**VARIATIONAL PRINCIPLES
WITH LINEAR GROWTH**

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VARIATIONAL PRINCIPLES WITH LINEAR GROWTH

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Dedicated to Ennio De Giorgi on his sixtieth birthday

Variational principles which exhibit only linear growth arise in several contexts. As a paradigm for the questions which these principles suggest, we take up here the study of the problem

$$(0.1) \quad \inf_{v \in \mathcal{A}} \left\{ \int_{\Omega} \phi(\nabla v) dx - \int_{\Omega} f v dx \right\}.$$

Here $\phi : \mathbf{R}^n \rightarrow \mathbf{R}$ is a non-negative convex sufficiently differentiable function satisfying

$$(0.2) \quad \phi(0) = 0, \quad \phi_p(0) = 0, \quad \text{and} \quad |p| - \lambda \leq \phi \leq |p|$$

for some $\lambda > 0$, subject to

$$(0.3) \quad \lim_{t \rightarrow \infty} \phi(tp)/t|p| = 1.$$

The competing v belong to a suitable class $\mathcal{A}$ of functions from Ω to $\mathbf{R}$ where Ω is a bounded domain with sufficiently smooth boundary and f is given.

Our discussion applies in particular to the integrand

$$(0.4) \quad \phi(p) = \begin{cases} \frac{1}{2}|p|^2 & \text{for } |p| \leq 1 \\ |p| - \frac{1}{2} & \text{for } |p| \geq 1 \end{cases}$$

which arises in the study of anti-planar shear in elastic/plastic deformation, $[HK_1]$, $[KT]$, $[ST]$.

A solution to (0.1) may be found direct methods or by resorting to a generalized principle of complementary energy. The solution found by direct methods needs not be unique and, pertinent here, its gradient may be only a measure. The solution of the dual problem of complementary energy is often unique and determines, in case of (0.4), the region associated to elastic and plastic behavior.

Our concerns here will include properties of the generalized stress, the solution of the dual problem, which lead to information about the direct problem and its consequent smoothness.

These issues were brought to our attention by the work of Anzellotti and Giaquinta ($[AG_1]$, $[AG_2]$) who treated the direct problem, and Strang and Temam $[ST]$ who investigated both the direct problem and the dual problem. As we have indicated, it was partly our desire to achieve a greater unity between these approaches which led us to take up this argument. Our method is transparent and may be summarized in a few words: We have attempted to apply what we have learned from our study of the work of Ennio De Giorgi. It is with special pleasure that we dedicate this paper to him.

Central to our work is the study of the $(n-1)$ density

$$\Theta(a) = \lim_{r \rightarrow 0} r^{1-n} \int_{B_r(a)} |Du|$$

for a local solution u of (0.1). In §1 we show that this density exists and is upper semi-continuous in Ω . In §2, we show that the local solution u is continuous at a point $a \in \Omega$ if and only if its $(n-1)$ density vanishes at that point. In §3 the behavior near points of positive density is investigated, and it is shown how the solution is, in a very strong sense, essentially two-sided continuous, with the exception of a set of $n-1$ dimensional measure zero. The curvature of the set of discontinuities, or slip set, is then determined by the inhomogeneous term f .

Some of the present work was announced in $[HK_2]$, the latter containing some discussion of open questions. We wish to point out some more recent work in this general area, $[AG_3]$, and dynamical problems, $[RS]$, $[AL]$.

Implicit in our introduction of the $(n-1)$ density $\Theta(a)$ and our subsequent discussion is the recognition that we shall be obliged to widen our class of admissible functions to $BV(\Omega)$. To further motivate this, we briefly review the dual problem associated to (0.1).

Assuming that $u_0 \in H^{1,1}(\Omega)$ and $f \in L^\infty(\Omega)$, set

$$\mathcal{A} = \{v \in H^{1,1}(\Omega) : v = u_0 \text{ on } \partial\Omega\}.$$

Defining

$$(0.5) \quad I(v) = \int_{\Omega} \phi(\nabla v) dx - \int_{\Omega} f v dx, \quad v \in \mathcal{A},$$

we may extend our functional to $H^{1,1}(\Omega)$ by [ET]

$$I(v) = \int_{\Omega} \phi(\nabla v) dx + I_{\mathcal{A}}(v) - F(v)$$

where

$$I_{\mathcal{A}}(v) = \begin{cases} 0 & v \in \mathcal{A} \\ \infty & v \notin \mathcal{A} \end{cases}$$

is the indicator function of $\mathcal{A}$ and

$$F(v) = \int_{\Omega} f v dx.$$

Our original variational principle (0.1) now becomes a problem

(P) *To find the solution or characteristic extremals of*

$$i = \inf_{H^{1,1}(\Omega)} \left\{ \int_{\Omega} \phi(\nabla v) dx + I_{\mathcal{A}}(v) - F(v) \right\}.$$

We may easily calculate the dual functional $I^*(\tau)$ for a vector-field τ . Indeed,

$$(0.6) \quad I^*(\tau) = \begin{cases} \int_{\Omega} \phi^*(-\tau) dx + \int_{\Omega} \tau \cdot \nabla u_0 dx + \int_{\Omega} f u_0 dx & \text{if } \operatorname{div} \tau = f, \\ +\infty & \text{otherwise.} \end{cases}$$

Several properties of the dual functional ϕ^* are worthy of note. First observe that (0.2) and the convexity of ϕ imply that

$$|\phi_p(p)| \leq 1 \quad \text{whenever } 0 \neq p \in \mathbf{R}^n.$$

Now

$$\phi^*(q) = \begin{cases} p \cdot \phi_p(p) - \phi(p) & \text{whenever } q = \phi_p(p) \text{ and } |q| \leq 1, \\ +\infty & \text{whenever } |q| > 1. \end{cases}$$

Again using (0.2), we obtain the estimate, useful later on,

$$0 \leq \phi^*(q) \leq \lambda \quad \text{if } |q| \leq 1.$$

Since (0.6) implies that

$$\int_{\Omega} \phi^*(-\tau) dx = +\infty \quad \text{if } \|\tau\|_{L^\infty(\Omega)} > 1,$$

we may further restrict the competitors which will occur in the dual principle. We arrive at the problem

(P^*) *To find or characterize extremals of*

$$(0.7) \quad i^* = \inf_{\operatorname{div} \tau = f, |\tau| \leq 1} \left\{ \int_{\Omega} \phi^*(-\tau) dx + \int_{\Omega} \tau \cdot \nabla u_0 dx + \int_{\Omega} f u_0 dx \right\}.$$

The theory of convex duality ([ET] or [B₃], p.8) assures us that whenever

$$\mathcal{A}^* = \{\tau \in L^\infty(\Omega)^n : |\tau| \leq 1 \text{ and } \operatorname{div} \tau = f\}$$

is not empty, there exists a $\sigma \in \mathcal{A}^*$ such that

$$i = i^* = \int_{\Omega} \phi^*(-\sigma) dx + \int_{\Omega} \sigma \cdot \nabla u_0 dx + \int_{\Omega} f u_0 dx.$$

Indeed, (P^*) is a variational inequality [S].

