**STEADY STATES OF AUSTENITIC–MARTENSITIC
DOMAINS IN THE GENZBURG–LANDAU THEORY
OF SHAPE MEMORY ALLOYS**

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STEADY STATES OF AUSTENITIC–MARTENSITIC DOMAINS IN THE GINZBURG–LANDAU THEORY OF SHAPE MEMORY ALLOYS

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Abstract. In this paper we investigate the steady state distributions of the austenitic and martensitic phases in a one-dimensional segment of a shape memory alloy which is clamped at both ends; i.e., we look for distributions remaining unchanged in time while the outside temperature is maintained constant, and no body forces, distributed heat sources or boundary stresses are applied.

§1. The problem. Following Falk [4] we assume the density of the free energy in the Landau–Ginzburg form

$$(1.1) \quad \tilde{F}(\tilde{\varepsilon}, \tilde{\varepsilon}_{\tilde{x}}, \tilde{\theta}) = \tilde{F}_0(\tilde{\theta}) + \gamma(\tilde{\theta} - \tilde{\theta}_M)\tilde{\varepsilon}^2 - \beta\tilde{\varepsilon}^4 + \alpha\tilde{\varepsilon}^6 + \delta\tilde{\varepsilon}_{\tilde{x}}^2,$$

where $\tilde{\varepsilon}$ stands for the shear strain ($\tilde{\varepsilon} = \frac{d\tilde{u}}{d\tilde{x}}$, $\tilde{u}$ is the displacement and $\tilde{x}$ is the space coordinate) and $\tilde{\theta}$ is the absolute temperature. The quadratic term of the strain gradient $\tilde{\varepsilon}_{\tilde{x}}$ takes interfacial energies into account. $\tilde{\theta}_M$ plays the role of the critical temperature in the Landau theory of martensitic phase transitions (see [3]), and $\alpha, \beta, \gamma, \delta$ are positive physical constants.

In the sequel the variables characterized by a tilde are measured in physical units while variables without the tilde are referred to rescaled dimensionless units.

For the particular shape memory alloy $Au_{23}C_{30}Zn_{47}$ a complete set of data have been supplied in [4]:

$$(1.2) \quad \begin{aligned} \tilde{\theta}_M &= 208K, \quad \alpha = 7.5 \times 10^6 Jcm^{-3}, \quad \beta = 1.5 \times 10^5 Jcm^{-3}, \\ \gamma &= 24 Jcm^{-3} K^{-1}, \quad \delta = 10^{-12} Jcm^{-1}. \end{aligned}$$

Notice that for fixed $\tilde{\varepsilon}_{\tilde{x}}$ and $\tilde{\theta}$ close to $\tilde{\theta}_M$, the free energy density $\tilde{F}(\cdot, \tilde{\varepsilon}_{\tilde{x}}, \tilde{\theta})$ is not convex as a function of the shear strain $\tilde{\varepsilon}$.

Introducing

$$(1.3) \quad \tilde{\theta}_E = \tilde{\theta}_M + \frac{\beta^2}{4\alpha\gamma}, \quad \tilde{\theta}_A = \tilde{\theta}_M + \frac{\beta^2}{3\alpha\gamma},$$

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we note the following facts:

- 1°: For $\tilde{\theta} > \tilde{\theta}_A$, $\tilde{F}(\cdot, \tilde{\varepsilon}_{\tilde{x}}, \tilde{\theta})$ has only one minimum at $\tilde{\varepsilon} = 0$, corresponding to the high temperature phase austenite.
- 2°: For $\tilde{\theta} < \tilde{\theta}_M$ there exist two minima, corresponding to the low temperature phase martensite existing in two orientations ("twinning").
- 3°: For $\tilde{\theta}_M < \tilde{\theta} < \tilde{\theta}_A$ there are three minima so that the three phases may coexist; for $\tilde{\theta} = \tilde{\theta}_E$ the minima of $\tilde{F}$ (as a function of $\tilde{\varepsilon}$) have equal depth.

For a detailed account of the related physical background we refer to [3] [4] [5].

Following [4] we introduce the rescaled dimensionless variables (with $\tilde{l}$ denoting the length of the sample, measured in physical units):

$$(1.4) \quad \begin{aligned} x &= \frac{\beta}{(\alpha\delta)^{1/2}} \tilde{x}, \quad u = \left(\frac{\beta}{\delta}\right)^{1/2} \tilde{u}, \quad \varepsilon = \left(\frac{\alpha}{\beta}\right)^{1/2} \tilde{\varepsilon}, \\ T &= \frac{\alpha\gamma}{\beta^2} (\tilde{\theta} - \tilde{\theta}_M) - \frac{1}{4}, \quad a = \frac{\beta}{(\alpha\delta)^{1/2}} \tilde{l}. \end{aligned}$$

Notice that $T = -\frac{1}{4}, 0, \frac{1}{12}$ correspond to $\tilde{\theta} = \tilde{\theta}_M, \tilde{\theta}_E, \tilde{\theta}_A$ in that order.

