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OF MIXED TYPE**

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IMA Preprint Series # 546

June 1989

SOME QUALITATIVE PROPERTIES OF 2×2 SYSTEMS OF CONSERVATION LAWS OF MIXED TYPE.

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Abstract. We study qualitative features of the initial value problem $z_t + F(z)_x = 0$, $z(x, 0) = z_0(x)$, $x \in \mathbb{R}$, where $z(x, t) \in \mathbb{R}^2$, with Riemann initial data, viz. $z_0(x) = z_l$ if $x < 0$ and $z_0(x) = z_r$ if $x > 0$. In particular we are interested in the case when the system changes type when the eigenvalues of the Jacobian dF become complex. It is proved that if z_l and z_r are in the elliptic region, and the elliptic region is convex, then part of the solution has to be outside the elliptic region. If both z_l and z_r are in the hyperbolic region, then the solution will not enter the elliptic region. We show with an explicit example that the latter property is not true for general Cauchy data. This example is investigated numerically.

Key words. conservation laws, mixed type, Riemann problems

AMS(MOS) subject classifications. 35L65, 35M05, 76T05

1. Introduction. In this note we analyze certain qualitative properties of the 2×2 system of partial differential equations in one dimension on the form

$$(1.1) \quad \frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} f(u, v) \\ g(u, v) \end{pmatrix} = 0$$

with $u = u(x, t)$, $v = v(x, t)$, $x \in \mathbb{R}$. In particular we are interested in the initial value problem with Riemann initial data, i.e.

$$(1.2) \quad \begin{pmatrix} u(x, 0) \\ v(x, 0) \end{pmatrix} = \begin{cases} \begin{pmatrix} u_l \\ v_l \end{pmatrix}, & \text{for } x < 0 \\ \begin{pmatrix} u_r \\ v_r \end{pmatrix}, & \text{for } x > 0 \end{cases}$$

where u_l, u_r, v_l, v_r are constants.

The system (1.1),(1.2) arises as a model for a diverse range of physical phenomena from traffic flow [2] to three-phase flow in porous media [1]. Common for these applications is that one obtains from very general physical assumptions a system of mixed type, i.e. there is a region $E \subset \mathbb{R}^2$ of phase space where the 2×2 matrix

$$(1.3) \quad dF = \begin{pmatrix} f_u(u, v) & f_v(u, v) \\ g_u(u, v) & g_v(u, v) \end{pmatrix}$$

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H.H. would like to thank Barbara Keyfitz, Michael Shearer and the Institute for Mathematics and its Applications for organizing a very stimulating workshop and for the invitation to present these results.

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has no real eigenvalues. The system is then called elliptic in E .

Consider e.g. the case of three-phase flow in porous media where the unknown functions u and v denote saturations, i.e. relative volume fractions, of two of the phases, e.g. oil and water respectively. A recent numerical study [1] gave as a result with realistic physical data that there in fact is a small compact region E in phase space, and quite surprisingly the Riemann problem (1.1),(1.2) turned out to be rather well-behaved numerically in this situation.

Subsequent mathematical analysis [25], [9], [16], [27] showed that one in general has to expect mixed type behavior in this case. Also in applications to elastic bars and van der Waal fluids [14], [28], [22], [23], [24] there is mixed type behavior. See also [20], [10], [11], [12], [13], [15], [17], [18], [19].

Parallel to this development there has been a detailed study of certain model problems with very simple flux functions (f, g) with elliptic behavior in a compact region E which has revealed a very complicated structure of the solution to the Riemann problem [7], [8]. In general one must expect nonuniqueness of the solution for Riemann problems, see [5].

We prove two theorems for general 2×2 conservation laws of mixed type. Specifically the flux function is *not* assumed to be quadratic. The first theorem states that if z_l is in the elliptic region E , then z_l is the only point on the Hugoniot locus of z_l inside E provided E is convex. In the second theorem we show that one cannot connect a left state outside E via an intermediate state inside E to a right state outside E if we only allow shocks with viscous profiles as defined by (2.21).

This latter theorem has also been proved independently by Azevedo and Marchesin (private communication).

Combining these two theorems we see that if $z_l, z_r \notin \bar{E}$ then also the solution $z(x, t) \notin E$ for all $x \in \mathbb{R}, t > 0$. Finally we explicitly show that this property is *not* valid for the general Cauchy problem. The consequences of this for the Glimm's scheme is discussed by Pego and Serre[21] and Gilquin[3]. For the most recent result on conservation laws of mixed type we refer to the other contributions to these proceedings.