In the special case (0.4) the dual function is

$$\phi^*(\sigma) = \begin{cases} \frac{1}{2} |\sigma|^2 & \text{for } |\sigma| \leq 1 \\ \infty & \text{for } |\sigma| > 1, \end{cases}$$

and the dual functional is

$$I^*(\tau) = \frac{1}{2} \int_{\Omega} |\tau|^2 dx + \int_{\Omega} \tau \cdot \nabla u_0 dx + \int_{\Omega} f u_0 dx \text{ whenever } \tau \in \mathcal{A}^*.$$

However, there need not be $u \in \mathcal{A}$ whose stress achieves this infimum, and thus we are led to search for u in the wider class $BV(\Omega)$ of functions whose gradients are measures.

A second motivation is the absence of a suitable Lagrangian functional which enables us to pair vectorfields τ and v in $H^{1,1}(\Omega)$. Determining the validity of the Lagrangian in an extended sense is one of the subjects treated by Kohn and Temam [KT]. Here we limit ourselves to the relationship between the solution of (0.1) or (0.7) and its own stress vector.

The important question of when $\mathcal{A}^*$ is not empty or when $I(v)$ is bounded below on $\mathcal{A}$ may be resolved in several ways, c.f. [AG₁], [KT], [HK₁]. For example, given Ω there is a constant C_{Ω} such that

$$\|\zeta\|_{L^1(\Omega)} \leq C_{\Omega} \|\zeta\|_{BV(\Omega)} \text{ for } \zeta \in BV_0(\Omega).$$

Thus $I(v)$ is bounded below whenever

$$\|f\|_{L^{\infty}(\Omega)} < C_{\Omega}^{-1}.$$

Methods of limit analysis offer an alternative view, [T₁].

In our motivation we have discussed a variational principle with Dirichlet boundary conditions. Neumann conditions may be considered as well.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with Lipschitz boundary. Given $v \in BV(\Omega)$, we decompose its gradient measure Dv into its absolutely continuous and singular parts with respect to Lebesgue measure:

$$Dv = \nabla v dx + D^s v.$$

We define

$$(0.8) \quad I(v) = \int_{\Omega} \phi(Dv) = \int_{\Omega} \phi(\nabla v) dx + \int_{\Omega} |D^s v|.$$

This definition is motivated primarily by the fact that there exist $v_{\delta} \in C^{\infty}(\bar{\Omega})$, $0 < \delta < 1$, such that (v_{δ}) is contained in a bounded set in $BV(\Omega)$ and

$$v_{\delta} \rightarrow v \text{ in } L^{\frac{n}{n-1}} \cap L^1(\partial\Omega) \text{ and } I(v_{\delta}) = \int_{\Omega} \phi(\nabla v_{\delta}) dx \rightarrow I(v)$$

as $\delta \rightarrow 0$, c.f. [AG₁], [HK₁, Th.A.2]. The *stress associated to v* is defined by

$$\sigma = \sigma(v) = \begin{cases} \phi_p(\nabla v) & \text{in } \Omega_a \\ D^s v / |D^s v| & \text{in } \Omega_s \end{cases}$$

where $D^s v / |D^s v|$ is the Radon-Nikodym derivative of $D^s v$ with respect to its total variation measure $|D_s v|$ and $\Omega = \Omega_a \cup \Omega_s$ is the decomposition of Ω with respect the mutually singular measures dx and $|D^s v|$.

We have already noticed that since ϕ is convex and (0.2) holds,

$$|\phi_p(p)| \leq 1 \text{ and } \phi(p) \leq \phi_p(p) \cdot p.$$

It follows that

$$|\sigma(v)| \leq 1 \text{ and}$$

$$(0.9) \quad \phi(Dv) = \phi(\nabla v) + |D^s v| \leq \phi_p(\nabla v) \cdot \nabla v + |D^s v| = \sigma(v) \cdot Dv.$$

1. Local solutions and variational formulas. We say that a function $u \in BV(\Omega)$ is a *local solution* in Ω provided that

$$\int_{\Omega} \phi(Du) - \int_{\Omega} f u dx \leq \int_{\Omega} \phi(D(u + \zeta)) - \int_{\Omega} f(u + \zeta) dx \text{ for } \zeta \in BV_0(\Omega).$$

For any $\zeta \in BV_0(\Omega)$ satisfyng

$$(1.1) \quad D^s \zeta \ll |D^s u|,$$

we have the first variation formula [HK₁, 2.4]

$$(1.2) \quad \int_{\Omega} \sigma \cdot \nabla \zeta dx + \int_{\Omega} \sigma \cdot \xi |D^s u| = \int_{\Omega} f \zeta dx,$$

where $\sigma = \sigma(u)$ and ξ is the Radon-Nikodym derivative of $D^s \zeta$ with respect to $|D^s u|$. Indeed, usually we shall write (1.2) as

$$(1.3) \quad \int_{\Omega} \sigma \cdot D\zeta dx = \int_{\Omega} f \zeta dx.$$

The absolute continuity (1.1) is necessary for the validity of (1.3) as a simple example illustrates. One may take $n = 1, \Omega = (0, 3)$, $u \equiv 0$, and $\zeta = \chi_{(1,2)} = \text{characteristic function of } (1, 2)$ to see that $\int_{\Omega} \phi(Du + t\zeta) = 2|t|$ is not differentiable at $t = 0$ so that the first variation formula makes no sense.

A local variational principle follows from (1.1) for the functional

$$(1.4) \quad I_{r,a}(v) = \int_{B_{r(a)}} \phi(Dv) - \int_{\partial B_{r(a)}} \sigma \cdot \frac{x}{|x|} v dS - \int_{B_{r(a)}} f v dx.$$

Proposition 1.1 *The function $u \in BV(\Omega)$ is a local solution if and only if*

$$(1.5) \quad I_{r,a}(u) \leq I_{r,a}(v)$$

for every $v \in BV(\Omega)$ and $r < \text{dist}(a, \partial\Omega)$. Moreover,

$$(1.6) \quad I_{r,a}(u) = - \int_{B_{r(a)}} \phi^*(\sigma) dx.$$

Here and in the sequel, $\sigma = \sigma(u)$ where u is the local solution under discussion.

The proof is facilitated by an elementary lemma.

Lemma 1.2. *If $v \in BV(\Omega)$ and $D^s v \ll |D^s u|$, then*

$$\int_{B_{r(a)}} \sigma \cdot Dv = \int_{\partial B_{r(a)}} \sigma \cdot \frac{x}{|x|} v dS + \int_{B_{r(a)}} f v dx.$$

PROOF. We assume $a = 0$. For fixed $0 < \rho < r$, choose $\zeta = \eta$ where

$$\eta(x) = \begin{cases} 1 & \text{in } B_{\rho} \\ (r - \rho)^{-1}(r - |x|) & \text{in } B_r \setminus B_{\rho} \\ 0 & \text{in } \Omega \setminus B_r. \end{cases}$$

By (1.1),

$$\begin{aligned} \int_{B_r} \eta \sigma \cdot Dv &= (r - \rho)^{-1} \int_{B_r \setminus B_{\rho}} \sigma \cdot \frac{x}{|x|} v dx = \\ &= (r - \rho)^{-1} \left\{ \int_{B_r} \sigma \cdot \frac{x}{|x|} v dx - \int_{B_{\rho}} \sigma \cdot \frac{x}{|x|} v dx \right\}. \end{aligned}$$

Then, letting $\rho \rightarrow r$, the conclusion follows.