The following drawing describes the shape and extrema of the rescaled free energy function $\tilde{F}(\varepsilon, 0, T)$:

FIGURE 1

In the rescaled dimensionless variables the steady state (Euler) equation

$$\frac{\partial^2}{\partial \tilde{x}^2} \frac{\partial \tilde{F}}{\partial \tilde{\varepsilon}_{\tilde{x}}} - \frac{\partial}{\partial \tilde{x}} \frac{\partial \tilde{F}}{\partial \tilde{\varepsilon}} = 0$$

becomes (see [5])

$$(1.5) \quad u^{(4)} - \left(T + \frac{1}{4}\right) u'' + 6u'^2 u'' - 15u'^4 u'' = 0, \quad 0 \leq x \leq a;$$

the boundary conditions are

$$(1.6) \quad u''(0) = u''(a) = 0$$

and, if the sample is clamped at $x = 0$ and at $x = a$,

$$(1.7) \quad u(0) = 0, \quad u(a) = 0.$$

The main purpose of this paper is to study the nontrivial solutions of (1.5)–(1.7). In §2 we show that

(i) if

$$T > \frac{1}{12} - \frac{\pi^2}{a^2}$$

then the only solution of (1.5)–(1.7) is $u \equiv 0$;

(ii) for any $k = 1, 2, \dots$, there is an unbounded component γ_k of nontrivial solutions (T, u_T) of (1.5)–(1.7) passing through $(T_k, 0)$ where

$$T_k = -\frac{1}{4} - \frac{\pi^2 k^2}{a^2}.$$

The proof of (ii) is based on bifurcation theory and can be applied to much more general functions $F(\varepsilon, \varepsilon_x, T)$, as for instance in [7]. In §3 we apply another general method, based on a fixed point theorem, in order to get a “horizontal” view of the branches of solutions of (1.5)–(1.7).

In §4 we study (1.5)–(1.7) by the direct method of integration. We shall obtain information on nontrivial solutions of (1.5)–(1.7) for values of T not covered by (i), (ii). Indeed, we find that

(iii) if $0 < T < \frac{1}{12}$ then there exist no nontrivial solutions, but if $T < 0$ then nontrivial solutions do exist.

Falk [5] has studied solutions of (1.5), (1.6) satisfying

$$(1.8) \quad u(0) = 0, \quad u'''(a) - \left(T + \frac{1}{4}\right) u'(a) + 2u'^3(a) - 3u'^5(a) = 0$$

(the last condition means total stress zero at $x = a$) by the direct method of integration. As we shall see in §4, there are interesting connections between the solutions of (1.5)–(1.7) and (1.5), (1.6), (1.8).

§2. Branches of solutions of (1.5)–(1.7).

THEOREM 2.1. *If $T > \frac{1}{12} - \frac{\pi^2}{a^2}$ then (1.5)–(1.7) has no nontrivial solutions.*

Proof. If u is a nontrivial solution then, multiplying (1.5) by u and integrating in x over $(0, a)$ we obtain, after integration by parts,

$$(2.1) \quad \int_0^a u''^2 dx + \int_0^a u'^2 \left(T + \frac{1}{4} - 2u'^2 + 3u'^4 \right) dx = 0.$$

We claim that

$$(2.2) \quad \int_0^a u''^2 dx \geq \frac{\pi^2}{a^2} \int_0^a u'^2 dx.$$

Indeed, let $u' = z$. Then $\int_0^a z dx = 0$ and thus it suffices to show that

$$(2.3) \quad \lambda \equiv \min_{z \in \mathcal{A}} \left\{ \int_0^a z'^2 dx / \int_0^a z^2 dx \right\} \geq \frac{\pi^2}{a^2}$$

where

$$\mathcal{A} = \left\{ \zeta \in H^1(0, a), \quad \int_0^a \zeta dx = 0 \right\}.$$

Suppose ζ_0 is a minimizer. Then

$$\int_0^a \zeta_0' \zeta' dx = \lambda \int_0^a \zeta_0 \zeta dx, \quad \forall \zeta \in \mathcal{A}.$$

It follows that

$$\int_0^a (\zeta_0'' + \lambda \zeta_0) \zeta dx = [\zeta_0' \zeta]_0^a.$$

Hence $\zeta_0'' + \lambda \zeta_0 = \text{const.} = c$ and $\zeta_0'(0) = \zeta_0'(a) = 0$. Finally,

$$ca = \int_0^a (\zeta_0'' + \lambda \zeta_0) dx = \lambda \int_0^a \zeta_0 dx = 0, \quad \text{i.e., } c = 0,$$

and we conclude that

$$\zeta_0 = c_0 \cos \sqrt{\lambda} x, \quad \sqrt{\lambda} a = \pi,$$

so that $\lambda = \pi^2/a^2$.

