ANALYSIS OF A DYNAMIC, DECENTRALIZED EXCHANGE ECONOMY

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17	David Wolkind	A Nonlinear Stability Analysis of a Model Equation for Alloy Solidification
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19	Henry C. Simpson Scott J. Spector	On the Modified Bessel Functions of the First Kind On Barrelling for a Material in Finite Elasticity
20	George R. Sell	Linearization and Global Dynamics
21	P. Constantin C. Foias	Global Lyapunov Exponents, Kaplan-Yorke Formulas and the Dimension of the Attractors for 2D Navier-Stokes Equations
22	Milan Miklavcic	Stability for Semilinear Parabolic Equations with Noninvertible Linear Operator
23	P. Collet, H. Epstein G. Gallavotti	Perturbations of Geodesic Flows on Surfaces of Constant Negative Curvature and their Mixing Properties
24	J.E. Dunn, J. Serrin	On the Thermomechanics of Interstitial working
25	Scott J. Spector	On the Absence of Bifurcation for Elastic Bars in Uniaxial Tension
26	W.A. Coppel	Maps on an Interval
27	James Kirkwood	Phase Transitions in the Ising Model with Traverse Field
28	Luis Magalhaes	The Asymptotics of Solutions of Singularly Perturbed Functional Differential Equations: and Concentrated Delays are Different
29	Charles Tresser	Homoclinic Orbits for Flow in $\mathbb{R}^3$
30	Charles Tresser	About some Theorems by L.P. Sil'nikov
31	Michael Aizenmann	On the Renormalized Coupling Constant and the Susceptibility in ψ_4 Field Theory and the Ising Model in Four Dimensions
32	C. Eugene Wayne	The KAM Theory of Systems with Short Range Interactions I

(continued on back cover)

ANALYSIS OF A DYNAMIC, DECENTRALIZED EXCHANGE ECONOMY*

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A dynamic exchange economy model is presented. Similarly to the Walrasian equilibrium problem, each consumer is characterized by a feasible set and by an instantaneous demand function, that depends on the price vector, time, and the commodity holding. The commodity holding of each consumer varies according to his instantaneous demand function at each moment. We show that the market can choose prices that will lead the commodity holding of each consumer to remain in his consumption set while the aggregate commodity holding satisfies the scarcity constraints of the market.

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1. Introduction

We present a model for a Dynamic Exchange Economy. We show that the market, at each moment, can adjust prices in such a way as to lead the consumers to make choices satisfying the scarcity constraints. The model is decentralized in the sense that each consumer makes his decisions independent of the rest of the consumers; the only information the consumer receives from the market is the price vector.

This result is well known in the framework of a Static Equilibrium. The model there is described by a set of available commodity bundles $M \subset \mathbb{R}^m$ and by n consumers $i = 1, \dots, n$. The set of normalized positive prices is $S = \{p \in \mathbb{R}_+^m \mid \|p\| = 1\}$ ¹, where $\mathbb{R}_+^m$ denotes the positive orthant. For each price vector the total revenue, defined as $r(p) := \sup_{w \in M} \langle p, w \rangle$, is allotted to the consumers; consumer i receives an income $r_i(p)$, where $r(p) = \sum_{i=1}^n r_i(p)$. Each consumer is represented by his demand function $d_i(p, r)$, which to each price vector and to each income associates a basket of goods (the demand). This demand function satisfies the Walras law: for each price vector and for each income the value of the demand does not exceed the income, i.e., $\langle p, d_i(p, r) \rangle \leq r$. The problem is then to find a price vector $\bar{p} \in S$, called a Walrasian equilibrium, such that the aggregate demand satisfies the scarcity constraints of the market, i.e., $\sum_{i=1}^n d_i(\bar{p}, r_i(\bar{p})) \in M$. It is possible to show that if M is closed and convex with $M = M - \mathbb{R}_+^m$, and the functions $p \rightarrow d_i(p, r_i(p))$ are continuous on S , then such a Walrasian equilibrium exists (see [2], p. 378).

In the pure exchange economy (with free disposal), $M = w - \mathbb{R}_+^m$, where $w = \sum_{i=1}^n x_i^0$ and x_i^0 is the endowment of consumer i . Then $r_i(p) = \langle p, x_i^0 \rangle$ and $r(p) = \langle p, w \rangle$ for each price vector p .

¹ Later on the set S will be redefined to be the same set but without its boundary.

We propose a non tâtonnement process that in some sense corresponds to the dynamic version of the above static model. Our main result is then a dynamic analogue to the existence theorem of a Walrasian equilibrium. The model we study is essentially the same as that described by Aubin [4]. In his model, each consumer is represented by a feasible set, and by an instantaneous demand function, that for each commodity holding and each price vector gives the desired change in the commodity holding, in a unit of time. Our model is more flexible; it also allows the instantaneous demand functions to depend on time. He shows that if the instantaneous demand functions are linear in the price vector, at each moment the market can set prices that will make the aggregate commodity holding satisfy the scarcity constraints, and on the other hand, will make the commodity holding of each consumer remain in his feasible set. His economic model can be summarized by the following evolution inclusion:

$$\left\{ \begin{array}{ll} \frac{dx}{dt}(t) \in D(x(t)) & \text{a.e. } (0, T) \\ x(t) \in K & \text{for all } t \in [0, T] \\ x(0) = x^0 & (x^0 \in K \text{ given}) \end{array} \right. .$$

To show the existence of a solution for the above evolution inclusion, also called trajectory, Aubin, Cellina and Nohel [1] require K to be a convex and compact subset of $\mathbb{R}^m$, D to be an upper hemicontinuous multifunction from K into $\mathbb{R}^m$ with nonempty closed convex images, and $D(x) \cap T_K(x) \neq \emptyset$ for each $x \in K$, where $T_K(x)$ denotes the tangent cone to K at x . In Aubin's model, linearity of the instantaneous demand functions in the price vector is the only way to guarantee that D has convex images. We have experimented with some small-scale problems and found out that consumers with this type of instantaneous demand functions behave "irrationally"; they lose all of their assets as time goes to infinity. Our aim is then to construct a solution to the above evolution inclusion, without assuming that the instantaneous demand functions are linear in the price vector. In fact, we will dispense with most of the convexity assumptions in [4].

Aubin [3] discusses other possibilities. To relax the convexity assumption on the images of D , he requires that D be continuous (upper and lower semicontinuous) on D and that $D(x) \subset T_K(x)$ for all $x \in K$. But this last condition amounts to requiring that the aggregate instantaneous demand never point in an inadmissible direction for the set of available commodity bundles M . This is the same as asking that each consumer know the behavior of the other consumers and the boundary of M .

In this setting, we show the existence of a solution to the above evolution equation without requiring D to have convex images but also without assuming that $D(x) \subset T_K(x)$ for all $x \in K$. Moreover, we will show that there exist price mechanisms $p: K \rightarrow S$ with a unique associated trajectory. In this sense, these price mechanisms act as regulatory mechanisms, prompting consumers to respect the scarcity constraints of the market.

We state the model and its assumptions in Section 2. In Section 3 we prove the existence of a solution and study the behavior of the admissible set of prices at each $x \in K$ and $t \geq 0$, as a function of the state of the economy, i.e., of (x, t) . Improving upon these results, in Section 4 we show that there is a continuous selection of prices, as a function of (x, t) , such that for any starting point $x^0 \in K$, there is a unique trajectory x with $x(0) = x^0$. Finally, in Section 5 we study the asymptotic behavior of the solutions to our model, in the particular case that each consumer uses the myopic strategy of utility maximization at each moment.

2. The model: notation and assumptions

The model we present here is closely related to that described by Aubin in [4]. Some of the assumptions have been relaxed and the linearity condition on the instantaneous demand functions has been removed. The linearity condition has the unfortunate consequence that the instantaneous demand functions remain bounded, even when prices approach 0. Assumptions ix) of this section and I of the next section are not found in his model. Assumption ix) is similar to the classical assumption that the demand goes to $+\infty$ when the price goes to 0. We will discuss these assumptions further at the end of Section 4.

The economic model consists of n consumers and a set of available commodity bundles. We may think of it as a stock exchange market. Each consumer holds a subset of the commodities and, by making transactions in a market, is able to change his holding. The market specifies prices at which commodities may be exchanged.

Let $\mathbb{R}^m$ be the commodity space and $M \subset \mathbb{R}^m$ be the set of available commodity bundles. Consumer i is characterized by his consumption set $L_i \subset \mathbb{R}^m$ and his instantaneous demand function $d_i : L_i \times [0, \infty) \times S \rightarrow \mathbb{R}^m$, where

$$S = \left\{ p \in \mathbb{R}^m \mid p_j > 0 \quad j = 1, \dots, m \quad \text{and} \quad \|p\| = 1 \right\}.$$

For each commodity holding $x_i \in L_i$, time t , and price vector $p \in S$, $d_i(x_i, t, p)$ represents the desired change in the i -th agent's commodity holding, in a unit of time. We extend d_i to $L_i \times [0, \infty) \times \text{int} \mathbb{R}_+^m$ by defining $d_i(x_i, t, \alpha p) := d_i(x_i, t, p)$ for each $x_i \in L_i$, $t \geq 0$, $p \in S$, and $\alpha > 0$.