Proof of Proposition 1.1. Suppose first that u is a local solution and that $v \in C^1(\Omega)$; hence, $D^s v = 0 \ll |D^s u|$. By (0.2) and Lemma 1.2,

$$\begin{aligned} \int_{B_r(a)} \phi(Dv) - \int_{B_r(a)} \phi(Du) &= \\ &= \int_{B_r(a)} (\phi(\nabla v) - \phi(\nabla u)) dx - \int_{B_r(a)} |D^s u| \\ &\geq \int_{B_r(a)} \sigma \cdot (\nabla v - \nabla u) dx - \int_{B_r(a)} |D^s u| \\ &= \int_{B_r(a)} \sigma \cdot D(v - u) = \int_{\partial B_r(a)} \sigma \cdot \frac{x}{|x|} (v - u) dS. \end{aligned}$$

Thus $I_{r,a}(u) \leq I_{r,a}(v)$ for $v \in C^1(\Omega)$.

For an arbitrary $v \in BV(\Omega)$, we may find a sequence $v_k \in C^1(\Omega)$ such that, for almost every r ,

$$\limsup_{k \rightarrow \infty} \int_{B_r} \phi(Dv_k) \leq \int_{B_r} \phi(Dv) \quad \text{and} \quad v_k \rightarrow v \text{ in } L^1(\partial B_r).$$

Conversely, suppose that (1.3) holds. If $\zeta \in BV(\Omega)$ has $D^s \zeta \ll |D^s u|$ and $\text{supp } \zeta \subset B_r(a) \subset \Omega$, then

$$\int_{B_r(a)} \phi(D(u + \zeta)) - \int_{B_r(a)} \phi(Du) = I_{r,a}(u + \zeta) - I_{r,a}(u) \geq 0;$$

hence, $\int_{\Omega} \sigma \cdot D\zeta dx = 0$ by our previous reasoning. For an arbitrary $\zeta \in BV_0(\Omega)$ with $D^s \zeta \ll |D^s u|$, we may use a partition of unity to deduce the equation $\int_{\Omega} \sigma \cdot D\zeta dx = 0$ and then obtain from convexity the desired inequality

$$\int_{\Omega} \phi(Du) = \int_{\Omega} \phi(Du) + \int_{\Omega} \sigma \cdot D\zeta \leq \int_{\Omega} \phi(Du + D\zeta).$$

The final assertion, the duality relation localized, is a direct calculation.

The duality relation is a useful test for minima.

Theorem 1.3 (Monotonicity). *Let $f \in L^\infty(\Omega)$ and let u be a local solution.*

(i) *Then,*

$$\rho^{1-n} \int_{\mathbf{B}_\rho(a)} |Du| \leq e^{\Lambda(r-\rho)} r^{1-n} \int_{\mathbf{B}_r(a)} |Du| + C_n(r-\rho),$$

for $0 < \rho < r < \text{dist}(a, \partial\Omega)$, where $\Lambda = C_n \|f\|_{L^\infty(\Omega)}$ and C_n is a dimensional constant.

(ii) *If $u \in L^\infty(\Omega)$, then*

$$\begin{aligned} \rho^{1-n} \int_{\mathbf{B}_\rho(a)} |Du| &\leq e^{\Lambda(r-\rho)} r^{1-n} \int_{\mathbf{B}_r(a)} |Du| + \\ &\quad + C_n(\lambda + \|f\|_{L^\infty(\Omega)} \|u\|_{L^\infty(\Omega)})(r-\rho), \end{aligned}$$

(iii) *In particular, the $(n-1)$ density*

$$\Theta(a) = \lim_{r \rightarrow 0} r^{1-n} \int_{\mathbf{B}_r(a)} |Du|$$

exists, is finite, and is uniformly bounded on compact subset of Ω .

We remark immediately that we prove in the next section that any local solution is locally bounded, so that (ii) always holds on compact subset of Ω .

PROOF. We again set $a = 0$. For $0 < r < \text{dist}(0, \partial\Omega)$, define

$$w(x) = \begin{cases} u(r \frac{x}{|x|}) & \text{in } \mathbf{B}_r \\ u(x) & \text{in } \Omega \setminus \mathbf{B}_r. \end{cases}$$

For this r , we infer that

$$I_{r,0}(u) \leq I_{r,0}(w).$$

Combining this with (0.2) gives that

$$\begin{aligned} \int_{\mathbf{B}_r} |Du| &\leq \int_{\mathbf{B}_r} \phi(Du) + \lambda |\mathbf{B}_r| \\ (1.7) \quad &\leq \int_{\mathbf{B}_r} \phi(Dw) - \int_{\mathbf{B}_r} f(w-u) dx + \lambda |\mathbf{B}_r| \\ &\leq \int_{\mathbf{B}_r} |Dw| - \int_{\mathbf{B}_r} f(w-u) dx + \lambda |\mathbf{B}_r|. \end{aligned}$$

Now by a well-known calculation, cf.[DCP] for example,

$$\int_{\mathbf{B}_r} |Dw| \leq (n-1)^{-1} r \int_{\partial \mathbf{B}_r} |Du|.$$

Now we may estimate the middle term in (1.7) two ways. If u is assumed bounded, then

$$\left| \int_{\mathbf{B}_r} f(w-u) dx \right| \leq 2 \|f\|_{L^\infty} \|u\|_{L^\infty} |\mathbf{B}_r|.$$

Thus

$$\int_{\mathbf{B}_r} |Du| \leq (n-1)^{-1} r \int_{\partial \mathbf{B}_r} |Du| + C_n (\lambda + \|f\|_{L^\infty} \|u\|_{L^\infty}) r^n$$

or

$$(d/dr) \{ r^{1-n} \int_{\mathbf{B}_r} |Du| + C_n (\lambda + \|f\|_{L^\infty} \|u\|_{L^\infty}) r \} \geq 0,$$

which implies (ii).

If we do not assume that u is bounded, then, since $w - u = 0$ on $\partial \mathbf{B}_r$,

$$\begin{aligned} \left| \int_{\mathbf{B}_r} f(w-u) dx \right| &\leq \|f\|_{L^\infty} \|w-u\|_{L^1(\mathbf{B}_r)} \\ &\leq C_n r \|f\|_{L^\infty} \left(\int_{\mathbf{B}_r} |Du| + \int_{\mathbf{B}_r} |Dw| \right), \end{aligned}$$

from which (i) follows.

Corollary 1.4. *The density $\Theta(x)$ is upper semi-continuous on Ω .*

PROOF. We assume that $\mathbf{B}_{2r} \subset \Omega$ and prove upper semi-continuity at 0. First note that $\mathbf{B}_\rho(x) \subset \mathbf{B}_{2r}$ whenever $x \in \mathbf{B}_r$ and $0 < \rho < r$. Thus by 1.3

$$\begin{aligned} \Theta(x) &\leq \rho^{1-n} \int_{\mathbf{B}_\rho(x)} |Du| + C_n \rho \\ &\leq \rho^{1-n} (\rho + |x|)^{n-1} (\rho + |x|)^{1-n} \int_{\mathbf{B}_{\rho+|x|}} |Du| + C_n (\rho + |x|) \\ &= (1 + \rho^{-1}|x|)^{n-1} (\rho + |x|)^{1-n} \int_{\mathbf{B}_{\rho+|x|}} |Du| + C_n (\rho + |x|). \end{aligned}$$

Holding ρ fixed and letting $x \rightarrow 0$ yields that

$$\limsup_{x \rightarrow 0} \Theta(x) \leq \rho^{1-n} \int_{\mathbf{B}_\rho} |Du| + C_n \rho.$$

Then letting $\rho \rightarrow 0$ gives

$$\limsup_{x \rightarrow 0} \Theta(x) \leq \Theta(0).$$

One may easily derive alternative formulas for the density based on the local functional (1.2) and the minimality of u .

