**A VARIATIONAL INEQUALITY ASSOCIATED
WITH A LUBRICATION PROBLEM**

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§0. Introduction. In this paper, we study a variational inequality which describes a lubrication problem in a journal bearing. For the physical background, we refer the reader to Bayada[3] and Bayada and Chambat[4] for details. Given a constant $a \in (0, 1)$ and a smooth function $\theta(y)$ defined in $[0, 1]$ such that

$$(1 - a)/(1 + a) < \theta(y) < 1, \forall y \in [0, 1],$$

let $h(x) = 1 + a \cos x$, $g(x, y) = h(x) - (1 + a)\theta(y)$, and introduce the sets (see Figure 1)

$$\begin{aligned} \Omega &= (0, 2\pi) \times (0, 1), \quad R_1 = (0, \pi) \times (0, 1), \quad R_2 = (\pi, 2\pi) \times (0, 1), \\ \Omega_1 &= \{(x, y) \in R_1; g(x, y) > 0\}, \quad \Omega_2 = \Omega \setminus \overline{\Omega_1}, \quad T = \partial\Omega_1 \cap \Omega, \\ \partial_0\Omega &= \{0\} \times (0, 1), \quad \partial_1\Omega = \partial\Omega \setminus \partial_0\Omega. \end{aligned}$$

Let

$$K_0 = \{\phi \in H^1(\Omega); \phi \geq 0 \text{ a.e. in } \Omega_2, \phi = 0 \text{ on } \partial_1\Omega\}.$$

Figure 1

Note that $T \subset R_1$ and $g = 0$ on T by the assumption on θ . The following classical formulation of the lubrication problem was posed in [4].

PROBLEM(P_c). Find a non-negative continuous function $u(x, y)$ in $\bar{\Omega}$ and two disjoint free boundaries S_1 and S_2 such that the following conditions hold:

$$S_1 \subset \Omega_1, S_2 \subset R_2, \quad (0.1)$$

$$-\nabla \cdot (h^3 \nabla u) = a \sin x \quad \text{in } \Omega^+ \equiv \{u > 0\}, \quad (0.2)$$

$$u = 0 \quad \text{on } \partial\Omega, \quad (0.3)$$

$$u = 0 \quad \text{on } S \equiv \partial\Omega^+ \cap \Omega = S_1 \cup S_2, \quad (0.4)$$

$$h^3 \frac{\partial u}{\partial \nu} = g \nu_x \quad \text{on } S_1, \quad (0.5)$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } S_2, \quad (0.6)$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$, ν is the unit outer normal to $\partial\Omega^+$, $\nu = (\nu_x, \nu_y)$, and $\partial u/\partial \nu$ is defined from the right-hand side of S_1 and from the left-hand side of S_2 .

Hereafter $A^+(u)$ or A^+ will denote the set $A \cap \{u > 0\}$. It is anticipated that the set $\{u = 0\}$ lies to the left of S_1 and to the right of S_2 . To solve this problem, Bayada and Chambat [4] considered the following variational inequality which is a weak formulation of Problem(P_c).

PROBLEM(P). Find $u \in K_0$ such that

$$\int_{\Omega} h^3 \nabla u \cdot \nabla(\phi - u) \geq \int_{\Omega_1} I_{\Omega_1^+(u)} g(\phi - u)_x + \int_{\Omega_2} g(\phi - u)_x, \quad (0.7)$$

for all $\phi \in K_0$, where I_A denotes the characteristic function of A .

In this paper the subscript x or y will always represent the partial derivative of functions with respect to that variable. Note that we do not require functions in K_0 vanish on $\partial_0\Omega$. In [4], the authors proved existence of a solution to the ϵ -problem (see Lemma 1.1 in §1). However, they have not been able to use it to establish existence of a solution to Problem(P) (unless they a priori assume that the free boundary is Lipschitz). We instead start with a weaker formulation:

PROBLEM(P_0). Find (u, γ) , $u \in K_0$, $\gamma \in L^\infty(\Omega_1)$, $0 \leq \gamma \leq 1$ a.e. in Ω_1 and $\gamma \subset H(u)$, such that

$$\int_{\Omega} h^3 \nabla u \cdot \nabla(\phi - u) \geq \int_{\Omega_1} \gamma g(\phi - u)_x + \int_{\Omega_2} (a \sin x)(\phi - u), \quad (0.8)$$

for all $\phi \in K_0$, where $H(u)$ is the Heaviside graph of u (i.e., $H(t) = 1$ if $t > 0$, $= 0$ if $t < 0$, and $= [0, 1]$ if $t = 0$).

We shall establish existence of a solution to Problem(P_0) and prove a regularity theorem for solutions to Problem(P_0) in §1. In §2 we shall be using, among other methods, the

comparison argument of [4] to prove, under some condition on θ , that any solution of Problem(P_0) vanishes on all of $\partial\Omega$; we shall also derive some properties of the free boundary and the non-coincidence set Ω^+ . In §3 we prove that $\gamma = I_{\Omega_1^+(u)}$ for any solution (u, γ) of Problem(P_0) and establish some regularity results of the free boundary. Since $g = 0$ on T by assumption, the second integrals on the right-hand sides of (0.7) and (0.8) are equal. Therefore, the two problems (P_0) and (P) are equivalent. Finally, in §4 we shall prove that there exists only one solution to Problem(P_0).