2. Qualitative properties. We write (1.1) as

$$(2.1) \quad z_t + F(z)_x = 0$$

where $z = \begin{pmatrix} u \\ v \end{pmatrix}$ and $F = \begin{pmatrix} f \\ g \end{pmatrix}$, with Riemann initial data

$$(2.2) \quad z(x, 0) = \begin{cases} z_l, & \text{for } x < 0 \\ z_r, & \text{for } x > 0. \end{cases}$$

We assume that f and g are real differentiable functions such that the Jacobian dF has real eigenvalues except in components of \mathbb{R}^2 , each of which are convex. Let

$$(2.3) \quad E = \left\{ z \in \mathbb{R}^2 : \lambda_j(z) \notin \mathbb{R} \right\}.$$

A shock solution is a solution of the form

$$(2.4) \quad z(x, t) = \begin{cases} z_l, & \text{for } x < st \\ z_r, & \text{for } x > st. \end{cases}$$

where the shock speed s must satisfy the Rankine–Hugoniot relation [29]

$$(2.5) \quad s(z_l - z_r) = F(z_l) - F(z_r).$$

The Hugoniot locus of z_l is the set of points satisfying

$$(2.6) \quad H_{z_l} = \left\{ z \in \mathbb{R}^2 : \exists s \in \mathbb{R}, s(z_l - z) = F(z_l) - F(z) \right\}.$$

For $z \in E$ we let E_z denote the convex component of E containing z . Then we have

THEOREM 2.1. *Let $z_l \in E$ and assume that E_{z_l} is convex, then*

$$(2.7) \quad H_{z_l} \cap E_{z_l} = \{z_l\}$$

and if $z_r \in E$ and E_{z_r} is convex, then

$$(2.8) \quad H_{z_r} \cap E_{z_r} = \{z_r\}.$$

Proof. We will show (2.7), (2.8) then follows by symmetry. Let $z_r \in H_{z_l}$ and assume that

$$(2.9) \quad z_r \in E_{z_l}.$$

Then the straight line connecting z_l and z_r is contained in E_{z_l} , viz.

$$(2.10) \quad \alpha(t) = tz_r + (1 - t)z_l \in E_{z_l}$$

for $t \in [0, 1]$ by convexity. Let

$$(2.11) \quad \beta(t) = F(\alpha(t)).$$

Then

$$(2.12) \quad \beta'(t) = dF(\alpha(t))(z_r - z_l).$$

We want to show the existence of $k \in \mathbb{R}$ and of $\tilde{t} \in [0, 1]$ such that

$$(2.13) \quad \beta'(\tilde{t}) = k(z_r - z_l).$$

Assuming (2.13) for the moment we obtain by combining (2.12) and (2.13)

$$(2.14) \quad \left[dF(\alpha(\tilde{t})) - k \right] (z_r - z_l) = 0$$

which contradicts (2.10).

To prove (2.13) we consider the straight line passing through $F(z_l)$ in the direction $z_l - z_r$. By assumption

$$(2.15) \quad s(z_r - z_l) = F(z_r) - F(z_l).$$

Using this we see that this line passes through $F(z_r)$ and that there is a $\tilde{t} \in [0, 1]$ such that $\beta'(\tilde{t}) \parallel (z_r - z_l)$ proving (2.13). \square

This implies that if $z_l \in E$ and $\{z_l, z_r\}$ are the initial values of a Riemann problem, then, the state immediately adjacent to z_l (z_r) in the solution will be outside of E_{z_l} (E_{z_r}). This is so since this state must either be a point on a rarefaction or a shock. Rarefaction curves do not enter E , and we have just shown that neither does the Hugoniot locus.

The other basic ingredient in the solution of the Riemann problem is rarefaction waves. These are smooth solutions of the form $z = z(x/t)$ that satisfy (2.1). The value $z(\xi)$ must be an integral curve of r_j , $j = 1, 2$ where r_j is a right eigenvector of dF corresponding to λ_j . ξ is the speed of the wave; $\xi = \lambda_j(z(x/t))$, therefore λ_j has to increase with ξ as z moves from left to right in the solution of the Riemann problem. Note that no rarefaction wave can intersect E since the eigenvectors are not defined there.