Noticing that $\frac{\pi^2}{a^2} + T + \frac{1}{4} - 2u'^2 + 3u'^4 > 0$ for $T > \frac{1}{12} - \frac{\pi^2}{a^2}$, we conclude the assertion of the theorem from (2.1) and (2.2).

REMARK 2.1. For the particular shape memory alloy $Au_{23}C_{30}Zn_{47}$ we obtain from (1.2), (1.4) that

$$(2.4) \quad a = \frac{\beta}{(\alpha\delta)^{1/2}} \tilde{l} \cong 0.5 \times 10^8 \text{ cm}^{-1} \tilde{l}.$$

Consequently, for samples of unit length the quantity π^2/a^2 is negligible compared to $\frac{1}{12}$, and thus for temperatures above and slightly below $\tilde{\theta}_A$ pure austenite is the only possible steady state.

We now proceed to establish existence of nontrivial solutions, by considering (1.5)–(1.7) as a nonlinear eigenvalue problem with T as parameter. A solution corresponding to T will be denoted by u_T , and (T, u_T) will denote the eigenpair.

For $k = 1, 2, 3, \dots$ and $\nu \in \{+1, -1\}$, set

$$(2.5) \quad H_k^\nu = \{y \in C^2[0, a]; y(0) = y(a) = 0, y \text{ has exactly } k+1 \text{ simple zeros in } [0, a], \text{ and } \lim_{x \rightarrow 0+} \text{sgn } y(x) = \nu\},$$

$$(2.6) \quad T_k = -\frac{1}{4} - \frac{\pi^2 k^2}{a^2}.$$

THEOREM 2.2. For any $k \in \mathbb{N}$, $\nu \in \{+1, -1\}$, there emanates from the bifurcation point $(T_k, 0)$ an unbounded and connected branch $C_k^\nu \subset (\mathbb{R} \times C^4[0, a])$ of eigenpairs (T, u_T) of (1.5)–(1.7), and

$$(2.7) \quad u_T'' \in H_k^\nu \quad \text{whenever} \quad (T, u_T) \in C_k^\nu \quad \text{and} \quad T \neq T_k;$$

moreover, for $T \neq T_k$, T sufficiently close to T_k , u_T' has exactly k simple zeros in $[0, a]$.

Proof. We shall work with the function $w = u''$. Since $u(0) = u(a) = 0$, we can solve u uniquely in terms of w ,

$$(2.8) \quad u(x) = \int_0^x \int_0^\xi w(t) dt d\xi - \frac{x}{a} \int_0^a \int_0^\xi w(t) dt d\xi.$$

The function u satisfies (1.5)–(1.7) if and only if w satisfies

$$(2.9) \quad \begin{aligned} -w'' = & -\left(T + \frac{1}{4}\right)w + \left[6 \left(\int_0^x w(\xi) d\xi - \frac{1}{a} \int_0^a \int_0^\xi w(t) dt d\xi\right)^2 \right. \\ & \left. - 15 \left(\int_0^x w(\xi) d\xi - \frac{1}{a} \int_0^a \int_0^\xi w(t) dt d\xi\right)^4 \right] w, \end{aligned}$$

$$(2.10) \quad w(0) = w(a) = 0.$$

Define an operator $S : C[0, a] \rightarrow C[0, a]$ by

$$(2.11) \quad \begin{aligned} (Sw)(x) = & 6 \left(\int_0^x w(\xi) d\xi - \frac{1}{a} \int_0^a \int_0^\xi w(t) dt d\xi \right)^2 \\ & - 15 \left(\int_0^x w(\xi) d\xi - \frac{1}{a} \int_0^a \int_0^\xi w(t) dt d\xi \right)^4. \end{aligned}$$

Then we can rewrite (2.9) as

$$(2.12) \quad -w'' = \lambda w + (Sw)(x) \cdot w, \quad \lambda = -\left(T + \frac{1}{4}\right),$$

and, obviously,

$$\|S(w)\|_{L^\infty(0,a)} = o(1) \quad \text{as} \quad \|w\|_{L^\infty(0,a)} \rightarrow 0.$$

