**PRESSURE DRIVEN FLOW OF PURE VAPOR
UNDERGOING LAMINAR FILM CONDENSATION
BETWEEN PARALLEL PLATES**

By

A. Narain

IMA Preprint Series # 542

June 1989

PRESSURE DRIVEN FLOW OF PURE VAPOR UNDERGOING LAMINAR FILM CONDENSATION BETWEEN PARALLEL PLATES*

A. Narain

Department of Aerospace Engineering and Mechanics†
University of Minnesota
Minneapolis, MN 55455

Abstract: The flow of pure saturated vapor between horizontal parallel plates, with film condensation on the bottom plate, is considered and an approximate solution is obtained. The mixed differential-integral approach of this paper predicts film-thickness profile, condensation rate, wall heat transfer rate, pressure variations and other quantities of interest. The solutions for Freon-113 show that an increase in mass flow rate (Reynold's number), with other control parameters being constant, leads to a thinner condensate, increased pressure drops, reduced interfacial mass transfers per unit inlet mass flux, and increased heat removal rates. At the same mass flow rate, a drop in temperature difference between the vapor and the wall leads to smaller heat removal rates with a thinner condensate.

*The research was supported partially by NASA - Lewis grant NAG 3-711, IMA at the University of Minnesota, and an NSF visiting grant from the University of Minnesota.

†On Leave of Absence From:
Department of Mechanical Engineering and Engineering Mechanics, Michigan Technological University, Houghton, MI 49931

Nomenclature

h	gap between the plates
h_1	enthalpy of liquid phase (h_f)
h_2	enthalpy of vapor phase (h_g)
(χ, y)	physical distances along and across the flow
(x, y)	non-dimensional (χ, y)
U	characteristic speed appearing in the inlet velocity profile
V_{av}	average inlet vapor velocity
k	thermal conductivity
C_p	specific heat
p	physical pressure
p_0	inlet value of physical pressure
π	non-dimensional pressure
u	component of physical velocity in the x direction
v	component of physical velocity in the y direction
u	non-dimensional velocity in the x direction
v	non-dimensional velocity in the y direction
$\mathcal{T}$	physical temperature
T	non-dimensional temperature
$\dot{m}$	physical value of the local condensation rate
$\dot{m}$	non-dimensional value of local condensation rate
$u_f(x)$	non-dimensional value of the x-component of speed at the interface
h_{fg}	latent heat ($h_g - h_f$)
Ja	Jacob Number, $h_{fg}/C_{p1} \mathcal{T}_w$
Δb_2	error in b_2
Δu_f	error in u_f
Re	Reynold's Number
Re_o	Reynold's Number, Uh/v_2
Re^*	Reynold's Number, $V_{av}h/v_2$
C_f	non-dimensional shear stress on the bottom plate
Nu_x	Nusselt number, $h\chi/k_1$
h	Film heat transfer coefficient
Pr	Prandtl number

Greek or Mathematical Symbols

∇_s	surface gradient on the interface
∇	gradient operator
ρ	density
σ	surface tension
μ	viscosity
ν	kinematic viscosity
α	thermal diffusivity
Δ	physical value of condensate film thickness
δ	non-dimensional value of condensate film thickness
τ_w^1	physical value of bottom wall shear stress

Superscripts

i	value of a variable at the interface of vapor and liquid.
---	---

Subscripts

1	liquid
2	vapor
s	saturation condition or surface kinematics
w	wall

1. Introduction

Flow involving pure vapor with film condensation on walls have received extensive attention in engineering literature. Some of the more classical results for these flows have been limited to condensing external flows over flat plates (Nusselt [1], Rohsenow [2], Sparrow and Gregg [3], etc.) or over drops (Ford and Lekic [4], Jacobs and Cook [5], etc.). While there are several studies ([6], [7], etc.) of condensing flows which deal with the more advanced topics of presence of non-condensables and turbulence in the vapor phase, a look at the existing literature on laminar film condensation in ducts ([8], [9], [10], [11], [12]) suggest a need for more accurate solution techniques along with more accurate experimental measurements. Here we focus on developing a more refined predictive capacity for the flow under consideration.

We begin with an updated review of the general flow equations modeling condensation and then we present the standard approximations for the specific flow situation being studied. We study a pressure driven laminar flow of pure saturated vapor undergoing film condensation between horizontal parallel plates (see Figure 1). For a simplified analysis, it is assumed that the vapor and the upper plate are at a saturation temperature $\mathcal{T}_s(p_0)$ determined by the inlet pressure p_0 . The bottom plate is assumed to be at a constant temperature $\mathcal{T}_w < \mathcal{T}_s(p_0)$. The choice of a horizontal configuration is also motivated by a desire to make the prediction of the analysis useful in estimating condensate flow rates in a gravity free environment. This paper implements an integral-differential scheme which exploits the essentially linear equations of motion that govern the sluggish flow of thin condensates and, at the same time, satisfies all the essential differential interface conditions at the phase change boundary. The classical integral approach ([13], [14]) lies in choosing vapor velocity profiles in the cross flow direction while determining the flow evolution in the downstream direction. We find that we can reduce the resulting problem to a set of coupled non-linear ordinary differential equations with the help of modern computer capabilities of symbolic manipulations. This is necessary because the method involves lengthy algebra and therefore human implementation is neither attractive nor efficient. The formulation presented here

is estimated to be quite accurate for predicting the unknown functions appearing in the exact solution forms of the condensate flow. As a result, the predictions of related bulk observables (condensation rate, heat transfer rate, pressure variation, etc.) and location of the interface are believed to be good. The determination of the unknown interface by this approach can also be of further use in more detailed studies of vapor velocity profiles by means of Finite Element or Finite Difference type schemes.

As examples, we present numerical solutions for some typical flow situations involving Freon-113 (see Table 1). The computations are restricted to a range of inlet Reynold's number and downstream distances that are, according to rough experimental estimates ([15]), below critical Reynold's numbers at which the interface becomes unstable or wavy. In section 6, we have presented and discussed the computational predictions of interface location, heat transfer rates, condensation rate, pressure drop, and wall shear. The numerical accuracy of the physically consistent predictions are discussed in section 7. The computations are seen to be robust in the sense that large changes in initial conditions, due to substantial changes in inlet velocity profile (from uniform to parabolic) at a constant mass flux and temperature difference ($\mathcal{T}_s(p_0) - \mathcal{T}_w$), do not lead to significant differences in the predicted values of the bulk observables.

Another test of the validity of the numerical results in this paper lies in a reasonable agreement with a more approximate earlier solution [16] which employed an independent computer implementation. This comparison is discussed in section 6 and in the captions of Figures 2 and 7.

In the absence of reliable experimental data for this problem, we are not able to make a comparison of our predictions with those of experiments. However some preliminary results of a related ongoing experiment is discussed in section 6.

2. General Flow Equations

Consider a flow of pure vapor (phase I=2) undergoing film condensation (liquid phase I=1) over a cold surface. We assume absence of non-condensables in the vapor and insignificant molecular diffusion associated with evaporation at the vapor-liquid interface. Let, in either phase at any point $\mathbf{x}$ at time t , $\mathbf{T}_I$ be the Cauchy Stress, $\mathbf{v}_I$ be the velocity field, μ_I be the viscosity, κ_I be the usually ignorable *expansion viscosity*, p_I be the thermodynamic pressure field, ρ_I be the density field, k_I be the thermal conductivity, $\hat{u}_I$ be the internal energy per unit mass, $\mathcal{T}_I$ be the temperature field, σ be the surface tension on the interface between the two fluids, $\nabla \mathcal{T}_I$ be the temperature gradient field, and $\mathbf{D}_I$ be the field associated with the symmetric part of the velocity gradient $\nabla \mathbf{v}_I$. Then, for I=1 or 2, the stress constitutive relations are

$$\mathbf{T}_I = -p_I \mathbf{1} + \mathbf{S}_I, \text{ where } \mathbf{S}_I \equiv 2\mu_I \mathbf{D}_I + \kappa_I (\text{div } \mathbf{v}_I) \mathbf{1}, \quad (1)$$

and the heat flux ($\mathbf{q}_I$) relations are

$$\mathbf{q}_I = -k_I \nabla \mathcal{T}_I. \quad (2)$$

The Continuity, Momentum, and Energy equations in either of the phases are

$$\begin{aligned} \frac{\partial \rho_I}{\partial t} + \nabla \cdot (\rho_I \mathbf{v}_I) &= 0, \\ \rho_I \frac{d\mathbf{v}_I}{dt} &= \rho_I \mathbf{g} + \text{div } \mathbf{T}_I, \end{aligned} \quad (3)$$

and

$$\rho_I \frac{d\hat{u}_I}{dt} = -\nabla \cdot \mathbf{q}_I + \text{tr} (\mathbf{T}_I \mathbf{D}_I).$$

Since in *local equilibrium*, spatial variations in the flow fields are significant only at distances much larger than the characteristic root mean square distances associated with molecular motions, we use the homogeneous process equations of state of the type

$$\hat{u}_I = \hat{u}_I(\rho_I, \mathcal{T}_I), \quad p_I = p_I(\rho_I, \mathcal{T}_I) \quad (4)$$

Similar characterizations

$$\mu_I = \mu_I(\rho_I, T_I), \kappa_I = \kappa_I(\rho_I, T_I), k_I = k_I(\rho_I, T_I) \quad (5)$$

are known to be adequate ([16]) for the viscosities μ_I and κ_I and conductivities k_I .

The interface conditions across a phase change boundary are given by Delhay [18]. Let $\mathbf{x} = \mathbf{x}(t)$ be the motion of a *fixed surface point* (see Scriven [19] or Moeckel [20]) on an interface located by the equation $I(\mathbf{x}, t) = 0$, $\mathbf{v}_s \equiv d\mathbf{x}/dt$ be the velocity of the interface, and $\mathbf{n}$ be a unit normal on the interface directed from the liquid to the vapor phase. Furthermore, denote the interfacial value of any arbitrary flow variable ϕ by ϕ^i . Under the above notations, the kinematic condition at the interface is

$$\mathbf{v}_s \cdot \mathbf{n} \equiv \frac{-\frac{\partial I}{\partial t}}{|\nabla I|}, \quad (6)$$

and the restriction of continuity of tangential velocities at a point on the interface is

$$\mathbf{v}_1^i \cdot \mathbf{t} = \mathbf{v}_2^i \cdot \mathbf{t} \equiv \mathbf{v}_s \cdot \mathbf{t} \equiv \mathbf{v}_t, \quad (7)$$

where $\mathbf{t}$ is any unit tangent vector on the interface. The equations (6) and (7) together completely specify the surface speed $\mathbf{v}_s$. The mass, momentum, and energy balance at the interface (see [18] or [21]) are respectively given as

$$\rho_1^i \{ (\mathbf{v}_1^i - \mathbf{v}_s) \cdot \mathbf{n} \} = \rho_2^i \{ (\mathbf{v}_2^i - \mathbf{v}_s) \cdot \mathbf{n} \} \equiv -\dot{m}(\mathbf{x}, t), \quad (8)$$

$$\begin{aligned} & \left[\dot{m}(\mathbf{x}, t) \mathbf{v}_1^i - \dot{m}(\mathbf{x}, t) \mathbf{v}_2^i \right] + \left[(p_2^i - p_1^i) \mathbf{n} \right] \\ & + \left[(\mathbf{S}_2^i - \mathbf{S}_1^i) \cdot \mathbf{n} \right] + \left[-\left\{ \nabla_s \sigma - (\nabla_s \cdot \mathbf{n}) \sigma \mathbf{n} \right\} \right] = 0, \end{aligned} \quad (9)$$

and

$$\begin{aligned}
& \left[\dot{m} (h_1^i - h_2^i) \right] + \left[\frac{1}{2} \dot{m} |v_1^i - v_s|^2 - \frac{1}{2} \dot{m} |v_1^i - v_s|^2 \right] \\
& + \left[(S_1^i n) \cdot (v_1^i - v_s) - (S_2^i n) \cdot (v_2^i - v_s) \right] \\
& + \left[-k_2 \frac{\partial T_2}{\partial n} \Big|_i + k_1 \frac{\partial T_1}{\partial n} \Big|_i \right] + \left[\frac{d\sigma}{dt} \right] = 0 .
\end{aligned} \tag{10}$$

The definition of $\dot{m}$ in (8) is such that $\dot{m} > 0$ when vapor condenses into liquid. In (10) above, $h_I (\equiv \hat{u} + p_I / \rho_I)$ denotes enthalpy for $I=1$ or 2 and $d\sigma / dt (\equiv \frac{\partial \sigma}{\partial t} \Big|_x + v_s \cdot \nabla_s \sigma)$ denotes rate change of surface energy per unit area. The reader should note that the contribution of surface energy in (10) above follows from the correct physical postulates given in Narain and Joseph [22] and does not follow from the postulates in Delhay [18] or Moeckel [20].

The conditions (8)–(10) above are now restricted to vapor flows of interest (Steam, Freon, etc.) by recalling that the saturation temperature ($T_s = T_s(p)$) for these fluids is determined by the pressure p alone. Therefore, the temperatures T_I^i and the enthalpies h_I^i are determined from

$$T_I^i = T_s(p_I^i) ,$$

and

$$h_I^i = h_I(p_I^i, T_s(p_I^i)) \tag{11}$$

found in standard tables ([23]) for common fluids in either phase ($I=1$ or 2 .)

In any specific condensation problem, equations (1)–(11) have to be solved subject to appropriate initial and boundary conditions (at wall, at infinity, or at inlet/outlet boundaries of the flow domain) relevant to the problem.