For a system of non-strictly hyperbolic conservation laws, the Riemann problem does not in general possess a unique solution, and by making the entropy condition sufficiently lax in order to obtain existence of a solution, one risks losing uniqueness. It is believed, see however [6], that the correct entropy condition which singles out the right physical solution is that the shock should be the limit as $\epsilon \rightarrow 0$ of the solution of the associated parabolic equation

$$(2.16) \quad z_t^\epsilon + F(z^\epsilon)_x = \epsilon z_{xx}^\epsilon \quad \epsilon > 0.$$

We then say that the shock has a viscous profile. Let now z_l, z_r be two states that can be connected with a shock of speed s . We seek solutions of the form

$$(2.17) \quad z^\epsilon = z^\epsilon \left(\frac{x - st}{\epsilon} \right) = z^\epsilon(\xi)$$

and then obtain

$$(2.18) \quad -s \frac{d}{d\xi} z^\epsilon + \frac{d}{d\xi} F(z^\epsilon) = \frac{d^2}{d\xi^2} z^\epsilon$$

which can be integrated to give

$$(2.19) \quad \frac{d}{d\xi} z^\epsilon = F(z^\epsilon) - s z^\epsilon + A$$

where A is a constant of integration. If $z^\epsilon(\xi)$ converges to the correct solution we must have

$$(2.20) \quad \lim_{\xi \rightarrow -\infty} z^\epsilon(\xi) = z_l \quad \lim_{\xi \rightarrow \infty} z^\epsilon(\xi) = z_r$$

(provided the derivatives decay sufficiently fast) which implies

$$(2.21) \quad \frac{d}{d\xi} z^\epsilon = (F(z^\epsilon) - F(z_l)) - s(z^\epsilon - z_l).$$

We see that z_l and z_r are fixpoints for this field, and if it admits an orbit *from* z_l *to* z_r we say that the shock is *admissible* and has a *viscous profile*. The associated eigenvalues of this field are

$$(2.22) \quad \lambda_j(z) - s \quad j = 1, 2.$$

THEOREM 2.2. *Assume that we have two admissible shocks, one connecting the left state z_l with a state z_m with speed s_l and one connecting z_m with z_r having speed s_r . If z_l and z_r are in the hyperbolic region, i.e. $z_l, z_r \notin \bar{E}$, then*

$$(2.23) \quad z_m \notin \bar{E}.$$

Proof. Assume that $z_m \in \bar{E}$. In \bar{E} the eigenvalues constitute a pair of complex conjugates. z_m is a source (sink) if $\text{Re}(\lambda_j(z_m)) - s_l > 0$ ($\text{Re}(\lambda_j(z_m)) - s_r < 0$), hence we obtain

$$(2.24) \quad s_l \geq \text{Re}(\lambda_j(z_m)) \geq s_r$$

which contradicts the fact that z_r is to the right of z_l unless $s_l = s_r$ in which case there is no z_m . \square

Combining Theorem 2.1 and Theorem 2.2 we obtain

COROLLARY 2.3. *Consider an admissible solution $z = z(x, t)$ of (2.1) with Riemann initial data (2.2). Assume that E is convex. Then*

- (1) *If $z_l, z_r \notin \bar{E}$, then also $z(x, t) \notin E$ for all $x \in \mathbb{R}, t > 0$.*
- (2) *If $z_l \in E$ or $z_r \in E$ and $z(\tilde{x}, \tilde{t}) \in E$ for some \tilde{x}, \tilde{t} , then $z(\tilde{x}, \tilde{t}) \in \{z_l, z_r\}$.*

The corollary states that if the initial values in a Riemann problem are inside the convex elliptic region, then the solution will contain values outside this region if the entropy condition is based on the “vanishing viscosity” approach. Furthermore if the initial values are outside the convex elliptic region, then the solution will not enter this region.

3. The Cauchy problem — a counterexample. Based on the results of the Riemann problem in the previous section it is natural to ask whether the same property is true for the general Cauchy problem: If

$$(3.1) \quad \begin{aligned} z_t + F(z)_x &= 0 \\ z(x, 0) &= z_0(x) \end{aligned}$$

and for all $x \in \mathbb{R}$

$$(3.2) \quad z_0(x) \notin E$$

is

$$(3.3) \quad z(x, t) \notin \bar{E}$$

for all $x \in \mathbb{R}$ and $t > 0$?