Corollary 1.5. *For u as above and $a \in \Omega$,*

$$\begin{aligned} \Theta(a) &= \lim_{r \rightarrow 0} r^{1-n} \int_{\mathbf{B}_{r(a)}} \phi(Du) \\ &= \lim_{r \rightarrow 0} r^{1-n} \int_{\partial \mathbf{B}_{r(a)}} \sigma \cdot \frac{x}{|x|} u dS = \lim_{r \rightarrow 0} r^{1-n} \int_{\mathbf{B}_{r(a)}} \sigma \cdot Du. \end{aligned}$$

PROOF. By (0.1),

$$0 \leq r^{1-n} \int_{\mathbf{B}_{r(a)}} |Du| - r^{1-n} \int_{\mathbf{B}_{r(a)}} \phi(Du) \leq \lambda r^{1-n} |\mathbf{B}_r| \rightarrow 0$$

as $r \rightarrow 0$, which implies the first equality. By (1.2), (1.4), and the boundedness of σ ,

$$\begin{aligned} r^{1-n} \int_{\mathbf{B}_{r(a)}} \phi(Du) - r^{1-n} \int_{\partial \mathbf{B}_{r(a)}} \sigma \cdot \frac{x}{|x|} u dS &= \\ &= -\frac{1}{2} r^{1-n} \int_{\mathbf{B}_{r(a)}} \phi^*(\sigma) dx \rightarrow 0 \end{aligned}$$

as $r \rightarrow 0$, since $0 \leq \phi^*(\sigma) \leq \lambda$, which implies the second equality. The third equality follows from Lemma 1.2 applied with $v = u$.

Recall that a measurable function u is *approximately continuous* at a if a is a *Lebesgue point* for u , that is, if

$$\lim_{\rho \rightarrow 0} \rho^{-n} \int_{\mathbf{B}_{\rho(a)}} |u - c| dx = 0 \quad \text{for some } c \in \mathbf{R}.$$

Corollary 1.6. *If a local solution is approximately continuous at a , then $\Theta(a) = 0$.*

PROOF. We apply Lemma 1.2 with $v = \eta \cdot (u - c)$ where c is as above and η is as in the proof of Lemma 1.2 with $r = 2\rho$. We conclude that

$$0 = \int_{B_{2\rho}(a)} \sigma \cdot Dv = \int_{B_{2\rho}(a)} \sigma \cdot [\eta Du + (u - c) \nabla \eta dx];$$

hence,

$$\int_{B_\rho(a)} |Du| \leq \rho^{-1} \int_{B_{2\rho}(a)} |u - c| dx.$$

By 1.5,

$$\begin{aligned} \Theta(a) &= \lim_{\rho \rightarrow 0} \rho^{1-n} \int_{B_\rho(a)} \sigma \cdot Du \leq \lim_{\rho \rightarrow 0} \rho^{1-n} \int_{B_\rho(a)} |Du| \\ &\leq \lim_{\rho \rightarrow 0} \rho^{-n} \int_{B_{2\rho}(a)} |u - c| dx = 0. \end{aligned}$$

2. The local boundedness. We establish here that a local solution is locally bounded and that it is continuous at each point $a \in \Omega$ where the density $\Theta(a) = 0$. Our first step is to estimate the measure of the set where a local solution exceeds a given value. We assume that

u is a local solution in Ω with $f \in L^\infty(\Omega)$, $0 \in \Omega$, and $\theta : \mathbf{R} \rightarrow \mathbf{R}$ is a bounded increasing piecewise differentiable function with $\theta(t) \leq 1$ for almost all t .

Suppose that $0 < \rho < r < \min\{1, \text{dist}(0, \partial\Omega)\}$ and let

$$\eta(x) = \begin{cases} 1 & \text{in } B_\rho \\ (r - \rho)^{-1}(r - |x|) & \text{in } B_r \sim B_\rho \\ 0 & \text{in } \Omega \sim B_r, \end{cases}$$

exactly as in Lemma 1.2. We may apply the variational formula (1.1) with $\zeta = \eta\theta(u - c)$, where c is a constant, to obtain that

$$\begin{aligned} (2.1) \quad & \int_{B_r} \eta \sigma \cdot D[\theta(u - c)] = \\ &= (r - \rho)^{-1} \int_{B_r \sim B_\rho} \sigma \cdot \frac{x}{|x|} \theta(u - c) dx + \int_{B_r} \eta \theta(u - c) f dx. \end{aligned}$$

Observe that

$$D[\theta(u - c)] = \theta'(u - c)Du \quad \text{and} \quad |D\theta(u - c)| = \theta'(u - c)|Du|.$$

Thus, by (0.2) and (0.8),

$$\begin{aligned} \int_{\mathbf{B}_r} \eta |D[\theta(u - c)]| &\leq \int_{\mathbf{B}_r} \eta \theta'(u - c) |Du| \\ &\leq \int_{\mathbf{B}_r} \eta \theta'(u - c) \phi(Du) + \lambda \int_{\mathbf{B}_r} \eta \theta'(u - c) dx \\ &\leq \int_{\mathbf{B}_r} \eta \theta'(u - c) \sigma \cdot Du + \lambda \int_{\mathbf{B}_r} \eta \theta'(u - c) dx \\ &= \int_{\mathbf{B}_r} \eta \sigma \cdot D[\theta(u - c)] + \lambda \int_{\mathbf{B}_r} \eta \theta'(u - c) dx. \end{aligned}$$

Applying (2.1) and recalling that $|\sigma| \leq 1$, we now obtain the basic estimate

$$\begin{aligned} (2.2) \quad \int_{\mathbf{B}_\rho} \eta |D[\theta(u - c)]| &\leq (r - \rho)^{-1} \int_{\mathbf{B}_r \sim \mathbf{B}_\rho} |\theta(u - c)| dx + \lambda |A| \\ &\quad + \|f\|_{L^\infty(\mathbf{B}_r)} \int_{\mathbf{B}_r} |\theta(u - c)| dx \end{aligned}$$

where $A = \text{supp } \eta \theta(u - c)$.

Now we let $0 < h < k < \infty$ and choose

$$(2.3) \quad \theta(t) = \begin{cases} 0 & \text{for } t \leq k \\ t - k & \text{for } k < t < h \\ h - k & \text{for } t \geq h. \end{cases}$$

Note that

$$\text{supp } (\eta \theta(u - c)) \subset \mathbf{B}_r \cap \{u - c \geq k\} = A(k, r).$$

Thus (2.2) implies that

$$(2.4) \quad \int_{\mathbf{B}_\rho} |D[\theta(u - c)]| \leq (C + (r - \rho)^{-1}) \int_{\mathbf{B}_\rho} |\theta(u - c)| dx + \lambda |A(k, r)|,$$

$$C = \|f\|_{L^\infty(\Omega)}.$$

Let $0 < r_0 < \text{dist}(0, \partial\Omega)$ be given and choose k_0 so large that

$$|A(k_0, \rho)| \leq \frac{1}{2} |\mathbf{B}_\rho| \quad \text{for} \quad \frac{1}{2} r_0 \leq \rho \leq r_0.$$