The eigenvalue problem (2.12), (2.10) belongs to the class of eigenvalue problems considered in Theorem 4.11 in [6], except that instead of $wS(w)$ there appears (in [6]) the term $h(x, w, w', \lambda)$. Nevertheless the proof of Theorem 4.11 can be adapted with obvious modifications to the present problem. We thus infer that for any $k \in \mathbb{N}, \nu \in \{+1, -1\}$ there exists an unbounded and connected branch $\tilde{C}_k^\nu$ of solutions (λ, w_λ) of (2.12), (2.10) which emanates from $(\lambda_k, 0)$ where $\lambda_k = \pi^2 k^2 / a^2$ is the k -th eigenvalue of the linear problem

$$-w'' = \lambda w \quad \text{in} \quad (0, a), \quad w(0) = w(a) = 0.$$

Moreover,

$$\tilde{C}_k^\nu \subset (\mathbb{R} \times H_k^\nu) \cup \{(\lambda_k, 0)\}$$

and the mapping $\lambda \rightarrow \|w_\lambda\|_{L^\infty}$ is continuous in a neighborhood of λ_k . This yields the existence assertion of Theorem 2.2. It remains to show that for $T \neq T_k, |T - T_k|$ small, u'_T has exactly k simple zeros, where $(T, u_T) \in \tilde{C}_k^\nu$.

From the standard theory of bifurcation from simple eigenvalues we have

$$(2.13) \quad w_T(x) = \rho(T) \sin \frac{k\pi}{a} x + o(\rho(T)) \quad , \quad \rho(T) = \|w_T\|_{L^\infty},$$

and

$$\rho \equiv \rho(T) \rightarrow 0 \quad \text{if} \quad T \neq T_k, T \rightarrow T_k.$$

From (2.8) we then get

$$(2.14) \quad u'_T(x) = -\rho \frac{a}{k\pi} \cos \frac{k\pi x}{a} + c + o(\rho).$$

Integrating (2.14) over $(0, a)$ and using (1.7) we get $c = o(\rho)$. We can now deduce from (2.14) (with $c = o(\rho)$) that u'_T has k zeros which are close to the zeros of $\cos \frac{k\pi x}{a}$. These zeros are simple and u'_T has no other zeros in $[0, a]$, since (by (2.13)) $u''_T \neq 0$ in a neighborhood of the zeros of $\cos \frac{k\pi x}{a}$.

REMARK 2.2. From Theorem 2.2 it follows that for values T near T_k there exist small (in $\|\cdot\|_{L^\infty}$ sense) nontrivial solutions u_T such that u'_T has k simple zeros. This means that for temperatures $\tilde{\theta}$ near $\tilde{\theta}_k = \tilde{\theta}_M - \frac{\beta^2}{\alpha\gamma} \frac{k^2\pi^2}{a^2}$ steady states with k martensite-martensite interfaces exist. For the shape memory alloy $Au_{23}Cu_{30}Zn_{47}$ we obtain from (1.2) and (1.4) for a sample of unit length $\tilde{l} = 1cm$:

$$\frac{\beta^2\pi^2}{\alpha\gamma a^2} = \frac{\pi^2}{24} \times 10^{-12} K.$$

Consequently, already $1K$ below the critical temperature $\tilde{\theta}_M = 208K$ steady states with up to 10^6 interfaces may exist. Such a finely twinned microstructure should be observed in experiments.

§3. The horizontal point of view. In §2 we have explored the $T - u_T$ relation in the vertical direction, i.e., we asked for which T a nontrivial solution u_T of (1.5)–(1.7) exists. An answer was given in Theorem 2.2. Now we take the horizontal point of view: We ask, for any prescribed value of $r > 0$, how many solutions u_T exist with $\|u''_T\|_{L^\infty} = r$. We shall employ a method developed in [2] and [8] in a similar context.

Set

$$(3.1) \quad H_k^\nu(r) = \{y \in H_k^\nu, \|y\|_{L^\infty(0,a)} = r\}$$

where H_k^ν is defined in (2.5).