At time $t = 0$ consumer i starts with a commodity holding $x_i^0 \in L_i$ (the endowment). We shall try to construct n functions $x_i : [0, T] \rightarrow L_i$, $i = 1, \dots, n$, where $x_i(t)$ represents the commodity holding of consumer i at time t , and a function $p : [0, T] \rightarrow S$ such that, at each moment, $\dot{x}_i(t)$ is equal to $d_i(x_i(t), t, p(t))$, and the aggregate commodity holding $\sum_{i=1}^n x_i(t)$ remains in the feasible set M . The functions x_i are called trajectories. $T > 0$ is a given, arbitrary number.

Let Z be a subset of $\mathbb{R}^m$. Then $\text{conv } Z$ will denote the convex hull of Z , and $\text{aff } Z$ will denote the affine hull of Z . We will denote by $B(Z, \epsilon)$ the ϵ -neighborhood of Z , i.e.,

$$B(Z, \epsilon) := \left\{ x \in \mathbb{R}^m \mid N(x, Z) < \epsilon \right\} ,$$

where

$$N(x, Z) := \inf \left\{ \|x - z\| \mid z \in Z \right\} .$$

The Hausdorff distance $\Delta(Y, Z)$ between two subsets Y and Z of $\mathbb{R}^m$ is

$$\Delta(Y, Z) := \max \left\{ \sup_{y \in Y} N(y, Z), \sup_{z \in Z} N(z, Y) \right\} .$$

Let us define the linear function $A : (\mathbb{R}^m)^n \rightarrow \mathbb{R}^m$ by $Ax := \sum_{i=1}^n x_i$, where $x = (x_1, \dots, x_n)$ and $x_i \in \mathbb{R}^m$, and the sets $L := \prod_{i=1}^n L_i$, $R := L \times [0, \infty)$, $K := \{x \in L \mid Ax \in M\}$, and for $\epsilon > 0$,

$$S(\epsilon) := \left\{ p \in S \mid p_j \geq \epsilon \text{ for all } j = 1, \dots, n \right\} .$$

We will assume:

- i) M is closed.
- ii) There exists $\omega \in \mathbb{R}$ and $\eta \in \mathbb{R}_+^m$ such that $M \subset \left\{ z \in \mathbb{R}^m \mid \langle \eta, z \rangle \leq \omega \right\}$.
- iii) $M - \mathbb{R}_+^m = M$.
- iv) L_i is closed and bounded below (i.e., there exists $\xi_i \in \mathbb{R}^m$ such that $L_i \subset \xi_i + \mathbb{R}_+^m$) for each $i = 1, \dots, n$.
- v) $A(L) \cap M \neq \emptyset$
- vi) $d_i : L_i \times [0, \infty) \times S \rightarrow \mathbb{R}^m$ is continuous for each $i = 1, \dots, n$.
- vii) for each $x_i \in L_i$, $t \geq 0$, and $p \in S$,

$$\liminf_{h \downarrow 0} \frac{N(x_i + h d_i(x_i, t, p), L_i)}{h} = 0 .$$

viii) $d : R \times S \rightarrow (\mathbb{R}^m)^n$, defined by

$$d(x, t, p) := \begin{bmatrix} d_1(x_1, t, p) \\ \vdots \\ d_n(x_n, t, p) \end{bmatrix},$$

satisfies the Instantaneous Collective Walras Law, i.e.,

$\langle p, Ad(x, t, p) \rangle \leq 0$ for every $(x, t) \in R$ and $p \in S$.

ix) There exists $\gamma > 0$ such that for every $i = 1, \dots, n$, the j -th component of $d_i(x_i, t, p)$ is nonnegative for each $(x, t) \in R$ and $p \in S$ such that $p_j \leq \gamma$.

If each consumer's instantaneous demand function satisfies the Instantaneous Walras Law $\langle p, d_i(x_i, t, p) \rangle \leq 0$ for every $x_i \in L_i$, $t \geq 0$, and $p \in S$, then the Instantaneous Collective Walras Law is satisfied. The Instantaneous Walras Law is a budget constraint; it requires that the value of the commodity holding after making a transaction be no greater than the value of the commodity holding before the transaction. This law is the equivalent of the Walras Law for the Static Equilibrium Problem.

Assumptions i), ii) and iv) imply that K is compact. Assumption vii) asks for the instantaneous demand functions to be compatible with the consumption sets. If a consumer has a commodity holding that is on the boundary of his consumption set, his instantaneous demand function must always point into an admissible commodity holding, no matter what the price vector is. Finally, assumption ix) will allow us to only consider prices in the compact subset $S(\gamma)$ of S .

We finish this section with an example of a dynamic exchange economy in which all the instantaneous demand functions come from utility maximization, and are independent of the time. Suppose that L_i is convex and that each consumer has a utility function $u_i : L_i \rightarrow \mathbb{R}$ to represent his preferences in L_i , which is continuous and strictly quasiconcave, i.e., $u_i(\lambda x^2 + (1 - \lambda)x^1) > u_i(x^1)$ for each $\lambda \in (0, 1)$, whenever $x^1, x^2 \in L_i$ are such that $x^1 \neq x^2$ and $u_i(x^2) \geq u_i(x^1)$. For the time being we are concerned with only one consumer so the subindex in L and u will be omitted. (Only for the rest of this section L will be the consumption set

of a consumer, from the beginning of the next section L will again denote the set $\prod_{i=1}^n L_i$.) Given an endowment $x \in L$ and a price vector $p \in S$ consider the following problem:

$$\begin{aligned} P(x, p) \quad & \max u(x + \delta) \\ \text{s.t.} \quad & \langle p, \delta \rangle \leq 0 \\ & x + \delta \in L. \end{aligned} \tag{1}$$

If $\delta(x, p)$ denotes the optimum for this problem (we will make assumptions that guarantee the existence and uniqueness of the solution), we will assume that the consumer has an instantaneous demand function proportional to $\delta(x, p)$, i.e., there exists $\alpha > 0$ (independent of x and p) such that $d(x, p) = \alpha \delta(x, p)$ (we omit the variable t since d is independent of t). More generally, it could be assumed that there exists a continuous function $a : L \times S \rightarrow \mathbb{R}$ such that $d(x, p) = a(x, p) \delta(x, p)$ and $0 < a(x, p) \leq \alpha$ for all $x \in L$ and $p \in S$. In any case $x + h d(x, p) \in L$ for all $h \in [0, 1/\alpha]$; this is stronger than condition vii). Also, this instantaneous demand function satisfies the Instantaneous Walras Law, i.e., $\langle p, d(x, p) \rangle \leq 0$ for each $x \in L$ and $p \in S$; this condition implies viii). Below we will study the continuity of δ on $L \times S$; this is condition vi).

Let us take $L = \mathbb{R}_+^2$ and consider the following three examples:

1) The Cobb-Douglas utility function [16] is $u(x_1, x_2) = x_1^a x_2^{1-a}$ where $a \in (0, 1)$. The solution to problem (1) is:

$$\delta_1(x, p) = \frac{a p_2 x_2 - (1-a) p_1 x_1}{p_1}, \quad \delta_2(x, p) = \frac{(1-a) p_1 x_1 - a p_2 x_2}{p_2}$$

2) The C.E.S. (Constant Elasticity of Substitution) utility function [16] is $u(x_1, x_2) = (x_1^\beta + x_2^\beta)^{1/\beta}$, $\beta \neq 0$. The solution we get for the maximization problem is in this case:

$$\delta_1(x, p) = \frac{p_2 x_2 - (p_1 p_2)^{-\beta} \frac{1}{1-\beta} x_1}{p_1 + (p_1 p_2)^{-\beta} \frac{1}{1-\beta}}, \quad \delta_2(x, p) = \frac{p_1 x_1 - (p_1 p_2)^{-\beta} \frac{1}{1-\beta} x_2}{p_2 + (p_1 p_2)^{-\beta} \frac{1}{1-\beta}}.$$

An important special case here is when $\beta = -1$. Then $u(x_1, x_2) = -\left(\frac{1}{x_1} + \frac{1}{x_2}\right)$ (we have used here the fact that if $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function then the preferences can also be represented by the function $\phi \circ u$; take $\phi(z) = -z^{-1}$) and:

$$\delta_1(x, p) = \frac{p_2 x_2 - \sqrt{p_1 p_2} x_1}{p_1 + \sqrt{p_1 p_2}}, \quad \delta_2(x, p) = \frac{p_1 x_1 - \sqrt{p_1 p_2} x_2}{p_2 + \sqrt{p_1 p_2}}.$$

3) Consider the utility function $u(x_1, x_2) := \log \left[\frac{x_1 x_2}{1+x_1} \right]$. This utility function is concave on $\mathbb{R}_+^2$. The solution of the maximization problem is in this case:

$$\delta_1(x, p) := -(x_1 + 1) + \left[\frac{p_1 x_1 + p_2 x_2}{p_1} \right]^{1/2}, \quad \delta_2(x, p) := -\frac{p_1 \delta_1(x, p)}{p_2}.$$