3. Approximate Model Equations for Flow Between Parallel Plates

We consider laminar pressure driven flow of pure vapor between two horizontal parallel plates (see Figure 1). The distance between the plates is h . It is assumed that the upper plate and

the incoming vapor are at a constant temperature equal to the saturation temperature ($T_s(p_0)$) determined by the inlet pressure ($p=p_0$). The constant bottom plate temperature T_w is lower than the saturation temperature at any location to allow filmwise condensation on the plate. The inlet vapor velocity profile at the onset of condensation (at $x=0$) is assumed known. For *typical* flows (e.g., see Table 1), the flow slightly away from the entrance is adequately modeled under the following assumptions:

- (i) The maximum temperature differences in the liquid and the vapor phases over $0 \leq x \leq L$ are small in the sense that $(T_s(p_0) - T_w)/T_w \ll 1$ and $(T_s(p_2(L)) - T_s(p_0))/T_w \ll 1$. Furthermore, the pressure gradient in the vapor flow is small enough to keep the vapor Mach Number at a value less than 0.3. Under these conditions, one can adequately treat the densities ρ_I , viscosities μ_I , conductivities k_I , and specific heats $(C_p)_I$ as approximate constants over each phase ($I=1$ or 2).
- (ii) At a typical $x \sim L$, the film thickness is small ($\Delta(L)/L \ll 1$) and the gap is narrow ($h/L \ll 1$). This means velocity and temperature variations perpendicular to the direction of flow outweigh velocity and temperature variations along the flow.
- (iii) The order of magnitude of viscous forces ($\mu_I \nabla^2 v_I$) are comparable to the order of magnitude of inertia forces ($\rho_I dv_I/dt$) in the flow of the vapor and its liquid condensate. The order of pressure force ($-\nabla p_I$) in either phase could take any value from zero up to the order of inertia or viscous force in that phase.
- (iv) Energy associated with viscous dissipation is small compared to the convective and conductive transport of energy in the energy equation for each of the two phases.

Figure 1 The geometry of the flow is indicated above. In the figure $\tau_w^2(x)$, $\tau_f(x)$, and $\tau_w^1(x)$ respectively indicate the values of shear stresses at the upper wall, the interface, and the bottom plate.

- (v) At a typical $\chi \sim L$, the liquid condensate thickness profile is both thin and small in curvature and the local mass transfer at the interface is typically small. This means one can ignore the surface tension and mass transfer effects in the momentum interface condition and this condition is then adequately modeled by a continuity of pressure in the normal direction and a continuity of shear in the tangential direction.
- (vi) The mean pressure difference in the flow over the length of interest is assumed moderate and since the latent heat h_{fg} does not vary much with interfacial pressure variations encountered in typical flows (Table 1), the latent heat h_{fg} can be treated as an approximate constant for the flow.
- (vii) The vapor flow rate (and therefore pressure gradients) are moderate enough to ignore reversible pressure working (see [24], p. 114) as compared to convective and conductive transport terms in the energy equation for either of the two phases.

The governing equations (1)–(11) can now be scaled under the above assumptions to arrive at a simplified, but realistic, set of equations similar to the one given by Sparrow and Gregg [3] for flow over a flat plate. The justification of these equations by scaling analysis can be found in a report [16] or in the appendix of a paper by Kocamustafaogullari [25]. Here we present the simplified equations for the two dimensional flow

$$\mathbf{v}_I \equiv u_I(\chi, y) \mathbf{i} + v_I(\chi, y) \mathbf{j} \quad (12)$$

by non-dimensionalizing the physical variables through the relations

$$\begin{aligned}
\chi &\equiv hx , \\
y &\equiv hy , \\
\Delta(\chi) &\equiv h\delta(x) , \\
u_I(\chi, y) &\equiv U u_I(x, y) , \\
v_I(\chi, y) &\equiv U v_I(x, y) , \\
T_I(\chi, y) &\equiv T_w T_I(x, y) ,
\end{aligned} \tag{13}$$

and

$$p_I(\chi, y) \equiv \rho_I U^2 \pi_I(x, y) + p_0 - \rho_I g h y.$$

In (13) above, p_0 is the centerline inlet pressure (at $\chi=0$) and U is a characteristic speed (defined later through (26) and (39)) occurring in the vapor velocity profile at the inlet. The resulting non-dimensional form of the scaled mass, momentum (x and y directions), and energy balance equations for the two phases ($I=1$ or 2) are respectively

$$\frac{\partial u_I}{\partial x} + \frac{\partial v_I}{\partial y} = 0 , \tag{14}$$

$$u_I \frac{\partial u_I}{\partial x} + v_I \frac{\partial u_I}{\partial y} = - \frac{\partial \pi_I}{\partial x} + \frac{1}{Re_I} \frac{\partial^2 u_I}{\partial y^2} , \tag{15}$$

$$\frac{\partial \pi_I}{\partial y} = 0 , \text{ and} \tag{16}$$

$$u_I \frac{\partial T_I}{\partial x} + v_I \frac{\partial T_I}{\partial y} = \frac{1}{Pr_I Re_I} \frac{\partial^2 T_I}{\partial y^2} ; \tag{17}$$

where

$$v_I \equiv \mu_I / \rho_I , Re_I \equiv U h / v_I \quad \alpha_I \equiv k_I / \rho_I C_{pI} \text{ and } Pr_I \equiv v_I / \alpha_I .$$

The conditions approximating normal and tangential components of interfacial momentum balance, continuity of tangential speeds at the interface, mass balance at the interface, energy balance at the interface, and the saturation temperature condition at the interface are respectively

$$\pi(x) \equiv \pi_2(x, \delta(x)) = \frac{\rho_1}{\rho_2} \pi_1(x, \delta(x)) + \frac{gh}{U^2} \left(1 - \frac{\rho_1}{\rho_2}\right) \delta(x) \equiv \frac{\rho_1}{\rho_2} \pi_1(x, \delta(x)), \quad (18)$$

$$\frac{\partial u_2}{\partial y}(x, \delta(x)) = \frac{\mu_1}{\mu_2} \frac{\partial u_1}{\partial y}(x, \delta(x)), \quad (19)$$

$$u_1(x, \delta(x)) = u_2(x, \delta(x)) \equiv u_f(x), \quad (20)$$

$$\dot{m}(x) \equiv \frac{\dot{m}}{\rho_1 U} = \left[u_1(x, \delta(x)) \frac{d\delta}{dx} - v_1(x, \delta(x)) \right] = \frac{\rho_2}{\rho_1} \left[u_2(x, \delta(x)) \frac{d\delta}{dx} - v_2(x, \delta(x)) \right], \quad (21)$$

$$\dot{m}(x) = \left\{ \frac{1}{Re_1} \cdot \frac{1}{Pr_1} \cdot \frac{1}{Ja} \right\} \left\{ \frac{\partial T_1}{\partial y}(x, \delta(x)) - \frac{k_2}{k_1} \frac{\partial T_2}{\partial y}(x, \delta(x)) \right\}, \quad (22)$$

and

$$\begin{aligned} T_1(x, \delta(x)) &= T_2(x, \delta(x)) \\ &= \frac{1}{\mathcal{T}_w} \left\{ \mathcal{T}_s (\rho_2 U^2 \pi(x) + p_0 - \rho_2 \delta(x) h g) \right\} \\ &\equiv T_s(\pi(x)), \end{aligned} \quad (23)$$

where

$$Ja \equiv h_{fg}/Cp_1 \mathcal{T}_w.$$

Note that the Sparrow and Gregg [3] type interface pressure condition (18) also ignores gravity forces $\left\{ (gh/U^2) \left(1 - \frac{\rho_1}{\rho_2}\right) \frac{d\delta}{dx} \right\}$ in comparison to the pressure force $(d\pi/dx)$ and this is a good approximation for either small values of $d\delta/dx$ or small values of g (microgravity environment).

The conditions at the wall $y=0$ are

$$u_1(x, 0) = v_1(x, 0) = 0,$$

and

$$T_1(x, 0) = 1. \quad (24)$$

The conditions at infinity in Sparrow and Gregg [3] are replaced by wall conditions at $y=1$, which are

$$u_2(x, 1) = v_2(x, 1) = 0 ,$$

and

$$T_2(x, 1) = T_s(p_0)/T_w = \text{constant} . \quad (25)$$

The inlet conditions at $x=0$ consist of a prescribed inlet velocity profile $\tilde{u}(y)$ defined as

$$u_2(0, y) = Uu_2(0, y) \equiv U\tilde{u}(y) \quad (26)$$

for $0 \leq y \leq 1$, a constant inlet temperature

$$T_2(0, y) = T_s(p_0)/T_w , \quad (27)$$

and compatibility restrictions

$$\delta(0) = 0 ,$$

$$u_1(x, \delta(x))|_{x=0} = u_2(x, \delta(x))|_{x=0} = 0 \quad (28)$$

asserting onset of condensation at $x=0$ and the no slip condition at $x=0$ and $y=0$.

4. Further Simplifications and Integral Formulation

We note, from (25) and (27), that vapor enters at a saturation temperature determined by inlet pressure and stays at that temperature at the upper wall. If the temperature at the interface given by (23) does not vary much, it would follow that the vapor phase remains at nearly constant temperature and the energy equation (17) for $I=2$ is automatically satisfied in an approximate manner. We assume this to be the case for *typical* flow situations (see equation (A.14) in the Appendix and Table 1) under study. Because of an approximately constant vapor temperature, (22) simplifies to

$$\dot{m}(x) = \left\{ \frac{1}{Re_1} \cdot \frac{1}{Pr_1} \cdot \frac{1}{Ja} \right\} \left\{ \frac{\partial T_1}{\partial y}(x, \delta(x)) \right\}. \quad (29)$$

For flow situations of interest in this paper ($\rho_1/\rho_2 \geq 100$, $\mu_1/\mu_2 \geq 10$, $Pr_1 \geq 0.9$, $Ja \geq 1$, $Re_1 \geq 500$), the density ratio ρ_1/ρ_2 and viscosity ratio μ_1/μ_2 are sufficiently large to make the thin condensate flow very sluggish in response to the no slip condition at the bottom wall. In this situation, the effective Reynold's number for the condensate flow (not Re_1) is quite small for sufficient downstream distances and one can ignore the inertia term in (15) for $I=1$ and the convection term in (17) for $I=1$. This fact is particularly evident in Koh's [26] computations of film condensation over a flat plate as he finds linear velocity and temperature profiles for *typical* condensate flows despite the fact that full non-linearities of inertia and convective transport were retained in the governing equations. Ignoring gravity forces in comparison to the pressure force (see (18)), the equations (16) for $I=1$ and 2 together with (18) imply

$$\pi_2(x, y) = \pi_2(x, \delta(x)) \equiv \pi(x) \quad (30)$$

for $\delta(x) \leq y \leq 1$ and

$$\pi_1(x, y) = \pi_1(x, \delta(x)) = \frac{\rho_2}{\rho_1} \pi(x) \quad (31)$$

for $0 \leq y \leq \delta(x)$. For the sake of convenience, we introduce the symbols

$$Re_2 \equiv Re_0 \quad \text{and} \quad Re_1 \equiv \frac{v_2}{v_1} Re_0. \quad (32)$$

After dropping inertia, we solve (15) for $I=1$ subject to boundary conditions in (20) and (24). Using the notations introduced in (30) and (32), we find the condensate velocity profile solution to be

$$u_1(x, y) = \left(\frac{\mu_2}{\mu_1} \frac{Re_0}{2} \frac{d\pi}{dx}(x) \right) y(y - \delta(x)) + \frac{u_f(x)}{\delta(x)} y. \quad (33)$$

After dropping the convection term, we solve (17) for $I=1$ subject to boundary conditions (23) and (24). The resulting linear temperature profile is given by

$$T_1(x, y) = 1 + \frac{T_s(\pi(x)) - 1}{\delta(x)} y. \quad (34)$$

Note that the solution forms in (33) and (34) involve unknown functions $u_f(x)$, $\delta(x)$, $\pi(x)$, and $\zeta(x) \equiv d\pi/dx$. The aim of this paper is to find these functions to predict the bulk observables consisting of the heat removal rate from the bottom plate, the condensation rate, film thickness profile, wall shear stress, pressure drop, etc. The approach then consists of predicting the bulk observables as accurately as possible while predicting the vapor velocity profile, if possible, only in an approximate integral sense. To achieve this end, we integrate *some* of the equations in (14)–(28) to arrive at control-volume equations for mass and momentum balance for the vapor phase.

Integrating (14) (for $I=1$ and 2) subject to the wall conditions in (24) and (25), we find that (21) can be rewritten in the control-volume mass balance form

$$\dot{m}(x) = \frac{d}{dx} \left\{ \int_0^{\delta(x)} u_1(x, y) dy \right\} = -\frac{\rho_2}{\rho_1} \left\{ \frac{d}{dx} \int_{\delta(x)}^1 u_2(x, y) dy \right\}. \quad (35)$$

Substituting (33) in (35), we rewrite this condition as

$$\begin{aligned} \frac{\rho_2}{\rho_1} \frac{d}{dx} \left\{ \int_{\delta(x)}^1 u_2(x, y) dy \right\} &= -\dot{m}(x) \\ &= B \delta^3(x) \frac{d\zeta}{dx} + 3B \delta(x)^2 \zeta(x) \frac{d\delta}{dx}(x) - \left(\frac{1}{2} u_f(x) \right) \frac{d\delta}{dx} - \left(\frac{1}{2} \delta(x) \right) \frac{du_f}{dx}, \end{aligned} \quad (36)$$

where $\zeta(x) \equiv \frac{d\pi}{dx}(x)$ and $B \equiv \frac{1}{12} \frac{\mu_2}{\mu_1} \text{Re}_0$.

Similarly, integrating (15) for $I=2$ with the aid of (14) for $I=2$, (21) and (31); we find the usual control volume form of vapor momentum balance

$$\frac{d}{dx} \left\{ \int_{\delta(x)}^1 u_2^2(x, y) dy \right\} + \frac{\rho_1}{\rho_2} \dot{m}(x) u_f(x) = \frac{1}{Re_0} \left[\frac{\partial u_2}{\partial y}(x, 1) - \frac{\partial u_2}{\partial y}(x, \delta(x)) \right] - (1-\delta(x)) \frac{d\pi}{dx}. \quad (37)$$

Substituting for $\dot{m}(x)$ as given by the right side of (36) and $\frac{\partial u_2}{\partial y}(x, \delta)$ from (19) and (33), we rewrite (37) as

$$\begin{aligned} \frac{d}{dx} \left\{ \int_{\delta(x)}^1 u_2^2(x, y) dy \right\} = & \left\{ \frac{1}{Re_0} \frac{\partial u_2}{\partial y}(x, 1) - \frac{1}{2} \delta(x) \frac{d\pi}{dx}(x) - \frac{1}{Re_0} \frac{\mu_1}{\mu_2} \frac{u_f(x)}{\delta(x)} \right\} - (1-\delta(x)) \frac{d\pi}{dx} \\ & + \frac{\rho_1}{\rho_2} u_f(x) \left\{ B\delta^3 \frac{d\zeta}{dx} + 3B\delta^2(x)\zeta(x) \frac{d\delta}{dx}(x) - \frac{1}{2} u_f \frac{d\delta}{dx} - \frac{1}{2} \delta \frac{du_f}{dx} \right\}. \end{aligned} \quad (38)$$

The problem now reduces to seeking a solution for u_f , δ , π , and ζ such that the control volume equations (36) and (38), as well as the interface conditions, wall conditions, and the inlet conditions given in (18)–(28), are all satisfied.