§1. Existence and regularity. For any $\epsilon > 0$, let

$$H_\epsilon(t) = \begin{cases} 0 & \text{if } t < 0, \\ t/\epsilon & \text{if } 0 \leq t \leq \epsilon, \\ 1 & \text{if } t > \epsilon. \end{cases}$$

LEMMA 1.1. *There exists a $u_\epsilon \in K_0$, $u_\epsilon \geq 0$ in Ω , such that the inequality*

$$\int_{\Omega} h^3 \nabla u_\epsilon \cdot \nabla(\phi - u_\epsilon) \geq \int_{\Omega_1} H_\epsilon(u_\epsilon) g(\phi - u_\epsilon)_x + \int_{\Omega_2} (a \sin x)(\phi - u_\epsilon) \quad (1.1)$$

holds for all $\phi \in K_0$. Moreover,

$$\int_{\Omega} |\nabla u_\epsilon|^2 \leq C, \quad (1.2)$$

where C is a positive constant independent of ϵ .

The proof of Lemma 1.1 is given in [4].

Taking $\phi = u_\epsilon \pm \zeta$ with $\zeta \in C_0^\infty(\Omega_1)$ in (1.1), we deduce that u_ϵ is a weak solution of the equation

$$-\nabla \cdot (h^3 \nabla u_\epsilon) = -(H_\epsilon(u_\epsilon) g)_x \quad \text{in } \Omega_1. \quad (1.3)$$

Next, taking $\zeta \in C^\infty(\Omega_1)$, $\zeta = 0$ on $\partial\Omega_1 \setminus \partial_0\Omega$, we obtain

$$\int_{\partial_0\Omega} h^3 \frac{\partial u_\epsilon}{\partial x} \zeta = \int_{\partial_0\Omega} H_\epsilon(u_\epsilon) g \zeta,$$

or equivalently,

$$\frac{\partial u_\epsilon}{\partial x} = \frac{1 - \theta(y)}{(1 + a)^2} H_\epsilon(u_\epsilon) \quad \text{on } \partial_0\Omega. \quad (1.4)$$

From (1.3), (1.4) and $u_\epsilon = 0$ on $\partial_1\Omega$ it follows by elliptic regularity theory(cf. [7] and [10]) that

$$\|u_\epsilon\|_{C^\alpha(\overline{\Omega'})} \leq C, \forall \Omega' \subset \subset \overline{\Omega_1} \setminus T, \forall \alpha \in (0, 1), \quad (1.5)$$

where C is a positive constant independent of ϵ .

THEOREM 1.2. *There exists a solution (u, γ) of Problem(P_0).*

Proof. By (1.2) and (1.5), there is a sequence $\{\epsilon_n\}$, $\epsilon_n \downarrow 0$ as $n \rightarrow \infty$, such that

$$\begin{aligned} u_{\epsilon_n} &\rightarrow u \quad \text{weakly in } H^1(\Omega), \\ u_{\epsilon_n} &\rightarrow u \quad \text{uniformly on any compact subset of } \overline{\Omega_1} \setminus T, \\ u_{\epsilon_n} &\rightarrow u \quad \text{strongly in } L^2(\Omega) \quad \text{and a.e. in } \Omega, \\ H_{\epsilon_n}(u_{\epsilon_n}) &\rightarrow \gamma \quad \text{weakly in } L^2(\Omega_1) \quad \text{and weakly}^* \text{ in } L^\infty(\Omega_1). \end{aligned} \tag{1.6}$$

Also, $\gamma = 1$ a.e. in Ω_1^+ and $0 \leq \gamma \leq 1$ a.e. in Ω_1 .

For any $\phi \in K_0$ we have

$$\int_{\Omega} h^3 \nabla u_{\epsilon_n} \cdot \nabla \phi \rightarrow \int_{\Omega} h^3 \nabla u \cdot \nabla \phi, \tag{1.7}$$

$$\int_{\Omega_2} (a \sin x)(\phi - u_{\epsilon_n}) \rightarrow \int_{\Omega_2} (a \sin x)(\phi - u), \tag{1.8}$$

$$\int_{\Omega_1} H_{\epsilon_n}(u_{\epsilon_n}) g \phi_x \rightarrow \int_{\Omega_1} \gamma g \phi_x, \tag{1.9}$$

as $n \rightarrow \infty$, and

$$\int_{\Omega} h^3 |\nabla u|^2 \leq \liminf_{n \rightarrow \infty} \int_{\Omega} h^3 |\nabla u_{\epsilon_n}|^2. \tag{1.10}$$

Introduce the function

$$G_{\epsilon}(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t^2/(2\epsilon) & \text{if } 0 \leq t \leq \epsilon \\ t - \epsilon/2 & \text{if } t \geq \epsilon; \end{cases}$$

it satisfies $G'_{\epsilon}(t) = H_{\epsilon}(t)$. Clearly

$$\begin{aligned} |G_{\epsilon_n}(u_{\epsilon_n}) - u| &\leq |G_{\epsilon_n}(u_{\epsilon_n}) - G_{\epsilon_n}(u)| + |G_{\epsilon_n}(u) - u| \\ &\leq |u_{\epsilon_n} - u| + |G_{\epsilon_n}(u) - u|. \end{aligned} \tag{1.11}$$