For simplicity, let $d_i(x_i, p) = \delta_i(x_i, p)$ and consider the pure exchange economy (with free disposal) described by $M = w - \mathbb{R}_+^m$, where $w = \sum_{i=1}^n x_i^0$ and $x_i^0 \in L_i$ is the initial endowment of agent i . In this particular case we can exhibit a solution to the dynamic economic model. Choose $p(t) = p^*$ for all $t \in [0, \infty)$, where p^* is a Walrasian equilibrium price. Let $(x_1^*, \dots, x_n^*)$ be the Walrasian allocation associated to p^* . Then, for each $x_i \in L_i$ with $\langle p^*, x_i \rangle = \langle p^*, x_i^0 \rangle$ we obtain:

$$d_i(x_i, p^*) = x_i^* - x_i.$$

It is easy to check that the solution to the differential equation $x_i'(t) = d_i(x_i(t), p^*)$, with initial condition $x_i(0) = x_i^0$, is

$$x_i(t) = x_i^0 + (1 - e^{-t})(x_i^* - x_i^0) ,$$

and that

$$d_i(x_i(t), p^*) = e^{-t}(x_i^* - x_i^0) .$$

But then

$$\sum_{i=1}^n x_i(t) = \sum_{i=1}^n x_i^0 + (1 - e^{-t}) \sum_{i=1}^n (x_i^* - x_i^0) ,$$

and since by assumption $\sum_{i=1}^n (x_i^* - x_i^0) \leq 0$, we get

$$\sum_{i=1}^n x_i(t) \in W - \mathbb{R}_+^m = M.$$

While we refer to the functions d_i as instantaneous demand functions, the theorems proved below do not require that they satisfy the axiom of revealed preference. Thus, although the results hold for utility-generated demands, they are by no means limited to this case, and could have applications in more general settings. Also, the result could be generalized to admit multi-valued instantaneous demand functions.

We now turn to the question of the continuity of the function δ_i . Let us recall that a multifunction $F: \mathbb{R}^r \rightarrow \mathbb{R}^s$ is continuous at x if it is u.s.c. and l.s.c. there. Since $\mathbb{R}^r$ is a metric space, F is u.s.c. at x if and only if $F(x)$ is compact and for each $\epsilon > 0$ there exists a neighborhood W of x such that $F(y) \subset F(x) + \epsilon B$ for every $y \in W$. Finally, F is Lipschitz continuous if there exists a constant $\alpha > 0$ such that for each $x, y \in \mathbb{R}^r$, $\Delta(F(x), F(y)) \leq \alpha \|x - y\|$, that is, F considered as a function from $\mathbb{R}^r$ into $(P(\mathbb{R}^s), \Delta)$ is Lipschitz continuous.

Lemma 2.1: Let $L \subset \mathbb{R}^m$ be closed, convex and bounded below. Consider the multifunction from $L \times \text{int } \mathbb{R}_+^m$ into $\mathbb{R}^m$ defined by $F(x, p) := \left\{ \delta \in \mathbb{R}^m \mid x + \delta \in L, \langle p, \delta \rangle \leq 0 \right\}$. Assume that $x \in L$ and $p \in \text{int } \mathbb{R}_+^m$ satisfy:

$$H(p^0) \cap (\text{int } L - x^0) \neq \emptyset, \quad (2)$$

where $H(p^0) := \{z \in \mathbb{R}^m \mid \langle p^0, z \rangle \leq 0\}$. Then F is lower semicontinuous (l.s.c.) at (x^0, p^0) .

Note: If $x^0 \in \text{int } L$, $0 \in H(p^0) \cap (\text{int } L - x^0)$. Thus, F is l.s.c. at least on $\text{int } L \times \text{int } \mathbb{R}_+^m$.

Proof: Let us define the multifunction Σ from $L \times \text{int } \mathbb{R}_+^m$ into $(\mathbb{R}^m)^2$ by:

$$\Sigma(x, p) := \left\{ (y, z) \in \mathbb{R}^m \times L \mid y - z + x = 0 \text{ and } \langle p, y \rangle \leq 0 \right\}.$$

Then $F(x, p) = [I \ 0] \Sigma(x, p)$ and it is easy to see that if Σ is l.s.c. at (x, p) then F is l.s.c. at (x, p) . If (y, z) is any point in $\Sigma(x^0, p^0)$, one can check that condition (2) is equivalent to the regularity condition stated by Robinson in [12] for the system of inequalities defining the set $\Sigma(x^0, p^0)$ at the point (y, z) . Therefore, by Theorem 1 there, Σ is l.s.c. at (x^0, p^0) . ■

If $\text{int } L \neq \emptyset$, $x^0 \in L$, and $p^0 \in \mathbb{R}^m \setminus \{0\}$, condition (2) is equivalent to the following condition:

$$-p^0 \notin N_L(x^0), \quad (3)$$

where $N_L(x^0)$ is the normal cone to L at x^0 . Condition (3) has a simpler geometric interpretation; in particular it is easy to see that if $L = \mathbb{R}_+^m$, F is l.s.c. on $L \setminus \{0\} \times \text{int } \mathbb{R}_+^m$.

Lemma 2.2: Let L , F and H be defined as in the previous lemma. Then F is upper semicontinuous (u.s.c.) on $L \times \text{int } \mathbb{R}_+^m$ and the images of F are convex and compact.

Proof: Let $\xi \in \mathbb{R}^m$ be such that $\xi \leq x$ for every $x \in L$. Take $x^0 \in L$ and $p^0 \in \text{int } \mathbb{R}_+^m$, and define $z := \xi - x^0$, then $z \leq 0$. We will show that

$$F(x, p) \subset \left\{ y \in \mathbb{R}^m \mid z_j - 1 \leq y_j \leq \frac{3}{p_j} \sum_{i=1}^m p_i^0 (1 - z_i) \right\} =: G$$

for all $x \in L$ such that $x \leq x^0 + e$, and $p \in P(p^0)$, where $e := (1, \dots, 1) \in \mathbb{R}^m$ and

$$P(p^0) := \left\{ q \in \mathbb{R}^m \mid \frac{1}{2} p_j^0 \leq q_j \leq \frac{3}{2} p_j^0 \text{ for each } j = 1, \dots, m \right\}.$$

Let $p \in P(p^0)$, $x \in L$ with $x - x^0 \leq e$, and $y \in F(x, p)$. Then $x + y \in L$, so $x + y \geq \xi$ and $y \geq \xi - x \geq \xi - (x^0 + e)$; therefore $y_j \geq z_j - 1$. From $\langle p, y \rangle \leq 0$ we get that $p_j y_j \leq - \sum_{i \neq j} p_i y_i$. Each $p_i \geq \frac{1}{2} p_i^0 > 0$, so $-p_i y_i \leq -p_i(z_i - 1)$ for all $i = 1, \dots, m$. Finally, $-(z_i - 1) > 0$ and $p_i \leq \frac{3}{2} p_i^0$, so $0 < -p_i(z_i - 1) \leq -\frac{3}{2} p_i^0(z_i - 1)$ for all $i = 1, \dots, m$. Therefore

$$p_j y_j \leq -\frac{3}{2} \sum_{i \neq j} p_i^0(z_i - 1) < -\frac{3}{2} \sum_{i=1}^m p_i^0(z_i - 1) = \frac{3}{2} \sum_{i=1}^m p_i^0(1 - z_i)$$

and thus

$$y_j \leq \frac{3}{2p_j} \sum_{i=1}^m p_i^0(1 - z_i) \leq \frac{3}{p_j^0} \sum_{i=1}^m p_i^0(1 - z_i).$$

Hence $F(x, p) \subset G$. Let $\mathbb{B}$ denote the open unit ball in $\mathbb{R}^m$. Since G is a compact set, there exists $R > 0$ such that $F(x, p) \subset R\overline{\mathbb{B}}$ for all $p \in P(p^0)$ and all $x \in L$ with $x \leq x^0 + e$, and we can write:

$$F(x, p) = (L - x) \cap H(p) = [(L - x) \cap H(p)] \cap R\overline{\mathbb{B}}.$$

L and $H(p)$ are closed convex sets, so $F(x, p)$ is a convex compact set. It can be shown that the multifunction $(x, p) \rightarrow (L - x) \cap H(p)$ is closed, i.e., its graph is closed. But the multifunction $(x, p) \rightarrow R\overline{\mathbb{B}}$ is constant and therefore u.s.c. Hence, by Theorem VI.1.7 on page 112 in [6], the multifunction F is u.s.c. $\square$

Lemmas 2.1 and 2.2 combined together say that the multifunction F is continuous at any $(x, p) \in L \times \text{int } \mathbb{R}_+^m$ satisfying condition (2). The Maximum Theorem (see Theorem 1 in [11]) can be used to prove the following:

Theorem 2.3: If $L \subset \mathbb{R}^m$ is closed, convex and bounded below, and $u : L \rightarrow \mathbb{R}$ is continuous and strictly quasiconcave, problem (1) has a unique solution $\delta(x, p)$ for each $p \in \text{int } \mathbb{R}_+^m$ and each $x \in L$. In addition, δ is continuous at any

$(x, p) \in L \times \text{int } \mathbb{R}_+^m$ satisfying (2). In particular, δ is continuous on $\text{int } L \times \text{int } \mathbb{R}_+^m$. If we also assume that u satisfies local nonsatiation, then $\langle p, \delta(x, p) \rangle = 0$ for each $x \in L$ and each $p \in \text{int } \mathbb{R}_+^m$.