5. The Integral Solution Scheme

The objective is to seek a representation of $u_2(x, y)$ which approximates the features of its differential solution at the boundaries $y=\delta(x)$ and $y=1$ and is also expressible in terms of the unknown functions $u_f(x)$, $\delta(x)$, $\pi(x)$, and $\zeta(x) \equiv d\pi/dx$ occurring in the condensate solutions (33) and (34). The choice of $u_2(x, y)$ is also dictated by a need to satisfy the conditions (18)–(28) and the control volume equations (36) and (38) so as to yield a well-posed initial value (starting at $x=0$) problem in terms of four first order non-linear ordinary differential equations for the functions $u_f(x)$, $\delta(x)$, $\pi(x)$, and $\zeta(x) \equiv d\pi/dx$. It is expected that the accuracy of predicting the stated unknowns by this mixed differential-integral solution scheme should depend only on how accurately the forcing functions on the right side of (36) and (38) are known in terms of the four unknown variables stated earlier. From the right sides of (36) and (38), we observe that except for

the upper wall shear term $\frac{\partial u_2}{\partial y}(x, 1)$, all the terms are given in terms of the stated unknowns.

Therefore, if we choose a representation of $u_2(x, y)$ such that all the differential restrictions on u_2 are satisfied near $y=1$, we would expect a fairly accurate prediction of the lower dimensional unknown variables. It is now noted that narrow gap assumption (approximately valid for $x \geq 5$) leading to (15) for $I=2$ is not restrictive. If one were to keep the full viscous force term by replacing the right side of (15) by $(1/Re_2)(\partial^2 u_2/\partial x^2 + \partial^2 u_2/\partial y^2)$ and the left side of (19) by $\{\partial u_2/\partial y(x, \delta(x)) + \partial v_2/\partial x(x, \delta(x))\}$ for $x < 5$, we would still arrive at the same overall mass and momentum balance given later as (51) and (52). This means that this integral approach will be good for the lower dimensional unknowns even for $0 < x < 5$.

We now choose one possible representation of $u_2(x, y)$, in the spirit of the choice criteria used in Kármán-Pohlhausen type boundary layer flow techniques [27]. Let b_1 , b_2 , and b_3 be three unknown functions of x such that

$$u_2(x, y) = b_1(x) \theta(y) + b_2(x) (y-1) + b_3(x) \frac{(y-1)^2}{2} \quad (39)$$

where $x \geq 0$, $0 \leq y \leq 1$, and the approximately constant function $\theta(y)$ is

$$\theta(y) = [1 - \exp(-Ay) - \exp(-A(1-y)) + \exp(-A)] \quad (40)$$

for $A \geq 5$. The choice in (39) satisfies the no slip condition (25), at $y = 1$. Now note that the interface restrictions (19), (20), and the evaluation of (15) for $I=2$ at $y=1$ respectively give

$$\frac{\partial u_2}{\partial y}(x, \delta(x)) = \frac{\mu_1}{\mu_2} \frac{u_f(x)}{\delta(x)} + \left(\frac{1}{2} Re_0\right) \delta(x) \frac{d\pi}{dx}(x), \quad (41)$$

$$u_2(x, \delta(x)) = u_f(x), \quad (42)$$

and

$$\frac{\partial^2 u_2}{\partial y^2}(x, 1) = Re_0 \frac{d\pi}{dx}(x). \quad (43)$$

Substitution of (39) in (41)–(43) gives three independent linear algebraic relations in three unknowns b_1 , b_2 , and b_3 . It is readily seen that one would then obtain solutions of the type

$$b_1 = b_1 (u_f, \delta, \zeta) ,$$

$$b_2 = b_2 (u_f, \delta, \zeta) ,$$

and

$$b_3 = b_3 (u_f, \delta, \zeta) , \quad (44)$$

where $\zeta = d\pi/dx$. The functions b_1 , b_2 , b_3 are defined in (A.1)–(A.5) of the Appendix. Substituting for b_3 from (A.1), (39) can also be rewritten as

$$u_2(x, y) = b_1(x) \hat{\theta}(y) + b_2(x) (y-1) + (Re_0/2) (d\pi/dx) (y-1)^2 , \quad (45)$$

where

$$\hat{\theta}(y) \equiv \theta(y) + (A^2/2) (1 + \exp(-A)) (y-1)^2.$$

We consider two classes of inlet velocity profiles. In (39), if we choose

$$b_1(0) = 1, b_2(0) = 0, \text{ and } b_3(0) = 0; \quad (46)$$

then we have nearly uniform inlet vapor velocity profiles for $A \geq 5$. Alternately, in (45), if we choose

$$b_1(0) = 0, \frac{d\pi}{dx}(0) = -D_0 < 0, \text{ and } b_2(0) = - (Re_0 D_0/2); \quad (47)$$

then we have symmetric parabolic inlet vapor velocity profiles. The solution in (44) implies that (39) is of the form

$$u_2(x, y) = \Psi (u_f, \delta, \zeta ; y) , \quad (48)$$

and therefore

$$I_1 = \int_{\delta(x)}^1 u_2(x, y) dy \equiv I_1(u_f, \delta, \zeta), \quad (49)$$

and

$$I_2 = \int_{\delta(x)}^1 u_2^2(x, y) dy \equiv I_2(u_f, \delta, \zeta), \quad (50)$$

From (48)–(50), it is evident that (36) and (38) can respectively be written in the forms

$$A_{11}(u_f, \delta, \zeta) \frac{du_f}{dx} + A_{12}(u_f, \delta, \zeta) \frac{d\delta}{dx} + A_{13}(u_f, \delta, \zeta) \frac{d\zeta}{dx} = 0 \quad (51)$$

and

$$C_{11}(u_f, \delta, \zeta) \frac{du_f}{dx} + C_{12}(u_f, \delta, \zeta) \frac{d\delta}{dx} + C_{13}(u_f, \delta, \zeta) \frac{d\zeta}{dx} + C_{14}(u_f, \delta, \zeta) \frac{d\pi}{dx} = \frac{\phi_2(u_f, \delta)}{Re_0}. \quad (52)$$

Energy interface condition (29), after substituting for $\dot{m}$ from (36) and for T_1 from (34), can similarly be written as

$$D_{11}(u_f, \delta, \zeta) \frac{du_f}{dx} + D_{12}(u_f, \delta, \zeta) \frac{d\delta}{dx} + D_{13}(u_f, \delta, \zeta) \frac{d\zeta}{dx} = \phi_3(\pi(x), \delta(x)) \quad (53)$$

The coefficients and functions in (51)–(53) are defined in (A.1)–(A.14) of the Appendix. The reader should note that the determination of these coefficient functions requires lengthy algebra and therefore they were calculated from their definitions by a symbolic manipulation software MAPLE (a software comparable to MACSYMA) and the generated coefficient functions were directly processed on the computer into Fortran form for use in the main program.

Here we note that the definition

$$\frac{d\pi}{dx} \equiv \zeta , \quad (54)$$

and the equations (51)–(53) can be written in the form

$$\mathbf{A} \frac{d\mathbf{y}}{dx} = \mathbf{g}(\mathbf{y}) , \quad (55)$$

where

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 0 & 0 & 1 \\ A_{11} & A_{12} & A_{13} & 0 \\ C_{11} & C_{12} & C_{13} & C_{14} \\ D_{11} & D_{12} & D_{13} & 0 \end{bmatrix}, \quad \mathbf{y} \equiv \begin{bmatrix} u_f(x) \\ \delta(x) \\ \zeta(x) \\ \pi(x) \end{bmatrix}, \quad \text{and} \quad \mathbf{g}(\mathbf{y}) \equiv \begin{bmatrix} \zeta \\ 0 \\ \frac{1}{Re_0} \phi_2(u_f, \delta) \\ \phi_3(\pi, \delta) \end{bmatrix}.$$

For computational convenience, (55) is rewritten as

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(\mathbf{y}) \equiv \mathbf{A}^{-1} \mathbf{g}(\mathbf{y}), \quad (56)$$

where $\mathbf{A}^{-1}$ is symbolically made explicit by Kramer's rule and the non-singularity condition $\det \mathbf{A} \neq 0$ is verified computationally at each step. The initial conditions for (56) come from prescribed inlet mass and momentum fluxes. The approximately uniform inlet conditions (46), when substituted in (A.1)–(A.5) of the Appendix, are seen to be satisfied if and only if

$$\lim_{x \rightarrow 0} \frac{u_f(x)}{\delta(x)} = \frac{\mu_2}{\mu_1} A (1 - e^{-A}),$$

and

$$\zeta(0) = -\frac{1}{Re_0} A^2 (1 + e^{-A}). \quad (57)$$

Recalling that p_0 is the inlet pressure, and using the definitions in (13) and (30), we find

$$\pi(0) = 0. \quad (58)$$

Because of (57), (58) and (28), we impose the initial conditions at a preassigned arbitrarily small $x=\epsilon_\delta$ as

$$y(\epsilon_\delta) = \begin{bmatrix} \frac{\mu_2}{\mu_1} A (1-\exp(-A)) \delta^* \\ \delta^* \\ -A^2(1+\exp(-A))/Re_0 \\ 0 \end{bmatrix}, \quad (59)$$

where δ^* is also some preassigned arbitrarily small number. Numerical solution of (56) subject to (59) was obtained by a fifth and a sixth order Runge Kutta method until the two solutions agreed with each other up to a desired level of accuracy (six decimal places). This implementation was automatically done by an IMSL subroutine called DVERK. We also study the effects of change in inlet vapor velocity profile for a given mass flow rate. This is done either by varying A in (46) or by considering parabolic inlet vapor velocity profiles given by (47). With the help of (A.1) – (A.5); it can be verified that (47) is approximately satisfied at some small $x = \epsilon_\delta$ if $\delta(\epsilon_\delta) \equiv \delta^*$ is *sufficiently small* and

$$y(\epsilon_\delta) = \begin{bmatrix} \frac{0.5Re_0D_0}{\left\{ \frac{\mu_1}{\mu_2} \frac{1}{\delta^*} - \frac{1}{(\delta^*-1)} \right\}} \\ \delta^* \\ -D_0 \\ 0 \end{bmatrix}. \quad (60)$$

Numerical integration of (56) subject to (60) then solves the problem for parabolic inlet vapor velocity profiles.

6. Numerical Results and Discussions:

The numerical solution obtained for $u_f(x)$, $\delta(x)$, $\zeta(x)$, and $\pi(x)$ completely determine the condensate velocity and temperature profiles given by (33) and (34). This in turn gives the

condensation rate through (29) or the right side of (36). Other bulk observables of interest are the non-dimensional values of the bottom plate shear

$$\begin{aligned}
 C_f(x) &\equiv \frac{\tau_w^1(x)}{(\rho_1 U^2/2)} \equiv \frac{\mu_1 \left. \frac{\partial u_1}{\partial y} \right|_{y=0}}{(\rho_1 U^2/2)} \\
 &= -\frac{\rho_2}{\rho_1} \left\{ \frac{d\pi}{dx}(x) \delta(x) \right\} + \frac{1}{Re_0} \frac{v_1}{v_2} \frac{2u_f(x)}{\delta(x)} ;
 \end{aligned} \tag{61}$$

non-dimensional values of local heat transfer rates given through the definitions

$$q_w = k_1 \left. \frac{\partial T_1}{\partial y} \right|_{y=0} \tag{62}$$

$$= \frac{k_1}{h} \frac{T_w}{\delta(x)} \left. \frac{\partial T_1}{\partial y} \right|_{y=0} \equiv h (T_s(p_0) - T_w),$$

$$\text{and } Nu_x \equiv \frac{h x}{k_1} = \frac{x \{ T_s(\pi(x)) - 1 \}}{\delta(x)} \cdot \left\{ \frac{T_w}{T_s(p_0) - T_w} \right\} ;$$

and the fraction of total mass flow rate condensed into liquid mass flow rate as given by

$$\begin{aligned}
 \% \text{ condensed } (x) &\equiv 100 \frac{\rho_1 \int_0^{\Delta(x)} u_1(x, y) dy}{\rho_2 \int_0^h u_2(0, y) dy} \\
 &= 100 \frac{\rho_1 \int_0^{\delta} u_1(x, y) dy}{\rho_2 \int_0^1 u_2(0, y) dy} .
 \end{aligned} \tag{63}$$

At this point, we would like to relate the Reynold's number $Re_0 \equiv Uh/v_2$ to characteristic Reynold's number

$$\text{Re}^* \equiv V_{\text{av}} h / \nu_2,$$

where

$$\begin{aligned} V_{\text{av}} &\equiv \frac{1}{h} \int_0^h u_2(0, y) \, dy \\ &= U \int_0^1 u_2(0, y) \, dy \end{aligned} \quad (64)$$

Using the definition in (64), it can be verified that approximately uniform inlet profiles (46) give

$$\text{Re}^* = \text{Re}_0 \left\{ 1 - \frac{2}{A} + e^{-A} \left(1 - \frac{2}{A} \right) \right\}, \quad (65)$$

and parabolic inlet velocity profiles (47) give

$$\text{Re}^* = \text{Re}_0 \left\{ \frac{1}{12} \text{Re}_0 D_0 \right\}. \quad (66)$$

We now consider a few typical flow situations given in Table 1 for R-113. Other flows involving R-12, Steam etc. can be similarly studied but they are not presented here for brevity. In Table 1, cases *c1a* and *c1b* are identical except for a small change in the value of the parameter A . Similarly, cases *c1a* and *c1ap* are identical except for significantly different inlet velocity profiles. The cases *c1a* and *c1c* are expected to indicate changes in bulk observables due to a change in $(\mathcal{T}_s(p_0) - \mathcal{T}_w) / \mathcal{T}_w$ while all other parameters remain the same. The cases *c1a*, *c2*, and *c3* are chosen to show the more significant effects of increasing Reynold's number (Re^*). The predictions of bulk observables $\delta(x)$, $u_f(x)$, $\pi(x)$, $\dot{m}(x)$, $C_f(x)$, Nu_x , % condensed (x), and $u_1(x, y)$ for the first five cases in Table 1 are respectively shown in Figures 2-9.