Hence, using the fact that $g = 0$ on T , we get

$$\begin{aligned} - \int_{\Omega_1} H_{\epsilon_n}(u_{\epsilon_n}) g u_{\epsilon_n, x} &= - \int_{\Omega_1} G'_{\epsilon_n}(u_{\epsilon_n}) u_{\epsilon_n, x} g \\ &= - \int_{\Omega_1} \frac{\partial}{\partial x} [G_{\epsilon_n}(u_{\epsilon_n})] g \\ &= \int_{\Omega_1} G_{\epsilon_n}(u_{\epsilon_n}) g_x + \int_{\partial_0 \Omega} G_{\epsilon_n}(u_{\epsilon_n}) g \\ &\rightarrow \int_{\Omega_1} u g_x + \int_{\partial_0 \Omega} u g \quad \text{as } n \rightarrow \infty, \end{aligned} \tag{1.12}$$

by (1.6) and (1.11). Also,

$$\int_{\Omega_1} u g_x + \int_{\partial_0 \Omega} u g = - \int_{\Omega_1} g u_x = - \int_{\Omega_1} \gamma g u_x, \quad (1.13)$$

since $\gamma = 1$ in $\{u > 0\}$. Letting $\epsilon \rightarrow 0$ in (1.1) and using (1.7-10) and (1.12-13), we conclude that (u, γ) is a solution of Problem(P_0). $\square$

We shall denote by $B_r(P)$ the ball with center at P and radius r .

THEOREM 1.3. *For any solution (u, γ) of Problem(P_0), $u \in C^\alpha(\bar{\Omega})$ for all $\alpha \in (0, 1)$ and u is real analytic in Ω^+ .*

Proof. Let (u, γ) be a solution of Problem(P_0). Substituting $\phi = u^+ \equiv \max(u, 0)$ in (0.8), we obtain that $u \geq 0$ in Ω . Let

$$K = \{\phi \in K_0; \phi \geq 0 \text{ a.e. in } \Omega\}.$$

Then (u, γ) satisfies the variational inequality (0.8) for all $\phi \in K$. Since $\tilde{g} \equiv \gamma g I_{\Omega_1}$ and $\tilde{h} \equiv a \sin x I_{\Omega_2}$ are functions in $L^\infty(\Omega)$, it follows from Theorem 2.9 in [9] that

$$u \in C^\alpha(\bar{\Omega}), \forall \alpha \in (0, 1).$$

Note that u satisfies the equation

$$-\nabla \cdot (h^3 \nabla u) = a \sin x \quad \text{in } \Omega_1^+ \cup \Omega_2^+. \quad (1.14)$$

Let $P \in T$ be such that $u(P) > 0$. Then there is a positive number r_0 such that $u > 0$ in $B = B_{r_0}(P)$. For any $\zeta \in C_0^\infty(B)$, the function $\phi = u \pm \epsilon \zeta$ is in K_0 for any small positive ϵ . By (0.8)

$$\int_B h^3 \nabla u \cdot \nabla \zeta = \int_{\Omega_1 \cap B} g \zeta_x + \int_{\Omega_2 \cap B} (a \sin x) \zeta = \int_B (a \sin x) \zeta. \quad (1.15)$$

It follows that u satisfies (1.14) also in B ; and consequently,

$$-\nabla \cdot (h^3 \nabla u) = a \sin x \quad \text{in } \Omega^+ \quad (1.16)$$

and by elliptic regularity theory u is analytic in Ω^+ . $\square$

§2. Preliminaries. Let (u, γ) be a solution of Problem(P_0). We shall denote the free boundary by $S = S(u, \gamma)$, i.e., $S = \partial\Omega^+ \cap \Omega$. Let

$$\tilde{K} = \{\phi \in H_0^1(\Omega); \phi \geq 0 \text{ a.e. in } \Omega\}.$$

We recall the classical Reynolds variational inequality:

PROBLEM($\tilde{P}$). Find $\tilde{u} \in \tilde{K}$ such that

$$\int_{\Omega} h^3 \nabla \tilde{u} \cdot \nabla (\phi - \tilde{u}) \geq \int_{\Omega} (a \sin x)(\phi - \tilde{u}), \forall \phi \in \tilde{K}. \quad (2.1)$$

It is well-known that there exists a unique solution $\tilde{u}$ of Problem($\tilde{P}$) such that $\tilde{u} \in W^{2,p}(\Omega) \cap W_{\text{loc}}^{2,\infty}(\Omega)$ for any $1 < p < \infty$. Let $\tilde{\Omega} = \{\tilde{u} > 0\}$ and $\tilde{S} = \partial\tilde{\Omega} \cap \Omega$. The following lemma shows that the free boundary $\tilde{S}$ is not empty.