The following example shows this not always to be the case. Let

$$(3.4) \quad f(u, v) = \frac{1}{2} \left(\frac{u^2}{2} + v^2 \right) + v \quad g(u, v) = uv.$$

Then

$$(3.5) \quad E = \left\{ (u, v) \in \mathbb{R}^2 \mid \frac{u^2}{16} + \left(v + \frac{1}{2} \right)^2 < \frac{1}{4} \right\}.$$

Making the *ansatz*

$$(3.6) \quad u(x, t) = \alpha(x)\beta(t) \quad v(x, t) = \gamma(t)$$

we easily find

$$(3.7) \quad u(x, t) = \frac{2c_1x + c_2}{c_1t + c_3}, \quad v(x, t) = \frac{c_4}{(c_1t + c_3)^2},$$

for constants $c_i \in \mathbb{R}$, $i = 1, \dots, 4$. Choosing

$$(3.8) \quad c_1 = c_3 = 1, \quad c_2 = 0, \quad c_4 = -2,$$

we find

$$(3.9) \quad u_0(x) = 2x, \quad v_0(x) = -2,$$

and

$$(3.10) \quad u(x, t) = \frac{2x}{t+1}, \quad v(x, t) = \frac{-2}{(t+1)^2}.$$

For this choice (3.2) is valid but (3.3) fails for some $x \in \mathbb{R}$ for $t > \sqrt{2} - 1$, see figures 1 and 2. This and other [21] examples of solutions entering the elliptic region do however have the property that the solutions u and v are also solutions to the viscous equations since $u_{xx} = v_{xx} = 0$, as well as to hyperbolic equations without any elliptic regions since we have that $v_x = 0$.

Comparing the general properties of the Riemann problem and the example just presented, it is clear that the Glimm's scheme [4] will be highly unstable when the system is of mixed type since one in this scheme replaces the general Cauchy problem by a series of Riemann problems. This has recently been discussed by Pego and Serre [21], where another counterexample is provided and by Gilquin [3].

It was found that difference schemes also exhibit instabilities in this mixed type problem. The scheme used for the numerical examples was itself a mixed scheme: If both eigenvalues had positive (negative) real part a upwind (downwind) scheme was used, else a Lax–Friedrichs scheme was used. A pure Lax–Friedrichs scheme will have the same kind of oscillations, but they appear at a much small Δx .

In figures 3–5 we see the numerical solution to the initial value problem

$$(3.11) \quad u_0(x) = \frac{2x}{10}, \quad v_0(x) = \frac{-11}{10}$$

at times $t = 0.0$, $t = 1.5$ and $t = 3.0$ respectively. This is (3.7) with $c_1 = 1$, $c_2 = 0$, $c_3 = 10$ and $c_4 = -110$, and the exact solution enters the elliptic region at $t = \sqrt{110} - 10 \approx 0.4881$. In all the examples $\Delta x = 0.01$ and $\Delta t = 0.002$. In figures 6–7 we see numerical solutions to initial value problems with perturbations of these initial values at $t = 2.0$.

Figure 6: $\bar{v}_0 = v_0 - 0.02x$, $\bar{u}_0 = u_0$.

Figure 7: $\bar{v}_0 = v_0 + 0.03 \sin \frac{\pi}{10} x$, $\bar{u}_0 = u_0$.

These examples indicate that the solutions do enter the elliptic region, but this is difficult to determine due to the oscillations.