Table 1: Some Flow Situations Involving R-113.
The constants c_{01} and c_{02} are defined in (A-14) of the Appendix.

case	Inlet Vapor Profile	Flow Conditions Inlet Pressure p_0 (kPa), Saturation Temp. T_s (p0) (K), Wall Temperature T_w (K)	Re_0 (Uh/ν_2)	Re^* ($\nabla_{av} h/\nu_2$)	Ja ($h_{fg}/C_p T_w$)	$\frac{\rho_2}{\rho_1}$	$\frac{\mu_2}{\mu_1}$	c_{01}	c_{02}
<i>cla</i>	Uniform, $A = 10$	101.3 kPa, 320.56 K, 290.57 K	5371.16	4296.92	0.5055	4.95×10^{-3}	0.02087	7.14×10^{-6}	1.103
<i>clb</i>	Uniform, $A = 15$	101.3 kPa, 320.56 K, 290.57 K	4958.36	4296.92	0.5055	4.95×10^{-3}	0.02087	7.14×10^{-6}	1.103
<i>clc</i>	Uniform, $A = 10$	101.3 kPa, 320.56 K, 300.57 K	5371.16	4296.92	0.5055	4.95×10^{-3}	0.02087	7.14×10^{-6}	1.103
<i>c2</i>	Uniform, $A = 10$	101.3 kPa, 320.56 K, 290.56 K	7521.61	6017.29	0.5055	4.95×10^{-3}	0.02087	1.4×10^{-5}	1.103
<i>c3</i>	Uniform, $A = 10$	101.3 kPa, 320.56 K, 297.56 K	10282.15	8225.7	0.4913	4.95×10^{-3}	0.02087	2.62×10^{-5}	1.103
<i>clap</i>	Parabolic, $D_0 = 10^{-3}$, $A = 5.0$	101.3 kPa, 320.56 K, 290.57 K	7190.74	4296.92	0.5055	4.95×10^{-3}	0.02087	3.26×10^{-5}	1.103

Fig. 2: The condensate thickness profiles $\delta(x)$ for different cases in Table 1. For case $c3$ above, the values of δ at x equal to 1, 5, and 8 are respectively 0.0133, 0.0223, and 0.0316. This agrees within 29% with a more approximate earlier prediction [16] of δ of 0.0194, 0.030, and 0.029 at the same respective values of x .

Fig. 3: The speed at the interface $u_1(x, \delta(x)) = u_f(x)$ for different cases in Table 1.

Fig. 4: Decreasing non-dimensional pressure $\pi(x)$ for different cases in Table 1.

Fig. 5: The interfacial local condensation rate $\dot{m}(x)$ for different cases in Table 1.

Fig. 6: The non-dimensional wall shear stress for the bottom plate for different cases in Table 1.

Fig. 7: The Nusselt number (see (62)) for different cases in Table 1. For case $c3$ above, the values of Nu_x at x equal to 1, 5, and 8 are respectively 72.69, 221.77, and 306.22. This agrees within 33% with a more approximate earlier prediction [16] of Nu_x of 54.84, 166.67, and 264.51 at the same respective values of x .

Fig. 8: The percentage fraction of inlet mass flow rate condensed into liquid flow rate.

Fig. 9: The liquid condensate velocity profile $u_1(x, y)$ for case $c1a$ of Table 1 at different fixed values of x and $0 \leq y \leq \delta(x)$.

As expected, in Fig. 2, we find that the condensate thins with increasing Reynold's number. We also find (compare cases *c1a* and *c1c*) that lowering the temperature difference ($\mathcal{T}_s(p_0) - \mathcal{T}_w$) tends to thin the condensate layer for the cases under consideration. Thinning of the condensate layer implies that the no-slip condition dominates and the interfacial speed $u_f(x)$ reduces whenever $\delta(x)$ reduces (see Fig. 3). We also find, in Fig. 4, that increasing Reynold's number (cases *c1a*, *c2*, and *c3*) associated with thinner condensate layers lead to a smaller drop in the *non-dimensional* pressure $\pi(x)$ but a higher drop in the *physical* pressure p . However, decreasing condensate thickness (cases *c1a* and *c1c*) due to a drop in temperature difference ($\mathcal{T}_s(p_0) - \mathcal{T}_w$) seems to accompany a very slight increase (within numerical error) in pressure drop. In Figure 5, we find that increasing Reynold's number leads to lower non-dimensional mass transfer rates at the interface.

The wall friction on the bottom plate as measured by $C_f(x)$ in Fig. 6 is seen to remain fairly constant with x and it decreases with increasing Reynolds number. This feature of $C_f(x)$ is consistent with the expression in (61). In Figure 9, the slope $\partial u_1 / \partial y \equiv u_f(x) / \delta(x)$ is approximately independent of x as the pressure gradient $\zeta(x)$ does not significantly affect the velocity profile of liquid. This means an almost constant value of physical shear stress across and along the condensate flow.

The reader should be careful in interpreting Nusselt number graph in Fig. 7 for the different cases. If the temperature difference ($\mathcal{T}_s(p_0) - \mathcal{T}_w$) is same, as in cases *c1a* and *c2*, an increase in Nu_x denotes increase in heat transfer rate. However, although Nu_x for case *c1c* is larger than the case *c1a*, the physical heat transfer rates calculated from (62) gives, a smaller value for the case *c1c*.

The fraction of total mass flux condensed into liquid form at any x increases with the thickness and the associated increase in the speed at the interface (see Figure 3). The curves in Figure 8 are consistent with this observation.

The proximity of the curves for cases *c1a* and *c1b* in all the Figures 2-8 indicates the regularity that a small change in inlet vapor velocity profile for the same mass flow rate (Re^*) leads to small changes in the values of the bulk observables.

Because of unavailability of reliable experimental data for this problem, we are unable to make a firm comparison of our predictions with experiments. However we note that some ongoing experiments [15] deal with condensing flow of pure R-113 vapor on the bottom plate of a horizontal duct of *rectangular* cross-section. Since the above mentioned experiments are still being repeated, we only mention that currently available data has enough scatter to have no clear trends with regard to effects of changes in Reynold's number and temperature difference ($\mathcal{T}_s(p_0) - \mathcal{T}_w$). Perhaps, as a result, we only find an order of magnitude agreement (within 120%) with regard to film thickness, heat transfer rates, etc. with this related, but different, experiment [15]. Since some other cases show considerable scatter, we do not attempt a comparison and a comprehensive parametric study until published reliable experimental data is available.

To indicate the reasonableness of the results in Figures 2-7, we have compared the predictions of film thickness (see caption of Figure 2) and heat transfer rates (see caption of Figure 7) with an earlier more approximate solution [16]. The earlier solution [16] ignored the effects of pressure drop in the flow by assuming pressure gradient to be zero. This, naturally, led to physically unrealistic vapor profiles. Despite this, perhaps because the flow of the heavier condensate is not significantly affected by the pressure gradient (Figure 9), we find a reasonable agreement between the two numerical solutions with regard to film thickness and heat transfer rates. We also find an order of magnitude agreement between the two solutions with regard to other bulk observables.

7. Numerical Accuracies:

It was numerically verified that our computations satisfied all the differential equations (51)-(54) to an acceptable accuracy. In particular, numerically evaluated values of the left side of

the mass balance equation (51) is shown in Table 2. Despite this, an integration of (51) led to a value of I_1 in (49) which actually increased five folds over $0 < x < 5$ and only for $x > 5$, I_1 remained approximately constant in accord with (35) and values of $\dot{m}$ in Figure 5.

The reason for the above mentioned inaccuracy lies in the fact that our predictions of u_f , δ , ζ , and π could not be improved beyond an accuracy of a fourth or a fifth decimal place (10^{-4} ~ 10^{-5}). This is partly because of limitations associated with round-off errors in very long expressions occurring in the function subprograms (see Appendix) and partly due to limitations on the fineness of the step size. While this limitation retains the accuracy of our predictions in the previous section to within 10-20% of what the model should theoretically predict, the error in function b_2 appearing in (39) is large. In Table 3, we see that b_2 itself is an order one quantity (with a boundary layer behavior near $x = 0$), and its derivatives are large and therefore the approximate magnitude of its error (Δb_2) as given by

$$O(\Delta b_2) \sim O\left(\frac{\partial b_2}{\partial u_f} \Delta u_f\right) \quad (67)$$

is also order one. In (67) above, the error Δu_f associated with u_f is order 10^{-4} to 10^{-5} . This explains the stated inaccuracy in the integral behavior of $u_2(x, y)$ under representation (39). However it must be noted that order one error in b_1 or b_2 does not affect our predictions of the vector y in (55) in any significant way. This is because b_1 and b_2 enter the solution scheme only through some of the terms in the coefficients of the matrix A in (55). These terms in turn are affected only through the derivatives defined in (A.9) of the Appendix-A and their values for a typical case *cl a* of Table 1 is shown in Tables 4 and 5. Since the derivatives in Tables 4 and 5 are affected by *less* than 1% due to the limited accuracy of 10^{-4} to 10^{-5} in u_f , δ , ζ , or π ; our predictions of bulk observables in section 6 remain good within 10-20%.

To further exhibit the robustness of our scheme with regard to its accuracy in predicting the bulk observables (such as the interface $\delta(x)$, etc.), we consider totally different inlet conditions

(cases *cla* and *clap*) and associated initial conditions (replacing (59) by (60)) at the same mass flow rate (or Reynold's number Re^*). From physical considerations, one expects that the condensation and heat transfer rates should not be too far apart for these two cases. We found that this is indeed the case and, for brevity, we present here the comparisons (within 15% for most x) for two of the significant flow variables in Fig. 10.

Table 2: Table illustrating approximate satisfaction of the Mass Balance Condition (51). The column $(LHS)_{51}$ indicates the value of left hand side of equation (51).

x	$(LHS)_{51}$
0.1000E-07	0.6159E-09
0.1000E-03	0.3563E-09
0.1000E-01	-0.2047E-12
0.2000E-01	-0.2627E-12
0.3000E-01	+0.2176E-13
0.4000E-01	+0.1599E-13
0.5000E-01	-0.1310E-12
0.1000E+01	+0.4501E-15
0.5000E+01	-0.3096E-15
0.1000E+02	-0.1605E-15
0.2000E+02	+0.9020E-16
0.3000E+02	0.1648E-16

Table 3: Values of b_2 and its derivatives for case $cl a$ of Table 1.

x	b_2	$\partial b_2 / \partial u_f$	$\partial b_2 / \partial \delta$	$\partial b_2 / \partial \zeta$
0.1000E-07	-0.0399	-0.5979E+05	0.1244E+05	-0.6680E+03
0.1000E-03	-0.3390	-0.4390E+05	0.9912E+04	-0.6668E+03
0.1000E-01	-2.1168	-0.1266E+05	0.4400E+04	-0.6565E+03
0.2000E-01	-2.5837	-0.1036E+05	0.3886E+04	-0.6534E+03
0.3000E-01	-2.8980	-0.9175E+04	0.3601E+04	-0.6513E+03
0.4000E-01	-3.1430	-0.8434E+04	0.3416E+04	-0.6494E+03
0.5000E-01	-3.3472	-0.7895E+04	0.3277E+04	-0.6484E+03
0.1000E+01	-8.1742	-0.3176E+04	0.1782E+04	-0.6226E+03
0.2000E+01	-10.3030	-0.2552E+04	0.1518E+04	-0.6144E+03
0.5000E+01	-14.2833	-0.1897E+04	0.1209E+04	-0.6027E+03
0.1000E+02	-18.5706	-0.1504E+04	0.1004E+04	-0.5940E+03
0.2000E+02	-24.3799	-0.1184E+04	0.8207E+03	-0.5867E+03
0.3000E+02	-28.6389	-0.1025E+04	0.7233E+03	-0.5840E+03

Table 4: The derivatives of I_1 as defined in (A.9) of the Appendix. The fourth column involving $\partial I_1/\partial \zeta$ was evaluated explicitly as a functions of u_f, δ, ζ and therefore it does not depend on the values of b_1 and b_2 .

x	$\partial I_1/\partial u_f$	$\partial I_1/\partial \delta$	$\partial I_1/\partial \zeta$
0.1000E-07	0.8991E+04	-0.2148E+04	0.5768E+02
0.1000E-03	0.6598E+04	-0.1743E+04	0.5750E+02
0.1000E-01	0.1891E+04	-0.8110E+03	0.5594E+02
0.2000E-01	0.1545E+04	-0.7188E+03	0.5548E+02
0.3000E-01	0.1366E+04	-0.6675E+03	0.5515E+02
0.4000E-01	0.1255E+04	-0.6340E+03	0.5492E+02
0.5000E-01	0.1173E+04	-0.6086E+03	0.5472E+02
0.1000E+01	0.4625E+03	-0.3356E+03	0.5079E+02
0.5000E+01	0.2700E+03	-0.2331E+03	0.4767E+02
0.1000E+02	0.2109E+03	-0.1976E+03	0.4621E+02
0.2000E+02	0.1628E+03	-0.1671E+03	0.4482E+02
0.3000E+02	0.1389E+03	-0.1513E+03	0.4413E+02

Table 5: The derivatives of I_2 as defined in (A.9) of the Appendix.

x	$\partial I_2/\partial u_f$	$\partial I_2/\partial \delta$	$\partial I_2/\partial \zeta$
0.1000E-07	-0.4122E+09	-0.5927E+06	0.1593E+05
0.1000E-03	-0.2761E+09	-0.4396E+06	0.1452E+05
0.1000E-01	-0.4707E+08	-0.1228E+06	0.8522E+04
0.2000E-01	-0.3469E+08	-0.9855E+05	0.7662E+04
0.3000E-01	-0.2878E+08	-0.8606E+05	0.7171E+04
0.4000E-01	-0.2526E+08	-0.7827E+05	0.6842E+04
0.5000E-01	-0.2279E+08	-0.7260E+05	0.6591E+04
0.1000E+01	-0.5382E+07	-0.2479E+05	0.3850E+04
0.5000E+01	-0.2446E+07	-0.1383E+05	0.2956E+04
0.1002E+02	-0.1766E+07	-0.1100E+05	0.2722E+04
0.2000E+02	-0.1292E+07	-0.8956E+04	0.2581E+04
0.3000E+02	-0.1079E+07	-0.8008E+04	0.2535E+04

Fig. 10: The two lower curves in the Figure above compares $u_f(x)$ for cases *c1a* and *c1ap* in Table 1. The upper two curves compares $\delta(x)$ for the same two cases.