LEMMA 2.1. *There is no positive solution for the Dirichlet problem*

$$\begin{aligned} -\nabla \cdot (h^3 \nabla u) &= a \sin x \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (D)$$

Proof. Suppose that there exists a solution u of (D) such that $u > 0$ in Ω . Integrating the partial differential equation over Ω and applying the divergence theorem, we obtain

$$-\int_{\partial\Omega} h^3 \frac{\partial u}{\partial \nu} = \int_{\Omega} a \sin x = 0. \quad (2.2)$$

Since $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, we have

$$\frac{\partial u}{\partial \nu} \leq 0 \quad \text{on } \partial\Omega, \quad (2.3)$$

and, by the strong maximum principle,

$$\frac{\partial u}{\partial \nu} < 0 \quad \text{on } \partial\Omega \cap \partial R_1, \quad (2.4)$$

since $-\nabla \cdot (h^3 \nabla u) \geq 0$ in R_1 . It follows that the left-hand side of (2.2) is positive, a contradiction. $\square$

THEOREM 2.2. *The solution $\tilde{u}$ of Problem($\tilde{P}$) is symmetric with respect to $y = 1/2$ and satisfies*

$$\tilde{u}_y \leq 0 \quad \text{in } Q \equiv (0, 2\pi) \times (1/2, 1).$$

The free boundary $\tilde{S}$ is non-empty and is contained in R_2 . Consequently, $\tilde{S} \cap Q$ is an x -graph which is given by

$$y = \tilde{\rho}(x) = \sup\{\eta \in (1/2, 1); \tilde{u}(x, \eta) > 0\}$$

for $x \in (\pi, 2\pi)$ such that $u(x, 1/2) > 0$, where $\tilde{\rho}$ is real analytic in $\{x; \tilde{\rho}(x) < 1\}$. Moreover, $\tilde{u}(x_0, 1/2) = 0$ for some $x_0 \in (\pi, 2\pi)$ implies that $\tilde{u} \equiv 0$ in $[x_0, 2\pi] \times [0, 1]$, i.e., $\tilde{S} \cap \{y = 1/2\}$ is a single point.

Proof. The first part of the theorem can be proved by using a reflection argument with the aid of the maximum principle. Utilizing a method used in [8](see also the proof of

Theorem 3.3 below), one can obtain the Lipschitz continuity of $\tilde{\rho}$. Then from the regularity results for the free boundary of variational inequality (cf. [6, Chap.2]) the analyticity of $\tilde{\rho}$ follows. Finally, the last statement of the theorem follows from the maximum principle. $\square$

Now, we introduce the function

$$\sigma(x) = \int_0^x \frac{a(1 + \cos \xi)}{h^3(\xi)} d\xi. \quad (2.5)$$

Note that $\sigma(0) = 0$, $\sigma(x) > 0$ if $x > 0$, and σ satisfies the equation

$$-\nabla \cdot (h^3 \nabla \sigma) = a \sin x.$$

By the strong maximum principle

$$\tilde{u} < \sigma \quad \text{in } \Omega, \quad \frac{\partial \tilde{u}}{\partial x} < \frac{\partial \sigma}{\partial x} = \frac{2a}{(1+a)^3} \quad \text{on } \partial_0 \Omega, \quad (2.6)$$

since $\tilde{u} = \sigma = 0$ on $\partial_0 \Omega$ and $-\nabla \cdot (h^3 \nabla (\tilde{u} - \sigma)) = 0$ in $\tilde{\Omega}$. We define a function $\tilde{\theta}(y)$ on $(0, 1)$ by

$$\frac{\partial \tilde{u}}{\partial x}(0, y) = \frac{1 - \tilde{\theta}(y)}{(1+a)^2}. \quad (2.7)$$

Then from (2.6) and (2.7) it follows that $\tilde{\theta}$ satisfies the condition

$$\tilde{\theta} > \frac{1-a}{1+a}. \quad (2.8)$$

Let

$$\tilde{g}(x, y) = h(x) - (1+a)\tilde{\theta}(y). \quad (2.9)$$

We claim that $\tilde{u}$ satisfies the variational inequality:

$$\int_{\Omega} h^3 \nabla \tilde{u} \cdot \nabla (\phi - \tilde{u}) \geq \int_{\Omega} \tilde{g}(\phi - \tilde{u})_x, \quad \forall \phi \in K_0. \quad (2.10)$$

Indeed, since $\tilde{S}$ is smooth, for any $\phi \in K_0$, we have

$$\begin{aligned} \int_{\Omega} h^3 \nabla \tilde{u} \cdot \nabla (\phi - \tilde{u}) &= \int_{\tilde{\Omega}} h^3 \nabla \tilde{u} \cdot \nabla (\phi - \tilde{u}) \\ &= \int_{\tilde{\Omega}} (a \sin x)(\phi - \tilde{u}) - \int_{\partial_0 \Omega} (1+a)^3 \frac{\partial \tilde{u}}{\partial x} \phi. \end{aligned} \quad (2.11)$$

On the other hand, the right-hand side of (2.10) can be computed as

$$\begin{aligned} \int_{\Omega} \tilde{g}(\phi - \tilde{u})_x &= \int_{\Omega} (a \sin x)(\phi - \tilde{u}) - \int_{\partial_0 \Omega} (1+a)(1 - \tilde{\theta})\phi \\ &= \int_{\tilde{\Omega}} (a \sin x)(\phi - \tilde{u}) + \int_{\Omega \setminus \tilde{\Omega}} (a \sin x)\phi - \int_{\partial_0 \Omega} (1+a)(1 - \tilde{\theta})\phi. \end{aligned} \quad (2.12)$$

Since $\sin x < 0$ for $x \in (\pi, 2\pi)$ and $\phi \geq 0$ in R_2 , (2.10) follows from (2.7), (2.11) and (2.12).

The following theorem is as same as Theorem 5.1 in [4]. Here we give a different proof.

THEOREM 2.3. If $\theta \leq \tilde{\theta}$, then $u \leq \tilde{u}$ in Ω and hence $u = 0$ on $\partial\Omega$.