THEOREM 3.1. *For any $k \in \mathbb{N}$, $\nu \in \{+1, -1\}$, $r > 0$ there exists an eigenpair $(T_k^\nu(r), u_k^\nu(r))$ of (1.5)–(1.7) such that*

$$(3.2) \quad (u_k^\nu(r))'' \in H_k^\nu(r);$$

moreover

$$(3.3) \quad \frac{\pi^2 k^2}{a^2} - \frac{27}{2} a^2 r^2 \leq - \left(T_k^\nu(r) + \frac{1}{4} \right) \leq \frac{\pi^2 k^2}{a^2} + \frac{1215}{16} a^4 r^4,$$

and for r sufficiently small $(u_k^\nu(r))'$ has exactly r simple zeros in $[0, a]$.

Proof. We shall work with the formulation (2.12), (2.10) of (1.5)–(1.7). Let

$$K_r = \{w \in C[0, a], \|w\|_{L^\infty} \leq r\}.$$

Clearly K_r is a closed, bounded and convex subset in $L^\infty(0, a)$. The operator S defined in (2.11) maps K_r continuously into a bounded subset M of $C[0, a]$; in fact, since

$$(3.4) \quad \left| \int_0^x w(\xi) d\xi - \frac{1}{a} \int_0^a \int_0^\xi w(t) dt d\xi \right| \leq \frac{3}{2} ar,$$

we can choose

$$M = \left\{ w \in C[0, a], -\frac{1215}{16} a^4 r^4 \leq w(x) \leq \frac{27}{2} a^2 r^2 \right\}.$$

From the maximum–minimum variational principle [1] we infer that for any $w \in K_r$ the linear Sturm–Liouville problem (with $\lambda = -(T + \frac{1}{4})$)

$$(3.5) \quad \begin{aligned} -y'' &= \lambda y + (Sw)(x)y & \text{in } (0, a), \\ y(0) &= y(a) = 0 \end{aligned}$$

has an infinite sequence of simple eigenvalues $\lambda_k(w)$ with associated (uniquely determined) eigenfunctions $y_k^\nu(w) \in H_k^\nu(r)$, $k \in \mathbb{N}$. Moreover the mapping $w \rightarrow \lambda_k(w)$ is continuous on K_r and, since $Sw \in M$, the comparison results on eigenvalues show that

$$(3.6) \quad \frac{\pi^2 k^2}{a^2} - \frac{27}{2} a^2 r^2 \leq \lambda_k(w) \leq \frac{\pi^2 k^2}{a^2} + \frac{1216}{16} a^4 r^4.$$

In particular, the set

$$\Lambda_k^r = \{\lambda_k(w), w \in K_r\}$$

is bounded.

Fix $k \in \mathbb{N}$ and $\nu \in \{+1, -1\}$ and consider the mapping W which assigns to each $w \in K_r$ the unique eigenfunction $y_k^\nu(w)$. If we can prove that W has a fixed point then the asserted existence result of the theorem follows.

Observe that $W(K_r) \subset H_k^\nu(r) \subset K_r$. Also, from the boundedness of Λ_k^r and $S(K_r)$ we easily infer that $W(K_r)$ is a relatively compact subset of $C[0, a]$. If we can show that W is continuous, then the Schauder fixed point theorem can be applied to conclude that W has a fixed point.

To prove the continuity of W , let $\{w_n\} \subset K_r$, $\|w_n - w\|_{L^\infty} \rightarrow 0$ for some $w \in K_r$, and let $(\lambda_k(w_n), y_k(w_n))$ and $(\lambda_k(w), y_k(w))$, respectively, denote the corresponding k -th eigenpairs of (3.6). Clearly $\lambda_k(w_n) \rightarrow \lambda_k(w)$ as $n \rightarrow \infty$. Denoting by $G(x, t)$ the Green function associated with $\{-y'', y(0) = 0, y(a) = 0\}$, we can write

$$(3.7) \quad y_k(w_n)(x) = \int_0^a G(x, t) [\lambda_k(w_n) + S(w_n)(t)] y_k(w_n)(t) dt.$$

Now take any convergent subsequence of $\{y_k(w_n)\}$, that is

$$\|y_k(w_n) - \bar{y}\|_{L^\infty} \rightarrow 0 \quad \text{if} \quad k = k_j \rightarrow \infty.$$

Taking $k = k_j \rightarrow \infty$ in (3.7) we get

$$\bar{y}(x) = \int_0^a G(x, t) [\lambda_k(w) + S(w)(t)] \bar{y}(t) dt.$$

Since also $\|\bar{y}\|_{L^\infty} = r$, $\bar{y}$ is an eigenfunction of (3.6) associated with $\lambda_k(w)$. Noting that $\lim_{x \rightarrow 0+} \text{sgn} \bar{y}(x) = \nu$, it follows, by the simplicity of $\lambda_k(w)$, that $\bar{y} = y_k'(w)$. Since $\bar{y}$ is independent of the subsequence k_j , we conclude that the full sequence y_n is convergent in $L^\infty[0, a]$ to $y_k'(w)$.