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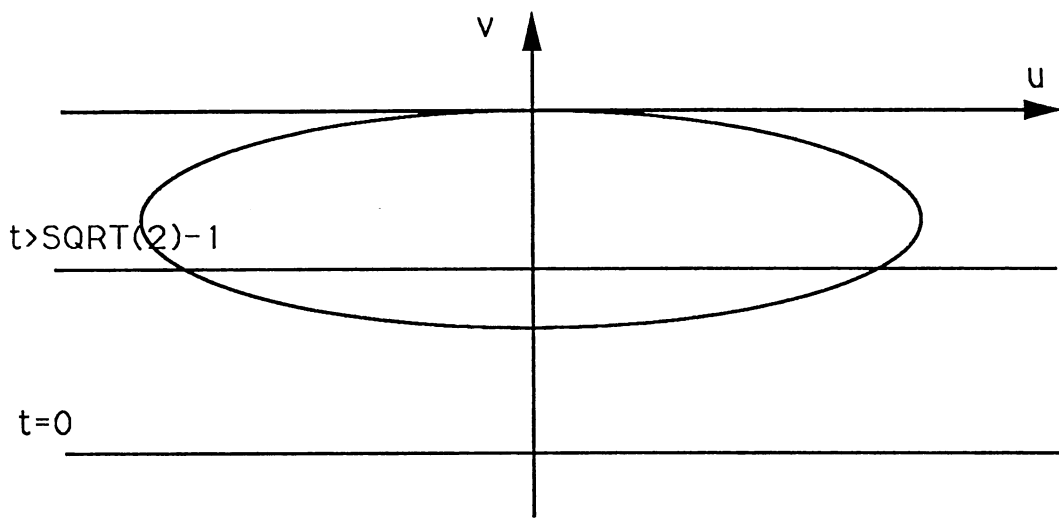


Fig. 1 The solution at $t=0$ and $t > \sqrt{2}-1$ in the z -plane

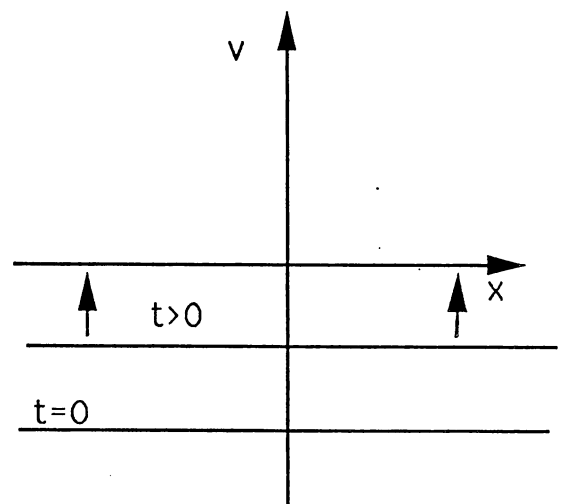
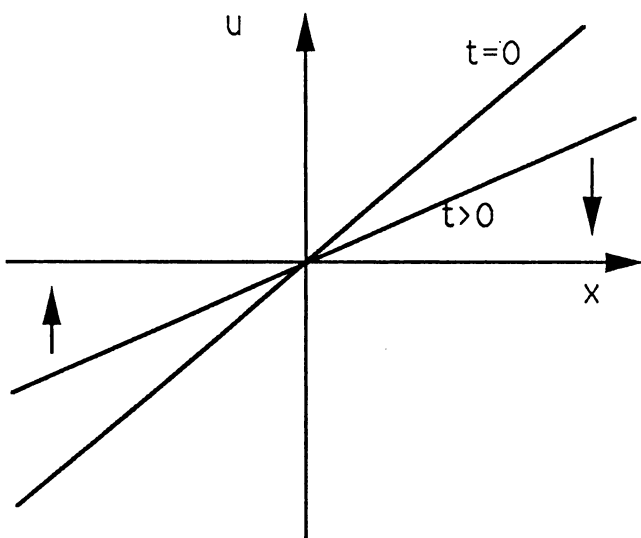


Fig. 2. The solution for $t=0$ and for $t > 0$.