Proof. Recall that u satisfies

$$\int_{\Omega} h^3 \nabla u \cdot \nabla(\phi - u) \geq \int_{\Omega_1} \gamma g(\phi - u)_x + \int_{\Omega_2} g(\phi - u)_x, \forall \phi \in K_0. \quad (2.13)$$

Since $\phi = u \pm (u - \tilde{u})^+ \in K_0$, $\gamma = 1$ if $u > 0$ and $(u - \tilde{u})^+ = 0$ if $u = 0$, we get

$$\int_{\Omega} h^3 \nabla u \cdot \nabla(u - \tilde{u})^+ = \int_{\Omega} g[(u - \tilde{u})^+]_x. \quad (2.14)$$

Also, taking $\phi = \tilde{u} + (u - \tilde{u})^+ \in K_0$ in (2.10),

$$\int_{\Omega} h^3 \nabla \tilde{u} \cdot \nabla(u - \tilde{u})^+ \geq \int_{\Omega} \tilde{g}[(u - \tilde{u})^+]_x. \quad (2.15)$$

Subtracting (2.15) from (2.14) we obtain

$$\int_{\Omega} h^3 |\nabla(u - \tilde{u})^+|^2 \leq \int_{\Omega} (g - \tilde{g})[(u - \tilde{u})^+]_x. \quad (2.16)$$

Since $g \geq \tilde{g}$ by assumption and $(g - \tilde{g})_x = 0$, an integration by parts of the integral on the right-hand side of (2.16) gives

$$\int_{\Omega} (g - \tilde{g})[(u - \tilde{u})^+]_x = - \int_{\partial_0 \Omega} (g - \tilde{g})u \leq 0.$$

Hence $(u - \tilde{u})^+ = 0$ and the theorem follows. $\square$

Remark 2.4. Note that $0 < \tilde{\theta} < 1$ was proved in [4]. However, (2.8) must be verified in order to ensure the existence of θ in the hypotheses of Theorem 2.3.

Next, we shall prove some properties of the free boundary S . Let

$$K = \{\psi \in H^1(\Omega_2); \psi = u \text{ on } T, \psi = 0 \text{ on } \partial\Omega_2 \setminus T, \psi \geq 0 \text{ a.e. in } \Omega_2\}.$$

Notice that the set K is a closed convex subset of $H^1(\Omega_2)$. By the matching lemma (cf. [6, p.31]), the function

$$\phi = \begin{cases} u & \text{in } \Omega_1 \\ \psi & \text{in } \Omega_2 \end{cases}$$

is in K_0 for any $\psi \in K$. Hence, by (0.8),

$$\int_{\Omega_2} h^3 \nabla u \cdot \nabla(\psi - u) \geq \int_{\Omega_2} (a \sin x)(\psi - u)$$

for all $\psi \in K$. From general regularity results for variational inequalities(cf. [6, p.44]) it follows that

$$u \in C_{\text{loc}}^{1,1}(\Omega_2). \quad (2.17)$$

LEMMA 2.5. *Let $\Omega_{21} = \{(x, y) \in R_1; g(x, y) < 0\} = R_1 \setminus \overline{\Omega_1}$. Then $u > 0$ in $\Omega_{21} \cup \{x = \pi\}$ and consequently S is a union of two disjoint sets S_1 and S_2 such that $S_1 \subset \Omega_1 \cup T$ and $S_2 \subset R_2$.*

Proof. In Ω_{21} , u satisfies the inequality

$$-\nabla \cdot (h^3 \nabla u) \geq a \sin x > 0.$$

From the non-negativity of u and the strong maximum principle it follows that

$$u > 0 \quad \text{in} \quad \Omega_{21}.$$

Hence S is a union of two disjoint sets S_1 and S_2 such that $S_1 \subset \Omega_1 \cup T$ and $S_2 \subset R_2 \cup \{x = \pi\}$. We claim that $S_2 \subset R_2$. If there is a point $(x_0, y_0) \in S_2$ with $x_0 = \pi$, then $u(x_0, y_0) = 0$ and $\nabla u(x_0, y_0) = 0$ by (2.17). On the other hand, by the Hopf boundary point lemma, we have

$$\frac{\partial u}{\partial x} < 0 \quad \text{at} \quad (x_0, y_0),$$

which is a contradiction. $\square$

PROPOSITION 2.6. *If $\theta \leq \tilde{\theta}$ then $S_2 \neq \emptyset$.*

Proof. This is a simple consequence of Theorems 2.2 and 2.3. $\square$

PROPOSITION 2.7. *If $\theta \leq \tilde{\theta}$ and θ is not identically equal to $\tilde{\theta}$, then $S_1 \neq \emptyset$.*

Proof. Suppose that $S_1 = \emptyset$. Then $u > 0$ in Ω_1 and $\gamma \equiv 1$ in Ω_1 . For any $\phi \in \tilde{K} \subset K_0$ we have

$$\int_{\Omega} h^3 \nabla u \cdot \nabla (\phi - u) \geq \int_{\Omega_1} g(\phi - u)_x + \int_{\Omega_2} (a \sin x)(\phi - u). \quad (2.18)$$

