DYNAMICS OF MEASURED VALUED SOLUTIONS TO A BACKWARD-FORWARD HEAT EQUATION

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DYNAMICS OF MEASURED VALUED SOLUTIONS TO A BACKWARD-FORWARD HEAT EQUATION*

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Abstract. This paper examines the asymptotic behavior of measure valued solutions to the initial value problem for the nonlinear heat conduction equation

$$\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{q}(\nabla u), \qquad x \in \Omega, \;\; t > 0$$

in a bounded domain $\Omega \subset \mathbf{R}^N$ with boundary condtions of the form

$$u = 0 \text{ on } \partial\Omega \quad \text{or} \quad \mathbf{q}(\nabla u) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

In particular, use of the Young measure representation of composite weak limits allows proof of a general trend to equilibrium. No linearity or monotonicity is assumed for $\mathbf{q}$; the only major restriction on $\mathbf{q}$ is that it satisfies the Fourier inequality $\mathbf{q}(\lambda) \cdot \lambda \geq 0$ for all $\lambda \in \mathbf{R}^N$. Applications are given to problems where $\mathbf{q}$ is not monotone.

0. Introduction. The purpose of this paper is to investigate the dynamics of the heat conduction equation

$$\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{q}(\nabla u) \qquad x \in \Omega, \; t > 0 \tag{0.1}$$

in a bounded domain $\Omega \subseteq \mathbf{R}^N$. The boundary $\partial\Omega$ of Ω is assumed to be smooth. On $\partial\Omega$ impose either homogeneous Dirichlet boundary values

$$u = 0 \quad \text{on} \quad \partial\Omega, \;\; t > 0 \tag{0.2}$$

or no flux insulated boundary values

$$\mathbf{q}(\nabla u) \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega, \; t > 0 \tag{0.3}$$

where $\mathbf{n}(x)$ denotes the exterior unit normal at $x \in \partial\Omega$. Also u will satisfy the initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega. \tag{0.4}$$

For convenience call system (0.1), (0.2), (0.4) P_D and call (0.1), (0.3), (0.4) P_N.

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The main interest of this paper is that $\mathbf{q}$ will only be assumed to satisfy smoothness and growth conditions and Fourier's inequality

$$\lambda \cdot \mathbf{q}(\lambda) \geq 0. \tag{0.5}$$

No monotonicity requirements such as $(\mathbf{q}(\lambda_1) - \mathbf{q}(\lambda_2)) \cdot (\lambda_1 - \lambda_2) \geq 0$ for $\lambda_1, \lambda_2 \in \mathbf{R}^N$ will be imposed.

Inequality (0.5) is consistent with classical Clausius-Duhem inequality (Truesdell, 1984, p. 116). For example when $N = 1$ the Fourier inequality says q must have its graph in the first and third quadrants. Thus (0.5) does not preclude q' being negative for values of λ, $\lambda \neq 0$. In such cases (0.1) becomes a backwards-forwards heat equation. Examples of two such possibilities are given in Figure 1 and 2.

Other assumptions on $\mathbf{q}$ are that

(i) $\mathbf{q}$ is (at least) a continuous map : $\mathbf{R}^N \to \mathbf{R}^N$ satisfying growth conditions

(ii)

$$|\mathbf{q}(\lambda)| \leq c_1(1 + |\lambda|^\gamma), \quad 1 \leq \gamma < 2 \tag{0.6}$$

(iii)

$$c_2|\lambda|^2 - c_3 \leq |\lambda \cdot \mathbf{q}(\lambda)| \tag{0.7}$$

for all $\lambda \in \mathbf{R}^N$; here c_1, c_2, c_3 are positive constants.

(iv) $\mathbf{q}$ is the gradient of a $C^1(\mathbf{R}^N; \mathbf{R})$ potential Φ, i.e.

$$\mathbf{q}(\lambda) = \nabla\Phi(\lambda) \tag{0.8}$$

for all $\lambda \in \mathbf{R}^N$.

(v)

$$\begin{aligned} \ker\{\lambda \cdot \mathbf{q}(\lambda)\} &\overset{\text{def}}{=} \{\lambda \in \mathbf{R}^N; \lambda \cdot \mathbf{q}(\lambda) = 0\} \\ &\subset \{\lambda \in \mathbf{R}^N; |\lambda| < \rho_0\} \quad \text{for some} \quad \rho_0 > 0. \end{aligned} \tag{0.9}$$

We note that in the case $N = 1$ equation (0.1) becomes

$$u_t = q(u_x)_x \tag{0.10}$$

which was studied by Hollig (1983) and Hollig and Nohel (1983). In his paper Hollig showed that if q was piecewise affine then weak solutions exist to the Neumann problem associated with (0.10).

The approach taken here is in no sense as subtle as Hollig's. In fact it is rather straightforward and physically natural. First imbed P_D or P_N respectively in the singularly perturbed systems

$$\frac{\partial u^\epsilon}{\partial t} = \nabla \cdot (\mathbf{q}(\nabla u^\epsilon)) - \epsilon \Delta^2 u, \quad x \in \Omega, t > 0, \;\; \epsilon > 0, \tag{0.11}$$

with boundary conditions

$$u^\epsilon = 0, \Delta u^\epsilon = 0 \quad \text{on} \quad \partial\Omega, \;\; t > 0 \tag{0.12}$$

or

$$\mathbf{q}(\nabla u^\epsilon) \cdot \mathbf{n} = 0, \;\; \frac{\partial}{\partial \mathbf{n}}(\Delta u^\epsilon) = 0 \quad \text{on} \quad \partial\Omega, \;\; t > 0, \tag{0.13}$$

and initial data

$$u^\epsilon(x, 0) = u_0(x). \tag{0.14}$$

System (0.11), (0.12), (0.14) is termed $P_{D,\epsilon}$ while system (0.11), (0.13), (0.14) is termed $P_{N,\epsilon}$. The regularization (0.11) is not ad hoc and is in fact based on a higher order theory of heat conduction due to J. C. Maxwell (1876, eqns. (53), (54)); see also Truesdell and Noll (1965, p. 514).

Problems $P_{D,\epsilon}$ and $P_{N,\epsilon}$ admit two natural "energy" estimates which motivate an attempt to pass to a weak limit as $\epsilon \to 0^+$ and hence obtain a weak solution of P_D and P_N. Unfortunately the presence of the nonlinear terms $\mathbf{q}(\nabla u^\epsilon)$ prevents the success of this venture. However in the spirit of the work of L. Tartar (1979, 1982) and later work of R. DiPerna (1983 a, b, c, 1985), M. Schonbek (1982), and R. DiPerna and A. Majda (1987a, b) the following information on the sequence $\{\mathbf{q}(\nabla u^\epsilon)\}$ is known. Namely if for $0 < T < \infty$,

$$\nabla u^\epsilon \rightharpoonup \nabla \overline{u} \quad \text{in} \quad L^2(Q_T),$$

$Q_T = (0, T) \times \Omega$, where $\rightharpoonup$ denotes weak convergence then for q continuous satisfying growth condition (0.5) there is a subsequence $\{\nabla u^{\epsilon_k}\}$ so that

$$q(\nabla u^{\epsilon_k}) \rightharpoonup \langle q(\lambda), \nu_{x,t}(\lambda) \rangle$$

weakly in $L^1(Q_T)$ for a probability measure $\nu_{x,t}(\lambda)$,

$$\langle \mathbf{q}(\lambda), \nu_{x,t}(\lambda) \rangle = \int_{\mathbf{R}^N} \mathbf{q}(\lambda) d\nu_{x,t}(\lambda)$$

and ν is called a Young measure.

The above representation of weak limits of $\mathbf{q}(\nabla u^\epsilon)$ permits a passage to weak limits in $P_{D,\epsilon}$ and $P_{N,\epsilon}$. These limits satisfy a measure theoretic version of P_D and P_N and the associated $\overline{u}$ is called a "measure valued" solution of P_D or P_N (in the sense of DiPerna (1985)). The function $\overline{u}$ lies in $L^\infty((0,\infty);V)$ when $V = H_0^1(\Omega)$ for P_D and $H^1(\Omega)$ for P_N. Moreover it inherits a natural "energy" inequality from the regularized problems $P_{D,\epsilon}$ and $P_{N,\epsilon}$. This inequality can be exploited to establish the trend to equilibrium as $t \to \infty$ of $\overline{u}$ (see also Slemrod, 1989a, b, c). The main tools used here are elementary concepts from topological dynamics and a careful study of sequences of Young measures borrowed almost verbatim from the paper of J. Ball (1988). For example it will be shown that for the case $N = 1$ and q possessing the graph shown in Figure 1 that the measure valued solutions $\overline{u}$ of P_D converge to zero weakly in $H_0^1(\Omega)$ as $t \to \infty$ whereas the measure valued solutions of P_N converge weakly in $H^1(\Omega)$ to the set $\chi \in H^1(\Omega)$ which are measure valued equilibria and satisfy $0 \leq \frac{d\chi}{dx}(x) \leq \xi_1$ a.e. in Ω.