Corollary 2.4: If $L = \mathbb{R}_+^m$ and $u : L \rightarrow \mathbb{R}$ is continuous and strictly quasiconcave, $\delta(x, p)$ is continuous on $L \setminus \{0\} \times \text{int } \mathbb{R}_+^m$.

We can strengthen the assumptions in the above Theorem and show that $\delta(x, p)$ is locally Lipschitz continuous, and therefore, by Rademacher's Theorem, almost everywhere differentiable. This result does not make the usual strict complementarity slackness assumption. Suppose that $L := \{x \in \mathbb{R}^m \mid g_k(x) \leq 0 \text{ } k = 1, \dots, r\}$, where $g_k : \mathbb{R}^m \rightarrow \mathbb{R}$ is convex and twice continuously differentiable on $\mathbb{R}^m$ for all $k = 1, \dots, r$. Let $u : L \rightarrow \mathbb{R}$ be twice continuously differentiable, and for each $x \in L$ define

$$\Gamma(x) := \left\{ h \in \mathbb{R}^m \mid h \in T_L(x) \setminus \{0\} \text{ and } \langle \nabla u(x), h \rangle = 0 \right\},$$

where $T_L(x)$ denotes the tangent cone to L at x . We assume that

$$h^t \nabla^2 u(x) h < 0 \text{ for all } x \in L \text{ and } h \in \Gamma(x). \quad (4)$$

This condition implies that u is strictly quasiconcave and also that it is pseudo-concave, i.e., $u(x) \leq u(y)$ for all $x, y \in L$ satisfying $\nabla u(y)(x - y) \leq 0$.

Consider the problem:

$$P(x, p) \quad \max u(x + \delta)$$

$$\text{s. t.} \quad \langle p, \delta \rangle \leq 0$$

$$g_k(x + \delta) \leq 0 \quad k = 1, \dots, r.$$

If the gradients of the binding constraints are linearly independent at $\delta(x, p)$, δ is Lipschitz continuous in a neighborhood of (x, p) (see, for example, Cornet and Laroque [9] and Robinson [13]). For instance, if $L = \mathbb{R}_+^m$, define $X = \mathbb{R}_+^m \setminus \{0\}$; then $\delta : X \times \text{int } \mathbb{R}_+^m \rightarrow \mathbb{R}^m$ is locally Lipschitz continuous.

3. Existence theorem

We begin this section by sketching the proof for the existence theorem.

Define the multifunctions

$$D(x, t) := \{d(x, t, p) \mid p \in S\} \quad ,$$

$$D_\epsilon(x, t) := \{d(x, t, p) \mid p \in S(\epsilon)\} \quad \epsilon > 0 \quad ,$$

and

$$D_\epsilon^*(x, t) := \{v \in D_\epsilon(x, t) \mid Av \in \mathbb{R}_-^m\} \quad ,$$

from R into $(\mathbb{R}^m)^n$. Since $S(\epsilon)$ is compact, $D_\epsilon(x, t)$ and $D_\epsilon^*(x, t)$ are compact sets for each $(x, t) \in R$.

Given $T > 0$, the problem is then to find $x : [0, T] \rightarrow (\mathbb{R}^m)^n$ absolutely continuous and $p : [0, T] \rightarrow S$, such that:

$$\begin{cases} \frac{dx}{dt}(t) = d(x(t), t, p(t)) & \text{a.e. } (0, T) \\ x(t) \in K & \text{for all } t \in [0, T] \\ x(0) = x^0 & (x^0 \in K \text{ given}) \end{cases} \quad (P)$$

This problem is equivalent to finding a function $x : [0, T] \rightarrow (\mathbb{R}^m)^n$ absolutely continuous, such that

$$\begin{cases} \frac{dx}{dt}(t) \in D(x(t), t) & \text{a.e. } (0, T) \\ x(t) \in K & \text{for all } t \in [0, T] \\ x(0) = x^0 \end{cases} \quad (P')$$

To show the existence of a solution to problem (P') we will:

- show that $D_\gamma^*(x, t)$ is nonempty for each $(x, t) \in R$;
- show that D_γ^* is u.s.c.;
- show that D_γ^* is l.s.c. The additional Assumption I, stated below, is required here;

d) consider the problem of finding $x:[0, T] \rightarrow (\mathbb{R}^m)^n$ absolutely continuous such that

$$\begin{cases} \frac{dx}{dt}(t) \in D_\gamma^*(x(t), t) & \text{a.e. } (0, T) \\ x(t) \in K & \text{everywhere} \\ x(0) = x^0 \end{cases} \quad (P'')$$

Using a theorem of Aubin [3], show that (P'') has a solution that a fortiori is a solution of (P') .

Lemma 3.1: For each $(x, t) \in R$ and $r \in \mathbb{R}_+^m$, there exists $p \in S(\gamma)$ and $\alpha \geq 0$ such that $Ad(x, t, p) + \alpha r = 0$. In particular $D_\gamma^*(x, t)$ is nonempty and compact for each $(x, t) \in R$, that is, $\text{dom } D_\gamma^* \supset R$.

Proof: For any $r \in \mathbb{R}_+^m$, $r \neq 0$, define the function d^r on $R \times S$ by:

$$d^r(x, t, p) := Ad(x, t, p) + \alpha(x, t, p)r$$

where $\alpha(x, t, p) := -\langle p, Ad(x, t, p) \rangle / \langle p, r \rangle$. Since $\langle p, r \rangle > 0$ and $\langle p, Ad(x, t, p) \rangle \leq 0$, we have that $\alpha(x, t, p) \geq 0$. Also, for each $(x, t) \in R$ and $p \in S$, $\langle p, d^r(x, t, p) \rangle = 0$.

So the function $d^r(x, t, \cdot) : S(\gamma) \rightarrow \mathbb{R}^m$ defines a tangent vector field on $S(\gamma)$. But, because of assumption ix), $d^r(x, t, p)$ points inward on the boundary of $S(\gamma)$.

Therefore, there exists $p \in S(\gamma)$ such that $d^r(x, t, p) = 0$, that is, $Ad(x, t, p) + \alpha(x, t, p)r = 0$. Thus, $D_\gamma^*(x, t) \neq \emptyset$, and we have already seen that $D_\gamma^*(x, t)$ is compact. $\square$

Assumption I: For any $(x, t) \in R$ and $r \in \mathbb{R}_+^m$, the line $\Phi(r) := \{\alpha r \mid \alpha \in \mathbb{R}\}$ intersects the set $A(D(x, t)) = \{Ad(x, t, p) \mid p \in S\}$ at most in one point.

Because of the Instantaneous Walras Law, $A(D(x, t)) \cap \mathbb{R}_+^m \setminus \{0\} = \emptyset$ for each $(x, t) \in R$. Therefore, if $(x, t) \in R$ and $r \in \mathbb{R}_+^m$, any intersection of $\Phi(r)$ with $A(D(x, t))$ must occur in $\mathbb{R}_-^m$. We have already seen in Lemma 3.1 that $\Phi(r) \cap A(D(x, t)) \neq \emptyset$, so this intersection must be a singleton for each $(x, t) \in R$ and $r \in \mathbb{R}_+^m$.

Lemma 3.2: The multifunction $D_\gamma^*: R \rightarrow (\mathbb{R}^m)^n$ is u.s.c.

Proof: We have that

$$D_\gamma^*(x, t) = D_\gamma(x, t) \cap A^{-1}(\mathbb{R}_-^m) \quad .$$

The constant multifunction that to each $x \in L$ associates the closed set $A^{-1}(\mathbb{R}_-^m)$ is closed.