Now $A(k, \rho) \subset A(k_0, \rho)$ for $k > k_0$, so by the isoperimetric inequality, the definition of θ , and (2.4),

$$\begin{aligned} (h - k) |A(h, \rho)|^{\frac{n-1}{n}} &\leq \left(\int_{\mathbf{B}_\rho} |\theta(u - c)|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \\ &\leq c_0 \int_{\mathbf{B}_\rho} |D[\theta(u - c)]| \\ &\leq c_0 [C + (r - \rho)^{-1}] \int_{\mathbf{B}_r \sim \mathbf{B}_\rho} |\theta(u - c)| dx + c_0 \lambda |A(k, r)| \\ &\leq c_0 [C + (r - \rho)^{-1}] (h - k) |A(k, r)| + c_0 \lambda |A(k, r)|. \end{aligned}$$

Hence,

$$|A(h, \rho)|^{\frac{n-1}{n}} \leq c_0 (C + \lambda) [1 + (r - \rho)^{-1} + (h - k)^{-1}] |A(k, r)|.$$

Recalling our restriction that $r_0 < 1$, we may rewrite the above as

$$\begin{aligned} (2.5) \quad |A(h, \rho)|^{\frac{n-1}{n}} &\leq \Lambda [(r - \rho)^{-1} + (h - k)^{-1}] |A(k, r)| \\ &\text{for} \quad \frac{1}{2} r_0 \leq \rho < r \leq r_0 \quad \text{and} \quad h > k \geq k_0, \end{aligned}$$

where the $\Lambda = \nu_0 + \nu_1 \|f\|_{L^\infty(\Omega)}$ for suitable constants ν_0, ν_1 .

The truncation estimate (2.5) differs from the more traditional one of De Giorgi and Stampacchia, cf. for a typical example [KS] p.63, in several notable ways. The first is the sum occurring on the right hand side which is usually a product. The second is the limitation on the range of ρ for which the estimate is valid.

At this point, we state our truncation lemma.

Lemma 2.1. *Suppose that $\{A(k, r) : \frac{1}{2} r_0 \leq r \leq r_0, k \geq k_0\}$ is any collection of subsets of $\mathbf{B}_{r_0}$ that satisfies (2.5) and that δ is a positive number with*

$$(2.6) \quad \delta \leq \frac{1}{2} |\mathbf{B}| \quad \text{and} \quad \delta^{\frac{1}{n}} \leq c_1^{-1} \quad \text{where} \quad c_1 = 2^{2n+3} c_0 (1 + \lambda).$$

if

$$(2.7) \quad |A(k_0, r_0)| \leq \delta r_0^n,$$

then

$$|A(k_0 + d, \frac{1}{2}r_0)| = 0 \quad \text{for } d = c_1 |A(k_0, r_0)|^{\frac{1}{n}}.$$

PROOF. For $i = 1, 2, \dots$, let

$$r_i = \frac{1}{2}r_0 + 2^{-i}r_0 \quad \text{and} \quad k_i = k_0 + d - 2^{-i}d.$$

Then

$$r_i - r_{i+1} = 2^{-i-1}r_0 \quad \text{and} \quad k_{i+1} - k_i = 2^{-i-1}d.$$

With

$$\alpha_i = |A(k_i, r_i)|^{\frac{n-1}{n}},$$

our goal is to prove inductively the estimate

$$(2.8) \quad \alpha_i \leq 2^{-ni} \alpha_1.$$

For this, we first note that

$$\alpha_1 \leq \delta^{\frac{n-1}{n}} r_0^{n-1} \quad \text{and} \quad d = \alpha_1^{\frac{1}{n-1}}.$$

Written in terms of the sequence α_i , the recursion relation (2.5) is

$$(2.9) \quad \alpha_{i+1} \leq 2^{i+1} c_0 (r_0^{-1} + d^{-1}) \alpha_i^{\frac{n}{n-1}}.$$

In particular, by the choice of δ and d ,

$$\begin{aligned} \alpha_2 &\leq 4c_0 (\alpha_1^{\frac{1}{n-1}} r_0^{-1} + \alpha_1^{\frac{1}{n-1}} d^{-1}) \alpha_1 \\ &\leq 4c_0 (\delta^{\frac{1}{n}} + c^{-1}) \alpha_1 \\ &\leq (2^{-2n-1} + 2^{-2n-1}) \alpha_1 = 2^{-2n} \alpha_1. \end{aligned}$$

Now assume inductively that $\alpha_i \leq 2^{-in}\alpha_1$. Then, by (2.9),

$$\begin{aligned}\alpha_{i+1} &\leq 2^{i+1}c_0(r_0^{-1} + d^{-1})\alpha_1^{\frac{n}{n-1}}(\alpha_i\alpha_1^{-1})^{\frac{n}{n-1}} \\ &\leq 2^i 4c_0(\alpha_1^{\frac{1}{n-1}}r_0^{-1} + \alpha_1^{\frac{1}{n-1}}d^{-1})\alpha_1(2^{-ni})^{\frac{n}{n-1}} \\ &\leq 2^i 2^{-2n}(2^{-ni})^{\frac{n}{n-1}}\alpha_1 \leq 2^{-(i+1)n},\end{aligned}$$

the latter inequality holding because the inequality

$$i - 2n - ni\left(\frac{n}{n-1}\right) \leq -(i+1)n$$

reduces to simply $n(n-1) + i \geq 0$. This completes the inductive proof of (2.8). The lemma now follows because

$$|A(k_0 + d, \frac{1}{2}r_0)| = \lim_{i \rightarrow \infty} \alpha_i^{\frac{n}{n-1}} = 0.$$

Note the hypotheses of the lemma implies that

$$(2.10) \quad d = c_1|A(k_0, r_0)|^{\frac{1}{n}} \leq c_1\delta^{\frac{1}{n}}r_0.$$

Consequently, whenever (2.5) and (2.7) are satisfied for a local solution u on $\mathbf{B}_{r_0}$,

$$u - c \leq k_0 + c_1\delta^{\frac{1}{n}} \quad \text{in } \mathbf{B}_{r_0/2}.$$

Since we may argue analogously with

$$\theta(t) = \begin{cases} h - k & \text{for } t \leq k \\ k - t & \text{for } k < t < h \\ 0 & \text{for } t \geq h, \end{cases}$$

we may conclude that

$$(2.11) \quad \|u - c\|_{L^\infty(\mathbf{B}_{r_0/2})} \leq k_0 + c_1\delta^{1/n}r_0$$

whenever (2.5) and (2.7) are satisfied for the sets

$$A(k, r) = \mathbf{B}_r \cap \{u \geq k\} \quad \text{and} \quad \tilde{A}(k, r) = \mathbf{B}_r \cap \{u \leq k\}.$$

Theorem 2.2. *If u is a local solution in Ω and $\mathbf{B}_{2r}(a) \subset\subset \Omega$, then there is a positive constant M depending only on $\|f\|_{L^\infty}$ so that*

$$(2.12) \quad \|u - \bar{u}_r\|_{L^\infty(\mathbf{B}_r(a))} \leq M \left\{ r^{1-n} \int_{\mathbf{B}_{2r}(a)} |Du| + r \right\}$$

where

$$\bar{u}_r = |\mathbf{B}_r|^{-1} \int_{\mathbf{B}_r(a)} u dx.$$

PROOF. We may assume that $a = 0$. We may also apply (2.11) to $u - \bar{u}_r$, with $r_0 = 2r$. By (2.5) and (2.11) it only remains to estimate k_0 .