Two further remarks are in order. First as regards nonhomogeneous boundry conditions. Consider for example the cases when the homogeneous boundary conditions (0.12) and (0.13) are replaced by nonhomogeneous boundary conditions

$$u^\epsilon(x) = \mathbf{M} \cdot x + K, \ \Delta u^\epsilon = 0 \quad \text{on} \quad \partial\Omega, t > 0 \tag{0.12$'$}$$

and

$$\mathbf{q}(\nabla u^\epsilon) \cdot \mathbf{n} = \mathbf{q}(\mathbf{M}) \cdot \mathbf{n}, \quad \frac{\partial}{\partial \mathbf{n}}(\Delta u^\epsilon) = 0 \quad \text{on} \quad \partial\Omega, \ t > 0 \tag{0.13$'$}$$

where K is a real scalar and $\mathbf{M}$ is a prescribed constant N-vector. It is then readily seen that $\mathbf{M} \cdot x + K$ is an equilibrium solution of (0.11), (0.12$'$) and (0.11), (0.13$'$) and hence the dependent variable $w^\epsilon(x,t) \overset{\text{def}}{=} u^\epsilon(x,t) - M \cdot x - K$ will satisfy the homogeneous equations (0.11), (0.12), (0.14) or (0.11), (0.13), (0.14) with u^ϵ replaced by w^ϵ, u_0 replaced by $u_0(x) - \mathbf{M} \cdot x - K$, and $\mathbf{q}(\lambda)$ replaced by $\mathbf{q}(\lambda + M) - \mathbf{q}(M)$. For example in the case $N = 1$ if q has the graph shown in Figure 2 with $q(M) = q(\xi^*)$, $q'(\xi^*) = 0$, imposing the boundary conditions (0.10$'$) or (0.11$'$) q has the effect of reducing our problem

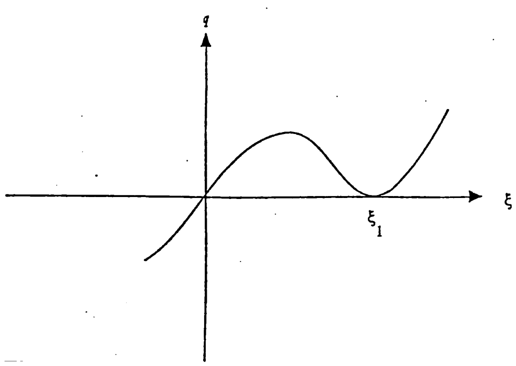

Figure 1

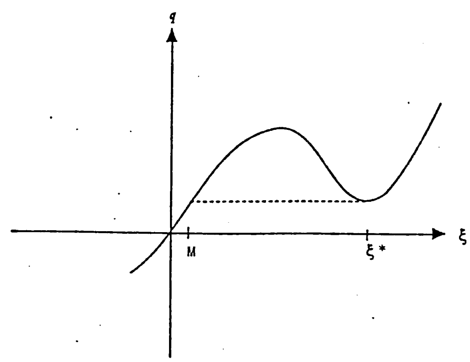

Figure 2

to the study of the q given by Figure 1. Of course, this procedure allows us to consider nonhomogeneous boundary value problems for the original $\epsilon = 0$ system (0.1) - (0.4) as well, i.e. with (0.2) replaced by

$$u(x) = \mathbf{M} \cdot x + K \quad \text{on} \quad \partial\Omega, \quad t > 0, \tag{0.2$'$}$$

or (0.3) replaced by

$$\mathbf{q}(\nabla u)\cdot\mathbf{n} = \mathbf{q}(M)\cdot\mathbf{n} \quad \text{on} \quad \partial\Omega, \quad t>0. \tag{0.3'}$$

The last observation is that in the case $N = 1$, $\Omega = (0,\ell)$, the change of variable $u^\epsilon_x = s^\epsilon$ and differentiation of (0.11) with respect to x yield the Cahn-Hilliard equation

$$\frac{\partial s^\epsilon}{\partial t} = \frac{\partial^2 q}{\partial x^2}(s^\epsilon) - \epsilon\frac{\partial^4 s^\epsilon}{\partial x^4}, \quad x\in\Omega,\ t>0. \tag{0.15}$$

The boundary conditions

$$q(s^\epsilon) = 0,\ s^\epsilon_{xx} = 0,\ x\in\partial\Omega,\ t>0 \tag{0.16}$$

are equivalent to (0.13). Hence the procedure given here is capable of computing the long time dynamics of measure value solutions of

$$\begin{aligned} \frac{\partial s}{\partial t} &= \frac{\partial^2 q(s)}{\partial x^2} \quad x\in\Omega,\ t>0 \\ q(s) &= 0 \quad x\in\partial\Omega,\ t>0 \end{aligned}$$

which are obtained as an $\epsilon\to 0$ limit of solutions of the Cahn-Hilliard system (0.13), (0.14).

On the other hand no-flux boundary conditions

$$q(s^\epsilon)_x = 0,\ s^\epsilon_{xxx} = 0 \quad x\in\partial\Omega,\ t>0 \tag{0.17}$$

are equivalent to

$$q(u^\epsilon_x)_x = 0,\ u^\epsilon_{xxxx} = 0 \quad x\in\partial\Omega,\ t>0$$

which from equation (0.11) means

$$\frac{\partial u^\epsilon}{\partial t} = 0 \text{ for } x\in\partial\Omega,\ t>0.$$

Hence $u^\epsilon(x,t) = u_0(x)$ for $x\in\partial\Omega$, $t>0$ and we see u^ϵ satisfies the Dirichlet boundary condition

$$u^\epsilon(x,t) = u_0(x) = \int_0^x s_0(\xi)d\xi + u_0(0)$$

so for $\Omega = (0,\ell)$ we have

$$\begin{aligned} u^\epsilon(0,t) &= u_0(0), \\ u^\epsilon(\ell,t) &= u_0(0) + \int_0^\ell s_0(\xi)d\xi \end{aligned} \tag{0.18}$$

which is of the form $u^\epsilon(x,t) = \mathbf{M}\cdot x + K$.

Also since we want $q(u^\epsilon_x)_x = 0$ on $\partial\Omega$ the additional restriction

$$u^\epsilon_{xx} = 0 \tag{0.19}$$

is sufficient though not necessary. Hence (0.11), (0.17) can be associated with a Dirichlet boundary value problem $P_{D,\epsilon}$.

The paper is divided into four sections after this one. The first section introduces the definition of measure valued solution. The second section reviews Ball's 1988 presentation of the fundamental theorem for Young measures. The third section establishes the existence of measure valued solutions for P_D and P_N and the basic energy "inequality" for measure valued solutions. Finally the fourth sections obtains the long term trend to equilibrium of measure valued solutions of P_D and P_N for $\mathbf{q}$ obeying the Fourier inequality (0.5).

Notation:

We *endow* $L^2(\Omega)$ with the usual inner product

$$(u,v) = \int_\Omega u(x)v(x)dx \quad \text{for } u,v \in L^2(\Omega),$$
$$\|u\|^2 = (u,u).$$

Q_T *denotes* the cylinder $\Omega \times (0,T)$ and for $u,v \in L^2(Q_T)$

$$(u,v)_{L^2(Q_T)} = \int_{Q_T} u(x,t)v(x,t)dxdt,$$
$$\|u\|^2_{L^2(Q_T)} = (u,u)_{L^2(Q_T)}.$$

For problem P_D we denote

$V = H^1_0(\Omega)$ where $H^1_0(\Omega)$ is endowed with the inner product

$$(u,v)_1 = \int_\Omega \nabla u \cdot \nabla v dx \quad \text{for} \quad u,v \in H^1_0(\Omega),$$
$$\|u\|^2_1 = (u,u)_1.$$

For problem P_N we denote

$V = H^1(\Omega)$ where $H^1(\Omega)$ is endowed with the inner product

$$(u,v)_1 = \int_\Omega \nabla u \cdot \nabla v dx + \left(\int_\Omega u dx\right)\left(\int_\Omega v dx\right) \text{ for } u,v \in H^1(\Omega),$$
$$\|u\|^2_1 = (u,u)^2_1$$

(see Temam (1988)).

For P_D we set

$$(u,v)_{H^1(Q_T)} = \int_{Q_T} \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + \nabla u \cdot \nabla v dx dt,$$
$$\|u\|^2_{H^1(Q_T)} = (u,u)_{H^1(Q_T)}.$$

For P_N we set

$$(u,v)_{H^1(Q_T)} = \int_{Q_T} \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + \nabla u \cdot \nabla v dx dt + \int_0^T \left(\int_\Omega u dx\right)\left(\int_\Omega v dx\right) dt,$$
$$\|u\|^2_{H^1(Q_T)} = (u,u)_{H^1(Q_T)}.$$

The subscript b will denote a uniformly bounded subset of an indicated set.

For the problem P_D we denote the space of *test functions* $W = C_0^\infty(\Omega)$.

For the problem P_N we denote the space of *test functions* $W = \{w \in C^\infty(\Omega); \frac{\partial w}{\partial \mathbf{n}} = 0\}$. Let $L_w^\infty(Q_T; M(\mathbf{R}^N))$ *denote the space of weak $*$ measurable mappings* $\mu : Q_T \mapsto M(\mathbf{R}^N)$ that are essentially bounded with norm

$$\|\mu\|_{\infty,M} = \text{ess} \sup_{x,t \in Q_T} \|\mu(x,t)\|_M < \infty.$$

(Recall μ is weak $*$ measurable if $\langle \mu(x,t), f\rangle$ is measurable with respect to $x,t \in Q_T$ for every $f \in C_0(\mathbf{R}^N)$.)

$M(\mathbf{R}^N)$ is the Banach space of bounded Radon measures over $\mathbf{R}^N$. For $\nu \in M(\mathbf{R}^N)$ we write

$$\|\nu\|_M = \int_{\mathbf{R}^N} d|\nu|.$$

Prob $(\mathbf{R}^N)$ is the Banach space of probablity measures over $\mathbf{R}^N$. For $\nu \in \text{Prob}\,(\mathbf{R}^n)$ we write

$$\|\nu\|_M = \int_{\mathbf{R}^N} d\nu.$$

$C_0(\mathbf{R}^N)$ denotes the Banach space of continuous functions $f : \mathbf{R}^N \to \mathbf{R}$ satisfying $\lim_{|\lambda|\to\infty} f(\lambda) = 0$,

$$\|f\|_{C_0(\mathbf{R}^N)} = \sup_{\lambda \in \mathbf{R}^N} |f(\lambda)|.$$

The arrows $\to, \rightharpoonup, \overset{*}{\rightharpoonup}$ denote strong, weak, and weak $*$ convergence respectively.

