)} |\xi|^\alpha |u-v| |||_q ds \\
(22) \quad &\leq C \int_0^t (t-s)^{-\frac{1}{2}-\frac{\mu\alpha}{2}} ||| |\xi|^\alpha w |||_{\beta q}^{\mu+k-1} ||| w |||_{\beta q}^{\beta-(\mu+k)} ||| |\xi|^\alpha (u-v) |||_{\beta q} ds \\
&\leq C \int_0^t (t-s)^{-\frac{1}{2}-\frac{\mu\alpha}{2}} ||| w |||_{p\gamma, \alpha}^{\beta-1} ||| |\xi|^\alpha (u-v) |||_{p\gamma} ds \\
&\leq C \int_0^t (t-s)^{-\frac{1}{2}-\frac{\mu\alpha}{2}} s^{-b\beta} ds (2\varepsilon)^{\beta-1} \sup_{t>0} t^b ||| u-v |||_{p\gamma, \alpha} \\
&= C(2\varepsilon)^{\beta-1} t^{-\frac{1}{2}+1-\frac{\mu\alpha}{2}-\beta b} \sup_{t>0} t^b ||| u-v |||_{p\gamma, \alpha} \\
&= C(2\varepsilon)^{\beta-1} \sup_{t>0} t^b ||| u-v |||_{p\gamma, \alpha}.
\end{aligned}$$

So we have got the estimate

$$(23) \quad ||| Fu - Fv |||_{q, k\alpha} \leq M((2\varepsilon)^{\gamma-1} + (2\varepsilon)^{\beta-1}) \sup_{t>0} t^b ||| u-v |||_{p\gamma, \alpha}.$$

Still we want to estimate $t^b ||| Fu - Fv |||_{p\gamma, \alpha}$. We have

$$\begin{aligned}
(24) \quad t^b ||| Fu - Fv |||_{p\gamma, \alpha} &\leq t^b \int_0^t ||| T(t-s)(f_0(u) - f_0(v)) |||_{p\gamma, \alpha} ds \\
&\quad + \sum_{i=1}^n t^b \int_0^t \left\| \left[\frac{\partial}{\partial x_i} T(t-s) \right] (f_i(u) - f_i(v)) \right\|_{p\gamma, \alpha} ds \\
&= I_1 + I_2.
\end{aligned}$$

$$\begin{aligned}
(25) \quad I_1 &= t^b \int_0^t ||| T(t-s)(f_0(u) - f_0(v)) |||_{p\gamma} ds + t^b \int_0^t ||| T(t-s)(f_0(u) - f_0(v)) |||_{p\gamma} ds \\
&= I_{11} + I_{12}.
\end{aligned}$$

$$\begin{aligned}
(26) \quad I_{11} &= t^b \int_0^t \|T(t-s)f'_0(w)(u-v)\|_{p\gamma} ds \quad w \in (u, v) \\
&\leq C_0 t^b \int_0^t \left\| T(t-s) \frac{||\xi|^\alpha w|^\lambda |w|^{\gamma-\lambda-1} |u-v|}{|\xi|^{\lambda\alpha}} \right\|_{p\gamma} ds \\
&\leq C_0 t^b \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} |||\xi|^\alpha w|^\lambda |w|^{\gamma-\lambda-1} |u-v|||_p ds \\
&\leq C_0 t^b \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} |||\xi|^\alpha w|^\lambda_{p\gamma} ||w|^{\gamma-\lambda-1}|_{p\gamma} ||u-v||_{p\gamma} ds \\
&\leq C_0 t^b \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} ||w|^{\gamma-1}|_{p\gamma, \alpha} ||u-v||_{p\gamma} ds \\
&\leq (2\varepsilon)^{\gamma-1} C t^{\frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2} + 1 - (\gamma-1)b} \sup_{t>0} t^b ||u-v||_{p\gamma} \\
&= C(2\varepsilon)^{\gamma-1} \sup t^b ||u-v||_{p\gamma}.
\end{aligned}$$

where we take

$$(27) \quad \frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2} + 1 - (\gamma-1)b = 0.$$