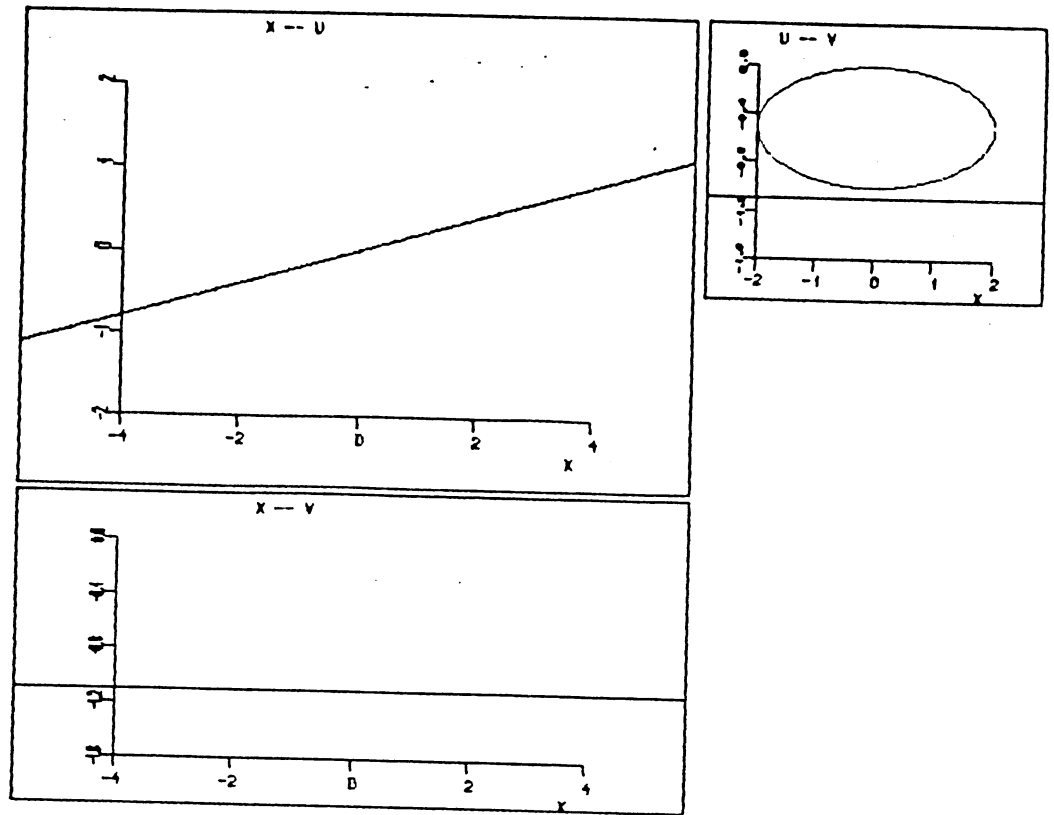


Figure 3.

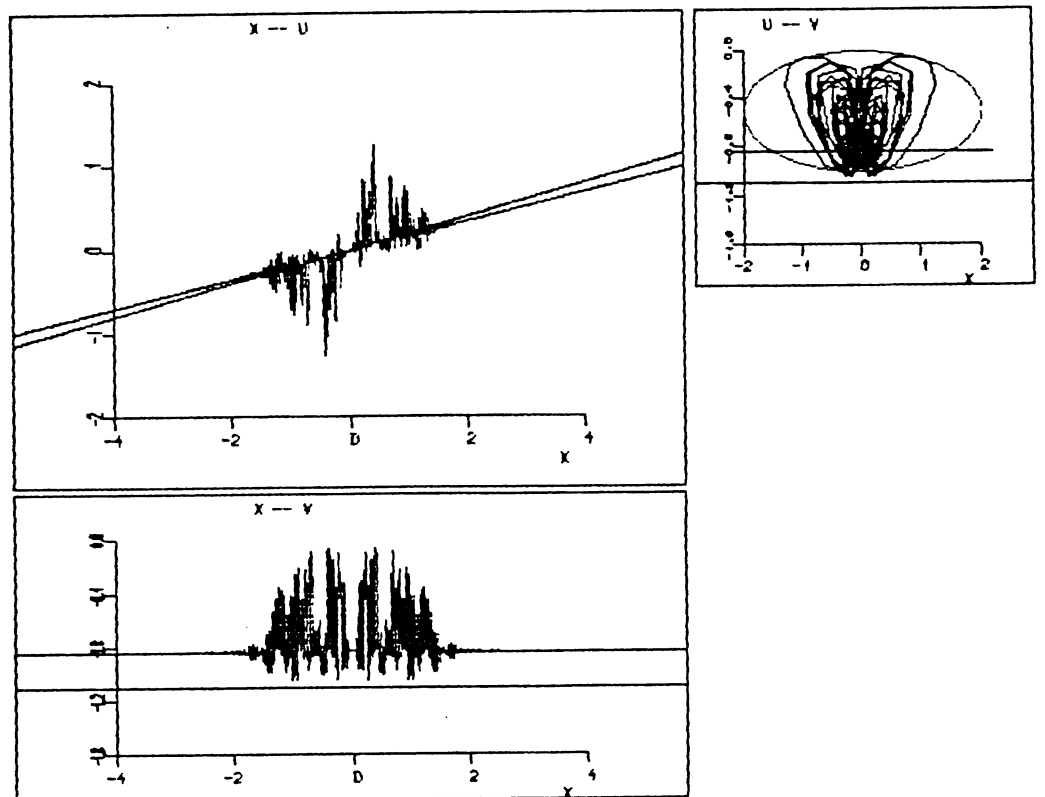


Figure 4.