1. Measure valued solutions. An element $\overline{u} \in H^1(Q_T) \cap C([0,T]; L^2(\Omega)) \cap L^\infty(0,T); V)$ is a *measure valued solution of P_D or P_N on Q_T* if there exists a measure valued map

$$\nu : x,t \mapsto \nu_{x,t} \in \text{Prob}\,(\mathbf{R}^N)$$

from the physical domain Q_T to Prob $(\mathbf{R}^N)$ the space of probability measures over the state space domain $\mathbf{R}^N$ so that

$$\frac{d}{dt}(\overline{u}, w) + (\langle q(\lambda), \nu_{x,t}(\lambda)\rangle, \nabla w) = 0 \tag{1.1}$$

for all $w \in W$ a.e. in $(0,T)$;

$$\nabla \overline{u} = \langle \lambda, \nu_{x,t}(\lambda)\rangle \;\text{ a.e. in } Q_T; \tag{1.2}$$

$$\overline{u}(x,0) = u_0(x) \quad x \in \Omega. \tag{1.3}$$

An element $\overline{u} \in V$ is a *measure valued equilibrium solution of P_D or P_N* if there exists a measure valued map $\nu : x,t \mapsto \nu_{x,t} \in$ Prob$(\mathbf{R}^N)$ from $\Omega \times [0,\infty)$ to Prob$(\mathbf{R}^N)$ so that

$$(\langle q(\lambda), \nu_{x,t}(\lambda), \nabla w) = 0$$

for all $w \in W$ a.e. on $(0,\infty)$ and

$$\nabla \overline{u}(x) = \langle \lambda, \nu_{x,t}(\lambda)\rangle \;\text{ a.e. in } \Omega \times [0,\infty).$$

2. The fundamental theorem for Young measures. We state the fundamental theorem for Young measures as given by J. M. Ball (1988).

THEOREM 2.1.. *Let $S \subset \mathbf{R}^n$ be Lebesgue measurable. Let $K \subset \mathbf{R}^m$ be closed and let $z^{(j)} : S \mapsto \mathbf{R}^n$, $j = 1, 2, \ldots$ be a sequence of Lebesgue measurable functions satisfying $z^{(j)}(\cdot) \to K$ in measure as $j \mapsto \infty$, i.e. given any open neighborhood U of K in $\mathbf{R}^m$*

$$\lim_{j\to\infty} \textit{meas}\,\{y \in S; z^{(j)}(y) \notin U\} = 0.$$

Then there exists a subsequence $z^{(\mu)}$ of $z^{(j)}$ and a family $\{\nu_y\}$, $y \in S$, of positive measures on $\mathbf{R}^m$, depending measurably on y, so that

(i) $\|\nu_y\|_M = \int_{\mathbf{R}^m} d\nu_y \le 1$ a.e. in $y \in S$;

(ii) supp $\nu_y \subset K$ for almost all $y \in S$;

(iii) $f(z^{(\mu)}) \overset{*}{\rightharpoonup} \langle \nu_y, f\rangle = \int_{\mathbf{R}^m} f(\lambda) d\nu_y(\lambda)$ in $L^\infty(S)$ for each continuous function $f \in C_0(\mathbf{R}^m)$.

Suppose further that $\{z^{(\mu)}\}$ satisfies the boundedness condition

$$\lim_{k\to\infty} \sup_\mu \text{ meas}\{y \in S \cap B_R : |z^{(\mu)}(y)| \ge k\} = 0 \tag{2.1}$$

for every $R > 0$ where $B_R = \{y \in \mathbf{R}^m; |y| \leq R\}$. Then $\|\nu_y\|_M = 1$ for $y \in S$ (i.e. ν_y is a probability measure) and given any measurable subset A of S

$$f(z^{(\mu)}) \rightharpoonup \langle \nu_y, f \rangle \text{ in } L^1(A) \tag{2.2}$$

for any continuous function $f : \mathbf{R}^m \to \mathbf{R}$ such that $\{f(z^{(\mu)})\}$ is sequentially weakly relatively compact in $L^1(A)$.

As noted in Ball (1988) condition (2.1) is very weak and is equivalent to the following: given any $R > 0$ there exists a continuous nondecreasing function $g_R : [0, \infty) \to \mathbf{R}$ with $\lim_{t\to\infty} g_R(t) = \infty$, such that

$$\sup_{\mu} \int_{S\cap B_R} g_R(|z^{(\mu)}(y)|)dy < \infty. \tag{2.3}$$

Furthermore Ball notes that if A is bounded, the condition that $\{f(z^{(\mu)})\}$ be sequentially weakly relatively compact in $L^1(A)$ is satisfied if and only if

$$\sup_{\mu} \int_A \psi(|f(z^{(\mu)})|)dy < \infty \tag{2.4}$$

for some continuous function $\psi : [0, \infty) \to \mathbf{R}$ with $\lim_{t\to\infty} \frac{\psi(t)}{t} = \infty$ (de la Vallée Poussin's criterion; cf. MacShane (1947), Dellacherie & Meyer (1975), Natanson (1955)).

We can use Theorem 2.1 to prove the following lemma which will prove useful.

LEMMA 2.2.. *Let $\Omega \subset \mathbf{R}^N$ and let $\{z^{(j)}\}$ be a sequence: $S(= \mathbf{R}^+ \times \Omega) \to \mathbf{R}^N$ which lies in a bounded subset of $L^\infty((0, \infty); L^2(\Omega))$. Then the following are true:*

(i) There exists $z \in L^\infty((0, \infty); L^2(\Omega))$ and a subsequence $\{z^{(\mu)}\}$ of $\{z^{(j)}\}$ so that $z^{(\mu)} \stackrel{*}{\rightharpoonup} z$ in $L^\infty((0, \infty); L^2(\Omega))$.

(ii) $\{z^{(j)}\}$ satisfies condition (2.1),

(iii) For every $f(\lambda)$ satisfying

$$|f(\lambda)| \leq \text{ const. } (1 + |\lambda|^\gamma), \;\; 0 < \gamma < 2, \;\; \lambda \in \mathbf{R}^N,$$

and every bounded set $A \subseteq \mathbf{R}^+ \times \Omega$ (2.2) is satisfied for some probability measure $\nu_{x,t}$, $x \in \Omega$, $t \in \mathbf{R}^+$.

Proof.

(i) This is a statement relative sequential weak $*$ compactness of bounded sets in $L^\infty((0, \infty); L^2(\Omega))$.

(ii) Check instead the equivalent condition (2.3) with $g_R(t) = t^2/R$. Then

$$\begin{aligned}\sup_{j} \int_{S\cap B_R} g_R(|z^{(j)}(y)|)dy &\leq \sup \int_0^R \frac{1}{R} \|z^{(j)}\|^2 dt \\ &\leq \text{ const.}\end{aligned}$$

(iii) *Apply de la Vallée Poussin's criterion (2.4) with* $\psi(t) = t^{2/\gamma}$ *to conclude* $\{f(z^{(\mu)})\}$ *is sequentially weakly relatively compact in* $L^1(A)$. *Now use Theorem 2.1.*

Existence of measure valued solutions.

Before discussing the existence of measure valued solutions to P_D or P_N we must ascertain the existence of solutions to $P_{D,\epsilon}$ and $P_{N,\epsilon}$. Since it is not the object of this paper to obtain "optimal" existence results for these regularized problems we will be content here to note that if $\mathbf{q}$ is "locally" nonlinear, i.e. $\mathbf{q}$ is smooth with several bounded partial derivatives on $\mathbf{R}^N$ then $P_{D,\epsilon}$ and $P_{N,\epsilon}$ possess global smooth solutions for smooth data. Sharper results could certainly be obtained using the methods of R. Temam (1988) and B. Nicolaenko, B. Scheuer, and R. Temam (1988). We note the results of this section do not use (0.5), (0.6), (0.7), (0.9).

We follow the discussion of Pazy (1983, p.195). Consider the abstract nonlinear evolution equation

$$\frac{du}{dt} + Au = f(t, u), \tag{3.1}$$

$$u(t_0) = u_0. \tag{3.2}$$

We assume $-A$ is the infinitesimal generator of an analytic semigroup e^{-At} on a Banach space X, $\|e^{-At}\|_{\mathcal{L}(X)} \leq M$ for some positive constant M for $t \geq 0$, and that $-A$ is invertible. (Of course if $-A$ is in the infintesimal generator of an analytic semigroup for $\lambda > 0$ large enough $-A - \lambda I$ is invertible and generates a bounded analytic semigroup. Hence we can always reduce the case where $-A$ is the infinitesimal generator of an analytic semigroup to the case where the semigroup is bounded and $-A$ is invertible.)

Under the above assumptions on A we can define the fractional power A^α for $0 \leq \alpha \leq 1$ and A^α is a closed linear invertible linear operator with domain $D(A^\alpha)$ dense in X. The closedness of A^α implies that $D(A^\alpha)$ endowed with graph norm of A^α, i.e. the norm $|||u||| = \|u\|_X + \|A^\alpha u\|_X$ is a Banach space. Since A^α is invertible its graph norm $|||\ \ |||$ is equivalent to the norm $\|u\|_\alpha = \|A^\alpha u\|_X$. Thus $D(A^\alpha)$ equipped with norm $\|\ \ \|_\alpha$ is a Banach space denoted by X_α.

The main assumption on f in (3.1) will be:

Assumption (F). Let U be an open subset of $\mathbf{R}^+ \times X_\alpha$. The function $f : U \to X$ satisfies the assumption constants $L \geq 0$, $0 < \theta \leq 1$ such that

$$\|f(t_1, u_1) - f(t_2, u_2)\|_X \leq L\ \left(|t_1 - t_2|^\theta + \|u_1 - u_2\|_\alpha\right)$$

for all $(t_i, u_i) \in V$.

The following proposition is essentially due to Fujita and Kato (1964) and may be found in Pazy (1983).