$$\begin{aligned}
(28) \quad I_{12} &\leq t^b C_0 \int_0^t \|T(t-s)|w|^{\gamma-1}|u-v||x|^\alpha\|_{p\gamma} ds \\
&= C_0 t^b \int_0^t \left\| T(t-s) \frac{||\xi|^\alpha w|^\lambda |w|^{\gamma-\lambda-1} |\xi|^\alpha |u-v||x|^\alpha}{|\xi|^{(\lambda+1)\alpha}} \right\|_{p\gamma} ds \\
&\leq C_0 t^b \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} |||\xi|^\alpha w|^\lambda |w|^{\gamma-\lambda-1} |\xi|^\alpha |u-v||_p ds \\
&\leq C_0 t^b \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} |||\xi|^\alpha w|^\lambda_{p\gamma} ||w|^{\gamma-\lambda-1}|_{p\gamma} |||\xi|^\alpha |u-v||_{p\gamma} ds \\
&\leq C(2\varepsilon)^{\gamma-1} \sup_{t>0} t^b ||u-v||_{p\gamma, \alpha},
\end{aligned}$$

by (27).

$$\begin{aligned}
(29) \quad I_2 &= \sum_{i=1}^n t^b \int_0^t \left\| \left[\frac{\partial}{\partial x_i} T(t-s) \right] (f_i(u) - f_i(v)) \right\|_{p\gamma} \\
&\quad + \sum_{i=1}^n t^b \int_0^t \left\| \frac{\partial}{\partial x_i} T(t-s) (f_i(u) - f_i(v)) |x|^\alpha \right\|_{p\gamma} ds \\
&= I_{21} + I_{22}
\end{aligned}$$

$$\begin{aligned}
(30) \quad I_{21} &= t^b \sum_{i=1}^n \int_0^t \left\| \frac{\partial}{\partial x_i} T(t-s)(f'_i(w)(u-v)) \right\|_{p\gamma} ds \quad w \in (u, v) \\
&\leq t^b \sum_{i=1}^n C_i \int_0^t (t-s)^{-\frac{n}{2}-1} \left\| \int_{E_n} |x_i - \xi_i| e^{-\frac{(x-\xi)^2}{4(t-s)}} \frac{|w|^{\beta-1} |u-v| |\xi|^{\mu\alpha}}{|\xi|^{\mu\alpha}} \right\|_{\beta q} ds \\
&\leq C t^b \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2q} - \frac{\mu\alpha}{2} - \frac{1}{2}} \left\| |\xi|^{\mu\alpha} |w|^{\beta-1} |u-v| \right\|_q ds \\
&\leq C t^b \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2q} - \frac{\mu\alpha}{2} - \frac{1}{2}} s^{-b\beta} (s^b \|w\|_{\beta q, \alpha})^{\beta-1} (s^b \|u-v\|_{\beta q, \alpha}) ds \\
&\leq C(2\varepsilon)^{\beta-1} t^{\frac{n}{2p\gamma} - \frac{n}{2q} - \frac{\mu\alpha}{2} + \frac{1}{2} - (\beta-1)b} \sup_{t>0} t^b \|u-v\|_{p\gamma, \alpha} \\
&= C(2\varepsilon)^{\beta-1} \sup_{t>0} t^b \|u-v\|_{p\gamma, \alpha},
\end{aligned}$$

by (17), (21), (27).

$$\begin{aligned}
(31) \quad I_{22} &= t^b \sum_{i=1}^n \int_0^t \left\| \frac{\partial}{\partial x_i} T(t-s)(f'_i(w)(u-v)|x|^\alpha) \right\|_{p\gamma} ds \quad w \in (u, v) \\
&\leq t^b \sum_{i=1}^n C_i \int_0^t (t-s)^{-\frac{n}{2}-1} \left\| \int_{E_n} |x_i - \xi_i| e^{-\frac{(x-\xi)^2}{4(t-s)}} \times \right. \\
&\quad \times \frac{||\xi|^\alpha w|^\mu |w|^{\beta-\mu-1} |\xi|^\alpha |u-v| |x|^\alpha}{|\xi|^{(\mu+1)\alpha}} d\xi \left. \right\|_{\beta q} ds \\
&\leq t^b C \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2q} - \frac{\mu\alpha}{2} - \frac{1}{2}} \left\| |\xi|^\alpha w \right\|_{\beta q}^\mu \|w\|_{\beta q}^{\beta-\mu-1} \left\| |\xi|^\alpha (u-v) \right\|_{\beta q} ds \\
&= C t^b \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2q} - \frac{\mu\alpha}{2} - \frac{1}{2}} s^{-b\beta} ds (s^b \|w\|_{p\gamma, \alpha})^{\beta-1} (s^b \|u-v\|_{p\gamma, \alpha}) \\
&\leq C(2\varepsilon)^{\beta-1} t^{\frac{n}{2p\gamma} - \frac{n}{2q} - \frac{\mu\alpha}{2} - \frac{1}{2} - (\beta-1)b} \sup_{t>0} t^b \|u-v\|_{p\gamma, \alpha} \\
&= C(2\varepsilon)^{\beta-1} \sup_{t>0} t^b \|u-v\|_{p\gamma, \alpha},
\end{aligned}$$