8. Conclusions

A mixed differential-integral approach was used for predicting bulk observables (condensation rate, heat transfer rate, film thickness, etc.) in pressure driven laminar film condensation flows inside ducts. The method also seems promising for providing a good first guess of interface location for a Finite Difference or a Finite Element scheme.

An increase in Reynold's number (higher inlet mass flow rate) with temperature difference ($\mathcal{T}_s(p_0) - \mathcal{T}_w$) remaining constant typically leads to thinner condensates, larger drops in *physical* pressures, smaller drops in *non-dimensional* pressure, smaller non-dimensional interfacial mass transfer rates, and an increased heat removal rate from the bottom plate. A decrease in temperature difference ($\mathcal{T}_s(p_0) - \mathcal{T}_w$) at a constant Reynold's number (Re^*) leads to thinner condensates, reduced physical heat transfer rates, and increased Nusselt number (a non-dimensional heat transfer rate). The physical value of shear stress is found to be nearly constant across the condensate as well as along the condensate.

This method has the limitation that the non-linearities of the various functions are hidden in computer generated functions and therefore one does not know their behavior until after the computations. As a result, in section 7, we found that numerical accuracy in some variables ($u_2(x,y)$) had to be compromised for numerical accuracy in other variables. However the predicted variables have the robustness that they are not very sensitive to changes in the inlet vapor profile at a given mass flow rate while they are sensitive to changes in mass flow rate.

Appendix

Substituting (39) in (43), one obtains

$$b_3(x) = b_1(x) \{A^2(1 + \exp(-A))\} + \text{Re}_0 \zeta \equiv \theta_1 (b_1, \zeta) . \quad (\text{A.1})$$

Substituting (39) in (42) and using (A.1), we find

$$b_2(x) = \frac{1}{(\delta-1)} \left[u_f(x) - b_1(x) \exp(\delta) - \frac{\text{Re}_0}{2} \zeta (\delta-1)^2 \right] \equiv \theta_2 (u_f, \delta, \zeta, b_1) , \quad (\text{A.2})$$

where

$$\exp(\delta) \equiv \left[1 - \exp(-A\delta) - \exp(-A(1-\delta)) + \exp(-A) + (A^2/2) (1 + \exp(-A))(\delta-1)^2 \right] .$$

Substituting (39) in (41) and using (A.1)–(A.2) gives

$$b_1(x) = \frac{1}{\text{den}(\delta)} \left[\frac{1}{2} \text{Re}_0 \zeta(x) + \frac{\mu_1}{\mu_2} \frac{u_f(x)}{\delta(x)} - \frac{u_f(x)}{(\delta(x)-1)} \right] \equiv b_1 (u_f, \delta, \zeta) , \quad (\text{A.3})$$

where

$$\begin{aligned} \text{den}(\delta) \equiv & \left[A \{ \exp(-A\delta) - \exp(-A(1-\delta)) \} \right. \\ & - \frac{1}{(\delta-1)} \{ 1 - \exp(-A\delta) - \exp(-A(1-\delta)) + \exp(-A) \} \\ & \left. + A^2 (1 + \exp(-A)) ((\delta-1)/2) \right] \end{aligned}$$

Substituting (A.3) in (A.2), we find

$$\begin{aligned} b_2 &= \theta_2 (u_f, \delta, \zeta, b_1 (u_f, \delta, \zeta)) \\ &\equiv b_2 (u_f, \delta, \zeta) . \end{aligned} \quad (\text{A.4})$$

Substituting (A.3) in (A.1), we find

$$\begin{aligned}
b_3 &= \theta_1 (b_1 (u_f, \delta, \zeta), \zeta) \\
&\equiv b_3 (u_f, \delta, \zeta) .
\end{aligned} \tag{A.5}$$

Substituting (A.1) in (39), we write

$$\begin{aligned}
u_2(x, y) &= b_1(x) \{1 - \exp(-Ay) - \exp(-A(1-y)) + \exp(-A) + A^2 (1+\exp(-A))((y-1)^2/2)\} \\
&\quad + b_2(x) (y-1) + Re_0 \zeta (y-1)^2/2 \\
&\equiv \phi (b_1, b_2, \zeta ; y)
\end{aligned} \tag{A.6}$$

First substituting (A.6) in the left side of (49) and subsequently substituting (A.3) and (A.4) for b_1 and b_2 , we find

$$\begin{aligned}
I_1 &= \int_{\delta(x)}^1 \phi (b_1, b_2, \zeta ; y) dy \\
&\equiv \hat{I}_1 (b_1, b_2, \delta_1, \zeta) \\
&= \hat{I}_1 (b_1 (u_f, \delta, \zeta), b_2(u_f, \delta, \zeta), \zeta, \delta) \\
&\equiv I_1 (u_f, \delta, \zeta) .
\end{aligned} \tag{A.7}$$

Similarly (A.6) and (50) are related by

$$\begin{aligned}
I_2 &= \int_{\delta(x)}^1 \phi^2 (b_1, b_2, \zeta ; y) dy \\
&\equiv \hat{I}_2 (b_1, b_2, \zeta, \delta) \\
&= \hat{I}_2 (b_1 (u_f, \delta, \zeta), b_2(u_f, \delta, \zeta), \zeta, \delta) \\
&\equiv I_2 (u_f, \delta, \zeta) .
\end{aligned} \tag{A.8}$$

Denoting I_1 and I_2 by I_i , with $i=1$ or 2 , we have

$$\begin{aligned}
\frac{\partial I_i}{\partial u_f} &= \frac{\partial \hat{I}_i}{\partial b_1} \frac{\partial b_1}{\partial u_f} + \frac{\partial \hat{I}_i}{\partial b_2} \frac{\partial b_2}{\partial u_f} , \\
\frac{\partial I_i}{\partial \delta} &= \frac{\partial \hat{I}_i}{\partial b_1} \frac{\partial b_1}{\partial \delta} + \frac{\partial \hat{I}_i}{\partial b_2} \frac{\partial b_2}{\partial \delta} + \frac{\partial \hat{I}_i}{\partial \delta} ,
\end{aligned}$$

and

$$\frac{\partial I_i}{\partial \zeta} = \frac{\partial \hat{I}_i}{\partial b_1} \frac{\partial b_1}{\partial \zeta} + \frac{\partial \hat{I}_i}{\partial b_2} \frac{\partial b_2}{\partial \zeta} + \frac{\partial \hat{I}_i}{\partial \zeta}. \quad (\text{A.9})$$

It is easy to verify that the coefficients in (51) are given by

$$\begin{aligned} A_{11}(u_f, \delta, \zeta) &= \frac{\rho_2}{\rho_1} \frac{\partial I_1}{\partial u_f} + \frac{1}{2} \delta \\ A_{12}(u_f, \delta, \zeta) &= \frac{\rho_2}{\rho_1} \frac{\partial I_1}{\partial \delta} + \left\{ \frac{1}{2} u_f - 3B\delta^2 \zeta \right\} \\ A_{13}(u_f, \delta, \zeta) &= \frac{\rho_2}{\rho_1} \frac{\partial I_1}{\partial \zeta} - B\delta^3, \end{aligned} \quad (\text{A.10})$$

where $B \equiv \frac{1}{12} \frac{\mu_2}{\mu_1} \text{Re}_0$.

Similarly we find that coefficients in (52) are given by

$$\begin{aligned} C_{11}(u_f, \delta, \zeta) &\equiv \frac{\partial I_2}{\partial u_f} + \frac{\rho_1}{\rho_2} \left(\frac{1}{2} u_f \delta \right), \\ C_{12}(u_f, \delta, \zeta) &\equiv \frac{\partial I_2}{\partial \delta} - \frac{\rho_1}{\rho_2} (3B\delta^2 u_f \zeta) + \frac{\rho_1}{\rho_2} \left(\frac{1}{2} u_f^2 \right), \\ C_{13}(u_f, \delta, \zeta) &\equiv \frac{\partial I_2}{\partial \zeta} \frac{\rho_1}{\rho_2} - B\delta^3 u_f, \\ C_{14}(u_f, \delta, \zeta) &\equiv (1-\delta) - \frac{1}{\text{Re}_0} \phi_1(\delta), \end{aligned} \quad (\text{A.11})$$

where

$$\begin{aligned} B &\equiv \frac{1}{12} \frac{\mu_2}{\mu_1} \text{Re}_0, \text{exprn}(\delta) \equiv A(\exp(-A)-1) - A \exp(-A\delta) + A \exp(-A(1-\delta)), \\ \phi_1(\delta) &\equiv \frac{1}{2} \cdot \frac{\text{Re}_0}{\text{den}(\delta)} \{ \text{exprn}(\delta) + A^2(1+\exp(-A))(1-\delta) \} + \text{Re}_0 (1-\delta). \end{aligned}$$

Here $\text{den}(\delta)$ in the definition of ϕ_1 above is the same as in (A.3). Furthermore, using $\text{den}(\delta)$, we define ϕ_2 on the right side of (52) as

$$\phi_2 \equiv \frac{1}{\text{den}(\delta)} \left\{ \frac{\mu_1}{\mu_2} \frac{u_f}{\delta} - \frac{u_f}{(\delta-1)} \right\} \{ \text{exprn}(\delta) + A^2(1+\exp(-A))(1-\delta) \} \quad (\text{A.12})$$

The coefficients and functions in (53) are easily seen to be

$$\begin{aligned} D_{11}(u_f, \delta, \zeta) &\equiv \frac{1}{2}\delta, \\ D_{12}(u_f, \delta, \zeta) &\equiv \frac{1}{2}u_f - 3B\delta^2\zeta, \\ D_{13}(u_f, \delta, \zeta) &\equiv (-B\delta^3), \end{aligned}$$

and

$$\phi_3(\pi(x), \delta(x)) \equiv T_s(\pi(x)), \quad (\text{A.13})$$

where B is same as in (A.10) and $T_s(\pi(x))$ is same as in (23). An explicit form of $T_s(\pi(x))$ for flows of R-113 described in Table 1 can be obtained by curve-fitting the saturation temperature data ([23]). Assuming that the absolute pressure (p) varies in the range $0.3216 \text{ MPa} \leq p \leq 1.013 \text{ MPa}$, $T_s(p)$ is quadratically approximated around $p = p^* = 100 \text{ kPa}$ to yield

$$\begin{aligned} T_s(\pi(x)) &= \frac{267.02}{T_w} + \left(\frac{79.97}{T_w} \right) (c_{01}\pi(x) + c_{02}) \\ &\quad - \left(\frac{26.632}{T_w} \right) (c_{01}\pi(x) + c_{02})^2, \end{aligned} \quad (\text{A.14})$$

where T_w is in Kelvins, $c_{01} \equiv \rho_2 U^2 / p^*$, and $c_{02} \equiv p_0 / p^*$. Some typical values of C_{01} and C_{02} are given in Table 1.

We now complete the definition in (A.9) by attaching the computer generated functions under the following names.

$$\begin{aligned} \frac{\partial \hat{I}_1}{\partial b_1} &\equiv \text{I1DB1}, \frac{\partial \hat{I}_1}{\partial b_2} \equiv \text{I1DB2}, \frac{\partial \hat{I}_1}{\partial \zeta} \equiv \text{I1DZE}, \frac{\partial \hat{I}_1}{\partial \delta} \equiv \text{I1DD}, \frac{\partial \hat{I}_2}{\partial b_1} \equiv \text{I2DB1}, \frac{\partial \hat{I}_2}{\partial b_2} \equiv \text{I2DB2}, \\ \frac{\partial \hat{I}_2}{\partial \zeta} &\equiv \text{I2DZE}, \frac{\partial \hat{I}_2}{\partial \delta} \equiv \text{I2DD}, \frac{\partial b_1}{\partial u_f} \equiv \text{B1DUF}, \frac{\partial b_1}{\partial \zeta} \equiv \text{B1DZE}, \frac{\partial b_1}{\partial \delta} \equiv \text{B1DD}, \frac{\partial b_2}{\partial u_f} \equiv \text{B2DUF}, \\ \frac{\partial b_2}{\partial \zeta} &\equiv \text{B2DZE}, \frac{\partial b_2}{\partial \delta} \equiv \text{B2DD}, \end{aligned}$$

$$Re_0 \equiv REI, \mu_2/\mu_1 \equiv MU21, \delta \equiv D, \zeta \equiv ZE \quad (A.15)$$

$$u_f \equiv UF, b_1 \equiv B1, b_2 \equiv B2$$

Using the definitions in (A.15), we now list below the Double Precision Fortran Function Subprograms for the derivatives in (A.9).

```
DOUBLE PRECISION FUNCTION I1DB1(B1,B2,D,ZE,A,REI)
IMPLICIT DOUBLE PRECISION (A-Z)
I1DB1 = 1.D0/2.D0*(2.D0*A+2.D0*EXP(-A)+2.D0*EXP(-A)*A-2.D0)/A
+-1.D0/6.D0*(-6.D0*EXP(A*(D-1.D0))+
+A**3*EXP(-A)*D**3-3.D0*A**3*EXP(-A)*D**2
+-A**3+6.D0*EXP(-A)*D*A+3.D0*A**3*EX
+P(-A)*D+A**3*D**3+6.D0*EXP(-D*A)-A**3*EXP(-A)+3.D0
+A**3*D-3.D0*A**3*D**2+6.D0*D*A)/A
RETURN
END
```