Suppose that $r \leq \rho \leq 2r < 1$. Then we may average the inequality

$$|\bar{u}_r - \bar{u}_\rho| \leq |u(x) - \bar{u}_r| + |u(x) - \bar{u}_\rho|$$

over the smaller ball $\mathbf{B}_r$ to obtain

$$\begin{aligned} |\bar{u}_r - \bar{u}_\rho| &\leq |\mathbf{B}_r|^{-1} \int_{\mathbf{B}_r} |u - \bar{u}_r| dx + |\mathbf{B}_r|^{-1} |\mathbf{B}_\rho| |\mathbf{B}_\rho|^{-1} \int_{\mathbf{B}_\rho} |u - \bar{u}_\rho| dx \\ &\leq c_2 \left\{ r^{1-n} \int_{\mathbf{B}_r} |Du| + 2^n \rho^{1-n} \int_{\mathbf{B}_\rho} |Du| \right\} \\ (2.13) \quad &\leq c_2 \left\{ (1 + 2^n)^{\Lambda r} \rho^{1-n} \int_{\mathbf{B}_\rho} |Du| + c_n \rho \right\} \\ &\leq c_3 \left\{ \rho^{1-n} \int_{\mathbf{B}_\rho} |Du| + \rho \right\}, \end{aligned}$$

where we have employed a Poincaré inequality and the monotonicity inequality (Theorem 1.3).

Letting

$$A_r(k, \rho) = \mathbf{B}_\rho \cap \{|u - \bar{u}_r| \geq k\},$$

we use a Sobolev inequality and (2.13) to calculate

$$\begin{aligned} k |A_r(k, \rho)|^{\frac{n-1}{n}} &\leq \|u - \bar{u}_r\|_{L^{\frac{n}{n-1}}(\mathbf{B}_\rho)} \\ &\leq \|u - \bar{u}_\rho\|_{L^{\frac{n}{n-1}}(\mathbf{B}_\rho)} + \|\bar{u}_\rho - \bar{u}_r\|_{L^{\frac{n}{n-1}}(\mathbf{B}_\rho)} \\ &\leq c_4 \int_{\mathbf{B}_\rho} |Du| + c_3 \left\{ \rho^{1-n} \int_{\mathbf{B}_\rho} |Du| + \rho \right\} |\mathbf{B}|^{\frac{n-1}{n}} \end{aligned}$$

or

$$k(\rho^{-n}|A_r(k, \rho)|)^{\frac{n-1}{n}} \leq c_5 \{\rho^{1-n} \int_{B_\rho} |Du| + \rho\}.$$

Selecting

$$k_0 = \inf\{\ell : |A_r(k, \rho)| \leq \delta \rho^n \text{ for all } r \leq \rho \leq 2r, k \geq \ell\}$$

now permits us to apply (2.11) with $r_0 = 2r$ as well as to estimate

$$\begin{aligned} k_0 &\leq c_5 \delta^{\frac{n}{n-1}} \sup_{r \leq \rho \leq 2r} \{\rho^{1-n} \int_{B_\rho} |Du| + \rho\} \\ &\leq c_6 \{r^{1-n} \int_{B_r} |Du| + r\}. \end{aligned}$$

This, combined with (2.11), completes the proof.

Corollary 2.3. *The local solution u is continuous at $a \in \Omega$ if and only if*

$$\Theta(a) = \lim_{r \rightarrow 0} r^{1-n} \int_{B_r(a)} |Du| = 0.$$

Moreover,

$$(2.14) \quad \|u - u(a)\|_{L^\infty(B_r(a))} \leq N \{r^{1-n} \int_{B_{2r}(a)} |Du| + r\}$$

for some constant N depending only on Ω , f , and a .

PROOF. We again take $a = 0$. The necessity of the condition $\theta(a) = 0$ was established in 1.5. To use 2.2 to prove the sufficiency we need to control the averages $\bar{u}_\rho$ as $\rho \rightarrow 0$. Using 2.2, we see that, for each positive $\rho \leq r$,

$$\begin{aligned} |\bar{u}_\rho| &\leq |B_\rho|^{-1} \left| \int_{B_\rho} u dx \right| \\ &\leq |\bar{u}_r| + |B_\rho|^{-1} \int_{B_\rho} |u - \bar{u}_r| dx \\ &\leq |\bar{u}_r| + M \{r^{1-n} \int_{B_{2r}} |Du| + r\}. \end{aligned}$$

Thus the collection $\{\bar{u}_\rho : 0 < \rho \leq r\}$ is uniformly bounded. Moreover,

$$\begin{aligned} |\bar{u}_\rho - \bar{u}_r| &\leq \\ &\leq |\mathbf{B}_\rho|^{-1} \int_{\mathbf{B}_\rho} |u - \bar{u}_\rho| dx + |\mathbf{B}_\rho|^{-1} \int_{\mathbf{B}_\rho} |u - \bar{u}_r| dx \\ &\leq c_2 \rho^{1-n} \int_{\mathbf{B}_\rho} |Du| + M \{r^{1-n} \int_{\mathbf{B}_{2r}} |Du| + r\}. \end{aligned}$$

It follows that $\{\bar{u}_\rho\}$ is Cauchy as $\rho \rightarrow 0$ if $\Theta(0) = 0$. In this case, the limit $u(0)$ satisfies

$$|u - u(0)| \leq N \{r^{1-n} \int_{\mathbf{B}_{2r}} |Du| + r\} \quad \text{in } \mathbf{B}_r.$$

3. Points of positive density. From [F], 4.5.9(3) we recall that a BV function u has, at $\mathcal{H}^{n-1}$ almost all points a of its domain, finite *approximate upper and lower limits* $u^+(a) \geq u^-(a)$ satisfying

$$\begin{aligned} u^+(a) &= \inf \{k : \lim_{\rho \rightarrow 0^+} |\mathbf{B}_\rho|^{-1} |\mathbf{B}_\rho(a) \cap \{u > k\}| = 0\} \quad \text{and} \\ (3.1) \quad u^-(a) &= \sup \{k : \lim_{\rho \rightarrow 0^+} |\mathbf{B}_\rho|^{-1} |\mathbf{B}_\rho(a) \cap \{u < k\}| = 0\}. \end{aligned}$$

Moreover, using [F], 4.5.9(27), one finds that these limits differ at $\mathcal{H}^{n-1}$ almost all points a where the upper $n-1$ density

$$\limsup_{r \rightarrow 0} r^{1-n} \int_{\mathbf{B}_r(a)} |Du| > 0.$$

In this section we shall limit ourselves to the case where $f = 0$.