by (17), (21), (27).

So there exists constant N such that

$$t^b \|Fu - Fv\|_{p\gamma, \alpha} \leq N((2\varepsilon)^{\gamma-1} + (2\varepsilon)^{\beta-1}) \sup_{t>0} t^b \|u-v\|_{p\gamma, \alpha}.$$

So for ε sufficiently small, we can find $\delta, 0 < \delta < 1$, such that

$$(32) \quad \|Fu - Fv\|_X \leq \delta \|u-v\|_X, \quad 0 < \delta < 1.$$

Now from (17), (21), (27) we need to solve these equations

$$(17) \quad \frac{n}{2q} - \frac{n}{2p} - \frac{\lambda\alpha}{2} + 1 - \gamma b = 0,$$

$$(21) \quad \frac{1}{2} - \frac{\mu\alpha}{2} - b\beta = 0, \quad \beta q = p\gamma,$$

$$(27) \quad \frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2} + 1 - (\gamma - 1)b = 0,$$

with inequalities:

$$(33) \quad \begin{aligned} 0 \leq \lambda \leq \gamma - k, \quad 0 \leq \mu \leq \beta - k, \quad \beta \geq k \geq 1, \\ (\lambda + k)\alpha < n(1 - \frac{1}{p}), \quad (\mu + k)\alpha < n(1 - \frac{1}{q}). \end{aligned}$$

It is easy to find that

$$\begin{aligned} b &= \frac{1}{\gamma - 1} - \frac{n}{2p\gamma} - \frac{\lambda\alpha}{2(\gamma - 1)}, \\ q &= \frac{n(\gamma - 1)}{2 - \lambda\alpha}, \quad \beta = \frac{\gamma + 1 - \lambda\alpha - \mu\alpha(\gamma - 1)}{2 - \lambda\alpha}, \\ p &= \frac{n(\gamma - 1)}{\gamma} \cdot \frac{\gamma + 1 - \lambda\alpha - \mu\alpha(\gamma - 1)}{(2 - \lambda\alpha)^2}. \end{aligned}$$

If $\alpha = 0$, then $\beta = \frac{\gamma+1}{2}$. From $\gamma > \frac{2+\sqrt{n^2+4}}{n}$, (supposition) we get

$$p = \frac{n(\gamma^2 - 1)}{4\gamma} > 1.$$

So we find

$$1 < p < q < p\gamma.$$

All the inequalities (33) are satisfied with $k = 1$. Estimate (32) holds. So for small α , these inequalities are also satisfied.

Concretely, for $\frac{\gamma+1-\gamma\lambda_1\alpha_1}{2-\lambda_1\alpha_1} < \beta < \frac{\gamma+1-\lambda_0\alpha_0}{2-\lambda_0\alpha_0}$, the estimate (32) holds, where $\lambda_0 = \gamma - 1, \alpha_0 = \min\{\frac{1}{\gamma}(n - \frac{4\gamma}{\gamma^2-1}), \frac{1}{\gamma-1}\}, \lambda_1 = \frac{(\gamma-1)\alpha_1+2-\sqrt{(\gamma-1)^2\alpha_1^2+4}}{2\alpha_1}, \alpha_1 = \frac{2}{\gamma+1} \min\{\frac{4}{\gamma+1}, n - \frac{4\gamma}{\gamma^2-1}\}$.