*****:

```
DOUBLE PRECISION FUNCTION I1DB2(B1,B2,D,ZE,A,REI)
IMPLICIT DOUBLE PRECISION (A-Z)
I1DB2 = -1.D0/2.D0+1.D0/6.D0*(6.D0*D*A-3.D0*D**2*A)/A
RETURN
END
```

```
DOUBLE PRECISION FUNCTION I1DD(B1,B2,D,ZE,A,REI)
IMPLICIT DOUBLE PRECISION (A-Z)
I1DD = -1.D0/6.D0*(-6.D0*A*EXP(A*(D-1.D0))*B1+6.D0*B2
+*D*A+3.D0*A**3*EXP(-A)*D**2*B1
+-6.D0*A**3*EXP(-A)*D*B1+6.D0*EXP(-A)*A*B1+3.D0*A
+**3*EXP(-A)*B1+3.D0*A**3*D**2*
+B1+3.D0*REI*ZE*D**2*A-6.D0*REI*ZE*D*A+3.D0*REI
+*ZE*A-6.D0*A*EXP(-D*A)*B1-6.D0*B2*A
++3.D0*A**3*B1-6.D0*A**3*D*B1+6.D0*A*B1)/A
RETURN
END
```

```
DOUBLE PRECISION FUNCTION I1DZE(B1,B2,D,ZE,A,REI)
IMPLICIT DOUBLE PRECISION (A-Z)
I1DZE = 1.D0/6.D0*(3.D0*REI*D**2*A-REI
+*D**3*A+REI*A-3.D0*REI*D*A)/A
RETURN
END
```

```

DOUBLE PRECISION FUNCTION I2DB1(B1,B2,D,ZE,A,REI)
IMPLICIT DOUBLE PRECISION (A-Z)
I2BB1A = -1.D0/4.D0*1.D0/EXP(A)*(-8.D0*EXP(A)+A**4*EXP(A)+4.D0
+*A**2+4.D0*EXP(A)*A**2+A**4-8.D0)*B2/A**2+1.D0/4.D0*(4.D0*A**4*D
+*EXP(A*D)*EXP(A)+8.D0*EXP(A*D)**2*A*D-6.D0*A**4*D**2*EXP(A*D)
+*EXP(A)-8.D0*A*D*EXP(A)-A**4*D**4*EXP(A*D)+4.D0*A**4*D**3
+*EXP(A*D)*EXP(A)-8.D0*EXP(A)+4.D0*A**4*D**3*EXP(A*D)-8.D0
+*EXP(A*D)**2*A-A**4*D**4*EXP(A*D)*EXP(A)+8.D0*D*EXP(A*D)
+*A**2-8.D0*EXP(A*D)**2-4.D0*D**2*EXP(A*D)*A**2-6.D0*A**4*D**2*E
+XP(A*D)+4.D0*A**4*D*EXP(A*D)+8.D0*D*EXP(A*D)*A**2*EXP(A)-4.D0*D**2
+*EXP(A*D)*A**2*EXP(A)+8.D0*A*EXP(A))*B2/EXP(A*D)/A**2/EXP(A)
I2BB1B = 1.D0/30.D0*1.D0/EXP(A)*(3.D0*A**5*EXP(A)+60.D0-60.D0
+*EXP(A)+3.D0*A**5+10.D0*A**3+10.D0*EXP(A)*A**3)*REI*ZE/A**3+1.D0
+/30.D0*(60.D0*A**2*D*EXP(A)-30.D0*A**2*D**2*EXP(A)-3.D0*A**5*D**5
+*EXP(A*D)*EXP(A)+60.D0*EXP(A*D)**2-60.D0*EXP(A*D)**2*A*D+60.D0
+*EXP(A*D)**2*A-15.D0*A**5*D*EXP(A*D)*EXP(A)+30.D0*D**2*EXP(A*D)
+*A**3*EXP(A)-15.D0*A**5*D*EXP(A*D)+30.D0*EXP(A*D)**2*A
+*A**2-30.D0*EXP(A)*A**2-30.D0*D*EXP(A*D)*A**3*EXP(A)-60.D0*A*D
+*EXP(A)+60.D0*A*EXP(A)-10.D0*D**3*EXP(A*D)*A**3*EXP(A)+30.D0*A
+*A**5*D**2*EXP(A*D)*EXP(A)-30.D0*A**5*D**3*EXP(A*D)*EXP(
+A)+15.D0*A**5*D**4*EXP(A*D)*EXP(A)+15.D0*A**5*D**4*EXP(A*D)
+-60.D0*EXP(A)-30.D0*D*EXP(A*D)*A**3+30.D0*A**5*D**2*EXP(A*D)
+-10.D0*D**3*EXP(A*D)*A**3-30.D0*A**5*D**3*EXP(A*D)+30.D0*D**2*EX
+P(A*D)*A**3+30.D0*EXP(A*D)**2*A**2*D**2-60.D0*EXP(A*D)**2
+*A**2*D-3.D0*A**5*D**5*EXP(A*D))*REI*ZE/EXP(A*D)/A**3/EXP(A)
I2CB1 = 0.D0
I2A1B1 = 1.D0/30.D0*B1*(60.D0*EXP(-A)**2+3.D0*A**4*EXP(-A)**2
++6.D0*A**4*EXP(-A)+20.D0*A**2*EXP(-A)**2+20.D0*A**2*EXP(-A)
++3.D0*A**4)-1.D0/30.D0*B1*D*(60.D0*EXP(-A)**2+3.D0*A**4*D**4
+-15.D0*A**4*D**3+30.D0*A**4*EXP(-A)+30.D0*A**4*D**2+20.D0*A**2*EX
+P(-A)**2*D**2-30.D0*A**4*D+30.D0*A**4*EXP(-A)**2*D**2+15.D0*A
+*A**4*EXP(-A)**2+60.D0*A**2*EXP(-A)+60.D0*A**2*EXP(-A)**2
+-60.D0*A**4*EXP(-A)*D+3*A**4*EXP(-A)**2*D**4-30.D0*A**4*EXP(-A)*D
+*A**3+15.D0*A**4-30.D0*A**4*EXP(-A)**2*D-60.D0*A**2*EXP(-A)*D+20.D0
+*A**2*EXP(-A)*D**2+6.D0*A**4*EXP(-A)*D**4-60.D0*A**2*EXP(
+-A)**2*D+60.D0*A**4*EXP(-A)*D**2-15.D0*A**4*EXP(-A)**2*D**3)
I2A2B1 = 1.D0/EXP(A)**2*(2.D0*EXP(A)**2*A+4.D0*EXP(A)-3.D0*EXP
+(A)**2-1.D0+4.D0*A*EXP(A))*B1/A-(2.D0*D*EXP(A*D)**2*A*EXP(A)*
+*2+4.D0*EXP(A*D)*EXP(A)**2-4.D0*EXP(A*D)**3*EXP(A)-EXP(A)*
+*2+4.D0*D*EXP(A*D)**2*A*EXP(A)+EXP(A*D)**4)*B1
+/EXP(A*D)**2/A/EXP(A)**2
I2A3B1 = 2.D0/3.D0*1.D0/EXP(A)**2*(6.D0*A*EXP(A)-6.D0*EXP(A)+A
+*A**3*EXP(A)**2+A**3*EXP(A)+12.D0-6.D0*EXP(A)**2)*B1/A-2.D0/3.D0
+*(-3.D0*EXP(A*D)**2*A**2*D**2*EXP(A)-3.D0*A**3*D**2*EXP(A*D)*EXP
+(A)+A**3*D**3*EXP(A*D)*EXP(A)+A**3*D**3*EXP(A*D)*EXP(A)**2+3.D0
+*A**3*D*EXP(A*D)*EXP(A)+6.D0*EXP(A*D)**2*A**2*D*EXP(A)-3.D0
+*A**3*D**2*EXP(A*D)*EXP(A)**2+6.D0*D*EXP(A*D)*A*EXP(A)-6.D0*A**2
+*D*EXP(A)**2+3.D0*A**2*D**2*EXP(A)+3.D0*A**3*D*EXP(A*D)*EXP(A)
+*A**2-6.D0*EXP(A*D)**2*A*EXP(A)+3.D0*A**2*D**2*EXP(A)**2+6.D0*A*D
+*EXP(A)**2+6.D0*A*D*EXP(A)-6.D0*A**2*D*EXP(A)-3.D0*EXP(A*D)**2
+*A**2*D**2-3.D0*EXP(A*D)**2*A**2*EXP(A)+6.D0*EXP(A*D)**2*A
+*A**2*D+6.D0*EXP(A*D)**2*A*D+12.D0*EXP(A)+6.D0*EXP(A)**2-12.D0*EXP
+(A*D)**2+6.D0*D*EXP(A*D)**2*A*EXP(A)-6.D0*A*EXP(A)-6.D0*EXP(A)**2
+*A-6.D0*EXP(A*D)**2*EXP(A)-6.D0*EXP(A*D)**2*A+3.D0*A**2*EXP(A)*
+*2+3.D0*A**2*EXP(A)-3.D0*EXP(A*D)**2*A**2)*B1/EXP(A*D)/A/
+EXP(A)**2
I2DB1 = I2A1B1+I2A2B1+I2A3B1+I2BB1A+I2BB1B+I2CB1
RETURN
END

```



```

DOUBLE PRECISION FUNCTION I2DB2(B1,B2,D,ZE,A,REI)
IMPLICIT DOUBLE PRECISION (A-Z)
I2A1B2 = 0.D0
I2A2B2 = 0.D0
I2A3B2 = 0.D0
I2BB2 = -1.D0/60.D0*B1/EXP(A)*(15.D0*A**5-120.D0*A+15.D0*A**5*
+EXP(A)+60.D0*A**3*EXP(A)-120.D0*A*EXP(A)+60.D0*A**3)/A**3+1.D0
+/60.D0*B1*(-60.D0*A**3*D**2*EXP(A*D)*EXP(A)-90.D0*A**5*D
**2*EXP(A)*EXP(A*D)-60.D0*D**2*EXP(A*D)*A**3+60.D0*A**5*D*E
XP(A)*EXP(A*D)+60.D0*A**5*D*EXP(A*D)+120.D0*A**2*EXP(A)-15.D0*A
**5*D**4*EXP(A)*EXP(A*D)-90.D0*A**5*D**2*EXP(A*D)+60.D0*A**5*D**3
*EXP(A)*EXP(A*D)-120.D0*EXP(A*D)**2*A**2+60.D0*A**5*D**3*EXP(A*D
+)-15.D0*A**5*D**4*EXP(A*D)-120.D0*EXP(A*D)**2*A-120.D0*A*EXP(A
+)-120.D0*A**2*D*EXP(A)+120.D0*D*EXP(A*D)*A**3+120.D0*EXP(A*D
+)**2*A**2*D+120.D0*A**3*D*EXP(A*D)*EXP(A))/EXP(A)/EXP(A*D)
+/A**3
I2CB2 = REI*ZE*D**3+2.D0/3.D0*B2-1.D0/4.D0*REI*ZE*D**4-2.D0*B2*D
++2.D0*B2*D**2+REI*ZE*D-3.D0/2.D0*REI*ZE*D**2-1.D0/4.D0*REI*ZE-
+2.D0/3.D0*B2*D**3
I2DB2 = I2A1B2+I2A2B2+I2A3B2+I2BB2+I2CB2
RETURN
END

```

```

DOUBLE PRECISION FUNCTION I2DD(B1,B2,D,ZE,A,REI)
IMPLICIT DOUBLE PRECISION (A-Z)
I2A1D = -1.D0/60.D0*B1**2*(60.D0*EXP(-A)**2+3.D0*A**4*D**4-15.D0
+A**4*D**3+30.D0*A**4*EXP(-A)+30.D0*A**4*D**2+20.D0*A
**2*EXP(-A)**2*D**2-30.D0*A**4*D+30.D0*A**4*EXP(-A)**2
*D**2+15.D0*A**4*EXP(-A)**2+60.D0*A**2*EXP(-A)+60.D0*A**2*EXP(-A)
**2-60.D0*A**4*EXP(-A)*D+3.D0*A**4*EXP(-A)**2*D**4-30.D0*A**4
*EXP(-A)*D**3+15.D0*A**4-30.D0*A**4*EXP(-A)**2*D-60.D0*A**2*EXP(
-A)*D+20.D0*A**2*EXP(-A)*D**2+6.D0*A**4*EXP(-A)*D**4-60.D0*A
**2*EXP(-A)**2*D+60.D0*A**4*EXP(-A)*D**2-15.D0*A**4*EXP(-A)
**2*D**3)-1.D0/60.D0*B1**2*D*(12.D0*A**4*D**3-45.D0*A**4*D
**2+60.D0*A**4*D+40.D0*A**2*EXP(-A)**2*D-30.D0*A**4+60.D0*A**4
**2*EXP(-A)**2*D-60.D0*A**4*EXP(-A)+12.D0*A**4*EXP(-A)**2*D**3-90.D0
+A**4*EXP(-A)*D**2-30.D0*A**4*EXP(-A)**2-60.D0*A**2*EXP(-A)+40.D0
+A**2*EXP(-A)*D+24.D0*A**4*EXP(-A)*D**3-60.D0*A**2*EXP(-A)*
**2+120.D0*A**4*EXP(-A)*D-45.D0*A**4*EXP(-A)**2*D**2)
I2A2D = -1.D0/2.D0*(2.D0*EXP(A*D)**2*A*EXP(A)**2+4.D0*D*EXP(A
**2)*A**2*EXP(A)**2+4.D0*A*EXP(A*D)*EXP(A)**2-12.D0*EXP(A*D)
**3*EXP(A)*A+4.D0*EXP(A*D)**2*A*EXP(A)+8.D0*EXP(A*D)**2*A**2*D*EXP
(A)+4.D0*EXP(A*D)**4*A*B1**2/EXP(A*D)**2/A/EXP(A)**2+
(2.D0*D*EXP(A*D)**2*A*EXP(A)**2+4.D0*EXP(A*D)*EXP(A)**2-4.D0
*EXP(A*D)**3*EXP(A)-EXP(A)**2+4.D0*D*EXP(A*D)**2*A*EXP(A
+)+EXP(A*D)**4)*B1**2/EXP(A*D)**2/EXP(A)**2
I2A3D = A**2*D**2*B1**2/EXP(A*D)+A**2*D**2/EXP(A)*B1**2/EXP(A
**2)+A**2/EXP(A)*B1**2/EXP(A*D)+A**2*B1**2/EXP(A*D)-2.D0*A
**2*D*B1**2/EXP(A*D)-2.D0*A**2*D/EXP(A)*B1**2/EXP(A*D)
+)+EXP(A*D)*A**2*B1**2/EXP(A)**2+A**2*D**2*EXP(A*D)/EXP(A)
+B1**2-A**2*D**2/EXP(A)*B1**2+EXP(A*D)*A**2*D**2*B1**2/EXP(A)**2
+-2.D0*EXP(A*D)*A**2*D*B1**2/EXP(A)**2+EXP(A*D)*A**2/EXP(A)
+B1**2+2.D0*A**2*D*B1**2+2.D0*A**2*D/EXP(A)*B1**2-2.D0*D*A**2*E
XP(A*D)/EXP(A)*B1**2-A**2*D**2*B1**2-A**2*B1**2+2.D0*EXP(A*D)
+B1**2/EXP(A)**2-2.D0*1.D0/EXP(A)*B1**2-A**2/EXP(A)*B1**2+2.D0
*1.D0/EXP(A)*B1**2/EXP(A*D)