Since

$$\int_{\Omega_1} g(\phi - u)_x = \int_{\Omega_1} (a \sin x)(\phi - u),$$

using $g = 0$ on T and $\phi = u = 0$ on $\partial\Omega$, we conclude from (2.18) that u is a solution of Problem($\tilde{P}$). Therefore, $u \equiv \tilde{u}$ by the uniqueness of solution to Problem($\tilde{P}$). Since

$$\frac{\partial u}{\partial x} = \frac{1 - \theta}{(1 + a)^2} \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial x} = \frac{1 - \tilde{\theta}}{(1 + a)^2} \quad \text{on} \quad \partial_0 \Omega,$$

we obtain that $\theta \equiv \tilde{\theta}$, a contradiction. $\square$

LEMMA 2.8. *The set Ω_2^+ is connected.*

Proof. Suppose that there is a component G of Ω_2^+ such that $G \subset R_2$. Since

$$\begin{aligned} -\nabla \cdot (h^3 \nabla u) &= a \sin x < 0 \quad \text{in } G, \\ u &= 0 \quad \text{on } \partial G, \end{aligned}$$

it follows from the maximum principle that $u < 0$ in G , a contradiction. $\square$

LEMMA 2.9. *$\gamma_x \geq 0$ in Ω_1 in the distribution sense.*

Proof. The idea of proof is essentially due to Alt[1]. For any $\zeta \in C_0^\infty(\Omega_1)$, the function $\phi = u \pm \zeta$ is in K_0 . Hence we obtain

$$\int_{\Omega_1} h^3 \nabla u \cdot \nabla \zeta = \int_{\Omega_1} \gamma g \zeta_x. \quad (2.19)$$

It follows that

$$-\nabla \cdot (h^3 \nabla u) + \gamma g_x + \gamma_x g = 0 \quad \text{in } \Omega_1 \quad (2.20)$$

in the distribution sense.

For any $\zeta \in C_0^\infty(\Omega_1)$, $\zeta \geq 0$ and $\epsilon > 0$, we have

$$\int_{\Omega_1} h^3 \nabla u \cdot \nabla [\min(u, \epsilon \zeta)] = \int_{\Omega_1} \gamma g [\min(u, \epsilon \zeta)]_x,$$

or

$$\epsilon \int_{\Omega_1 \cap \{u > \epsilon \zeta\}} h^3 \nabla u \cdot \nabla \zeta + \int_{\Omega_1 \cap \{u \leq \epsilon \zeta\}} h^3 |\nabla u|^2 = \int_{\Omega_1} g [\min(u, \epsilon \zeta)]_x, \quad (2.21)$$

since $u > 0$ implies that $\gamma = 1$. The right-hand side of (2.21) can be computed as

$$\begin{aligned} \int_{\Omega_1} g [\min(u, \epsilon \zeta)]_x &= - \int_{\Omega_1} g_x [\min(u, \epsilon \zeta)] \\ &= - \int_{\Omega_1^+} g_x [\min(u, \epsilon \zeta)] \\ &= -\epsilon \int_{\Omega_1^+} g_x \zeta + \int_{\Omega_1^+} g_x [\epsilon \zeta - \min(u, \epsilon \zeta)] \\ &= -\epsilon \int_{\Omega_1^+} \gamma g_x \zeta + \epsilon \int_{\Omega_1^+} g_x (\zeta - u/\epsilon)^+. \end{aligned}$$

Therefore, we obtain

$$\int_{\Omega_1 \cap \{u > \epsilon \zeta\}} h^3 \nabla u \cdot \nabla \zeta + \int_{\Omega_1^+} \gamma g_x \zeta \leq \int_{\Omega_1^+} g_x (\zeta - u/\epsilon)^+. \quad (2.22)$$

Using $g = 0$ on T , the right-hand side of (2.25) is equal to

$$\int_B (a \sin x) \zeta.$$

Hence we obtain from (2.25)

$$\int_{T \cap B} h^3 \left(\frac{\partial u^l}{\partial n} - \frac{\partial u^r}{\partial n} \right) \zeta \geq 0,$$

i.e.,

$$\frac{\partial u^l}{\partial n} - \frac{\partial u^r}{\partial n} \geq 0 \text{ on } T \cap B.$$

This is a contradiction, since $\partial u^l / \partial n < 0$ and $\partial u^r / \partial n > 0$ at P by the Hopf boundary point lemma. Hence $u(P) > 0$.

Defining

$$\begin{aligned} \rho_1(y_0) &= \inf \{x; u(x, y_0) > 0, (x, y_0) \in \Omega_1\} \quad \text{if } (x_0, y_0) \in \Omega_1^+ \text{ for some } x_0, \\ &= \rho(y_0) \quad \text{if } u(x, y_0) = 0, \forall (x, y_0) \in \Omega_1, \end{aligned}$$

we conclude that (2.24) holds. Finally, the upper semi-continuity of ρ_1 follows from the continuity of u . $\square$

As a consequence of Lemmas 2.8 and 2.10, we have:

THEOREM 2.11. *The set Ω^+ is connected and*

$$S_1 = \{(x, y); x = \rho_1(y), y \in (0, 1) \text{ such that } \rho_1(y) > 0\}.$$

§3. Regularity of free boundary. In the rest of this paper we shall assume that

$$\theta < \tilde{\theta}; \tag{3.1}$$

then by Theorem 2.3, Propositions 2.6 and 2.7 $u = 0$ on $\partial\Omega$, $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$.