In fact,

- 1) if $\frac{\gamma+1}{2} < \beta < \frac{\gamma+1-\lambda_0\alpha_0}{2-\lambda_0\alpha_0}$, then we may take $\lambda = \lambda_0, 0 < \alpha < \alpha_0, \mu = 0, k = 1, \beta = \frac{\gamma+1-\lambda_0\alpha}{2-\lambda_0\alpha}, p = \frac{n(\gamma-1)}{\gamma} \frac{\gamma+1-\lambda_0\alpha}{(2-\lambda_0\alpha)^2}$. We have $p > \frac{n(\gamma^2-1)}{4\gamma} > \frac{n}{n-\gamma\alpha} = \frac{n}{n-(\lambda_0+1)\alpha}, \beta < \gamma$. All the inequalities are satisfied;

2) if $\frac{\gamma+1-\gamma\lambda_1\alpha_1}{2-\lambda_1\alpha_1} < \beta \leq \frac{\gamma+1}{2}$, then we may take $\lambda = \lambda_1, 0 < \alpha < \alpha_1, \lambda = \mu, k = 1, \beta = \frac{\gamma+1-\gamma\lambda_1\alpha_1}{2-\lambda_1\alpha_1}, p = \frac{n(\gamma-1)(\gamma+1-\gamma\lambda_1\alpha_1)}{\gamma(2-\lambda_1\alpha_1)^2}$. We have $\beta < \frac{\gamma+1}{2} < \gamma, \lambda_1 + 1 = \frac{\gamma+1-\gamma\lambda_1\alpha_1}{2-\lambda_1\alpha_1} < \frac{\gamma+1}{2}$, so $\lambda_1\alpha < \lambda_1\alpha_1 < \frac{4}{\gamma+1}$. So $p > \frac{n(\gamma^2-1)}{4\gamma}$. On the other hand, $(\lambda_1 + k)\alpha = (\lambda_1 + 1)\alpha < \frac{\gamma+1}{2}\alpha_1 \leq n - \frac{4\gamma}{\gamma^2-1}$, so $p > \frac{n}{n-(\lambda_1+k)\alpha}$, i.e. $(\lambda_1 + k)\alpha < n(1 - \frac{1}{p})$. All the inequalities are satisfied. So we have (32).

Still we need to prove for any $u \in X, Fu \in X$ also. In fact, we see

$$\begin{aligned} t^b \|T(t)\phi\|_{p\gamma} &\leq C t^b t^{\frac{n}{2p\gamma} - \frac{n}{2q}} \|\phi\|_q = C \|\phi\|_q, \\ t^b \|T(t)\phi \cdot |x|^\alpha\|_{p\gamma} &= t^b \left\| |x|^\alpha T(t) \frac{|\xi|^\alpha \phi}{|\xi|^\alpha} \right\|_{p\gamma} \\ &\leq C t^b t^{\frac{n}{2p\gamma} - \frac{n}{q}} \| |\xi|^\alpha \phi \|_q = C \| |\xi|^\alpha \phi \|_q \leq C \|\phi\|_{q, k\alpha}. \end{aligned}$$

So

$$(34) \quad t^b \|T(t)\phi\|_{p\gamma, \alpha} \leq 2C \|\phi\|_{q, k\alpha}.$$

It is easy to get

$$(35) \quad \|T(t)\phi\|_{q, k\alpha} \leq 2C \|\phi\|_{q, k\alpha}.$$

So if $\|\phi\|_{q, k\alpha}$ is sufficiently small, $T(t)\phi \in X$ also.

For the estimation of Fu , we have

$$\begin{aligned} t^b \|Fu\|_{p\gamma, \alpha} &\leq t^b \|T(t)\phi\|_{p\gamma, \alpha} + t^b \int_0^t \|T(t-s)f_0(u)\|_{p\gamma, \alpha} ds \\ &\quad + \sum_{i=1}^n t^b \int_0^t \left\| \frac{\partial}{\partial x_i} T(t-s)f_i(u) \right\|_{p\gamma, \alpha} ds. \end{aligned}$$

Similar to the estimation of I_1, I_2 , we have

$$\begin{aligned} t^b \int_0^t \|T(t-s)f_0(u)\|_{p\gamma, \alpha} ds &\leq N(\sup_{t>0} t^b \|u\|_{p\gamma, \alpha})^\gamma, \\ t^b \int_0^t \left\| \frac{\partial}{\partial x_i} T(t-s)f_i(u) \right\|_{p\gamma, \alpha} ds &\leq N(\sup_{t>0} t^b \|u\|_{p\gamma, \alpha})^\beta. \end{aligned}$$