Let us check that $D_\gamma: R \rightarrow (\mathbb{R}^m)^n$ is u.s.c. Let $(x, t) \in R$; then for each $p \in S(\gamma)$ and $\epsilon > 0$, there exists $\delta(p, \epsilon) > 0$ (δ also depends on (x, t) , but (x, t) is fixed here) such that $\|d(x, t, p) - d(y, s, q)\| < \epsilon$ for every $q \in (p + \delta(p, \epsilon)\mathbb{B}) \cap S$ and every $(y, s) \in ((x, t) + \delta(p, \epsilon)\mathbb{B}) \cap R$. But $S(\gamma)$ is compact, therefore there exist $p^1, \dots, p^r \in S(\gamma)$ such that $S(\gamma) \subset \bigcup_{\alpha=1}^r (p^\alpha + \delta(p^\alpha, \epsilon)\mathbb{B})$. Let $\delta^* = \min \{\delta(p^\alpha, \epsilon) \mid \alpha = 1, \dots, r\}$. Then, if $(y, s) \in ((x, t) + \delta^*\mathbb{B}) \cap R$ and $p \in S(\gamma)$, there exists an α such that $p \in p^\alpha + \delta(p^\alpha, \epsilon)\mathbb{B}$, and $\|(x, t) - (y, s)\| \leq \delta(p^\alpha, \epsilon)$. Therefore $\|d(x, t, p^\alpha) - d(y, s, p)\| < \epsilon$, i.e., $D_\gamma(y, s) \subset D_\gamma(x, t) + \epsilon\mathbb{B}$ for all $(y, s) \in ((x, t) + \delta^*\mathbb{B}) \cap R$, and D_γ is u.s.c. on R .

Since the intersection of a closed multifunction with a u.s.c. multifunction is a u.s.c. multifunction, D_γ^* is u.s.c. $\blacksquare$

Lemma 3.3: The multifunction $D_\gamma^*: R \rightarrow (\mathbb{R}^m)^n$ is l.s.c.

Proof: The proof is by contradiction. Assume that for some $(x, t) \in R$ and $v \in D_\gamma^*(x, t)$, there exist an $\epsilon > 0$ and a sequence $\{(x^k, t^k)\}$ in R converging to (x, t) , such that $D_\gamma^*(x^k, t^k) \cap (v + \epsilon\mathbb{B}) = \emptyset$ for each k . Let $p \in S(\gamma)$ be such that $v = d(x, t, p)$. For each k there exists a price $p^k \in S(\gamma)$ such that $d(x^k, t^k, p^k) = \alpha^k d(x, t, p)$ for some $\alpha^k \geq 0$. Without loss of generality we can assume that p^k converges to some price $q \in S(\gamma)$. Then, by continuity, $d(x, t, q) = \alpha d(x, t, p)$ for some $\alpha \geq 0$. By Assumption I, this implies that $d(x, t, q) = d(x, t, p)$, and thus, for sufficiently large k , $d(x^k, t^k, p^k) \in (v + \epsilon\mathbb{B})$, which is a contradiction. $\blacksquare$

Let U and V be the two Hilbert spaces, and $T > 0$. Assume that the sets $Z \subset U$, $Y_0 \subset V$ and $Y \subset V$ are nonempty and closed, and that $Y \subset \text{int } Y_0$. Let A be a continuous linear operator from U to V , and define the sets

$$X := \{x \in Z \mid Ax \in Y\} \quad \text{and} \quad X_0 := \{x \in Z \mid Ax \in Y_0\} \quad .$$

Let $F : Z \times [0, T] \rightarrow U$ be a multifunction with nonempty compact images, and $P : Z \rightarrow Z$ be a multifunction with closed images, satisfying

- a) $x \in P(x)$ for each $x \in Z$
- b) $P(y) \subset P(x)$ for each $x \in Z$ and $y \in P(x)$.

Using a technique devised by Filippov [10], Aubin has proven a theorem that contains the following as a special case (see Theorem 5.1 in [3]).

Theorem 3.4: Assume that:

- (1) X is nonempty and X_0 is compact.
- (2) F is continuous.
- (3) P is Lipschitz continuous.
- (4) For each $x \in X_0$, $t \in [0, T]$, and $v \in F(x, t)$,

$$\liminf_{h \downarrow 0} \frac{N(x + hv, P(x))}{h} = 0 \quad ,$$

and if $x \in X$, $Av \in T_Y(Ax)$, where $T_Y(z)$ denotes the tangent cone to Y at z , in the sense of Clarke.

Then, for each $x^0 \in X$ and $v^0 \in F(x^0, 0)$, there exists an absolutely continuous function $x : [0, T] \rightarrow U$ satisfying:

- (5) $x'(t) \in F(x(t), t)$ a. e. in $(0, T)$, $x(0) = x^0$, $x'(0) = v^0$.
- (6) For each $t \in [0, T]$, $x(t) \in P(x(s))$ for all $s \leq t$, and $Ax(t) \in Y$.

Furthermore, x' is regulated (i. e., is a uniform limit of step functions).

Remarks: (a) In the theorem proved in [3], F does not depend on t . However, it is easy to check that the same proof applies to this case as well.

(b) In [3] the following tangential condition is assumed:

(4') For each $x \in X_0$, $t \in [0, T]$, and $v \in F(x, t)$, $v \in T_{P(x)}(x)$, and if $x \in X$, $Av \in T_Y(Ax)$.

This condition is stronger than the tangential condition (4). However, it is only required to show that the function

$$a_0(h) := \sup \left\{ \frac{1}{h} N((x, x+hv), G(P)) \mid x \in X, t \in [0, T], v \in F(x, t) \right\}$$

converges to 0 when $h \rightarrow 0$, where $G(P)$ is the graph of P . Aubin [5] has improved a result by Clarke and Aubin [8] that shows that $a_0(h)$ converges to 0 when $h \rightarrow 0$ under the weaker assumption (4). Incidentally, Lemma 5.1 in [3] needs the following minor corrections. One should consider the graphs $G(P_0) \subset X \times U$ and $G(F) \subset X \times U$ of P_0 and F , where $P_0(x) := P(x) \cap X_0$.

We make $U = (\mathbb{R}^m)^n$, $V = \mathbb{R}^m$, $Z = L$, $Y = M$, $Y_0 = \{z \in \mathbb{R}^m \mid \langle \eta, z \rangle \leq \omega + 1\}$ (recall assumption iii)), $X = K$, $F = D_\gamma^*$, and define $P(x) := L$ for each $x \in L$. We only need to check that for each $x \in K$, $t \geq 0$ and $v \in D_\gamma^*(x, t)$, $Av \in T_M(Ax)$. But by the definition of D_γ^* , we have that $Av \in \mathbb{R}_-^m$, and by assumption ii), $M + \mathbb{R}_-^m \subset M$, so $Av \in \mathbb{R}_-^m \subset T_M(Ax)$. Thus we have the following theorem.

Theorem 3.5: For each $x^0 \in K$ and $v^0 \in D_\gamma^*(x^0, 0)$, and for any $T > 0$, there exists a trajectory solving (P'') and such that $x(0) = x^0$ and $x'(0) = v^0$. Besides, x' is regulated.

Since $x(T) \in K$, we could apply this theorem again with the initial condition $x^0 = x(T)$. This way we can always extend the solution to the whole interval $[0, \infty)$.

With the approach we have followed in this section, the information about prices is somehow missing. But we know that for each $x \in K$ and $t \geq 0$, if $p \in S$ is such that $d(x, t, p) \in D_\gamma^*(x, t)$, p is an admissible price at (x, t) , in the sense that if p is chosen when a trajectory passes through x at time t , the aggregate consumption is "pushed" inside M .

Let us define the multifunction $P : R \rightarrow S(\gamma)$ by:

$$P(x, t) := \left\{ p \in S(\gamma) \mid \text{Ad}(x, t, p) \in \mathbb{R}_-^m \right\}.$$

It is easy to check that for each $x \in K$ and $t \geq 0$, the admissible subset of prices $P(x, t)$ is nonempty and compact. Moreover, let us replace condition ix) by:

ix') There exists $\gamma > 0$ such that for every $i = 1, \dots, n$, the j -th component of $d_i(x_i, t, p)$ is (strictly) positive for each $(x, t) \in R$ and $p \in S$ with $p_j \leq \gamma$.

Then $P(x, t) = \left\{ p \in S \mid \text{Ad}(x, t, p) \in \mathbb{R}_-^m \right\}$, that is, if $p \in S$ and $(x, t) \in R$ are such that $\text{Ad}(x, t, p) \in \mathbb{R}_-^m$, necessarily $p \in S(\gamma)$. The next theorem also requires the following assumption.

Assumption II: For each $(x, t) \in R$, the function $\text{Ad}(x, t, \cdot) : S(\gamma) \rightarrow \mathbb{R}^m$ is injective.

Theorem 3.6: The multifunction $P : R \rightarrow S(\gamma)$ is continuous.

For the trajectory $x : [0, \infty) \rightarrow K$ that we have constructed, the price $p(t)$ associated to $x(t)$ plays the role of a feedback control, in the sense that $p(t) \in P(x(t), t)$ for almost all $t \geq 0$. However, we cannot use the ideas in [4] to construct a continuous selection of P , that is, a continuous function $p : R \rightarrow S$ such that $p(x, t) \in P(x, t)$ for each $(x, t) \in R$, from Theorem 3.6. In the next section, we will construct continuously differentiable selections of P .