LEMMA 3.1. *Let $0 < y_1 < y_2 < 1$ and $x_0 > 0$ such that*

- (a) $x_0 \leq \min_{y \in [y_1, y_2]} \rho(y),$
- (b) $u = 0 \quad \text{on} \quad (0, x_0) \times \{y_1, y_2\}.$

Then

$$\int_Q (-h^3 u_x + \gamma g) \leq 0.$$

Proof. We adopt a method from [5]. Introduce $Q \equiv (0, x_0) \times (y_1, y_2)$. Then $Q \subset \Omega_1$ by assumption (a). Let $\zeta \in H^1(Q) \cap C^0(\bar{Q})$ be a non-negative function such that $\zeta = 0$ on $x = x_0$. Then for any $\epsilon > 0$ the function

$$\phi = \begin{cases} u \pm \min(u, \epsilon\zeta) & \text{in } Q \\ u & \text{elsewhere} \end{cases}$$

is in K_0 by assumption (b). Hence

$$\int_Q h^3 \nabla u \cdot \nabla [\min(u, \epsilon\zeta)] = \int_Q \gamma g[\min(u, \epsilon\zeta)]_x,$$

or

$$\epsilon \int_{Q \cap \{u > \epsilon\zeta\}} h^3 \nabla u \cdot \nabla \zeta + \int_{Q \cap \{u \leq \epsilon\zeta\}} h^3 |\nabla u|^2 = \int_Q g[\min(u, \epsilon\zeta)]_x,$$

since $u > 0$ implies that $\gamma = 1$. Therefore, we obtain

$$\int_{Q \cap \{u > \epsilon\zeta\}} h^3 \nabla u \cdot \nabla \zeta - \int_Q g\zeta_x \leq - \int_Q g[(\zeta - u/\epsilon)^+]_x. \quad (3.2)$$

The right-hand side of (3.2) can be computed as

$$\begin{aligned} - \int_Q g[(\zeta - u/\epsilon)^+]_x &= - \int_{y_1}^{y_2} \int_0^{x_0} g[(\zeta - u/\epsilon)^+]_x dx dy \\ &= \int_{y_1}^{y_2} \int_0^{x_0} g_x(\zeta - u/\epsilon)^+ dx dy \\ &\quad - \int_{y_1}^{y_2} [g(\zeta - u/\epsilon)^+](x_0, y) dy + \int_{y_1}^{y_2} [g(\zeta - u/\epsilon)^+](0, y) dy \\ &\leq \int_{y_1}^{y_2} g(0, y) \zeta(0, y) dy, \end{aligned}$$

using the facts that $\zeta = 0$ on $x = x_0$ and $g_x < 0$ in Q . Letting $\epsilon \rightarrow 0$ in (3.2), we obtain

$$\int_Q h^3 \nabla u \cdot \nabla \zeta - \int_Q g\zeta_x \leq \int_{y_1}^{y_2} g(0, y) \zeta(0, y) dy. \quad (3.3)$$

Now, let $\sigma(x) = x_0 - x$ in $[0, x_0]$ and $= 0$ for $x > x_0$. Define

$$\eta_\delta(y) = \begin{cases} 1 & \text{if } y \in [y_1 + \delta, y_2 - \delta], \\ 0 & \text{if } y \notin (y_1, y_2), \\ \text{linear} & \text{if } y \in [y_1, y_1 + \delta] \cup [y_2 - \delta, y_2], \end{cases}$$

for any small positive constant δ . Then the function

$$\phi_{\pm}^{\delta}(x, y) = u(x, y) \pm \sigma(x)\eta_{\delta}(y)$$

is in K_0 . Let

$$\zeta^{\delta}(x, y) = \sigma(x)[1 - \eta_{\delta}(y)].$$

Then $\zeta^{\delta} \in H^1(Q) \cap C^0(\bar{Q})$ and satisfies

$$\zeta^{\delta} \geq 0 \quad \text{in } Q, \quad \zeta^{\delta} = 0 \quad \text{on } x = x_0.$$

Therefore, we obtain

$$\begin{aligned} \int_Q (-h^3 u_x + \gamma g) &= \int_Q (h^3 \nabla u \cdot \nabla \sigma - \gamma g \sigma_x) \\ &= \int_Q [h^3 \nabla u \cdot \nabla (\sigma \eta_{\delta}) - \gamma g (\sigma \eta_{\delta})_x] \\ &\quad + \int_Q [h^3 \nabla u \cdot \nabla [\sigma(1 - \eta_{\delta})] - \gamma g [\sigma(1 - \eta_{\delta})]_x]. \end{aligned} \tag{3.4}$$

The first integral on the right-hand side of (3.4) vanishes since $\phi_{\pm}^{\delta} \in K_0$. From (3.3) it follows that the second integral on the right-hand side of (3.4) is less than or equal to

$$\int_{y_1}^{y_2} g(0, y) x_0 [1 - \eta_{\delta}(y)] dy + \int_Q (1 - \gamma) g [\sigma(1 - \eta_{\delta})]_x. \tag{3.5}$$

Since the second integral in (3.5) is less than or equal to zero and the first integral in (3.5) tends to zero as δ tends to zero, the lemma follows. $\square$