Proposition 3.1. Let $0 \in \rho(-A)$ (the resolvent set of $-A$) and let $-A$ be the infinitesimal generator of an analytic semigroup e^{-At} satisfying $\|e^{-At}\| \leq M$ for $t \geq 0$. Let $f : [t_0, \infty) \times X_\alpha \to X$ satisfy (F). If there is a continuous nondecreasing real valued function $k(t)$ such that

$$\|f(t,u)\|_X \leq k(t)(1 + \|u\|_\alpha) \quad \text{for} \quad t \geq t_0, u \in X_\alpha,$$

then for every $u_0 \in X_\alpha$, the initial value problem (3.1) has a unique solution u which exists for all $t \geq t_0$ where $u \in C([t_0,\infty); X) \cap C^1((t_0,\infty); X)$. By solution we mean a function u which is differentiable a.e. on $[t_0, \infty)$ such that $\frac{du}{dt} \in L^1((t_0, T); X)$ for each $t_0 < T$, $u(t_0) = u_0$, and $\frac{du}{dt} = Au + f(u(t), t)$ a.e. on (t_0, ∞).

THEOREM 3.2. *There are positive integers s_0, s_1 so that if q possesses at least s_0 continues bounded partial derivates on $\mathbf{R}^N$ and $u_0 \in C^{s_1+2}(\overline{\Omega})$ with $u_0 = \Delta u_0 = \cdots = \Delta^{s_1} u_0 = 0$ on $\partial\Omega$ then $P_{D,\epsilon}$ possesses a unique classical solution on $[0, \infty)$.*

THEOREM 3.3. *Let $\mathbf{q}$ be such that $\lambda \cdot \mathbf{n} = 0$ implies $\mathbf{q}(\lambda) \cdot \mathbf{n} = 0$ for $\lambda \in \mathbf{R}^N$, $\mathbf{n}$ the exterior unit normal to $\partial\Omega$. Then there are positive integers s_0, s_1 so that if $\mathbf{q}$ possesses at least s_0 continuous bounded partial derivatives on $\mathbf{R}^N$ and $u_0 \in C^{s_1+3}(\overline{\Omega})$ with*

$$\frac{\partial u_0}{\partial \mathbf{n}} = \frac{\partial(\Delta u_0)}{\partial \mathbf{n}} = \cdots = \frac{\partial(\Delta^{s_1} u_0)}{\partial \mathbf{n}} = 0 \ \text{ on } \ \partial\Omega$$

then $P_{N,\epsilon}$ possesses a unique classical smooth solution on $[0, \infty)$.

Remark. The extra assumption on $\mathbf{q}$ for $P_{N,\epsilon}$ arises from that fact that the proof of theorem will only assert $\frac{\partial u}{\partial \mathbf{n}} = 0$ on $\partial\Omega$. An example of a case where the extra assumption is satisfied is when

$$\mathbf{q}(\lambda) = q_0(\lambda)\lambda$$

for some scalar function $q_0 : \mathbf{R}^N \to \mathbf{R}$. Of course the extra assumption is always satisfied when $N = 1$ since $q(0) = 0$ is that case.

Proof. We will consider the Dirichlet problem $P_{D,\epsilon}$. (The Neumann problem follows analogously by setting $\overline{u}(t) = u(t)e^{-\lambda t}$, $\lambda > 0$ sufficiently large.) First define the linear operator

$$Au = \epsilon\Delta^2 u, \ D(A) = \{u \in H^4(\Omega) \cap H_0^1(\Omega); \Delta u = 0\}, \text{ on } \partial\Omega.$$

Hence $(Au, u) = \epsilon\|\Delta u\|^2$ and A is self adjoint on its domain. A well known result (see for example Kato (1966, p. 491)) implies $-A$ is the infinitesimal generator of an analytic semigroup e^{-At} on $L^2(\Omega)$, $\|e^{-At}\| \leq 1$.

Now let s be the smallest positive integer so that

$$4s > 4 + \frac{N}{2}.$$

In this case the Sobolev imbedding theorem implies $H^{4s}(\Omega) \subset C^4(\overline{\Omega})$. Now set $X = D(A^s)$ so that $X_{1/2} = D(A^{s+\frac{1}{2}})$. From the Hille-Yosida-Phillips Theorem and the above remarks on generation of analytic semigroups we know $-A$ is the infinitesimal generator of an analytic semigroup e^{-At} on X with $\|e^{-At}\|_X \leq M$ for $t > 0$.

Set

$$f(t,u) = \text{div } \mathbf{q}(\nabla u) \tag{3.3}$$

for $u \in D(A^s)$. In this case (F) will hold if there is a constant L so that

$$\|\text{div } (\mathbf{q}(\nabla u_1) - \mathbf{q}(\nabla u_2))\|_{D(A^s)} \leq L\|u_1 - u_2\|_{D(A^{s+\frac{1}{2}})}. \tag{3.4}$$

Since the graph norm of $D(A^s)$ is equivalent to $H^{4s}(\Omega)$ and the graph norm of $D(A^{s+1/2})$ is equivalent to $H^{4s+2}(\Omega)$ application of the chain rule and mean value theorem shows that when $\mathbf{q}$ possesses a sufficiently large number of bounded continuous derivatives on $\mathbf{R}^N$ such a constant L exists. Also setting $u_1 = u$, $u_2 = 0$ in (3.2) shows that all the conditions of the proposition hold and u is a classical smooth solution in

$$C([t_0,\infty); C^4(\overline{\Omega})) \cap C^1([t_0,\infty); C^4(\overline{\Omega}))$$

for data $u_0 \in D(A^s)$. Finally the standard trace theorem (see for example Temam (1988)) asserts the boundary conditions $u = \Delta u = 0$ on $\partial\Omega$ hold for $P_{D,\epsilon}$.

For the Neumann problem $P_{N,\epsilon}$ we again set $Au = \epsilon\Delta^2 u$ but with

$$D(A) = \{u \in H^4(\Omega); \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial}{\partial \mathbf{n}}(\Delta u) = 0 \quad \text{on} \quad \partial\Omega\}.$$

LEMMA 3.2. *Let u^ϵ be a classical smooth solution of either $P_{D,\epsilon}$ or $P_{N,\epsilon}$. Then for all $t, \tau \in \mathbf{R}^+$:*

$$\begin{aligned}\|u^\epsilon(t+\tau)\|^2 - \|u^\epsilon(t)\|^2 = &-2\int_0^\tau ((\Delta u^\epsilon(s+t), \mathbf{q}(\nabla u^\epsilon(s+t))))ds \\ &- 2\epsilon\int_0^\tau \|\Delta u^\epsilon\|^2 ds\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\epsilon\|\Delta u^\epsilon(t)\|^2 &+ \int_\Omega \Phi(\nabla u^\epsilon(x,t))dx \\ &+ \int_0^t \|\frac{\partial u^\epsilon(s)}{\partial t}\|^2 dx = \epsilon\|\Delta u_0\|^2 + \int_\Omega \Phi(\nabla u_0(x))dx.\end{aligned} \tag{3.6}$$

Furthermore for $P_{N,\epsilon}$

$$\int_\Omega u^\epsilon(s,t)dx = \int_\Omega u_0(x)dx \tag{3.7}$$

for all $t \in \mathbf{R}^+$.

Proof. First, multiply (0.11), by u^ϵ, integrate over Ω, apply the divergence theorem, and then integrate from t to $t+\tau$; (3.5) then follows. Second, multiply (0.11) by $\frac{\partial u^\epsilon}{\partial t}$ and apply the same procedure; (3.6) follows. Finally, integrate (0.11) over Ω and apply the divergence theorem; (3.7) follows.

LEMMA 3.5. *Let u^ϵ be a classical smooth solution of either $P_{D,\epsilon}$ or $P_{N,\epsilon}$. Then for any $T > 0$*

$$\begin{aligned}
&\{u^\epsilon\} \subset L_b^\infty((0,\infty); V) \cap H_b^1(Q_T),\\
&\{\frac{\partial u^\epsilon}{\partial t}\} \subset L_b^2((0,\infty); L^2(\Omega)),\\
&\{\epsilon^{1/2}\Delta u^\epsilon\} \subseteq L_b^\infty((0,\infty); L^2(\Omega)).
\end{aligned} \tag{3.8}$$

Furthermore there exists a subsequence of $\{u^\epsilon\}$ also denoted by $\{u^\epsilon\}$ and $\overline{u}$,

$$\begin{aligned}
&\overline{u} \in L^\infty((0,\infty); V)) \cap H^1(Q_T) \cap C([0,T]; L^2(\Omega)),\\
&\frac{\partial \overline{u}}{\partial t} \in L^2((0,\infty); L^2(\Omega)),
\end{aligned}$$

so that

(a) $u^\epsilon \overset{*}{\rightharpoonup} \overline{u}$ *in* $L^\infty((0,\infty); V)$;

(b) $\nabla u^\epsilon \overset{*}{\rightharpoonup} \nabla\overline{u}$ *in* $L^\infty((0,\infty); L^2(\Omega))$;

(c) $\frac{\partial u^\epsilon}{\partial t} \rightharpoonup \frac{\partial \overline{u}}{\partial t}$ *in* $L^2((0,\infty); L^2(\Omega))$; *(3.9)*

(d) $u^\epsilon \rightharpoonup \overline{u}$ *in* $H^1(Q_T)$;

(e) $u^\epsilon \to \overline{u}$ *in* $C([0,T]; L^2(\Omega))$;

(f) $\overline{u}(t) \to u_0$ *in* $L^2(\Omega)$ *as* $t \to 0^+$;

Proof. The growth condition on Φ and energy equality (3.6) imply $\sup_{t\geq 0}\|\nabla u^\epsilon\| \leq$ const.. For $P_{D,\epsilon}$ this implies $\sup_{t\geq 0}\|u^\epsilon\|_1 \leq$ const. while for $P_{N,\epsilon}$ we use in addition (3.7) to see $\sup_{t\geq 0}\|u^\epsilon\|_1 \leq$ const.. Also from (3.6) we see $\{\frac{\partial u^\epsilon}{\partial t}\} \subseteq L_b^2((0,\infty); L^2(\Omega))$ and $\{\epsilon^{1/2}\Delta u\} \subseteq L_b^\infty((0,\infty); L^2(\Omega))$; (3.8) then follows. Extracting subsequences in the standard fashion we see (3.9a, b, c) follow immediately.