4. A continuously differentiable price mechanism

In what follows, it will be simpler to consider prices in the unit simplex $S_1 := \{p \in \mathbb{R}^m \mid p_j > 0 \ j = 1, \dots, m, \langle e, p \rangle = 1\}$ rather than in S ($e = (1, \dots, 1) \in \mathbb{R}^m$.) For $\epsilon > 0$ we define accordingly:

$$S_1(\epsilon) := \{p \in S_1 \mid p_j \geq \epsilon \ j = 1, \dots, m\} \quad .$$

Condition ix') implies that for some positive number, which we denote γ again, the j -th component of $d_i(x_i, t, p)$ is positive for each $i = 1, \dots, n$, for all $p \in S_1$ with $p_j \leq \gamma$, and all $(x, t) \in \mathbb{R}$. Let us define the multifunction $P_1 : \mathbb{R} \rightarrow S_1(\gamma)$ by

$$P_1(x, t) := \{p \in S_1(\gamma) \mid \text{Ad}(x, t, p) \in \mathbb{R}^m\} \quad .$$

Condition ix') implies again that $P_1(x, t) = \{p \in S_1 \mid \text{Ad}(x, t, p) \in \mathbb{R}^m\}$ for each $(x, t) \in \mathbb{R}$. Let us change condition vi) to:

vi') For each $i = 1, \dots, n$, $d_i : L_i \times [0, \infty) \times \text{int } \mathbb{R}_+^m \rightarrow \mathbb{R}^m$ is continuously differentiable,

and replace Assumptions I and II by:

Assumption III: For each $r \in \mathbb{R}^m$ define $\bar{\Phi}(r) := \{\lambda r \mid \lambda \in \mathbb{R}\}$. There exists a continuously differentiable function $g : \mathbb{R} \rightarrow S_1$ such that, for each $(x, t) \in \mathbb{R}$ and $p \in S_1$ solving $\text{Ad}(x, t, p) + \alpha g(x, t) = 0$ for some $\alpha \geq 0$, $\bar{\Phi}(g(x, t)) + \text{Im}(A J_p d(x, t, p)) = \mathbb{R}^m$, where $J_p d(x, t, p)$ denotes the Jacobian of d with respect to p at the point (x, t, p) .

We note that $A J_p d(x, t, p) = J_p(\text{Ad})(x, t, p)$. Consider the square matrix $C = A J_p d(x, t, p)$; in the following lemma we state some equivalent conditions related to Assumption III.

Lemma 4.1: Let C be an $m \times m$ matrix, $g \in \mathbb{R}^m$ with $g \neq 0$, and let $\bar{\Phi}$ be defined as above. Then the following conditions are equivalent:

- (1) $\bar{\Phi}(g) + \text{Im } C = \mathbb{R}^m$.
- (2) $(\bar{\Phi}(g))^\perp \cap \ker C^t = \{0\}$.

- (3) There exists $r \in \mathbb{R}^m$ such that $\langle r, g \rangle > 0$ and $(\Phi(r))^\perp \subset \text{Im } C$.
- (4) There exists $r \in \mathbb{R}^m$ such that $\langle r, g \rangle > 0$ and $\ker C \subset \Phi(r)$.

In particular, if $g \in S_1$ and there exists $r \in \mathbb{R}_+^m$ such that $(\Phi(r))^\perp \subset \text{Im } C$, condition (3) is satisfied. Thus we could impose the stronger, but simpler, following assumption.

Assumption III': For each $(x, t) \in \mathbb{R}$ and $p \in P_1(x, t)$, there exists $r \in \mathbb{R}_+^m$ such that $(\Phi(r))^\perp \subset \text{Im}(AJ_p d(x, t, p))$.

Note that since d is homogeneous of degree 0 in p , for each $(x, t) \in \mathbb{R}$ and $p \in S_1$, $AJ_p d(x, t, p) \cdot p = 0$. Thus Assumption III and condition (3) of Lemma 4.1 imply that for each $(x, t) \in \mathbb{R}$ and $p \in S_1$ satisfying $Ad(x, t, p) + \alpha g(x, t) = 0$ for some $\alpha \geq 0$, there exists $r \in \mathbb{R}^m$ such that $\langle r, g(x, t) \rangle > 0$ and $(\Phi(r))^\perp = \text{Im}(AJ_p d(x, t, p))$.

Theorem 4.2: Assume that L is path connected. Let $g: \mathbb{R} \rightarrow S_1$ be the function of Assumption III. Then there exist two continuously differentiable functions $p: \mathbb{R} \rightarrow S_1$ and $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ such that $Ad(x, t, p(x, t)) + \alpha(x, t)g(x, t) = 0$ and $\alpha(x, t) \geq 0$ for each $(x, t) \in \mathbb{R}$.

Proof: Let $(x, t) \in \mathbb{R}$. If $p \in S_1$ and $\alpha \in \mathbb{R}$ are such that $Ad(x, t, p) + \alpha g(x, t) = 0$, necessarily $\alpha \geq 0$ (just premultiply by p^t the above equation). Let $\bar{p} \in S_1$ by any price vector such that $Ad(x, t, \bar{p}) + \bar{\alpha}g(x, t) = 0$ for some $\bar{\alpha} \geq 0$, and consider the system of equations $F(y, s, q, \beta) = 0$, where for each $(y, s) \in \mathbb{R}$, $q \in S_1$, and $\beta \in \mathbb{R}$, we define :

$$F(y, s, q, \beta) := \begin{bmatrix} Ad(y, s, q) + \beta g(y, s) \\ \langle e, q \rangle - 1 \end{bmatrix}.$$

We study this system in a neighborhood of $(x, t, \bar{p}, \bar{\alpha})$ with the help of the Implicit Function Theorem. We have that $F(x, t, \bar{p}, \bar{\alpha}) = 0$ and

$$G := J_{(p, \alpha)} F(x, t, \bar{p}, \bar{\alpha}) = \begin{bmatrix} AJ_p d(x, t, \bar{p}) & g(x, t) \\ e^t & 0 \end{bmatrix}.$$

We show that G is nonsingular. Let $(z, a) \in \mathbb{R}^m \times \mathbb{R}$ be such that $(z^t, a)G = 0$, that is,

$$z^t A J_p d(x, t, \bar{p}) + a e^t = 0$$

$$\langle g(x, t), z \rangle = 0 \quad .$$

Postmultiply the first equation by $\bar{p}$. Since d is homogeneous of degree 0 in p , $A J_p d(x, t, \bar{p}) \cdot \bar{p} = 0$, thus $a \langle e, \bar{p} \rangle = 0$. Therefore $a = 0$ and $z \in \left(\Phi(g(x, t)) \right)^\perp \cap \ker \left((A J_p d(x, t, \bar{p}))^t \right)$. By Assumption III (and condition (2) of Lemma 4.1), $z = 0$. Therefore there exist neighborhoods U of (x, t) , V of $\bar{p}$, and W of $\bar{\alpha}$, and a continuously differentiable function $(p, \alpha) : U \cap R \rightarrow V \times W$ such that:

$$(y, x) \in U \cap R \quad , \quad (q, \beta) \in V \times W \quad , \quad \text{and} \quad F(y, s, q, \beta) = 0$$

is equivalent to:

$$(y, s) \in U \cap R \quad \text{and} \quad (q, \beta) = (p(y, s), \alpha(y, s)) \quad .$$

Without loss of generality, we can assume that $V \cap \{q \in \mathbb{R}^m \mid \langle e, q \rangle = 1\} \subset S_1$. But then, from an earlier remark, $p(y, s) \in S_1(\gamma)$ for each $(y, s) \in U \cap R$. Let $P_1(x, t, g) := \{q \in S_1(\gamma) \mid A d(x, t, q) + \alpha g(x, t) = 0 \text{ for some } \alpha \geq 0\}$; $P_1(x, t, g)$ is a compact subset of $P_1(x, t)$, and since $\bar{p} + V \cap P_1(x, t, g) = \{\bar{p}\}$, $P_1(x, t, g)$ contains only a finite number of points.

Let $(\bar{x}, \bar{t}) \in R$ and $\bar{p} \in P_1(\bar{x}, \bar{t}, g)$. Consider the component C of the graph of $P_1(\cdot, \cdot, g)$ that contains the point $(\bar{x}, \bar{t}, \bar{p})$. We will show that C is the graph of a continuously differentiable function $p : R \rightarrow S_1(\gamma)$.

For each $(x, t) \in R$, define $p(x, t) := \{q \in S_1(\gamma) \mid (x, t, q) \in C\}$. We first show that $p(x, t) \neq \emptyset$ for each $(x, t) \in R$. By contradiction, assume that $(x, t) \in R$ is such that $p(x, t) = \emptyset$. Let $(\xi, \tau) : [0, 1] \rightarrow R$ be a continuous function such that $(\xi, \tau)(0) = (\bar{x}, \bar{t})$ and $(\xi, \tau)(1) = (x, t)$. Let $a = \sup\{b \in [0, 1] \mid p(\xi(b), \tau(b)) \neq \emptyset\}$. Then $p(\xi(a), \tau(a)) = \emptyset$, otherwise we can use the Implicit Function Theorem to find prices $q(b)$ for b in a neighborhood of a , such that $(\xi(b), \tau(b), p(b)) \in C$, and this would contradict the fact that a is the supremum. For each $b < a$, choose a

price $q(b) \in p(\xi(b), \tau(b))$. Since $q(b) \in S_1(\gamma)$ for each b , there exists a strictly increasing sequence $\{b_n\}$ converging to a , and a point $s \in S_1(\gamma)$, such that $q(b_n) \rightarrow s$ as $n \rightarrow \infty$. Since $\text{Ad}(\xi(a), \tau(b), q(b)) \in \mathbb{R}_-^m$ for each b , $\text{Ad}(\xi(a), \tau(a), s) \in \mathbb{R}_-^m$, thus $s \in p(\xi(a), \tau(a))$ and this is a contradiction. Therefore $p(x, t) \neq \emptyset$ for each $(x, t) \in R$.