So we have

$$\sup_{t>0} \|t^b Fu\|_{p\gamma, \alpha} \leq 2C \|\phi\|_{q, k\alpha} + N(\sup_{t>0} t^b \|u\|_{p\gamma, \alpha})^\gamma + N(\sup_{t>0} t^b \|u\|_{p\gamma, \alpha})^\beta.$$

Put

$$2C \|\phi\|_{L_{q, k\alpha}} \leq \frac{\varepsilon}{2}, \quad N(\varepsilon^{\gamma-1} + \varepsilon^{\beta-1}) < \frac{1}{2}.$$

We get

$$(36) \quad \sup_{t>0} t^b \|Fu\|_{p\gamma, \alpha} < \varepsilon.$$

Again similar to the estimation of J_1, J_2 , we have

$$\|Fu\|_{q, k\alpha} \leq 2C\|\phi\|_{q, k\alpha} + M\{\sup_{t>0} t^b \|u\|_{p\gamma, \alpha}\}^\gamma + M\{\sup_{t>0} t^b \|u\|_{p\gamma, \alpha}\}^\beta.$$

So $Fu \in X$.

So $F : X \longrightarrow X$ is a contraction operator. We have a fixed point u , such that

$$(37) \quad u = Fu,$$

which is the desired global solution.

§4. Proof of Theorem 2. Now $f_i(u) \equiv 0, i = 1, \dots, n$, so the integral equation becomes

$$u = T(t)\phi - \int_0^t T(t-s)f_0(u)ds.$$

Put

$$(38) \quad Fu = T(t)\phi - \int_0^t T(t-s)f_0(u)ds,$$

$$(39) \quad \begin{aligned} q_1 &= \frac{n(\gamma-1)}{2-\gamma\lambda\alpha}, \quad 1 \leq k \leq \gamma, \quad 0 \leq \lambda \leq k-1, \\ b &= \frac{1}{\gamma-1} - \frac{n}{2p\gamma} - \frac{\lambda\alpha}{2(\gamma-1)}. \end{aligned}$$

It is easy to prove

$$0 < b < b\gamma < 1.$$

We need to choose p , such that

$$\frac{n}{n-(\lambda+1)\alpha} < p < q_1 < p\gamma.$$

Because now $\frac{n}{2}(\lambda-1) > 1$, so for small α , we have

$$\frac{n}{n-(\lambda+1)\alpha} < q_1,$$

so p actually exists.

For $u, v \in X$,

$$\begin{aligned}
\|Fu - Fv\|_{q_1, k\alpha} &\leq \int_0^t \|T(t-s)\{f_0(u) - f_0(v)\}\|_{q_1, k\alpha} ds \\
(40) \quad &= \int_0^t \|T(t-s)(f_0(u) - f_0(v))\|_{q_1} ds + \int_0^t \|T(t-s)(f_0(u) - f_0(v))|x|^{k\alpha}\|_{q_1} ds \\
&= J'_1 + J'_2
\end{aligned}$$

$$\begin{aligned}
J'_1 &= \int_0^t \|T(t-s)f'_0(w)(u-v)\|_{q_1} ds \quad w \in (u, v) \\
(41) \quad &\leq C_0 \int_0^t (t-s)^{\frac{n}{2q_1} - \frac{n}{2p}} \| |w|^{\gamma-1}(u-v) \|_p ds \\
&\leq C_0 \int_0^t (t-s)^{\frac{n}{2q_1} - \frac{n}{2p}} \|w\|_{p\gamma}^{\gamma-1} \|u-v\|_{p\gamma} ds \\
&\leq C(2\varepsilon)^{\gamma-1} \sup_{t>0} t^b \|u-v\|_{p\gamma},
\end{aligned}$$