Having completed the proof of existence, it remains to show that $(u_k'(r))'$ has k simple zeros in $[0, a]$, for sufficiently small r . But this can be established by an argument similar to that used in the proof of Theorem 2.2.

§4. The method of direct integration. From (1.5) we get

$$(4.1) \quad u^{(3)} - \left(T + \frac{1}{4}\right) u' + 2u'^3 - 3u'^5 \equiv \text{const.} = \frac{\gamma}{2}.$$

Set $z = u'$ and

$$\begin{aligned} F(z) &= \left(T + \frac{1}{4}\right) z - 2z^3 + 3z^5, \\ f(z) &= \left(T + \frac{1}{4}\right) z^2 - z^4 + z^6 = 2 \int F(z) dz. \end{aligned}$$

Then (4.1) becomes

$$(4.2) \quad z'' = F(z) + \frac{\gamma}{2}$$

and therefore

$$\frac{1}{2} \frac{d}{dx} (z'^2) = z'' z' = \left(F(z) + \frac{\gamma}{2}\right) z' = \frac{1}{2} \frac{d}{dx} (f(z) + \gamma z).$$

It follows that

$$(4.3) \quad \frac{dx}{dz} = \pm \frac{1}{\sqrt{f(z) + \gamma z + c}}, \quad c \text{ constant},$$

where “+” and “−” correspond to $z' > 0$ and $z' < 0$, respectively.

If z is a solution of (4.2) and u is defined by

$$(4.4) \quad u(x) = \int_0^x z(y) dy$$

then u satisfies (1.5)–(1.7) if and only if

$$(4.5) \quad \int_0^a z(y) dy = 0 ,$$

and

$$(4.6) \quad \left| \frac{dx}{dz} \right| = \infty \quad \text{at } x = 0 \text{ and } x = a$$

(i.e., $u'' = z' = 0$ at $x = 0$ and $x = a$).

We shall construct solutions of (4.2) by gluing together solutions of (4.3). For this purpose we shall define a *positive branch* to be a solution of

$$(4.7) \quad \frac{dx}{dz} = \frac{1}{\sqrt{f(z) + \gamma z + c}} \quad \text{for } z_1 < z < z_2,$$

where z_1, z_2 are such that

$$(4.8) \quad \begin{aligned} f(z) + \gamma z + c &> 0 & \text{if } z_1 < z < z_2, \\ f(z_i) + \gamma z_i + c &= 0 & \text{for } i = 1, 2. \end{aligned}$$

Similarly, we define a *negative branch* by

$$(4.9) \quad \frac{dx}{dz} = -\frac{1}{\sqrt{f(z) + \gamma z + c}} \quad \text{for } z_1 < z < z_2$$

subject to (4.8). On a positive branch $z' > 0$, and on a negative branch $z' < 0$.

Consider for example a positive branch starting with $x = 0$:

$$(4.10) \quad x(z) = \int_{z_1}^z \frac{d\zeta}{\sqrt{f(\zeta) + \gamma \zeta + c}} , \quad z \text{ goes from } z_1 \text{ to } z_2,$$

and denote its right x -endpoint by x_1 :

$$(4.11) \quad x_1 = \int_{z_1}^{z_2} \frac{d\zeta}{\sqrt{f(\zeta) + \gamma \zeta + c}} \equiv \alpha , \quad \alpha = \alpha(\gamma, c).$$

We can glue this branch at $x = x_1$ to a negative branch, given by

$$(4.12) \quad x(z) = x_1 + \int_z^{z_2} \frac{d\zeta}{\sqrt{f(\zeta) + \gamma\zeta + c}}, \quad z \text{ goes from } z_2 \text{ to } z_1.$$

Notice that $x(z)$ is continuously differentiable across x_1 (since $|dx/dz| \rightarrow \infty$ or $|dz/dx| \rightarrow 0$ as $x \rightarrow x_1 \pm 0$). Further, from the differential equation (4.2) we see that the second derivatives of z also agree from both sides of $x = x_1$. The branch (4.12) is defined for $x_1 \leq x \leq x_2$ where $x_2 = x_1 + \alpha = 2\alpha$.

Similarly we can glue a positive branch at $x = x_2$, defined for $x_2 \leq x \leq x_3$ where $x_3 = 3\alpha$, then a negative branch for $3\alpha \leq x \leq 4\alpha$, etc.