```

```

I2BD = 1.D0/2.D0*B1*(6.D0*A**2*D**2*B2*EXP(A)*EXP(A*D)+4.D0*A**2
+*D*REI*ZE*EXP(A*D)+6.D0*A**2*D**2*B2*EXP(A*D)-REI*ZE*A**2*EXP(A)
+*EXP(A*D)+4.D0*A**2*D*REI*ZE*EXP(A)*EXP(A*D)-A**2*D**4*REI*ZE
+*EXP(A)*EXP(A*D)-6.D0*A**2*D*B2*EXP(A*D)+4.D0*A**2*D**3*REI*ZE
+*EXP(A)*EXP(A*D)+4.D0*A**2*D**3*REI*ZE*EXP(A*D)+2.D0*REI*ZE*D**2
+*EXP(A*D)**2-2.D0*A**2*D**3*B2*EXP(A*D)-4.D0*REI*ZE*D*EXP(A*D)**2
+-6.D0*A**2*D**2*REI*ZE*EXP(A*D)-6.D0*A**2*D*B2*EXP(A)*EXP
+(A*D)-6.D0*A**2*D**2*REI*ZE*EXP(A)*EXP(A*D)+2.D0*B2*A*
+*2*EXP(A*D)+4.D0*B2*D*EXP(A*D)**2-2.D0*A**2*D**3*B2*EXP(A)*EXP(A*D
+)+2.D0*B2*A**2*EXP(A)*EXP(A*D)-REI*ZE*A**2*EXP(A*D)-A**2*D**4*REI
+*ZE*EXP(A*D)+4.D0*B2*EXP(A*D)+4.D0*REI*ZE*D*EXP(A)*EXP(A*D)-2.D0
+*REI*ZE*EXP(A*D)-2.D0*REI*ZE*D**2*EXP(A*D)-2.D0*REI*ZE*EXP(A)*
+EXP(A*D)-4.D0*REI*ZE*D*EXP(A)+2.D0*REI*ZE*D**2*EXP(A)-4.D0
+*B2*D*EXP(A)*EXP(A*D)+4.D0*REI*ZE*D*EXP(A*D)-2.D0*REI*ZE*D**2
+*EXP(A)*EXP(A*D)+4.D0*B2*D*EXP(A)-4.D0*B2*D*EXP(A*D)+4.D0*B2
+*EXP(A)*EXP(A*D)+2.D0*EXP(A*D)**2*REI*ZE+2*REI*ZE*EXP(A)-4.D0
+*B2*EXP(A)-4.D0*EXP(A*D)**2*B2)/EXP(A)/EXP(A*D)
I2CD = 3.D0*B2*REI*ZE*D**2-B2*REI*ZE*D**3-B2**2+2.D0*B2
+*2*D+B2*REI*ZE-3.D0*B2*REI*ZE*D+REI**2*ZE**2*D-1.D0/4.D0*REI
+*2*ZE**2*D**4-1.D0/4.D0*REI**2*ZE**2+REI**2*ZE**2*D**3-
+3.D0/2.D0*REI**2*ZE**2*D**2-B2**2*D**2
I2DD = I2A1D+I2A2D+I2A3D+I2BD+I2CD
RETURN
END

```

```

DOUBLE PRECISION FUNCTION I2DZE(B1,B2,D,ZE,A,REI)
IMPLICIT DOUBLE PRECISION (A-Z)
I2A1ZE = 0.D0
I2A2ZE = 0.D0
I2A3ZE = 0.D0
I2BZE = -1.D0/60.D0*B1/EXP(A)*(-6.D0*A**5*REI*EXP(A)+120.D0
+*REI*EXP(A)-20.D0*REI*EXP(A)*A**3-20.D0*REI*A**3-6.D0*A**5*REI
+-120.D0*REI)/A**3+1.D0/60.D0*B1*(60.D0*EXP(A*D)**2*REI*A
+*2+60.D0*EXP(A*D)**2*A**2*D**2*REI-6.D0*A**5*D**5*REI*EXP(
+A*D)+60.D0*REI*D**2*EXP(A)*EXP(A*D)*A**3-20.D0*REI
+*D**3*EXP(A*D)*A**3-120.D0*EXP(A*D)**2*A*D*REI-6.D0*A**5*D
+*5*REI*EXP(A)*EXP(A*D)-20.D0*REI*D**3*EXP(A)*EXP(A*D)*A**3
+-60.D0*A**2*D**2*REI*EXP(A)-120.D0*A*D*REI*EXP(A)-60.D0
+*REI*D*EXP(A*D)*A**3-120.D0*EXP(A*D)**2*A**2*D*REI+120.D0
+*EXP(A*D)**2*REI*A+60.D0*REI*D**2*EXP(A*D)*A**3+120.D0*REI
+*EXP(A)*A-30.D0*A**5*D*REI*EXP(A*D)+120.D0*A**2*D*REI*EXP(A)
++30.D0*A**5*D**4*REI*EXP(A*D)+60.D0*A**5*D**2*REI*EXP(A)*EXP(A*D
+)+60.D0*A**5*D**2*REI*EXP(A*D)-60.D0*REI*EXP(A)*A**2-60.D0
+*A**5*D**3*REI*EXP(A*D)-60.D0*A**5*D**3*REI*EXP(A)
+*EXP(A*D)-30.D0*A**5*D*REI*EXP(A)*EXP(A*D)-60.D0*REI*D*EXP(A)
+*EXP(A*D)*A**3+30.D0*A**5*D**4*REI*EXP(A)*EXP(A*D)+120.D0*EXP(A*D
+)***2*REI-120.D0*REI*EXP(A))/EXP(A)/EXP(A*D)/A**3
I2CZE = B2*REI*D**3-1.D0/4.D0*B2*REI*D**4+B2*REI*D-3.D0/2.D0
+*B2*REI*D**2+REI**2*ZE*D**2-1.D0/10.D0*REI**2*ZE*D**5
+-1.D0/2.D0*REI**2*ZE*D+1.D0/2.D0*REI**2*ZE*D**4+
+1.D0/10.D0*REI**2*ZE-REI**2*ZE*D**3-1.D0/4.D0*B2*REI
I2DZE = I2A1ZE+I2A2ZE+I2A3ZE+I2BZE+I2CZE
RETURN
END

```

```

DOUBLE PRECISION FUNCTION B1DUF(UF,D,ZE,REI,MU21,A)
IMPLICIT DOUBLE PRECISION (A-Z)
B1DUF = 1.D0/(A*(EXP(-A*D)-EXP(-A*(1.D0-D))))-(1.D0-EXP(-A*D)
+-EXP(-A*(1.D0-D))+EXP(-A))/ (D-1.D0)+1.D0/2.D0*A**2*(1.D0+EXP(-A))
+*(D-1.D0))* (1.D0/MU21/D-1.D0/(D-1.D0))
RETURN
END

```

```

DOUBLE PRECISION FUNCTION B1DZE(UF,D,ZE,REI,MU21,A)
IMPLICIT DOUBLE PRECISION (A-Z)
B1DZE = 1.D0/2.D0*REI/(A*(EXP(-A*D)-EXP(-A*(1.D0-D))) - (1.D0
+-EXP(-A*D)-EXP(-A*(1.D0-D))+EXP(-A))/(D-1.D0)+1.D0/2.D0*A**2
+*(1.D0+EXP(-A))*(D-1.D0))
RETURN
END

```

```

DOUBLE PRECISION FUNCTION B1DD(UF,D,ZE,REI,MU21,A)
IMPLICIT DOUBLE PRECISION (A-Z)
DDEN = A*(EXP(-A*D)-EXP(-A*(1.D0-D))) - 1.D0
+/(D-1.D0)*(1.D0-EXP(-A*D)-EXP(-A*(1.D0
+-D))+EXP(-A))+1.D0/2.D0*A**2*(1.D0+EXP(-A))*(D-1.D0)
B23DD = -1.D0/MU21*UF/D**2+UF/(D-1.D0)**2
DDENDD = A*(-A*EXP(-A*D)-A*EXP(-A*(1.D0-D)))+1.D0/
+(D-1.D0)**2*(1.D0-EXP(-A*D)-EXP(-A*(1.D0-D))+EXP(-A))
+-1.D0/(D-1.D0)*(A*EXP(-A*D)-A*EXP(-A*(1.D0-D)))+1.D0/2.D0
+A**2*(1.D0+EXP(-A))
B23 = 1.D0/2.D0*REI*ZE+1.D0/MU21*UF/D-UF/(D-1.D0)
B1DD = 1.D0/DDEN*B23DD-(DDENDD*B23)/(DDEN**2)
RETURN
END

```

```

DOUBLE PRECISION FUNCTION B2DUF(UF,D,ZE,REI,MU21,A)
IMPLICIT DOUBLE PRECISION (A-Z)
B2DUF = 1.D0/(D-1.D0)*(1.D0-1.D0/(A*(EXP(-A*D)-EXP(-A*(1.D0-D)))
+- (1.D0-EXP(-A*D)-EXP(-A*(1.D0-D))+EXP(-A))/(D-1.D0)+1.D0
+/2.D0*A**2*(1.D0+EXP(-A))*(D-1.D0))*(1.D0/MU21/D-1.D0
+/(D-1.D0))*(1.D0-EXP(-A*D)-EXP(-A*(1.D0-D))+EXP(-A)
++1.D0/2.D0*A**2*(1.D0+EXP(-A))*(D-1.D0)**2))
RETURN
END

```

```

DOUBLE PRECISION FUNCTION B2DZE(UF,D,ZE,REI,MU21,A)
IMPLICIT DOUBLE PRECISION (A-Z)
B2DZE = 1.D0/(D-1.D0)*(-1.D0/2.D0*REI/(A*(EXP(-A*D)-EXP(-A
+*(1.D0-D))) - (1.D0-EXP(-A*D)-EXP(-A*(1.D0-D))+EXP(-A))/(D-1.D0)
++1.D0/2.D0*A**2*(1.D0+EXP(-A))*(D-1.D0))*(1.D0-EXP
+(-A*D)-EXP(-A*(1.D0-D))+EXP(-A)+1.D0/2.D0*A**2*(1.D0+EXP(-A))
+*(D-1.D0)**2)-1.D0/2.D0*REI*(D-1.D0)**2)
RETURN
END

```

```

DOUBLE PRECISION FUNCTION B2DD(UF,D,ZE,REI,MU21,A,EXPR1,TH1,TH2)
IMPLICIT DOUBLE PRECISION (A-Z)
DDEN = A*(EXP(-A*D)-EXP(-A*(1.D0-D))) - 1.D0
+/(D-1.D0)*(1.D0-EXP(-A*D)-EXP(-A*(1.D0
+-D))+EXP(-A))+1.D0/2.D0*A**2*(1.D0+EXP(-A))*(D-1.D0)
B23DD = -1.D0/MU21*UF/D**2+UF/(D-1.D0)**2
DDENDD = A*(-A*EXP(-A*D)-A*EXP(-A*(1.D0-D)))+1.D0/
+(D-1.D0)**2*(1.D0-EXP(-A*D)-EXP(-A*(1.D0-D))+EXP(-A))
+-1.D0/(D-1.D0)*(A*EXP(-A*D)-A*EXP(-A*(1.D0-D)))+1.D0/2.D0
+A**2*(1.D0+EXP(-A))
B23 = 1.D0/2.D0*REI*ZE+1.D0/MU21*UF/D-UF/(D-1.D0)
B1DD = 1.D0/DDEN*B23DD-(DDENDD*B23)/(DDEN**2)
B1 = B23/DDEN
B21DD = -UF/(D-1.D0)**2-1.D0/2.D0*REI*ZE
B22DD = (A*EXP(-A*D)-A*EXP(-A*(1.D0-D)))/
+(D-1.D0)-(1.D0-EXP(-A*D)-EXP(-A*(
+1.D0-D))+EXP(-A))/(D-1.D0)**2+1.D0/2.D0*A**2
+*(1.D0+EXP(-A))
B22 = (1.D0-EXP(-A*D)-EXP(-A*(1.D0-D))+EXP(-A))
+/(D-1.D0)+1.D0/2.D0*A**2*(1.D0+EXP(-
+A))*(D-1.D0)
B2DD = B21DD-B1*B22DD-B1DD*B22
RETURN
END

```

Acknowledgement:

We gratefully appreciate computing support from the Academic Computing Services at Michigan Technological University, Houghton and the University of Minnesota. **Y. Kizilyalli's** computational contribution to this paper is limited only to his M.S. thesis leading to an earlier effort [16] which neglected effects of pressure drop. I thank **Professor T.S. Lundgren** (University of Minnesota) for discussions with regard to significance of pressure variations in this problem.