Next we show that $p(x, t)$ is in fact a singleton. The argument is by contradiction. For example, assume that at the point $(\bar{x}, \bar{t}) \in R$ there are two price vectors $p_1, p_2 \in p(\bar{x}, \bar{t})$. Since C is connected, $(\bar{x}, \bar{t}, p_1)$ and $(\bar{x}, \bar{t}, p_2)$ must be in two different "branches" of C that intersect at some point, say (x, t, q) . But then we can again use the Implicit Function Theorem to conclude that in a neighborhood of (x, t, q) there is only one "branch" of C passing through (x, t, q) ; this is a contradiction. Therefore $p(x, t)$ is a singleton for each $(x, t) \in R$, and in fact, $p: R \rightarrow S_1(\gamma)$ is a continuously differentiable function. $\square$

Let $p: R \rightarrow S_1$ be the price mechanism constructed in Theorem 4.2. Consider the following system of ordinary differential equations:

$$\begin{cases} x' = f(x, t) := d(x, t, p(x, t)) & \text{a.e. } (0, T) \\ x(t) \in K & \text{for all } t \in [0, T] \\ x(0) = x^0 \end{cases} \quad (P''')$$

Since f is locally Lipschitz continuous, using again Theorem 3.4 with $F(x, t) = \{f(x, t)\}$ for each $(x, t) \in R$, we get the following theorem.

Theorem 4.3: There exists a continuously differentiable price mechanism $p: R \rightarrow S_1(\gamma)$ such that, for each $x^0 \in K$, there exists a unique trajectory $x: [0, \infty) \rightarrow \mathbb{R}^{m+n}$ solving (P''') . Thus the pair (x, q) , where $q(t) := p(x(t), t)$ for each $t \geq 0$, solves problem (P) .

Thus we have not only constructed a price mechanism that is continuous on R , but with this price mechanism we can predict the behavior of the market, since the trajectories are uniquely determined by the starting point x^0 . This is an important improvement over Theorem 3 in [4]. For example, if we consider

the well-known differential equation in one variable $x' = x^{1/3}$, and make $x^0 = 0$ and $K = (-\infty, 0]$, $x(t) = 0$ for each $t \geq 0$ is a solution and $x(t) \in K$ for all $t \geq 0$. However, $x(t) = (2t/3)^{3/2}$ is also a solution and $x(t) \notin K$ for each $t > 0$. Note that $p \cdot x'(t) \leq 0$ for each $p > 0$ and $x \in K$. Therefore the price mechanisms constructed in [4] are enough to produce solutions of the problem (P'''), but they do not guarantee that any trajectory will remain in K . Our price mechanism depends on $(x, t) \in R$ and in general only aggregate information of the state of the market is known. One would like to construct a price mechanism that depends on (Ax, t) instead. However, this is not possible in general.

Assumption III is related to Assumptions I and II. Loosely speaking, Assumption III is a local version of Assumptions I and II. Assumptions II and III rule out a certain sort of "bump" on the surface $A(D(x, t))$. If $(x, t) \in R$ and $p \in S_1$ are such that $Ad(x, t, p) + \alpha g(x, t) = 0$ for some $\alpha \geq 0$, Assumption III (together with condition (3) of Lemma 4.1) implies that there exist neighborhoods V of p in S_1 and W of $g(x, t)$ such that $Ad(x, t, \cdot)(V)$ intersects $\Phi(r)$ in exactly one point, for each $r \in W$. On the other hand, if we assume that d is twice continuously differentiable with respect to the price vector, and $(x, t) \in R$ and $p \in S_1$ are such that $Ad(x, t, p) + \alpha e = 0$ for some $\alpha \geq 0$, Assumption III' implies that there exists a neighborhood V of p in S_1 such that $Ad(x, t, \cdot) : V \rightarrow \mathbb{R}^m$ is injective.

5. Liapunov functions and stability of Pareto maxima

When the demand functions come from utility maximization, any Walrasian equilibrium of the static model described in the introduction is Pareto optimal. We will extend in some sense this result to the dynamic model. Roughly, we will show that when the instantaneous demand functions are generated by utility maximization, any trajectory solving problem (P') is Pareto improving and converges to a critical point of the multifunction D (defined in Section 3). With an additional assumption, we will prove that any trajectory converges to a Pareto optimal point.

For the rest of this section we will assume that the sets M and L_i , $i = 1, \dots, n$, are convex, and that the utility function $u_i : L_i \rightarrow [-\infty, +\infty)$ is strictly concave with $L_i \subset \text{ri dom } u_i$ for each $i = 1, \dots, n$. For $x_i \in L_i$ and $p \in S$, $\delta_i(x_i, p)$ will be the solution of the problem

$$\begin{aligned} P_i(x_i, p) \quad & \max u_i(x_i + \delta) \\ \text{s.t.} \quad & \langle p, \delta \rangle \leq 0 \\ & x_i + \delta \in L_i, \end{aligned}$$

and $d_i(x_i, p) = \alpha_i \delta_i(x_i, p)$, where $\alpha_i > 0$ for each $i = 1, \dots, n$. Since strict concavity is a stronger condition than strict quasiconcavity, $\delta_i(x_i, p)$ is well defined on $L_i \times S$.

The solutions of $0 \in D(\xi)$ are called the critical points of the multifunction D . If $\xi \in K$ is such a critical point, the constant trajectory $x(t) = \xi$ is obviously a solution of (P') (with initial condition $x(0) = \xi$).

Let V be a function from K into $\mathbb{R}^n$. A point $x \in K$ is a Pareto maximum of V on K if there is no $y \in K$ such that $V(y) \geq V(x)$ and $V(y) \neq V(x)$, i.e., if we set

$$\mathbb{T}(x) := \{y \in K \mid V(y) \geq V(x)\},$$

$x \in K$ is a Pareto maximum of V if and only if $\mathbb{T}(x) = V^{-1}V(x)$. If V is continuous on K , the set $\mathbb{T}(x)$ contains a Pareto maximal point for every $x \in K$.

Let $f: \mathbb{R}^n \rightarrow [-\infty, +\infty)$ be a concave function. The domain of f is defined by $\text{dom } f := \{x \in \mathbb{R}^n \mid f(x) > -\infty\}$. We will denote by $H(f)$ the subspace parallel to $\text{aff dom } f$, i.e., $H(f) = -x + \text{aff dom } f$, where x is any point in $\text{aff dom } f$. If $x \in \text{dom } f$ and $h \in \mathbb{R}^n$ is such that the segment $[x - \theta h, x + \theta h]$ is contained in the domain of f for some $\theta > 0$, the right derivative of f at x in the direction h , $Df(x)(h)$, exists, is positively homogeneous in h , and satisfies:

$$\frac{f(x) - f(x - \theta h)}{\theta} \geq Df(x)(h) \geq \frac{f(x + \theta h) - f(x)}{\theta}$$

In particular, if $x \in \text{ri dom } f$ and $h \in H(f)$, there exists a positive number θ such that the interval $[x - \theta h, x + \theta h]$ is contained in the domain of f .

Definition: Let $v: (\mathbb{R}^m)^n \rightarrow [-\infty, +\infty)$ be a concave function with $K \subset \text{ri dom } v$. Assume that $D(x) \subset H(v)$ for all $x \in K$. Then v has a derivative $Dv(x)(z)$ from the right at each point $x \in K$, in any direction $z \in D(x)$. Let us define

$$B(v, x) := \inf \{Dv(x)z \mid z \in D(x)\}.$$

We say that v is a Liapunov function on K for the multifunction D if $B(v, x) \geq 0$ for each $x \in K$.

Lemma 5.1: For $x = (x_1, \dots, x_n) \in (\mathbb{R}^m)^n$ define $v_i(x) \doteq u_i(x_i)$ and the function $V: (\mathbb{R}^m)^n \rightarrow \mathbb{R}^n$ by $V(x) := (v_1(x), \dots, v_n(x))$. Then each v_i is a Liapunov function on K for the multifunction D , and for every $\lambda \in \mathbb{R}_+^n$, $\sum_{i=1}^n \lambda_i v_i$ is also a Liapunov function for the multifunction D .