Observe that if $z_1 < z_2 \leq 0$ then $z(x) < 0$ for both positive and negative branches and, consequently, (4.5) cannot be satisfied. The same applies to the case $0 \leq z_1 < z_2$. We conclude:

LEMMA 4.1. *If (4.8) implies that either $z_1 < z_2 \leq 0$ or $0 \leq z_1 < z_2$ then there exist no nontrivial solutions of (1.5)–(1.7).*

From Figure 1 in §1 and Lemma 4.1 we deduce the following extension of Theorem 2.1.

THEOREM 4.2. *If $T > 0$ then (1.5)–(1.7) has no nontrivial solutions.*

Consider next the case $T < 0$. Then there certainly are values of γ, c with corresponding branches for which $z_1 < 0 < z_2$. Let us consider a solution $z(x)$ made up of n branches, starting with (4.10), (4.12). The right-endpoints of the branches are $x_j = \alpha j$ ($j = 1, 2, \dots, n$). For $0 < x < \alpha$,

$$(4.13) \quad \begin{aligned} u(x) - u(0) &= \int_0^x z(x) dx = \int_{z_1}^{z(x)} z \frac{dx}{dz} dz \\ &= \int_{z_1}^{z(x)} \frac{z dz}{\sqrt{f(z) + \gamma z + c}}; \end{aligned}$$

for $\alpha < x < 2\alpha$,

$$(4.14) \quad u(x) - u(\alpha) = - \int_{z_2}^{z(x)} \frac{z dz}{\sqrt{f(z) + \gamma z + c}},$$

etc. In particular,

$$(4.15) \quad \begin{aligned} u(\alpha) - u(0) &= \int_{z_1}^{z_2} \frac{z dz}{\sqrt{f(z) + \gamma z + c}} \equiv A, \\ u(2\alpha) - u(\alpha) &= A, \quad \text{etc.}; \end{aligned}$$

thus

$$(4.16) \quad u(j\alpha) = jA ; \quad A = A(\gamma, c).$$

Consequently, in order that $u(a) = u''(a) = 0$, we must have

$$(4.17) \quad A = 0 \quad \text{and} \quad a = n\alpha, \quad \text{for some} \quad n \in \mathbb{N}.$$

Consider the case $\gamma = 0$ first. Then, since $f(z) \equiv f(-z)$, (4.8) can only be satisfied if $z_1 = -z_2$. On the other hand, if $z_1 = -z_2$, then $A = 0$. We conclude:

THEOREM 4.3. *If $\gamma = 0$ then any n -branch is a solution of (1.5)–(1.7) provided $a = n\alpha$.*

When $\gamma = 0$, the integral in (4.10) can be expressed by elliptic integrals and the relation

$$(4.18) \quad a = n\alpha \quad (\alpha = \alpha(c))$$

can be solved for c in terms of a and n ; this was done by Falk [5]. As he has shown, for any given n and sufficiently large a there exists a c such that (4.18) holds.

Next we consider the case where $\gamma \neq 0$. We are going to prove that this implies that $A \neq 0$ whenever $T \leq -\frac{1}{4}$. Consequently, the n -branch solutions for $\gamma = 0$ (established in Theorem 4.3) are the only possible nontrivial solutions for $T \leq -\frac{1}{4}$. In particular, the solutions u_T obtained in Theorem 2.2 are, at least for T close to T_k , solutions of this type.

LEMMA 4.4. *Assume that $T \leq -\frac{1}{4}$. If $\gamma > 0$ then $A > 0$ and, similarly, if $\gamma < 0$ then $A < 0$.*

Proof. Set $g(z) = f(z) + c$. Then

$$g(z) > 0 \quad \text{if} \quad -\beta < z < \beta, \quad g(\pm\beta) = 0$$

and

$$\begin{aligned} g(z) + \gamma z > 0 \quad \text{if} \quad -\beta + \varepsilon_0 < z < \beta + \varepsilon, \\ \gamma(-\beta + \varepsilon_0) + g(-\beta + \varepsilon_0) = g(\beta + \varepsilon) + \gamma(\beta + \varepsilon) = 0 \quad \text{for some} \quad \varepsilon > 0, \varepsilon_0 > 0. \end{aligned}$$