References

1. W. Nusselt, Die Oberflächenkondensation des Wasserdampfes, *Z. Ver. Dt. Ing.* **60** (27), 541-546 (1916).
2. W. M. Rohsenow, Heat Transfer and Temperature Distribution in Laminar Film Condensation, *Transactions ASME* **78**, 1645-1648 (1956).
3. E.M. Sparrow and J.L. Gregg, A boundary layer treatment of laminar film condensation, *ASME J. Heat Transfer* **81**, 13-18 (1959).
4. J.D. Ford and A. Lekic, Rate of Growth of Drops During Condensation. *Int. J. Heat Mass Transfer* **16**, 61-66 (1973).
5. H.R. Jacobs and D.S. Cook, Direct Contact Condensation on a Non-Circulating Drop. Proceedings of the 6th Internat. Heat Transfer Conf., *Heat Transfer 1978* **3**, 389-393, Toronto, Canada (1978).
6. E.M. Sparrow, W.J. Minkowycz and M. Saddy, Forced Convection Condensation in the presence of noncondensables, interfacial resistance, superheating, variable properties, and diffusion, *Int. J. Heat Mass Transfer* **9**, 1125-1144 (1966).
7. C.Y. Wang and C.J. Tu, Effects of non-condensable gas on laminar film condensation in a vertical tube, *Int. J. Heat Mass Transfer* **31**, 2339-2345 (1988).
8. M. Soliman, J.R. Schuster and P.J. Berenson, A general heat transfer correlation for annular flow condensation, *J. Heat Transfer* **90**, 267-276 (1968).
9. J.B. Chaddock, Film Condensation of vapor in a horizontal tube, *Refrigerating Engineering* **65**, 36-41 (1957).

10. J.C. Chato, Laminar condensation inside horizontal tube, *ASHRAE Journal* **4**, 52-60 (1962).
11. C.E. Rufer and S.P. Kezios, Analysis of two-phase, one-component, stratified flow with condensation, *J. Heat Transfer*, **88** (2), 265-275 (1966).
12. K. Lucas and B. Moser, Laminar film condensation of pure vapors in tubes. *Int. J. of Heat and Mass Transfer* **22**, 431-435 (1979).
13. Theodore von Kármán, Über Laminare und turbulente Reibung, *ZAMM* **1**, 233-252 (1921). See also NACA Tech. Memo. 1092 (1946).
14. K. Pohlhausen, Zur näherungsweisen Integration der Differentialgleichung der laminaren Reibungsschicht, *ZAMM* **1**, 252-268 (1921).
15. C. Christodoulou, Condensation in a Horizontal Rectangular Duct. An Experimental Study. M.S. Thesis (adviser: N.V. Suryanarayana) Michigan Technological University, Houghton, MI-49931, USA (1988).
16. A. Narain and Y. Kizilyalli, The flow of pure vapor undergoing film condensation between parallel plates, IMA Preprint Series #502, University of Minnesota (1989).
17. F.M. White, *Viscous Fluid Flow*, McGraw Hill (1974).
18. J.M. Delhay, Jump Conditions and Entropy Sources in Two-Phase Systems: Local Instant Formulation. *Int. J. of Multiphase Flow* **1**, 395-409 (1974).
19. L.E. Scriven, Dynamics of a Fluid Interface. *Chemical Engineering Science* **2** (1960).
20. G.P. Moeckel, Thermodynamics of an Interface, *Arch. Rat. Mech. and Anal.*, **57** (3), 255-280 (1975).

21. J.M. Delhay, Local Instantaneous Equations in *Two-Phase Flows and Heat Transfer*, Vol. 1, Proceedings of NATO Advanced Study Institute, Aug. 16-27, 1976. Instabul, Turkey. Edited by S. Kakac and F. Mayinger (1976).
22. A. Narain and D.D. Joseph: Note on Balance of Energy at a Phase Change Interface, (submitted for publication) in *Int. J. of Multiphase Flow*. Also in IMA Preprint Series #518, Univ. of Minnesota (1989).
23. ASHRAE Handbook, Fundamentals SI Edition, American Society of Heating, Refrigerating and Air-Conditioning Engineers, Inc., Atlanta, GA (1985).
24. S. Whitaker, *Fundamental Principles of Heat Transfer*, Pergamon Press (1977).
25. G. Kocamustafaogullari, Two-Fluid Modeling in Analyzing the Interfacial Stability of Liquid Film Flows, *Int. J. Multiphase Flow*, **11** (1), 63-89 (1985).
26. J.C.Y. Koh, Film condensation in a forced-convection boundary-layer flow. *Int. J. Heat Mass Transfer* **5**, 941-954 (1962).
27. H. Schlichting, *Boundary Layer Theory*, Seventh Edition, McGraw Hill (1979).

#	Author/s	Title
464	Yisong Yang	Existence, Regularity, and Asymptotic Behavior of the Solutions to the Ginzburg-Landau Equations on $\mathbb{R}^3$
465	Chjan. C. Lim	On Symplectic Tree Graphs
466	Wilhelm I. Fushchich, Ivan Krivsky and Vladimir Simulik,	On Vector and Pseudovector Lagrangians for Electromagnetic Field
467	Wilhelm I. Fushchich,	Exact Solutions of Multidimensional Nonlinear Dirac's and Schrödinger's Equations
468	Wilhelm I. Fushchich and Renat Zhdanov,	On Some New Exact Solutions of Nonlinear D'Allembert and Hamilton Equations
469	Brian A. Coomes,	The Lorenz System Does Not Have a Polynomial Flow
470	J.W. Helton and N.J. Young,	Approximation of Hankel Operators: Truncation Error in an H^∞ Design Method
471	Gregory Ammar and Paul Gader,	A Variant of the Gohberg-Semencul Formula Involving Circulant Matrices
472	R.L. Fosdick and G.P. MacSithigh,	Minimization in Nonlinear Elasticity Theory for Bodies Reinforced with Inextensible Cords
473	Fernando Reitich,	Rapidly Stretching Plastic Jets: The Linearized Problem
474	Francisco Bernis and Avner Friedman,	Higher Order Nonlinear Degenerate Parabolic Equations
475	Xinfu Chen and Avner Friedman,	Maxwell's Equations in a Periodic Structure
476	Avner Friedman and Michael Vogelius	Determining Cracks by Boundary Measurements
477	Yuji Kodama and John Gibbons,	A Method for Solving the Dispersionless KP Hierarchy and its Exact Solutions II
478	Yuji Kodama,	Exact Solutions of Hydrodynamic Type Equations Having Infinitely Many Conserved Densities
479	Robert Carroll,	Some Forced Nonlinear Equations and the Time Evolution of Spectral Data
480	Chjan. C. Lim	Spanning Binary Trees, Symplectic Matrices, and Canonical Transformations for Classical N-body Problems
481	E.F. Assmus, Jr. and J.D. Key,	Translation Planes and Derivation Sets
482	Matthew Witten,	Mathematical Modeling and Computer Simulation of the Aging-Cancer Interface
483	Matthew Witten and Caleb E. Finch,	Re-Examining The Gompertzian Model of Aging
484	Bei Hu,	A Free Boundary Problem for a Hamilton-Jacobi Equation Arising in Ions Etching
485	T.C. Hu, Victor Klee and David Larman,	Optimization of Globally Convex Functions
486	Pierre Goossens,	Shellings of Tilings
487	D. David, D. D. Holm, and M.V. Tratnik,	Integrable and Chaotic Polarization Dynamics in Nonlinear Optical Beams
488	D. David, D.D. Holm and M.V. Tratnik,	Horseshoe Chaos in a Periodically Perturbed Polarized Optical Beam
489	Laurent Habsieger,	Linear Recurrent Sequences and Irrationality Measures
490	Laurent Habsieger,	MacDonald Conjectures and The Selberg Integral
491	David Kinderlehrer and Giorgio Vergara-Caffarelli,	The Relaxation of Functionals with Surface Energies
492	Richard James and David Kinderlehrer,	Theory of Diffusionless Phase Transitions
493	David Kinderlehrer,	Recent Developments in Liquid Crystal Theory
494	Niky Kamran and Peter J. Olver,	Equivalence of Higher Order Lagrangians 1. Formulation and Reduction
495	Lucas Hsu, Niky Kamran and Peter J. Olver,	Equivalence of Higher Order Lagrangians II. The Cartan Form for Particle Lagrangians
496	D.J. Kaup and Peter J. Olver,	Quantization of BiHamiltonian Systems
497	Metin Arik, Fahrünisa Neyzi, Yavuz Nutku, Peter J. Olver and John M. Verosky	Multi-Hamiltonian Structure of the Born-Infeld Equation
498	David H. Wagner,	Detonation Waves and Deflagration Waves in the One Dimensional ZND Model for High Mach Number Combustion
499	Jerrold R. Griggs and Daniel J. Kleitman,	Minimum Cutsets for an Element of a Boolean Lattice
500	Dieter Jungnickel,	On Affine Difference Sets
501	Pierre Leroux,	Reduced Matrices and q-log Concavity Properties of q-Stirling Numbers
502	A. Narain and Y. Kizilyalli,	The Flow of Pure Vapor Undergoing Film Condensation Between Parallel Plates
503	Donald A. French,	On the Convergence of Finite Element Approximations of a Relaxed Variational Problem

Recent IMA Preprints (Continued)

#	Author/s	Title
504	Yisong Yang,	Computation, Dimensionality, and Zero Dissipation Limit of the Ginzburg-Landau Wave Equation
505	Jürgen Sprekels,	One-Dimensional Thermomechanical Phase Transitions with Non-Convex Potentials of Ginzburg-Landau Type
506	Yisong Yang,	A Note On Nonabelian Vortices
507	Yisong Yang,	On the Abelian Higgs Models with Sources
508	Chjan. C. Lim,	Existence of Kam Tori in the Phase Space of Vortex Systems
509	John Weiss,	Bäcklund Transformations and the Painlevé Property
510	Pu Fu-cho and D.H. Sattinger,	The Yang-Baxter Equation for Integrable Systems
511	E. Bruce Pitman and David G. Schaeffer,	Instability and Ill-Posedness in Granular Flow
512	Brian A. Coomes,	Polynomial Flows on C^n
513	Bernardo Cockburn, Suchung Hou and Chi-Wang Shu,	The Runge-Kutta Local Projection Discontinuous Galerkin Finite Element Method for Conservation Laws IV: The Multidimensional Case
514	Peter J. Olver,	Invariant Theory, Equivalence Problems, and the Calculus of Variations
515	Daniel D. Joseph and Thomas S. Lundgren with an appendix by R. Jackson and D.A. Saville,	Ensemble Averaged and Mixture Theory Equations
516	P. Singh, Ph. Caussignac, A. Fortes, D.D. Joseph and T. Lundgren,	Stability of Periodic Arrays of Cylinders Across the Stream by Direct Simulation
517	Daniel D. Joseph,	Generalization of the Foscolo-Gibilaro Analysis of Dynamic Waves
518	A. Narain and D.D. Joseph,	Note on the Balance of Energy at a Phase Change Interface
519	Daniel D. Joseph,	Remarks on inertial radii, persistent normal stresses, secondary motions , and non-elastic extensional viscosities
520	D. D. Joseph,	Mathematical Problems Associated with the Elasticity of Liquids
521	Henry C. Simpson and Scott J. Spector,	Some Necessary Conditions at an Internal Boundary for Minimizers in Finite Elasticity
522	Peter Gritzmann and Victor Klee,	On the 0-1 Maximization of Positive Definite Quadratic Forms
523	Fu-Cho Pu and D.H. Sattinger,	The Yang-Baxter Equations and Differential Identities
524	Avner Friedman and Fernando Reitich,	A Hyperbolic Inverse Problem Arising in the Evolution of Combustion Aerosol
525	E.G. Kalnins, Raphael D. Levine and Willand Miller, Jr.,	Conformal Symmetries and Generalized Recurrences for Heat and Schrödinger Equations in One Spatial Dimension
526	Wang Jinghua and Gerald Warnecke,	On Entropy Consistency of Large Time Step Godunov and Glimm Schemes
527	C. Guillopé and J.C. Saut,	Existence Results for the Flow of Viscoelastic Fluids with a Differential Constitutive Law
528	H.L. Bodlaender, P. Gritzmann, V. Klee and J. Van Leeuwen	Computational Complexity of Norm-Maximization
529	Li Ta-tsien (Li Da-qian) and Yu Xin,	Life-Span of Classical Solutions to Fully Nonlinear Wave Equations
530	Jong-Shenq Guo,	A Variational Inequality Associated with a Lubrication Problem
531	Jong-Shenq Guo,	On the Semilinear Elliptic Equation $\Delta w - \frac{1}{2}y \cdot \nabla w + \lambda w - w^{-\beta}$ in R^n
532	Andrew E. Yagle,	Inversion of the Bloch transform in magnetic resonance imaging using asymmetric two-component inverse scattering
533	Bei Hu,	A Fiber Tapering Problem
534	Peter J. Olver,	Canonical Variables for BiHamiltonian Systems
535	Michael Renardy,	A Well-Posed Boundary Value Problem for Supercritical Flow of Viscoelastic Fluids of Maxwell Type
536	Michael Renardy,	Ill-Posedness Resulting from Slip As a Possible Explanation of Melt Fracture
537	Michael Renardy,	Compatibility Conditions at Corners Between Walls and Inflow Boundaries for Fluids of Maxwell Type
538	Rolf Rees,	The Spectrum of Restricted Resolvable Designs with $r = 2$
539	D. Lewis and J.C. Simo,	Nonlinear stability of rotating pseudo-rigid bodies
540	Robert Hardt and David Kinderlehrer,	Variational Principles with Linear Growth
541	San Yih Lin and Yisong Yang,	Computation of Superconductivity in Thin Films
542	A. Narain,	Pressure Driven Flow of Pure Vapor Undergoing Laminar Film Condensation Between Parallel Plates
543	P.J. Vassiliou,	On Local Equivalence for Vector Field Systems