We now show $\{u^\epsilon\}$ is a Cauchy sequence in $C([0,T]; L^2(\Omega))$. To see this let $U = u^{\epsilon_1} - u^{\epsilon_2}$. Note $U(x,0) = 0$ and hence for $0 \leq t \leq T$

$$U^2(x,t) = 2\int_0^t UU_t dt \leq 2\left(\int_0^t U^2 ds\right)^{1/2}\left(\int_0^t U_t^2 ds\right)^{1/2}.$$

Integrate over Ω to find

$$\|u(t)\|^2 \leq 2 \int_\Omega \left(\int_0^t U^2 ds \right)^{1/2} \left(\int_0^t U_t^2 ds \right)^{1/2} dx$$

which the Schwarz inequality shows

$$\sup_{0 \leq t \leq T} \|u^{\epsilon_1}(t) - u^{\epsilon_2}(t)\|^2 \leq 2\|u^{\epsilon_1} - u^{\epsilon_2}\|_{L^2(Q_T)} \|u^{\epsilon_1} - u^{\epsilon_2}\|_{H^1(Q_T)}. \tag{3.10}$$

From (3.9) we know $\|u^{\epsilon_1} - u^{\epsilon_2}\|_{H^1(Q_T)} \leq$ const. and $\|u^{\epsilon_1} - u^{\epsilon_2}\|_{L^2(Q_T)} \to 0$ via the compactness of the imbedding $H^1(Q_T) \to L^2(Q_T)$. Since $\{u^\epsilon\} \subseteq C([0,T]; L^2(\Omega))$ (3.10) shows $\{u^\epsilon\}$ is Cauchy in $C([0,T]; L^2(\Omega))$ and hence $\overline{u} \in C([0,T]; L^2(\Omega))$.

Finally note

$$\begin{aligned} \lim_{t \to 0+} \|\overline{u}(t) - u_0\| &\leq \lim_{t \to 0+} \|\overline{u}(t) - u^\epsilon(t)\| + \lim_{t \to 0+} \|u^\epsilon(t) - u_0\| \\ &\leq \sup_{0 \leq t \leq T} \|\overline{u}(t) - u^\epsilon(t)\| + \lim_{t \to 0+} \|u^\epsilon(t) - u_0\|. \end{aligned} \tag{3.11}$$

Given $\delta > 0$ arbitrary we have just shown that we can find $\epsilon(\delta) > 0$ so that the first term on the right hand side of (3.11) is less than δ. Next for this $\epsilon = \epsilon(\delta)$ the fact that u^ϵ is classical smooth solution of the initial value problem shows the second term on the right hand side of (3.11) is zero. Hence $\lim_{t \to 0+} \|\overline{u}(t) - u_0\| < \delta$ and since δ is arbitrary (f) follows.

LEMMA 3.6. *Let $\{u^\epsilon\}$ be the subsequence obtained in Lemma 3.3. There exists a further subsequence again denoted by $\{u^\epsilon\}$ and a probability measure $\nu_{x,t}$, $(x,t) \in \Omega \times \mathbf{R}^+$, so that for every bounded subset $A \subset \Omega \times \mathbf{R}^+$ and every $f(\lambda)$ satisfying*

$$|f(\lambda)| \leq const.(1 + |\lambda|^\gamma), \ \ 0 < \gamma < 2, \ \lambda \in \mathbf{R}^N$$

we have

$$f(\nabla u^\epsilon) \rightharpoonup \langle \nu_{x,t}, f \rangle \ \ in \ \ L^2(A), \tag{3.12}$$

and

$$\nabla \overline{u} = \langle \nu_{x,t}, \lambda \rangle \ \ a.e. \ in \ \ A. \tag{3.13}$$

Proof. Apply Lemma 2.2 to the sequence $\{\nabla u^\epsilon\}$.

THEOREM 3.7. *Let $\overline{u}$ be as given in Lemma 3.3. Then $\overline{u}$ is a measure valued solution of the relevant initial-boundary value problem P_D or P_N. Furthermore if*

$$\lambda \cdot \mathbf{q}(\lambda) \geq 0 \ \ for \ \ |\lambda| \geq a$$

set

$$g(\lambda) \overset{def.}{=} \lambda \cdot \mathbf{q}(\lambda) \quad |\lambda| \le a,$$

$$\frac{\lambda \cdot \mathbf{q}(\lambda) a^3}{|\lambda|^3} \quad |\lambda| > a,$$

then $\overline{u}$ *satisfies the "energy" inequality*

$$\|\overline{u}(t+T)\|^2 - \|\overline{u}(t)\|^2 \le -2 \int_0^T \int_\Omega \langle g(\lambda), \nu_{x,s+t} \rangle dt ds. \tag{3.14}$$

Proof. Since the u^ϵ are smooth solutions of the appropriate regularized initial-boundary value problems we know

$$(u^\epsilon(t), w) - (u_0, w) + \int_0^t ((\mathbf{q}(\nabla u^\epsilon), \nabla w)) ds \tag{3.15}$$
$$+ \epsilon \int_0^t (\Delta u^\epsilon, \Delta w) ds = 0$$

for all $w \in C_0^\infty(\Omega)$ for $P_{D,\epsilon}$ and for all $w \in C^\infty(\Omega)$ for which $\frac{\partial w}{\partial \mathbf{n}} = 0$ on $\partial\Omega$ for $P_{N,\epsilon}$. By Lemmas 3.5 and 3.6 $u^\epsilon \to \overline{u}$ in $C([0,T]; L^2(\Omega))$ as $\epsilon \to 0+$;

$$\int_0^t ((\mathbf{q}(\nabla u^\epsilon), \nabla w)) ds \to \int_0^t ((\langle \nu_{x,s}, \mathbf{q}(\lambda) \rangle, \nabla w)) ds$$

as $\epsilon \to 0+$; and

$$\epsilon \int_0^t (\Delta u^\epsilon, \Delta w) ds \to 0 \quad \text{as} \quad \epsilon \to 0+.$$

Hence $\overline{u}$ satisfies

$$(\overline{u}(t), w) - (u_0, w) + \int_0^t ((\langle \nu_{x,s}, \mathbf{q}(\lambda) \rangle, \nabla w)) ds = 0. \tag{3.16}$$

Since the above integral defines an absolutely continuous function (3.16) may be differentiated a.e. on $(0,T)$ to conclude $\overline{u}$ is a measure valued solution of P_D or P_N.

Finally if $\lambda \cdot \mathbf{q}(\lambda) \ge 0$ for $|\lambda| \ge a \ge 1$ then $g(\lambda) \le \lambda \cdot \mathbf{q}(\lambda)$ and (3.5) implies

$$\|u^\epsilon(t+T)\|^2 - \|u^\epsilon(t)\|^2 \le -2 \int_0^T \int_\Omega g(\nabla u^\epsilon(t+s)) dx ds. \tag{3.17}$$

Since $u^\epsilon \to \overline{u}$ in $C([0,T]; L^2(\Omega))$ for any T the left hand side of (3.17) approaches $\|\overline{u}(t+T)\|^2 - \|u(t)\|^2$ whereas $|g(\lambda)| \le \text{const.}(1 + |\lambda|^{\gamma-1})$, $-2\int_0^\tau \int_\Omega \langle g(\lambda), \nu_{x,s+\tau} \rangle dx ds$. The theorem is thus proven.

Remark. The idea of minorizing $\lambda \cdot \mathbf{q}(\lambda)$ by $g(\lambda)$ was given to the author by Professor E. Zuazua. It allows the application of the fundamental theorem on Young measure to the energy inequality (3.5) without putting additional growth restrictions on $\mathbf{q}$.

4. Asymptotic behavior of measure valued solutions. In what follows we shall assume (0.5) - (0.9) hold.

For problems $P_D(P_N)$ let u_0 be as given in Theorem 3.2 (Theorem 3.3). Then $\mathcal{O}^+(u_0) = \bigcup_{t\geq 0} \overline{u}(t; u_0)$ defines the *positive orbit* in V through u_0. From Lemma 3.5 we know $\overline{u} \in L^\infty((0,\infty); V)$ and so $\mathcal{O}^+(u_0) \subseteq B \subset V$, with a metrized weak-$V$ topology with metric d. Define *the w-distance* between two sets $B_1, B_2 \subseteq B$ by

$$\text{w-dist}(B_1, B_2) = \inf_{\substack{b_1 \in B_1 \\ b_2 \in B_2}} d(b_1, b_2).$$

Finally define the *weak ω-limit set* of $\mathcal{O}^+(u_0)$ by $\omega_w(u_0) = \{\chi \in V; \overline{u}(t_n; u_0) \rightharpoonup \chi$ in V as $n \to \infty$ for some sequence $\{t_n\}$, $t_n \to \infty\}$.

THEOREM 4.1. *For u_0 as given in Theorems 3.2 or 3.3 $\omega_w(u_0)$ is non-empty and w-dist$(\overline{u}(t; u_0), \omega_w(u_0)) \to 0$ as $t \to \infty$.*

Proof. Since $\mathcal{O}^+(u_0) \subseteq B$ and B is weakly sequentially relatively compact in V, $\omega_w(u_0)$ is non-empty. Furthermore if w-dist$(\overline{u}(t, u_0), \omega_w(u_0))$ does not approach zero as $t \to \infty$ there exists $\{t_j\}$, $\omega_w(u_0)) > \epsilon > 0$ for some ϵ. But since $\{\overline{u}(t_j, u_0)\} \subseteq B$ we may extract a further subsequence $\{t_k\}$, $t_k \to \infty$ so that $d(\overline{(}t_k, u_0); \overline{\chi}) \to 0$ as $k \to \infty$ for some $\overline{\chi} \in \omega_w(u_0)$. This contradiction proves the theorem.