Proof: We first show that $Du_i(x_i)(\delta_i(x_i, p)) \geq 0$ for each $x_i \in L_i$, $p \in S$ and $i = 1, \dots, n$. By definition we have that $\delta_i(x_i, p) \in -x_i + L_i \subset H(u_i)$ and $u_i(x_i) \leq u_i(x_i + \delta_i(x_i, p))$. Since $x_i \in L_i \subset \text{ri dom } u_i$, there exists a positive number θ such that the interval $[x_i - \theta \delta_i(x_i, p), x_i + \theta \delta_i(x_i, p)]$ is contained in $\text{ri dom } u_i$. Also, by strict concavity $u_i(x_i + \lambda \delta_i(x_i, p)) = u_i(\lambda(x_i + \delta_i(x_i, p)) + (1 - \lambda)x_i) \geq \lambda u_i(x_i + \delta_i(x_i, p)) + (1 - \lambda)u_i(x_i)$ for each $\lambda \in (0, 1)$, with strict inequality if $\delta_i(x_i, p) \neq 0$. Thus

$$\begin{aligned} Du_i(x_i)(\delta_i(x_i, p)) &= \lim_{\lambda \rightarrow 0^+} \frac{u_i(x_i + \lambda \delta_i(x_i, p)) - u_i(x_i)}{\lambda} \\ &\geq u_i(x_i + \delta_i(x_i, p)) - u_i(x_i) \geq 0, \end{aligned}$$

and both inequalities are strict if $\delta_i(x_i, p) \neq 0$.

We have that $K \subset \bigcap_{i=1}^n L_i \subset \bigcap_{i=1}^n \text{ri dom } u_i \subset \text{ri dom } v_i$ and $D(x) \subset \bigcap_{i=1}^n \alpha_i(-x_i + L_i) \subset \bigcap_{i=1}^n H(u_i) \subset H(v)$. Finally, for each $x \in \bigcap_{i=1}^n L_i$ and each $d = (d_1, \dots, d_n) \in D(x)$ there exists $p \in S$ such that $d_i = \alpha_i \delta_i(x_i, p)$, and therefore

$$Dv_i(x)(d) = Du_i(x_i)(d_i) = Du_i(x_i)(\alpha_i \delta_i(x_i, p)) = \alpha_i Du_i(x_i)(\delta_i(x_i, p)) \geq 0.$$

The inequality $\sum_{i=1}^n \lambda_i B(v_i, x) \leq B\left(\sum_{i=1}^n \lambda_i v_i, x\right)$ holds for any $\lambda \in \mathbb{R}_+^n$, so $\sum_{i=1}^n \lambda_i v_i$ is also a Liapunov function. $\blacksquare$

Theorem 5.2: If $x : [0, T] \rightarrow K$, together with $p : [0, T] \rightarrow S$, is a solution of the economic problem (P), the function $t \rightarrow w_i(t) = u_i(x_i(t))$ is absolutely continuous and nondecreasing for each $i = 1, \dots, n$. Each Pareto maximal point of V is a critical point of D , and $\bigcap(z)$ contains a critical point of D for each $z \in K$.

Proof: The function $x : [0, T] \rightarrow K$ is absolutely continuous; therefore x is differentiable almost everywhere in $(0, T)$. Also $K \subset \text{ri dom } v_i$, hence $x'(t) \in T_K(x(t)) \subset H(v_i)$ for almost all t in $(0, T)$. Therefore, the right derivative $Dv_i(x(t))(x'(t))$ exists for almost all t in $(0, T)$, and one can show that $Dv_i(x(t))(x'(t)) = D(v_i \circ x)(t)(1)$. On the other hand, $x'(t) = d(x(t), p(t))$ almost everywhere in $(0, T)$. But $D(v_i)(x(t))(d(x(t), p(t))) \geq 0$ for all $t \in [0, T]$; therefore $D(v_i)(x(t))(x'(t)) = Dw_i(t)(1) \geq 0$ for almost all t in $(0, T)$.

The function v_i being locally Lipschitz continuous on K (see Theorem 10.4 in [14]) and the function x being absolutely continuous imply that $w_i = v_i \circ x$ is absolutely continuous. Hence there exists a measurable function $g : [0, T] \rightarrow \mathbb{R}$

such that:

$$v_i \circ x(\tau) = v_i \circ x(0) + \int_0^\tau g(t) dt \quad \text{for all } \tau \in [0, T] .$$

It is possible to show that $g(t) = D(v_i \circ x)(t)(1)$ almost everywhere in $(0, T)$ (see, for example, Theorem 9 on p. 103 in [15]). So, if $0 \leq \sigma < \tau \leq T$,

$$w_i(\tau) = w_i(\sigma) + \int_\sigma^\tau D(w_i)(t)(1) dt \geq w_i(\sigma) ,$$

and w_i is nondecreasing.

Let $\xi \in K$ be a Pareto optimal point of V . The strict concavity of the functions u_i , $i = 1, \dots, n$, implies that $\Pi(\xi) = V^{-1}V(\xi) = \{\xi\}$, for if $z \in \Pi(\xi)$ is another point, $V(\lambda z + (1-\lambda)\xi) > \lambda V(z) + (1-\lambda)V(\xi) = V(\xi)$ for each $\lambda \in (0, 1)$; this is a contradiction because $\lambda z + (1-\lambda)\xi \in K$ for all $\lambda \in [0, 1]$. Let $x: [0, T] \rightarrow K$ together with $p: [0, T] \rightarrow S$ be a solution of the economic problem (P) with initial condition $x(0) = \xi$. Then $V(x(t)) \geq V(\xi)$ for all $t > 0$; therefore $x(t) = \xi$ for all $t > 0$ and

$$0 = x'(t) = d(x(t), p(t)) = d(\xi, p(t)) \quad \text{for all } t \geq 0 .$$

Hence $0 \in D(\xi)$. If $z \in K$, $\Pi(z)$ contains a Pareto maximal point and therefore a critical point of D . $\blacksquare$

We shall call limit point of the trajectory $x: [0, \infty) \rightarrow K$ any point z for which there exists an increasing sequence $\{t_\alpha\}$ such that $t_\alpha \rightarrow \infty$ and $x(t_\alpha) \rightarrow z$. Since K is compact, any trajectory x has limit points. Also, each $w_i(t) = u_i(x_i(t))$ is bounded, and by the previous theorem, nondecreasing. So $w_i(t)$ has a limit as $t \rightarrow \infty$. An easy argument shows that if z_1 and z_2 are two limit points of the trajectory x , $V(z_1) = V(z_2)$. Finally, if x is a trajectory such that $V(x(t))$ remains constant on $[0, T]$, the proof of Lemma 5.1 shows that $x(t) = x(0)$ for all $t \in [0, T]$, and $x(0)$ is a critical point of D .

Let $X_T(x^0)$ denote the set of all trajectories solving problem (P') with initial condition $x(0) = x^0$. We have shown in Section 3 that $X_T(x^0)$ is not empty for each $x^0 \in K$. The remarks made above show that the functions $\sum_{i=1}^n \lambda_i v_i$, $\lambda \in \text{Int } \mathbb{R}_+^n$, defined in Lemma 5.1, are Liapunov functions in the sense defined by Campsaur, Dreze and Henry [7]. They have proven the following theorem:

Theorem 5.3: If $X_T(x^0)$ is nonempty and compact in $C_U([0, T], (\mathbb{R}^m)^n)$ for every $x^0 \in K$, and if X_T is u.s.c., then any limit point of any trajectory is a critical point.

We have seen that $0 \leq \sum_{i=1}^n \lambda_i B(v_i, x) \leq B\left(\sum_{i=1}^n \lambda_i v_i, x\right)$ for any $\lambda \in \mathbb{R}_+^n$. If we assume that whenever x is not a Pareto maximum, there is a $\lambda \in \mathbb{R}_+^n \setminus \{0\}$ and $\epsilon > 0$ such that $B\left(\sum_{i=1}^n \lambda_i v_i, y\right) \geq \epsilon$ for every $y \in K$ satisfying $v_i(y) \leq v_i(x)$ for all $i = 1, \dots, n$, we can show the following theorem. This condition will not be satisfied if the utility functions satisfy local nonsatiation and there exists a critical point $x = (x_1, \dots, x_n) \in K$ of D such that $Ax \in \text{int } M$. In this situation x cannot be a Pareto maximal point for V in K . However, $B\left(\sum_{i=1}^n \lambda_i v_i, x\right) = 0$ for any $\lambda \in \mathbb{R}_+^n$, and for any $\epsilon > 0$, by continuity, $B\left(\sum_{i=1}^n \lambda_i v_i, y\right) < \epsilon$ for all y sufficiently close to x .

Theorem 5.4: Let x be a trajectory solving problem (P'). Then any limit point of x is a Pareto maximum of V .

The proof is exactly analogous to that given by Aubin, Cellina and Nohel [1] for the discrete counterpart of this theorem.

Definition: $Q \subset K$ is stable for the economic problem if for any neighborhood M of Q , there exists a neighborhood N of Q such that, for any initial value $x(0) \in N$, all the solutions of the economic problem starting at $x(0)$ remain in M .

Aubin, Cellina and Nohel have also shown that for any Pareto maximum $x \in K$, the subset $\mathbb{T}(x)$ is stable (see Theorem 3.3 on pp. 110-111 and the remark on p. 112 in [1]).

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