because

$$(42) \quad \frac{n}{2q_1} - \frac{n}{2p} - b\gamma + 1 = 0.$$

$$\begin{aligned}
J'_2 &= \int_0^t \|T(t-s)f'_0(w)(u-v)|x|^{k\alpha}\|_{q_1} ds \quad w \in (u, v) \\
(43) \quad &\leq C_0 \int_0^t \left\| T(t-s) \frac{\|\xi\|^\alpha w^{k-1} |w|^{\gamma-k} \|\xi\|^\alpha (u-v)}{\|\xi\|^{k\alpha}} |x|^{k\alpha} \right\|_{q_1} ds \\
&\leq C_0 \int_0^t (t-s)^{\frac{n}{2q_1} - \frac{n}{2p}} \| |\xi|^\alpha w^{k-1} |w|^{\gamma-k} \|\xi\|^\alpha (u-v) \|_p ds \\
&\leq C_0 \int_0^t (t-s)^{\frac{n}{2q_1} - \frac{n}{2p}} \| |\xi|^\alpha w \|_{p\gamma}^{k-1} \|w\|_{p\gamma}^{\gamma-k} \| |\xi|^\alpha (u-v) \|_{p\gamma} ds \\
&\leq C(2\varepsilon)^{\gamma-1} \sup_{t>0} t^b \|u-v\|_{p\gamma, \alpha}
\end{aligned}$$

$$\begin{aligned}
t^b \|Fu - Fv\|_{p\gamma, \alpha} &\leq t^b \int_0^t \|T(t-s)(f_0(u) - f_0(v))\|_{p\gamma, \alpha} ds \\
(44) \quad &= t^b \int_0^t \|T(t-s)(f_0(u) - f_0(v))\|_{p\gamma} ds \\
&\quad + t^b \int_0^t \|T(t-s)(f_0(u) - f_0(v))|x|^\alpha\|_{p\gamma} ds \\
&= I'_1 + I'_2.
\end{aligned}$$

$$\begin{aligned}
(45) \quad I'_1 &\leq C_0 t^b \int_0^t \|T(t-s)|w|^{\gamma-1}(u-v)\|_{p\gamma} ds \\
&= C_0 t^b \int_0^t \left\| T(t-s) \frac{||\xi|^\alpha w|^\lambda |w|^{\gamma-\lambda-1} |u-v|}{|\xi|^{\lambda\alpha}} \right\|_{p\gamma} ds \\
&\leq C_0 t^b \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} |||\xi|^\alpha w|^\lambda |w|^{\gamma-\lambda-1} |u-v|||_{p\gamma} ds \\
&\leq C_0 t^b \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} |||\xi|^\alpha w|^\lambda_{p\gamma} ||w|^{\gamma-\lambda-1}|u-v||_{p\gamma} ds \\
&\leq C_0 t^b \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} ||w|^{\gamma-1}|u-v||_{p\gamma, \alpha} ds \\
&\leq C(2\varepsilon)^{\gamma-1} t^{\frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2} + 1 - b\gamma} \sup_{t>0} t^b ||u-v||_{p\gamma} \\
&= C(2\varepsilon)^{\gamma-1} \sup_{t>0} t^b ||u-v||_{p\gamma}
\end{aligned}$$

because

$$(46) \quad \frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2} + 1 - b\gamma = 0.$$

$$\begin{aligned}
(47) \quad I'_2 &\leq C_0 t^b \int_0^t \|T(t-s)|w|^{\gamma-1}|u-v||x|^\alpha\|_{p\gamma} ds \\
&= C_0 t^b \int_0^t \left\| T(t-s) \frac{||\xi|^\alpha w|^\lambda |w|^{\gamma-\lambda-1} (|\xi|^\alpha |u-v|)|x|^\alpha}{|\xi|^{(\lambda+1)\alpha}} \right\|_p ds \\
&\leq C_0 t^b \int_0^t (t-s)^{\frac{n}{2p\gamma} - \frac{n}{2p} - \frac{\lambda\alpha}{2}} |||\xi|^\alpha w|^\lambda |w|^{\gamma-\lambda-1} |\xi|^\alpha |u-v|||_p ds \\
&\leq C(2\varepsilon)^{\gamma-1} \sup_{t>0} t^b ||u-v||_{p\gamma, \alpha},
\end{aligned}$$

by (46).