Theorem 3.1. *Suppose that $f = 0$ and u is a local solution in Ω , $a \in \Omega$, $\Theta(a) > 0$ and $u^+ > u^-$ where $u^\pm = u^\pm(a)$. For any $\epsilon > 0$, there is a $\sigma > 0$ so that*

$$u^- - \epsilon < u < u^+ + \epsilon \quad \text{in } \mathbf{B}_\sigma(a).$$

PROOF. We assume $a = 0$. For fixed $0 < \rho < r < \text{dist}(0, \partial\Omega)$, we again let η be as in 1.2, and, for fixed $h > k \geq 0$, let θ be as in (2.3). We now use the test function

$$\zeta(x) = \eta(x) \cdot \theta(u(x) - u^+)$$

in the variational formula (1.1). Since

$$\text{supp}\zeta \subset \mathbf{B}_r \cap \text{supp}(\theta(u - u^+)) = \mathbf{B}_r \cap \{u - u^+ \geq k\},$$

we deduce, exactly as in the proof of (2.4) and using that u is bounded, that

$$(3.2) \quad \int_{\mathbf{B}_\rho} |D[\theta(u - u^+)]| \leq (r - \rho)^{-1} \int_{\mathbf{B}_r \sim \mathbf{B}_\rho} |\theta(u - u^+)| dx + C|A^+(k, r)|$$

where C is a constant and

$$A^+(k, r) = \mathbf{B}_r \cap \{u - u^+ \geq k\}.$$

By (3.1), we may, for any positive number k_0 , choose a positive $r_0 < \min\{k_0, \text{dist}(0, \partial\Omega)\}$ so that

$$(3.3) \quad |A^+(k_0, r_0)| \leq \min\{\delta, 2^{-n-1}|\mathbf{B}|\}r_0^n$$

where δ is the constant from Lemma 2.1. Then, for $\frac{1}{2}r_0 \leq \rho \leq r_0$ and $k \geq k_0$,

$$|A^+(k, \rho)| \leq |A^+(k_0, r_0)| \leq 2^{-n-1}|\mathbf{B}_{r_0}|\mathbf{B}_\rho^{-1}|\mathbf{B}_\rho| \leq \frac{1}{2}|\mathbf{B}_{r_0}|.$$

Since

$$\mathbf{B}_\rho \cap \text{supp}(\theta(u - u^+)) \subset |A^+(k, \rho)|,$$

we infer from a Sobolev inequality and (3.2) that, for $\frac{1}{2}r_0 \leq \rho < r \leq r_0$ and $h > k \geq k_0$,

$$(h - k)|A^+(h, \rho)|^{\frac{n-1}{n}} \leq \left[\int_{A^+(h, \rho)} |\theta(u - u^+)|^{\frac{n}{n-1}} dx \right]^{\frac{n-1}{n}} \leq$$

$$\begin{aligned}
&\leq \left[\int_{B_\rho} |\theta(u - u^+)|^{\frac{n}{n-1}} dx \right]^{\frac{n-1}{n}} \leq c_0 \int_{B_r} |D[\theta(u - u^+)]| \\
&\leq c_0(r - \rho)^{-1} \int_{B_r \sim B_\rho} |\theta(u - u^+)| dx + c_0 C |A^+(k, r)| \\
&\leq c_0(r - \rho)^{-1}(h - k) |A^+(k, r)| + c_0 C |A^+(k, r)|.
\end{aligned}$$

Thus we find, as in our previous argument in §2 that for a constant C_0 ,

$$\begin{aligned}
(3.4) \quad &|A^+(h, \rho)|^{\frac{n-1}{n}} \leq C_0 [(r - \rho)^{-1} + (h - k)^{-1}] |A^+(k, r)| \\
&\text{for } \frac{1}{2}r_0 \leq \rho < r \leq r_0 \text{ and } h > k \geq k_0.
\end{aligned}$$

By (3.3) and (3.4), we may again apply Lemma 2.1 to conclude that

$$u - u^+ \leq k_0 + d = k_0 + c_1 |A(k_0, r_0)|^{\frac{1}{n}} \leq c_7 k_0 \text{ in } B_{r_0/2}.$$

Similarly, there is a positive s_0 such that

$$u - u^- \geq c_7 k_0 \text{ in } B_{s_0/2}.$$

Next we seek more precise information on the behavior of u near a point of positive density. From [F], 4.5.9(22), we recall that, a BV function u , is, at $\mathcal{H}^{n-1}$ almost all points a of positive upper density, *approximately two-sided continuous* in the sense that there is a unit vector $\nu(a)$ so that

$$\alpha^\pm(\rho) = [\rho^{-n} \int_{B_\rho^\pm(a)} |u - u^\pm|^{\frac{n}{n-1}} dx]^{\frac{n-1}{n}} \rightarrow 0 \text{ as } \rho \rightarrow 0$$

where $B_\rho^\pm(a) = B_\rho(a) \cap \{\pm(x - a) \cdot \nu(a) > 0\}$.

Theorem 3.2. *Suppose that, in addition to the hypotheses of Theorem 3.1, the local solution u is approximately two-sided continuous at a in the direction ν . For any $0 < \beta < \frac{1}{n-1}$ and any $\epsilon > 0$, there is a $\tau > 0$ so that*

$$\begin{aligned}
(3.5) \quad &|u - u^\pm| \leq \epsilon \\
&\text{in } E^\pm = B_\tau(a) \cap \{x : \pm(x - a) \cdot \nu \geq [\alpha^\pm(2|x - a|)]^\beta |x - a|\}
\end{aligned}$$

where $\alpha^\pm$ is as above. In particular, the two sets

$$\{x : u(x) > \frac{1}{2}(u^+ + u^-)\} \quad \text{and} \quad \{x : u(x) < \frac{1}{2}(u^+ + u^-)\}$$

contain open domains whose boundaries are tangent to the hyperplane $\{(x - a) \cdot \nu = 0\}$ at a . Moreover,

$$(3.6) \quad \lim_{\rho \rightarrow 0} \int_{B_\rho(a)} Du / \int_{B_\rho(a)} |Du| = \nu.$$

PROOF. We assume $a = 0$. Choose a positive number $s \leq \frac{1}{2}(c_1^{-1} \delta^{-\frac{1}{n}})$ so that $B_s \subset \Omega$ and

$$[\alpha^+(2t)]^{1-\beta(n-1)} \leq 4^{1-n}(\delta|B|)^{\frac{n-1}{n}} \epsilon \quad \text{for all } 0 < t \leq s.$$

Suppose $b \in B_s$ and $b \cdot \nu \geq [\alpha^+(2|b|)]^\beta |b|$. Setting

$$r_0 = [\alpha^+(2|b|)]^\beta |b| \quad \text{and} \quad k_0 = \frac{1}{2} \epsilon,$$

we see that

$$(3.7) \quad k_0^{-1} 4^{n-1} |B|^{\frac{n}{n-1}} [\alpha^+(2|b|)] (r_0^{-1} |b|)^{n-1} \leq \delta^{\frac{n-1}{n}}.$$

Next, for $\frac{1}{2} r_0 \leq \rho \leq r_0$ and $k \geq k_0$, we let

$$A_b(k, \rho) = B_\rho(b) \cap \{u < u^+ - k\},$$

note that $B_\rho(b) \subset B_{\rho+|b|}^+$ because $b \cdot \nu \geq \rho$, and use the definition of α^+ and (3.7) to estimate

$$\begin{aligned} k |A_b(k, \rho)|^{\frac{n-1}{n}} &\leq \left[\int_{A_b(k, \rho)} |u - u^+|^{\frac{n}{n-1}} dx \right]^{\frac{n-1}{n}} \\ &\leq \left[\int_{B_{\rho+|b|}^+} |u - u^+|^{\frac{n}{n-1}} dx \right]^{\frac{n-1}{n}} \\ &\leq [\alpha^+(\rho + |b|)] (\rho + |b|)^{n-1} \\ &\leq 2^{n-1} [\alpha^+(2|b|)] (\rho^{-1} |b|)^{n-1} \rho^{n-1} \\ &\leq 4^{n-1} [\alpha^+(2|b|)] (r_0^{-1} |b|)^{n-1} \rho^{n-1} \\ &\leq k_0 \delta^{\frac{n-1}{n}} \rho^{n-1} \leq k \delta^{\frac{n-1}{n}} \rho^{n-1}. \end{aligned}$$