LEMMA 4.2. *Let $\chi \in \omega_w(u_0)$ and $\overline{u}(t_n, u_0) \rightharpoonup \chi$ in V and $t_n \to \infty$. Then for any $T > 0$ the sequence*

$$\overline{u}_n(t) \overset{def}{=} \overline{u}(t + t_n; u_0)$$

satisfies

$$\begin{aligned} &\{\overline{u}_n\} \subset L^\infty_b((0,\infty); V) \cap H^1_b(Q_T), \\ &\{\frac{\partial \overline{u}_n}{\partial t}\} \subset L^2_b((0,\infty); L^2(\Omega)). \end{aligned} \tag{4.1}$$

Furthermore there exists subsequences of $\{t_n\}$ and $\{\overline{u}_n\}$ also denoted by $\{t_n\}$ and $\{\overline{u}_n\}$ and v,

$$\begin{aligned} &v \in L^\infty((0,\infty); V) \cap H^1(Q_T) \cap C([0,T]; L^2(\Omega)) \\ &\frac{\partial v}{\partial t} \in L^2((0,\infty); L^2(\Omega)), \end{aligned} \tag{4.2}$$

so that

(a) $u_n \overset{*}{\rightharpoonup} v$ *in* $L^\infty((0,\infty); V)$;

(b) $\nabla u_n \overset{*}{\rightharpoonup} \nabla v$ *in* $L^\infty((0,\infty); L^2(\Omega))$;

(c) $\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial v}{\partial t}$ *in* $L^2((0,\infty); L^2(\Omega))$;

(d) $u_n \rightharpoonup v$ in $H^1(Q_T)$; (4.3)

(e) $u_n \to v$ in $C([0,T]; L^2(\Omega))$;

(f) $v(t) \to \chi$ in $L^2(\Omega)$ as $t \to 0+$.

Proof. Since $\overline{u} \in L^\infty((0,\infty); V)$, we see $\{u_n\} \subset L^\infty_b((0,\infty); V) \cap H^1_b(Q_T)$, Extracting appropriate subsequences we see (a), (b), (c), (d) follow immediately. To prove (e) we write

$$U(t) = u^\epsilon(t+t_n) - u^\epsilon(t+t_m).$$

Then we see

$$\int_\Omega U^2(t)dx = \int_\Omega U^2(0)dx + 2\int_\Omega \int_0^t UU_t dtdx$$

and hence by the Schwarz inequality

$$\int_\Omega U^2(t)dx \leq \int_\Omega U^2(0)dx + \left(\int_\Omega \int_0^T U^2 dxdt\right)^{1/2} \left(\int_\Omega \int_0^T U_t^2 dxdt\right)^{1/2}.$$

Therefore we see

$$\|u^\epsilon(t+t_n) - u^\epsilon(t+t_n)\| \leq \|u^\epsilon(t_n) - u^\epsilon(t_m)\| + 2\|u^\epsilon(\cdot+t_n) - u^\epsilon(\cdot+t_m)\|_{L^2(Q_T)} \|u^\epsilon(\cdot+t_n) - \epsilon(\cdot+t_m)\|_{H^1(Q_T)}. \tag{4.4}$$

Since $\{u^\epsilon\} \subset L^\infty_b((0,\infty); V)$, then $\{u^\epsilon(\cdot+t_n)\} \subset L^\infty_b((0,\infty); V)$, $\{\frac{\partial u^\epsilon}{\partial t}(\cdot+t_n)\} \subset L^2_b((0,\infty); L^2(\Omega))$ and hence $\{u^\epsilon(\cdot+t_n)\} \subset H^1_b(Q_T)$. Now let $\epsilon \to 0+$. We know since $u^\epsilon \to \overline{u}$ in $C([0,T']; L^2(\Omega))$ for any $T' > 0$, (4.4) implies

$$\sup_{0\leq t\leq T} \|\overline{u}(t+t_n) - \overline{u}(t+t_m)\| \leq \|\overline{u}(t_n) - \overline{u}(t_m)\| + \text{const.} \lim_{\epsilon\to 0+} \|\overline{u}^\epsilon(\cdot+t_n) - u^\epsilon(\cdot+t_m)\|_{L^2(Q_t)}. \tag{4.5}$$

Now since $\{u^\epsilon(\cdot+t_n)\} \subset H^1_b(Q_T)$ we see there is a subsequence also denoted by $\{u^\epsilon(\cdot+t_n)\} \subset H^1_b(Q_T)$ which converges weakly in $H^1(Q_T)$ and hence strongly in $L^2(Q_T)$. Taking the limit on this subsequence in (4.5) we see

$$\lim_{n,m\to\infty} \sup_{0\leq t\leq T} \|\overline{u}(t+t_n) - \overline{u}(t+t_m)\| = 0$$

so $\{\overline{u}_n(t)\}$ is Cauchy in $C([0,T]; L^2(\Omega))$. Hence $v \in C([0,T]; L^2(\Omega))$. Since $\overline{u}_n(0) \to \chi$ in $L^2(\Omega)$ we see $v(0) = \chi$ and so $\lim_{t\to 0+} v(t) = \chi$ in $L^2(\Omega)$. This proves the lemma.

LEMMA 4.3. *Let $\{t_n\}$ be the subsequence given in Lemma 4.2. Define the sequence of measures $\nu^{(n)}_{x,t}(\lambda) \stackrel{\text{def}}{=} \nu_{x,t+t_n}(\lambda)$. Then $\{\nu^{(n)}_{x,t}\}$ belongs to the space $L^\infty_w(Q_T; M(\mathbf{R}^N))$. Furthermore there exists a subsequence $\{\nu^{(k)}_{x,t}\}$ of $\{\nu^{(n)}_{x,t}\}$ and an element $\overline{\nu}_{x,t}$ of $L^\infty_w(Q_T; M(\mathbf{R}^N))$ so that*

(i) $\overline{\nu}_{x,t} \geq 0$ a.e. *in* Q_T;

(ii) $\nu^{(k)} \stackrel{*}{\rightharpoonup} \overline{\nu}$ *in* $L^\infty_w(Q_T; M(\mathbf{R}^N))$;

(iii) for every $f \in C_0(\mathbf{R}^N)$, $\langle \nu^{(k)}_{x,t}, f\rangle \stackrel{*}{\rightharpoonup} \langle \overline{\nu}_{x,t}, f\rangle$ *in* $L^\infty(Q_T)$ *as* $k \to \infty$.

Proof. The proof is taken from an argument of Ball (1988). First since the fundamental theorem for Young measures tells us $\|\nu^{(n)}_{x,t}\|_M = \int_{\mathbf{R}^N} d\nu^{(n)}_{x,t} = 1$ and $\langle f, \nu^{(n)}_{x,t}\rangle$ is in $L^1(Q_T)$ for $f \in C_0(\mathbf{R}^N)$ when $\{\nu^{(n)}_{x,t}\} \subset L^\infty_w(Q_T; M(\mathbf{R}^N))$.

Next note that under the norm $\|\cdot\|_{\infty,M}$, $L^\infty_w(Q_T; M(\mathbf{R}^N))$ is a Banach space. Since $C_0(\mathbf{R}^N)$ is separable there is an isometric isomorphism between the dual space $L^1(Q_T; C_0(\mathbf{R}^N))$ and $L^\infty_w(Q_T; M(\mathbf{R}^N))$ obtained by associating with each $\mu \in L^\infty_w(Q_T; M(\mathbf{R}^N))$ the linear form

$$\psi \mapsto \int_{Q_T} \langle \mu(x,t), \psi(x,t); \cdot\rangle dx dt \tag{4.6}$$

on $L^1(Q_T; C_0(\mathbf{R}^N))$.

Since $C_0(\mathbf{R}^N)$ is separable so is $L^1(Q_T; C_0(\mathbf{R}^N))$ and hence by weak $*$ precompactness of bounded sets in $L^\infty_w(Q_T; M(\mathbf{R}^N))$ (see for example Dunford & Schwartz (1958, pp. 424-426)) there exists a subsequence $\nu^{(k)}$ of $\nu^{(n)}$ and an element $\overline{\nu} = \overline{\nu}_{x,t}$ of $L^\infty_w(Q_T; M(\mathbf{R}^N))$ so that $\nu^{(k)} \stackrel{*}{\rightharpoonup} \overline{\nu}$ in $L^\infty_w(Q_T; M(\mathbf{R}^N))$. This proves (ii).

By (4.6) this implies

$$\int_{Q_T} \langle \nu^{(k)}_{x,t}, \psi(x,t;\cdot)\rangle dx dt \to \int_{Q_T} \langle \overline{\nu}_{x,t}, \psi(x,t;\cdot)\rangle dx dt$$

as $k \to \infty$ for every $\psi \in L^1(Q_T; C_0(\mathbf{R}^N))$. In particular taking $\psi(x,t;\lambda) = \phi(x,t)f(\lambda)$ where $\phi(x,t) \in L^1(Q_T)$ and $f \in C_0(\mathbf{R}^N)$ we find

$$\int_{Q_T} \phi(x,t)\langle \nu^{(k)}_{x,t}, f\rangle dx dt \to \int_{Q_T} \phi(x,t)\langle \overline{\nu}_{x,t}, f\rangle dx dt$$

i.e. $\langle \nu^{(k)}_{x,t}, f\rangle \stackrel{*}{\rightharpoonup} \langle \overline{\nu}_{x,t}, f\rangle$ in $L^\infty(Q_T)$ as $k \to \infty$. This proves (iii). Also since $\nu^{(k)}_{x,t} \geq 0$ a.e. in Q_T we see $\overline{\nu}_{x,t} \geq 0$ a.e. in Q_T as well.