Notice that, since $T \leq -\frac{1}{4}$, c is a positive constant. It will suffice to prove that if $\gamma > 0$ then $A > 0$, i.e.,

$$(4.19) \quad \int_0^{\beta+\varepsilon} \frac{zdz}{\sqrt{g(z) + \gamma z}} > - \int_{-\beta+\varepsilon_0}^0 \frac{zdz}{\sqrt{g(z) + \gamma z}} ;$$

the right-hand side is equal to

$$\int_0^{\beta-\varepsilon_0} \frac{zdz}{\sqrt{g(z)-\gamma z}} = \int_0^{\beta+\varepsilon} \frac{1}{M^2} \frac{zdz}{\sqrt{g(z/M)-\gamma z/M}}$$

where

$$M = \frac{\beta + \varepsilon}{\beta - \varepsilon_0} > 1.$$

Hence in order to prove (4.19) it is sufficient to show that

$$M^2 \sqrt{g(z/M) - \gamma z/M} > \sqrt{g(z) + \gamma z},$$

or

$$h(z) \equiv M^4 g\left(\frac{z}{M}\right) - g(z) - (M^3 \gamma z + \gamma z) > 0 \quad \text{if } 0 < z < \beta + \varepsilon.$$

Notice that $h(0) = M^4 c - c > 0$, $h(\beta + \varepsilon) = 0$. Hence if we can show that

$$h'(z) < 0 \quad \text{for } 0 < z < \beta + \varepsilon$$

then $h(z) > 0$ and the lemma is proved. Setting $\lambda = -\left(T + \frac{1}{4}\right)$ we easily see that

$$h'(z) = -2M^2 \lambda z + 2\lambda z + 6 \frac{z^5}{M^2} - 6z^5 - (M^3 \gamma + \gamma) < 0,$$

since $\lambda \geq 0, \gamma > 0, M > 1$.

REMARK 4.1. If $T < 0$, then again $c > 0$. Thus we can proceed as in the above proof until the point where we have to show that $h'(z) < 0$ for $0 < z < \beta + \varepsilon$. But for this conclusion we need $\lambda \geq 0$, i.e., $T \leq -\frac{1}{4}$. In fact, if $\lambda < 0$ and $|\lambda| \ll 1, 0 < c \ll |\lambda|^2, 0 < \gamma \ll c$, then

$$6(\beta + \varepsilon)^4 < 2|\lambda|M^2$$

i.e., $h'(\beta + \varepsilon) > 0$, which means that $h(z) < 0$ for z near $\beta + \varepsilon$. Hence n -branch solutions for which $\gamma \neq 0$ solutions may exist for $-\frac{1}{4} < T < 0$. Numerical calculations using a shooting technique indicate that solutions actually do exist for $\gamma \neq 0$; a particular set of data admitting a nonzero numerical solution is given by $a = 1025$, $\lambda = -0.01$, $\gamma = 6.47262985 \times 10^{-4}$. Figure 2 shows plots of v, v', v'' , respectively, where $v(x) = u(a \cdot x)$, $0 \leq$

$x \leq 1$. Notice that according to (2.4) the dimensionless length $a = 1025$ corresponds to a sample length of $\tilde{l} = 2.1 \times 10^{-5}$ cm in physical units, i.e., to about 10^3 atomic diameters.

v for $a=1025$ $\Lambda=-0.01$

x

v' for $a=1025$ $\Lambda=-0.01$

v'' for $a=1025$, $\Lambda=-0.01$

x

FIGURE 2

REMARK 4.2. Let us consider briefly the case where the boundary conditions (1.7) are replaced by

$$(4.20) \quad u(a) = 0, \quad u^{(3)}(a) - \left(\left(T + \frac{1}{4} \right) u'(a) - 2u'^3(a) + 3u'^5(a) \right) = 0;$$

the second condition means that the total stress at $x = a$ is zero.

Introducing the function $z = u'$ we can reduce (1.5), (1.6), (4.20) to

$$(4.21) \quad \begin{aligned} -z'' &= - \left(T + \frac{1}{4} \right) z + 2z^3 - 3z^5, \quad 0 < x < a, \\ z'(0) &= z'(a) = 0, \end{aligned}$$

and we can recover u from z by $u(x) = \int_0^x z(y) dy$.

As in the proof of Theorem 2.1 one can show that the problem (1.5), (1.6), (4.20), or (4.21), has no nontrivial solutions if $T > \frac{1}{12}$. All the other results of §§2, 3 also extend to (1.5), (1.6), (4.20) (or (4.21)) with minor changes.

REMARK 4.3. The method of direct integration can be more effectively used for (4.21) than for (1.5)–(1.7), since it leads in this case to known elliptic integrals (see [5]). Theorem 4.3 shows that both problems (1.5)–(1.7) and (1.5), (1.6), (4.20) do have common solutions (made up of n branches). However, the solutions obtained by Falk [5] for $T > 0$ (austenite–martensite walls) are not solutions for problems (1.5)–(1.7).

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