Thus,

$$(3.8) \quad |A_b(k, \rho)| \leq \delta \rho^n \quad \text{for } \frac{1}{2}r_0 \leq \rho \leq r_0 \quad \text{and } k \geq k_0.$$

Now for fixed ρ and r with $\frac{1}{2}r_0 \leq \rho < r \leq r_0$, we let η be as in 1.2. Also for fixed h and k with $h > k \geq k_0$, we let

$$\theta(t) = \begin{cases} h - k & \text{for } t \leq -k \\ -t - k & \text{for } -h < t < -k \\ 0 & \text{for } t \geq -k. \end{cases}$$

Using $\zeta(x) = \eta(x) \cdot \theta(u(x) - u^+)$ in the variational formula (1.1) and noting that

$$\text{supp}(\eta\theta'(u)) \subset \text{supp}\zeta \subset A_b(k, r),$$

we deduce, as in the proof of (2.4), that

$$(3.9) \quad \int_{B_\rho} |D[\theta(u - u^+)]| \leq (r - \rho)^{-1} \int_{B_r \sim B_\rho} |\theta(u - u^+)| dx + C|A_b(k, r)|.$$

Since $\delta \leq \frac{1}{2}|\mathbf{B}|$, we may, as in §2, use a Sobolev inequality to find that

$$|A_b(h, \rho)|^{\frac{n-1}{n}} \leq \text{const.} [(r - \rho)^{-1} + (h - k)^{-1}] |A_b(k, r)|$$

for $\frac{1}{2}r_0 \leq \rho < r \leq r_0$ and $h > k \geq k_0$. Applying 2.1 now gives

$$|A_b(k_0 + d, \frac{1}{2}r_0)| = 0 \quad \text{where } d = c_1 |A_b(k_0, r_0)|^{\frac{1}{n}}.$$

Thus,

$$u - u^+ \geq -(k_0 + c_1 \delta^{\frac{1}{n}} r_0) \geq -(\frac{1}{2}\epsilon + c_1 \delta^{\frac{1}{2}} s) \geq \epsilon \quad \text{in } B_{r_0/2}(b).$$

Similarly, for $b \in \mathbf{B}_s$ with $b \cdot \nu \leq -[\alpha^-(2|b|)]^\beta |b|$, we use

$$r_0 = [\alpha^-(2|b|)]^\beta |b|$$

and find that

$$u - u^+ \leq -\epsilon \quad \text{in } B_{r_0/2}(b).$$

Combining these two inequalities with Theorem 3.1 gives the existence of a suitable τ .

Finally to obtain the formula for the vector ν , we write

$$\int_{\mathbf{B}_\rho} Du = \int_{\partial \mathbf{B}_\rho} u \frac{x}{|x|} dS,$$

where the right hand side is to be understood in the sense of BV trace theory [G], and consider the integrals over the three spherical regions $\partial \mathbf{B}_\rho \cap E^+$, $\partial \mathbf{B}_\rho \cap E^-$, and $\partial \mathbf{B}_\rho \sim (E^+ \cup E^-)$ separately.

To handle the last region, note that by Theorem 3.1, u is bounded on $\mathbf{B}_\rho$ for ρ sufficiently small. Thus,

$$\begin{aligned} \left| \int_{\partial \mathbf{B}_\rho \sim (E^+ \cup E^-)} u \frac{x}{|x|} dS \right| &\leq \\ &\leq \|u\|_{L^\infty(\mathbf{B}_\rho)} \mathcal{H}^{n-1}(\partial \mathbf{B}_\rho \sim (E^+ \cup E^-)) \\ &\leq c \|u\|_{L^\infty(\mathbf{B}_\rho)} \max\{[\alpha^+(2\rho)]^\beta, [\alpha^-(2|b|)]^\beta\} \rho^{n-1} = o(\rho^{n-1}). \end{aligned}$$

For the other two regions, we use the first conclusion of 3.2 to see that

$$\begin{aligned} \left| \int_{\partial \mathbf{B}_\rho \cap E^\pm} u \frac{x}{|x|} dS - \int_{\partial \mathbf{B}_\rho \cap E^\pm} u^\pm \frac{x}{|x|} dS \right| &\leq \\ &\leq \int_{\partial \mathbf{B}_\rho \cap E^\pm} |u - u^\pm| dS \\ &\leq \text{ess sup}_{\partial \mathbf{B}_\rho \cap E^\pm} |u - u^\pm| \mathcal{H}^{n-1}(\partial \mathbf{B}_\rho) = o(\rho^{n-1}). \end{aligned}$$

Moreover, since each set $E^\pm$ is symmetric in the directions orthogonal to ν ,

$$\int_{\partial \mathbf{B}_\rho \cap E^\pm} u^\pm \frac{x}{|x|} dS = \left(\int_{\partial \mathbf{B}_\rho \cap E^\pm} \frac{x}{|x|} \cdot \nu dS \right) u^\pm \nu.$$

We conclude that

$$\rho^{1-n} \int_{\mathbf{B}_\rho} Du = \gamma \nu + o(1) \quad \text{as } \rho \rightarrow 0,$$

where $\gamma = [(\int_{\partial \mathbf{B} \cap E^+} \frac{x}{|x|} \cdot \nu dS) u^+ + (\int_{\partial \mathbf{B} \cap E^-} \frac{x}{|x|} \cdot \nu dS) u^-]$ is a nonzero scalar. Then

$$\rho^{1-n} \int_{\mathbf{B}_\rho} |Du| = |\gamma| + o(1) \quad \text{as } \rho \rightarrow 0,$$

and we may divide and let $\rho \rightarrow 0$ to obtain the desired formula.

As a consequence of Theorem 3.2 and the characterization from [F], 4.5.9(22), the set E^+ defined in (3.5) is a set of finite perimeter and (3.6) holds on $\partial E^+ \cap B_\delta(a)$, $\delta < \tau$, $\mathcal{H}^{n-1}$ a.e. Thus for $f = 0$ and η smooth with $\text{supp } \eta \subset B_\delta(a)$,

$$(3.10) \quad D(\eta \chi_{E^+}) \ll |Du|.$$

Working directly with the variational principle for the functional, this leads in a standard way to

Theorem 3.3 *Suppose that $f = 0$ and u is a local solution in Ω , $a \in \Omega$, $\Theta(a) > 0$, and $u^+(a) > u^-(a)$. Suppose in addition that u is two-sided continuous at a in the direction ν . Then there is a neighborhood $\mathbf{B}_\tau(a)$ such that the set*

$$\partial\{x : u(x) > \frac{1}{2}(u^+ + u^-)\} \cap \mathbf{B}_\tau(a)$$

is an area minimizing hypersurface in $\mathbf{B}_\tau(a)$.

We adopt several standard notations: $\mathbf{B}_r(a)$ indicates the open ball $\{x \in \mathbf{R}^n : |x - a| < r\}$ and $\mathbf{B}_r = \mathbf{B}_r(0)$. The Lebesgue outer measure of a set A is written $|A|$, while $\mathcal{H}^{n-1}$ denotes $n - 1$ dimensional Hausdorff measure. When restricted to a sphere $\partial \mathbf{B}_r(a)$, the latter becomes ordinary surface measure and will be, under integrals signs, abbreviated as dS .

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