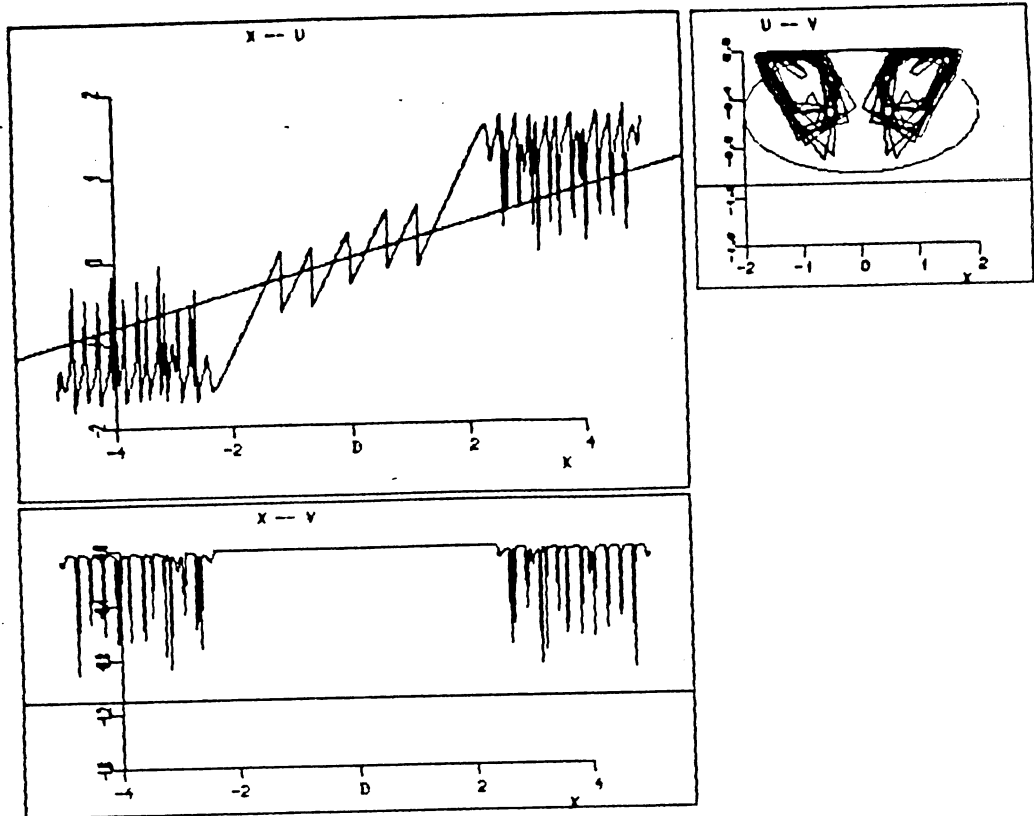


Figure 5.