LEMMA 4.4. *If* $\lambda \cdot \mathbf{q}(\lambda) \geq 0$ *then*

$$\text{supp}\, \overline{\nu}_{x,t} \subseteq \ker \lambda \cdot \mathbf{q}(\lambda) \quad \text{a.e. in} \quad Q_T.$$

Proof. Set $t = t_n$ in (3.14), i.e. we know

$$\|\overline{u}(T+t_n)\|^2 - \|\overline{u}(t_n)\|^2 \leq -2\int_0^T \int_\Omega \langle g(\lambda), \nu_{x,s+t_n}\rangle dx ds. \tag{4.7}$$

Now since $g(\lambda) \geq 0$, $\|\overline{u}(t)\|^2$ is a nonincreasing real valued function of t bounded from below. Hence $\lim_{t\to\infty} \|\overline{u}(t)\|$ exists. Let $n \to \infty$ for $\{t_n\}$ the sequence in Lemma 4.2. We see the left hand side of (4.7) goes to zero as $n \to \infty$ and hence

$$\lim_{n\to\infty} \int_0^T \int_\Omega \langle g(\lambda), \nu_{x,s+t_n}\rangle dx ds = 0.$$

Since $g \in C_0(\mathbf{R}^N)$ we can use Lemma 4.3 to see that

$$\int_0^T \int_\Omega \langle g(\lambda), \overline{\nu}_{x,t}(\lambda)\rangle dx dt = 0.$$

Since $g \geq 0$ we see supp $\overline{\nu}_{x,t} \subseteq \ker g$ a.e. in Q_T. But $\ker g = \ker \lambda \cdot \mathbf{q}(\lambda)$ and the result follows immediately.

LEMMA 4.5. *Assume (i) f is continuous: $\mathbf{R}^N \to \mathbf{R}$ satisfying $|f(\lambda)| \leq$ const.$(1 + |\lambda|^\gamma)$, $0 < \gamma < 2$. Then $\langle f(\lambda), \nu_{x,t}^{(k)}\rangle \overset{*}{\rightharpoonup} \langle f(\lambda), \overline{\nu}_{x,t}\rangle$ in $L^\infty(Q_T)$.*

Proof. Since we do not assume $f \in C_0(\mathbf{R}^N)$ we cannot appeal to Lemma 4.3 (iii). Instead we again paraphrase an argument of Ball (1988).

Define the function $\Theta^{(\rho)}(\lambda)$ as

$$\Theta^{(\rho)}(\lambda) = \begin{cases} 1 & |\lambda| \leq \rho, \\ 1+\rho-|\lambda| & \rho \leq |\lambda| \leq \rho+1, \\ 0 & |\lambda| \geq \rho+1. \end{cases}$$

We now show that for any $\phi \in L^\infty(Q_T)$

$$\lim_{\rho\to\infty} \int_{Q_T} \phi\langle f(\lambda)\Theta^{(\rho)}(\lambda), \nu_{x,t}^{(k)}\rangle dx dt = \int_{Q_T} \phi\langle f(\lambda), \nu_{x,t}^{(k)}\rangle dx dt \tag{4.7}$$

uniformly in k.

We prove (4.7) as follows: Set $p^{(\rho)}(\lambda) = f(\lambda)(\Theta^\rho(\lambda) - 1)$. Since $|p^{(\rho)}(\lambda)| \leq$ const.$(1 + |\lambda|^\gamma)$ and $0 < \gamma < 2$ the fundamental theorem on Young measures implies $p^{(\rho)}(\nabla u^\epsilon(x, t + t_n)) \rightharpoonup \langle p^{(\rho)}(\lambda), \nu_{x,t+t_n}\rangle$ in $L^1(Q_T)$ as $\epsilon \to 0+$.

Hence for any $\phi \in L^\infty(Q_T)$

$$
\begin{aligned}
&|\int_{Q_T} \phi\langle f(\lambda) - f(\lambda)\Theta^{(\rho)}(\lambda), \nu_{x,t}^{(k)}\rangle dxdt| \\
&\quad = |\lim_{\epsilon\to 0+} \int_{Q_T} \phi p^{(\rho)}(\nabla u^\epsilon(x,t+t_k))dxdt| \\
&\quad \leq \sup_{0\leq\epsilon\leq 1} \text{const.} \int_{Q_T} |p^{(\rho)}(\nabla u^\epsilon(x,t+t_k))|dxdt \\
&\quad \leq \sup_{0\leq\epsilon\leq 1} \text{const.} \int_{E_\rho^{\epsilon,k}} |\nabla u^\epsilon(x,t+t_k)|^\gamma dxdt,
\end{aligned}
$$

for $0 < \gamma < 2$ where

$$E_\rho^{\epsilon,k} = \{x,t \in Q_T; |\nabla u^\epsilon(x,t+t_k)| \geq \rho\}.$$

Therefore we see

$$
\begin{aligned}
&|\int_{Q_T} \phi\langle f(\lambda) - f(\lambda)\Theta^{(\rho)}(\lambda), \nu_{x,t}^{(k)}\rangle dxdt| \\
&\quad \leq \sup_{0\leq\epsilon\leq 1} \sup_k \text{const.} \int_{E_\rho^{\epsilon,k}} |\nabla u^\epsilon(x,t+t_k)|^\gamma dxdt. \qquad (4.8)
\end{aligned}
$$

Since $\{\nabla u^\epsilon\} \subset L_b^\infty((0,\infty);L^2(\Omega))$ we have $\{\nabla u^\epsilon(\cdot,\cdot+t_k)\} \subset L_b^\infty((0,\infty);L^2(\Omega))$ and hence $\{\nabla u^\epsilon(\cdot,\cdot+t_k)\} \subset L_b^2(Q_T)$. But a bounded subset of $L^2(Q_T)$ is weakly sequentially precompact in $L^1(Q_T)$ and hence we may apply the Dunford-Pettis Theorem to $\{\nabla u^\epsilon(\cdot,\cdot+t_k)\}$. We recall:

DUNFORD-PETTIS THEOREM. *(Dunford and Schwartz (1958), p.492). A subset K of $L^1(S,\Sigma,\mu)$ is weakly sequentially compact if and only if it is bounded and the countable additivity of the integrals $\int_E f(s)\mu(ds)$ is uniform with respect to f in K. The statement that the countable additivity of the integrals $\int_E f(s)\mu(ds)$ is uniform with respect to f in K means that for each decreasing sequence $\{E_n\}$ in Σ with void intersection the limit $\lim_{n\to\infty} \int_{E_n} f(s)\mu(ds) = 0$ is uniform for $f \in K$.*

To apply the Dunford-Pettis theorem to (4.8) we must show

$$\lim_{\rho\to\infty} \sup_{\epsilon,k} \; meas\{(x,t) \in Q_T; |\nabla u^\epsilon(x,t+t_k)| > \rho\} = 0.$$

But by the argument given in the proof of Lemma 2.2 (ii) this is known to be true and hence (4.7) is proved.

Now to conclude the proof of the lemma we write for $\phi \in L^\infty(Q_T)$:

$$
\begin{aligned}
&|\int_{Q_T} (\phi\langle f(\lambda), \overline{\nu}_{x,t}\rangle - \phi\langle f(\lambda), \nu_{x,t}^{(k)}\rangle)dxdt| \qquad (4.9) \\
&\quad \leq I(\rho) + II(\rho,k) + III(\rho,k)
\end{aligned}
$$

where

$$I(\rho) = |\int_{Q_T} \phi\langle f(\lambda), \overline{\nu}_{x,t}\rangle - \phi\langle f(\lambda)\Theta^{(\rho)}(\lambda), \overline{\nu}_{x,t}\rangle dxdt|,$$

$$II(\rho,k) = |\int_{Q_T} \phi\langle f(\lambda)\Theta^{(\rho)}(\lambda), \overline{\nu}_{x,t}\rangle - \phi\langle f(\lambda)\Theta^{(\rho)}(\lambda), \nu_{x,t}^{(k)}\rangle dxdt|,$$

$$III(\rho,k) = |\int_{Q_T} \phi\langle f(\lambda)\Theta^{(\rho)}(\lambda), \nu_{x,t}^{(k)}\rangle - \phi\langle f(\lambda), \nu_{x,t}^{(k)}\rangle dxdt|.$$

Without loss of generality assume $\rho \geq \rho_0$ (given by (0.9)). Then $I(\rho)$ vanishes since by Lemma 4.4 supp $\overline{\nu}_{x,t} \subseteq \ker \lambda \cdot \mathbf{q}(\lambda) \subseteq \{\lambda; |\lambda| \leq \rho_0\}$.

Now from (4.7) we know given $\delta > 0$ there exists $\overline{\rho}(\delta)$ so that $III(\rho,k) \subseteq \delta$ for $\rho \geq \overline{\rho}$ and all k. Set $\rho = \overline{\rho}$ in (4.9) and use Lemma 4.3 and (4.7) to see $\lim_{k\to\infty} III(\overline{\rho},k) = 0$. Hence

$$|\int_{Q_T} \phi\langle \lambda, \overline{\nu}_{x,t}\rangle - \langle f(\lambda), \nu_{x,t}^{(k)}\rangle dxdt| \leq \delta \quad \text{as} \quad k \to \infty.$$

Since δ is arbitrary Lemma 4.5 is proven.

LEMMA 4.6. *$\overline{\nu}_{x,t} \in Prob(\mathbf{R}^N)$ a.e. in Q_T.*

Proof. Take $f(\lambda) = 1$ in Lemma 4.5. We then see meas $E = \int_E \int_{\mathbf{R}^N} d\overline{\nu}_{x,t} dxdt$ for every measurable set E. Hence $\int_{\mathbf{R}^N} d\overline{\nu}_{x,t} = 1$ a.e. in Q_T.