So if ε is sufficiently small, we have

$$(48) \quad \|Fu - Fv\|_X \leq \delta \|u - v\|_X, \quad 0 < \delta < 1.$$

Still we need to prove for any $u \in X$, $Fu \in X$ also. Firstly we notice

$$(49) \quad t^b \|T(t)\phi\|_{p\gamma, \alpha} = t^b \|T(t)\phi\|_{p\gamma} + t^b \|T(t)\phi|x|^\alpha\|_{p\gamma}.$$

Now

$$\begin{aligned}
t^b \|T(t)\phi\|_{p\gamma} &= t^b \left\| T(t) \frac{|\xi|^{\lambda\alpha} \phi}{|\xi|^{\lambda\alpha}} \right\|_{p\gamma} \\
&\leq C t^b t^{\frac{1}{2p\gamma} - \frac{1}{2q_1} - \frac{\lambda\alpha}{2}} |||\xi|^{\lambda\alpha} \phi||_{q_1} \\
&= C |||\xi|^{\lambda\alpha} \phi||_{q_1} \leq C \|\phi\|_{q_1, k\alpha},
\end{aligned}$$

by (42), (46).

$$\begin{aligned}
t^b \|T(t)\phi|x|^\alpha\|_{p\gamma} &= t^b \left\| T(t) \frac{|\xi|^{(\lambda+1)\alpha}\phi}{|\xi|^{(\lambda+1)\alpha}} |x|^\alpha \right\|_{p\gamma} \\
&\leq C t^b t^{\frac{n}{2p\gamma} - \frac{n}{2q_1} - \frac{\lambda\alpha}{2}} \| |\xi|^{(\lambda+1)\alpha}\phi \|_{q_1} \\
&= C \| |\xi|^{(\lambda+1)\alpha}\phi \|_{q_1} \leq C \|\phi\|_{q_1, k\alpha}, \quad (\lambda + 1 \leq k)
\end{aligned}$$

by (42), (46).

So

$$(50) \quad t^b \|T(t)\phi\|_{p\gamma, \alpha} \leq 2C \|\phi\|_{q_1, k\alpha}.$$

It is easy to see

$$(51) \quad \|T(t)\phi\|_{q_1, k\alpha} \leq 2C \|\phi\|_{q_1, k\alpha}.$$

So

$$(52) \quad T(t)\phi \in X.$$

Hence

$$\begin{aligned}
(53) \quad t^b \|Fu\|_{p\gamma, \alpha} &\leq t^b \|T(t)\phi\|_{p\gamma, \alpha} + t^b \int_0^t \|T(t-s)f_0(u)\|_{p\gamma, \alpha} ds \\
&\leq 2C \|\phi\|_{q_1, k\alpha} + C_0 t^b \int_0^t \|T(t-s)|u|^\gamma\|_{p\gamma, \alpha} ds
\end{aligned}$$

Similar to the estimation of I'_1, I'_2 , we see

$$(54) \quad t^b \|Fu\|_{p\gamma, \alpha} \leq 2C \|\phi\|_{q_1, k\alpha} + M \left[\sup_{t>0} t^b \|u\|_{p\gamma, \alpha} \right]^\gamma.$$

Then

$$\sup_{t>0} t^b \|Fu\|_{p\gamma, \alpha} \leq 2C \|\phi\|_{q_1, k\alpha} + M \left[\sup_{t>0} t^b \|u\|_{p\gamma, \alpha} \right]^\gamma.$$

If we take

$$\begin{aligned}
2C \|\phi\|_{q_1, k\alpha} &\leq \frac{\varepsilon}{2}, \\
M 2^\gamma \varepsilon^{\gamma-1} &\leq \frac{1}{2},
\end{aligned}$$

then if

$$\sup_{t>0} t^b \|u\|_{p\gamma, \alpha} \leq \varepsilon,$$

we have

$$(55) \quad \sup t^b \|Fu\|_{p\gamma, \alpha} \leq \varepsilon.$$

Similarly we can prove

$$(56) \quad \|Fu\|_{q_1, k\alpha} \leq 2C\|\phi\|_{q_1, k\alpha} + M \left[\sup_{t>0} t^b \|u\|_{p\gamma, \alpha} \right]^\gamma$$

So $F : X \longrightarrow X$ is a contraction operator. There exists a fixed point u , such that

$$(57) \quad Fu = u.$$

The function u is the global solution (generalized) of the problem (4'),(2).

Acknowledgment. I am very grateful to IMA, University of Minnesota, and the Director, Professor Avner Friedman, for some results of this paper were obtained when the author was visiting the Institute.

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