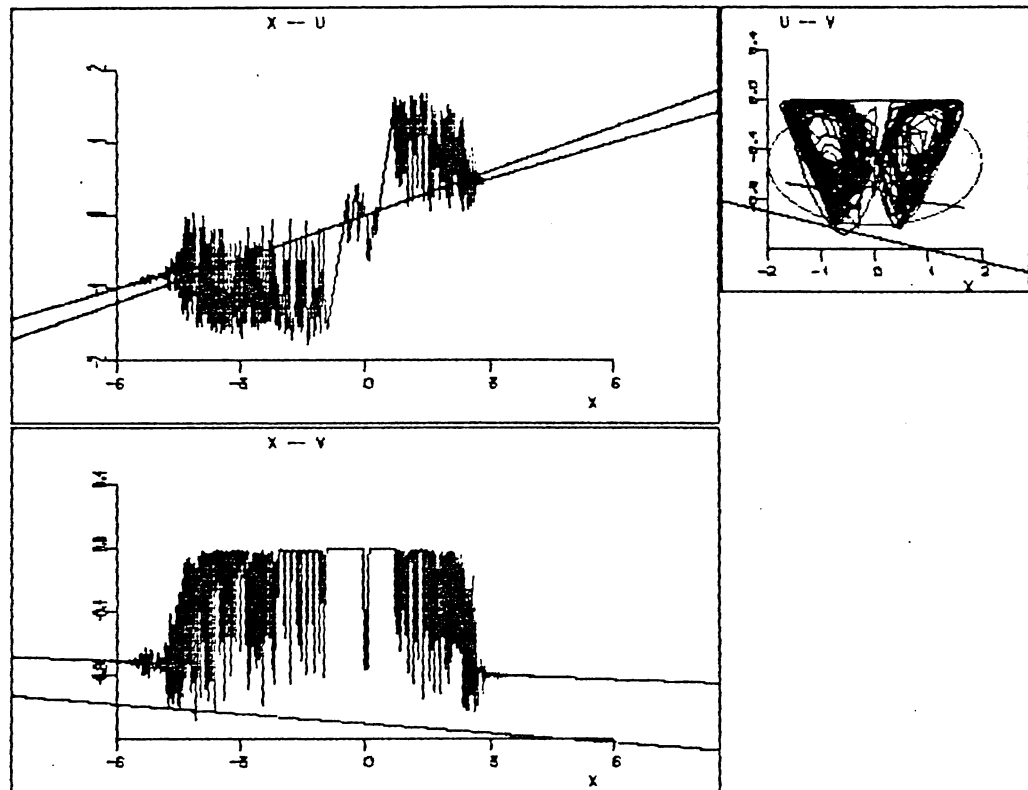


Figure 6.

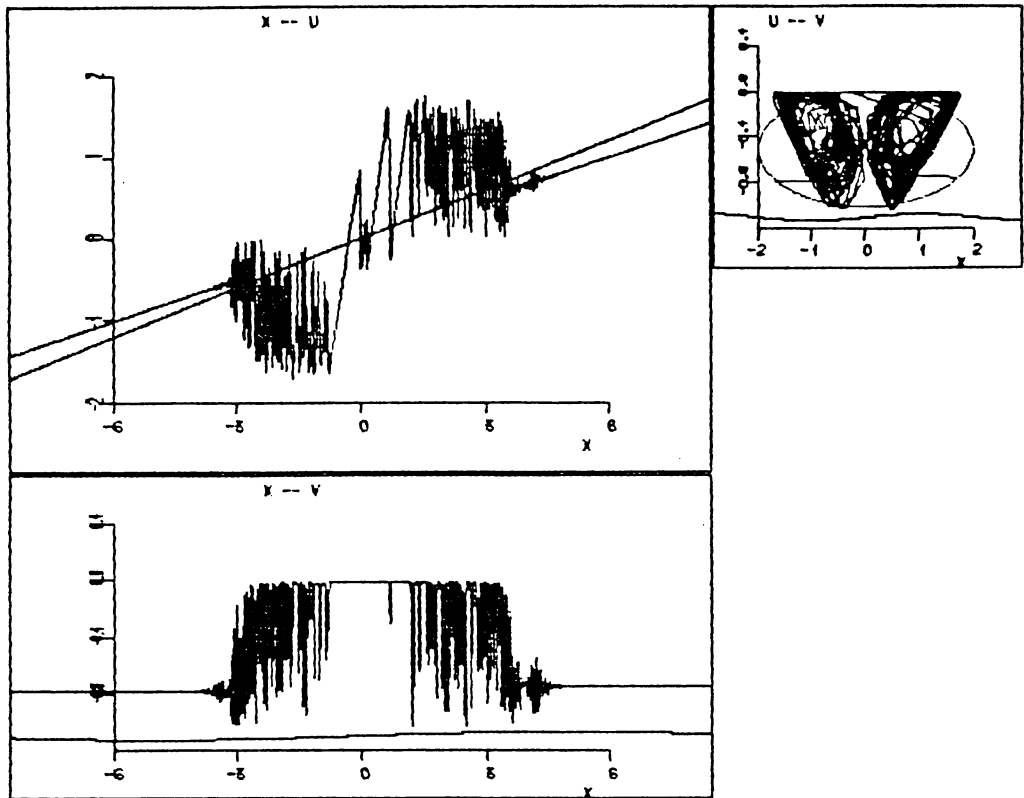


Figure 7.

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