LEMMA 4.7. *$\nabla v = \langle \lambda, \overline{\nu}_{x,t}\rangle$ a.e. in Q_T and $\langle \mathbf{q}(\lambda), \nu_{x,t}^{(k)}\rangle \overset{*}{\rightharpoonup} \langle \mathbf{q}(\lambda), \overline{\nu}_{x,t}\rangle$ in $L^\infty(Q_T)$.*

Proof. We know from (4.3) that $\nabla \overline{u}_n \overset{*}{\rightharpoonup} \nabla v$ in $L^\infty((0,T); L^2(\Omega))$ and since $\nabla \overline{u}_n = \langle \lambda, \nu_{x,t+t_n}\rangle = \langle \lambda, \nu_{x,t}^{(n)}\rangle$ we have $\langle \lambda, \nu_{x,t}^{(n)}\rangle \overset{*}{\rightharpoonup} \nabla v$ in $L^\infty((0,T); L^2(\Omega))$. But by Lemma 4.5 $\langle \lambda, \nu_{x,t}^{(k)}\rangle \overset{*}{\rightharpoonup} \langle \lambda, \overline{\nu}_{x,t}\rangle$ in $L^\infty(Q_T)$ and hence $\nabla v = \langle \lambda, \overline{\nu}_{x,t}\rangle$ a.e. in Q_T. Similarly we see $\langle \mathbf{q}(\lambda), \nu_{x,t}^{(x)}\rangle \overset{*}{\rightharpoonup} \langle \mathbf{q}(\lambda), \overline{\nu}_{x,t}\rangle$ in $L^\infty(Q_T)$ by again applying Lemma 4.5.

LEMMA 4.8. $\ker \lambda \cdot \mathbf{q}(\lambda) = \ker \mathbf{q}(\lambda)$.

Proof. Clearly $\ker \mathbf{q}(\lambda) \subseteq \ker \lambda \cdot \mathbf{q}(\lambda)$ so we need to prove the reverse set inclusion holds. First since $\lambda \cdot \mathbf{q}(\lambda) \geq 0$ for all $\lambda \in \mathbf{R}^N$ by setting all $\lambda_k = 0$ but one shows that $\lambda_i q_i(\lambda) \geq 0$ for all i. Hence $q_i(\lambda) \geq 0$ for $\lambda_1 \geq 0$ and $q_i(\lambda) \leq 0$ for $\lambda_i \leq 0$. By continuity of $\mathbf{q}$ this implies $q_i(\lambda) = 0$ when $\lambda_i = 0$ for all i. So if $\lambda \cdot \mathbf{q}(\lambda) = 0$, we know for each i $\lambda_i q_i(\lambda) = 0$ which means either $\lambda_i = 0$ (which implies $q_i = 0$) or $q_i = 0$. Either way we have $\lambda \in \ker \mathbf{q}$.

THEOREM 4.9. *Let $\chi \in \omega_w(u_0)$. Then χ is an equilibrium solution of the generalized evolution equations (1.1) - (1.3), i.e. there is positive probability measure $\overline{\nu}_{x,t}$ with*

$$supp\, \overline{\nu}_{x,t} \subseteq \ker \mathbf{q} \tag{4.10}$$

hence satisfying

$$\langle \mathbf{q}(\lambda), \overline{\nu}_{x,t} \rangle = 0 \quad \text{a.e. } \mathit{in} \quad Q_T \tag{4.11}$$

and

$$\nabla\chi(x) = \langle \lambda, \overline{\nu}_{x,t} \rangle \quad \text{a.e. } \mathit{in} \quad Q_T. \tag{4.12}$$

Moreover if $\ker q_i \subseteq [a_i, b_i]$, $i = 1, \dots, N$ *then*

$$a_i \leq (\nabla\chi)_i \leq b_i \quad \text{a.e. } \mathit{in} \quad \Omega, \quad i = 1, \dots, N. \tag{4.13}$$

Proof. We know

$$\frac{d}{dt}\langle \overline{u}_k, w \rangle + (\langle \mathbf{q}(\lambda), \nu_{x,t}^{(k)}(\lambda) \rangle, \nabla w) = 0$$

for all $w \in W$ a.e. in $(0,T)$. Hence for $0 < t < T$

$$\int_0^t \left(\frac{\partial \overline{u}_k}{\partial t}, w \right) ds + \int_0^t (\langle \mathbf{q}(\lambda), \nu_{x,s}^{(k)}(\lambda) \rangle, \nabla w) ds = 0.$$

Now use (4.3c) and Lemma 4.6 to pass to the limit $k \to \infty$ to obtain

$$\int_0^t \left(\frac{\partial \overline{v}}{\partial t}, w \right) ds + \int_0^t (\langle \mathbf{q}(\lambda), \overline{\nu}_{x,t}(\lambda), \nabla w) ds = 0.$$

Since $\operatorname{supp} \overline{\nu}_{x,t} \subseteq \ker \mathbf{q} = \ker \lambda \cdot \mathbf{q}(\lambda)$ we see (4.10), (4.11) trivially hold and $\frac{\partial \overline{v}}{\partial t} = 0$ a.e. in Q_T. But (4.3f) and Lemma 4.7 imply $\nabla v(t) = \langle \lambda, \overline{\nu}_{x,t} \rangle$ a.e. in Q_T and $v(t) \to \chi$ in $L^2(\Omega)$ as $t \to 0+$. Hence $v(t) = \chi$ on $(0,T)$ and (4.12) follows. Finally the fact that $\overline{\nu}_{x,t}$ is a probability measure yields (4.13).

LEMMA 4.10. *For problem* P_D *if for some* i, $1 \leq i \leq N$, $(\nabla\chi)_i = 0$ *a.e. in* Ω.

Proof. From Theorem 4.2 we know either $(\nabla\chi)_i \leq 0$ a.e. in Ω or $(\nabla\chi)_i \geq 0$ a.e. in Ω. Now apply the divergence theorem (see for example J. P. Aubin (1984), p. 289) to the N vector $\mathbf{J}(x)$, $J_k \equiv 0$, $k \neq i$, $J_i = \chi$ to see

$$\begin{aligned} \int_\Omega \frac{\partial \chi}{\partial x_i} dx &= \int_\Omega \operatorname{div} J dx \\ &= \int_{\partial\Omega} \mathbf{J} \cdot \mathbf{n} d\sigma = \int_{\partial\Omega} \chi n_i d\sigma. \end{aligned}$$

As $\chi \in H_0^1(\Omega)$ the last term above is zero and hence $(\nabla\chi)_i$ cannot be non-positive or non-negative a.e. in Ω.

THEOREM 4.11. *For Problem* P_D: *if for each* i $\ker q_i \subseteq R^-$ *or* $\ker q_i \subseteq R^+$, $1 \leq i \leq N$, *then* $\omega_w(u_0) = \{0\}$ *and for any* u_0 $\overline{u}(t, u_0) \rightharpoonup 0$ *as* $t \to \infty$ *in* $H_0^1(\Omega)$.

Proof. By Lemma 4.10 $\nabla\chi = 0$ and hence since $\chi \in H_0^1(\Omega)$ we must have $\chi = 0$. Now use Theorem 4.1.

THEOREM 4.12. *For Problem P_N: if* $\ker \mathbf{q} = 0$ *then* $\omega_w(u_0) = c$ *(a constant),*

$$c = (meas\ \Omega)^{-1} \int_\Omega u_0(x)dx,$$

and for any u_0 $\overline{u}(t,u_0) \rightharpoonup c$ *as* $t \to \infty$ *in* $H^1(\Omega)$.

Proof. By Theorem 4.9 $\nabla\chi = 0$ a.e. in Ω and since $\int_\Omega v(x,t)dx = \int_\Omega u_0(x)dx$, $t \geq 0$, $v = \chi$ we see $\chi = c$. Now use Theorem 4.1.

Examples.

1) Consider the case $N = 1$ and q possessing the graph shown in Figure 1.

For Problem P_D: Theorem 4.11 applies and $\overline{u}(t,u_0) \rightharpoonup 0$ as $t \to \infty$ in $H_0^1(\Omega)$.

For Problem P_N: Theorem 4.9 applies and weak $\text{dist}(\overline{u}(t,u_0), \omega_w(u_0)) \to 0$ in $H^1(\Omega)$ as $t \to \infty$ where $\omega_w(u_0) \subseteq \{\chi$; measure valued equilibrium solutions of P_N, $0 \leq \frac{d\chi}{dx}(x) \leq \xi_1$ a.e. in $\Omega\}$.

2) Consider the case $N = 1$ and q possessing the graph shown in Figure 3.

For Problems $P_D(P_N)$: Theorem 4.9 applies and weak-dist$(\overline{u}(t,u_0); \omega_w(u_0)) \to 0$ in V as $t \to \infty$ where $\omega_w(u_0) \subseteq \{\chi$; measure valued equilibrium solutions of $P_D(P_N)$, $\xi_0 \leq \frac{d\chi}{dx}(n) \leq \xi_1$ a.e. in $\Omega\}$.

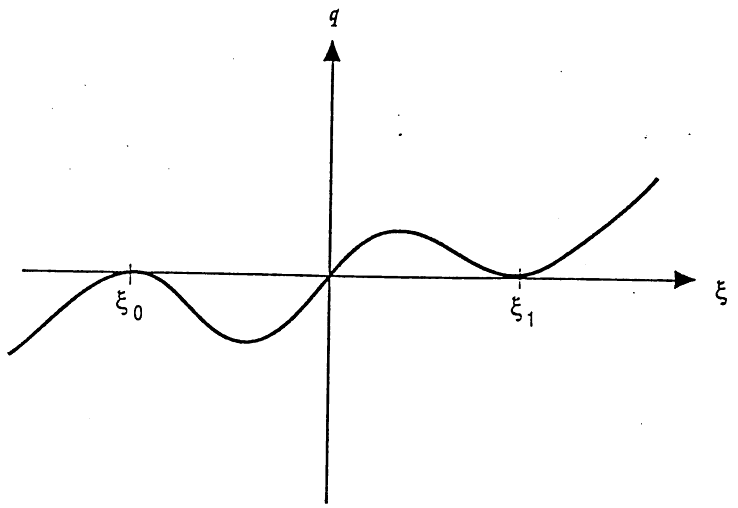

Figure 3

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