LEMMA 3.2. *If $u = 0$ in a ball $B = B_r(x_0, y_0)$ which is contained in Ω_1 for some $r > 0$, then $\gamma = 0$ a.e. in $\tilde{B}$, where*

$$\tilde{B} = \{(x, y); x < x_0, |y - y_0| < r\} \cup B.$$

Proof. For any $y_1 < y_2$ with $y_1 \geq y_0 - r$ and $y_2 \leq y_0 + r$, by Lemmas 2.10 and 3.1, we have

$$\int_Q (-h^3 u_x + \gamma g) \leq 0, \tag{3.6}$$

where $Q = (0, x_0 + \bar{x}) \times (y_1, y_2) \subset \tilde{B}$ and $\bar{x} = \min(\sqrt{r^2 - (y_2 - y_0)^2}, \sqrt{r^2 - (y_1 - y_0)^2})$. Since $u = 0$ in $\tilde{B}$ by Lemma 2.10, (3.6) implies that

$$\int_Q \gamma g \leq 0.$$

From the facts $g > 0$ and $\gamma \geq 0$ in Ω_1 it follows that $\gamma = 0$ a.e. in Q . Therefore, $\gamma = 0$ a.e. in $\tilde{B}$. $\square$

Let δ be any positive constant. For any set A , let

$$A^* = A \cup E, \quad \text{where } E = (-\delta, 0] \times (0, 1).$$

Extend u and γ to be zero in E . Notice that $u \in H_0^1(\Omega^*)$ by Theorem 2.3. For any $\zeta \in C_0^\infty(\Omega_1^*)$ the function $\phi = u \pm \zeta$ restricted to Ω is in K_0 . Hence

$$\int_{\Omega_1} h^3 \nabla u \cdot \nabla \zeta = \int_{\Omega_1} \gamma g \zeta_x.$$

Therefore, we obtain

$$\int_{\Omega_1^*} h^3 \nabla u \cdot \nabla \zeta = \int_{\Omega_1^*} \gamma g \zeta_x,$$

i.e., u satisfies the equation

$$-\nabla \cdot (h^3 \nabla u) + (\gamma g)_x = 0 \text{ in } \Omega_1^* \quad (3.7)$$

in the weak sense. Now, we are ready to prove the Lipschitz continuity of $S_1 \setminus T$.

THEOREM 3.3. *The free boundary $S_1 \setminus T$ is locally Lipschitz continuous.*

Proof. Let $P \in S_1 \setminus T$ and let r_0 be a positive number such that the ball $B_0 = B_{r_0}(P)$ is contained in Ω_1 . Adapting a method of Alt[2], we introduce the function

$$w(x, y) = \int_{-\delta}^x h^3(\xi) u(\xi, y) d\xi. \quad (3.8)$$

Then $u(x, y) = 0$ if and only if $w(x, y) = 0$. We have

$$\begin{aligned} w_x &= h^3 u, \\ w_{xx} &= h^3 u_x + 3 \frac{h'}{h} w_x, \\ w_y(x, y) &= \int_{-\delta}^x h^3(\xi) u_y(\xi, y) d\xi. \end{aligned}$$

We approximate γ by a sequence $\{\gamma_n\}$ of functions in $C^\infty(\Omega_1^*) \cap L^\infty(\Omega_1^*)$ such that $\gamma_n = 0$ in a neighborhood of $\{x = -\delta, 0 < y < 1\}$. For each n let u_n be the solution of

$$\begin{cases} -\nabla \cdot (h^3 \nabla u_n) + (\gamma_n g)_x = 0 & \text{in } \Omega_1^* \\ u_n = u & \text{on } \partial\Omega_1^*. \end{cases} \quad (3.9)$$

Note that $u_n \rightarrow u$ uniformly in $\overline{\Omega_1^*}$ and strongly in $H^1(\Omega_1^*)$. Let w_n be defined as in (3.8) for u being replaced by u_n . We compute using (3.9) and $u_{n,x}(-\delta, y) = \gamma_n(-\delta, y) = 0$

$$\begin{aligned} w_{n,yy}(x, y) &= \int_{-\delta}^x [h^3(\xi)u_{n,y}(\xi, y)]_y d\xi \\ &= - \int_{-\delta}^x [h^3(\xi)u_{n,\xi}(\xi, y)]_\xi d\xi + \int_{-\delta}^x (\gamma_n g)_\xi(\xi, y) d\xi \\ &= -w_{n,xx}(x, y) + 3\left(\frac{h'}{h}w_{n,x}\right)(x, y) + (\gamma_n g)(x, y). \end{aligned}$$

Hence we obtain

$$-\Delta w_n + 3\frac{h'}{h}w_{n,x} = -\gamma_n g \quad \text{in } \Omega_1^*. \quad (3.10)$$

Therefore, for any $\zeta \in C_0^\infty(\Omega_1^*)$ we have

$$\int_{\Omega_1^*} \nabla w_n \cdot \nabla \zeta + \int_{\Omega_1^*} 3\frac{h'}{h}w_{n,x}\zeta + \int_{\Omega_1^*} \gamma_n g \zeta = 0. \quad (3.11)$$

Letting $n \rightarrow \infty$ in (3.11), we obtain that w satisfies

$$-\Delta w + 3\frac{h'}{h}w_x = -\gamma g \quad \text{in } \Omega_1^* \quad (3.12)$$

in the weak sense. Since $\gamma \in L^\infty(\Omega_1^*)$, (3.12) holds in the strong sense. Recall that $0 \leq \gamma \leq 1$ and $\gamma = 1$ if $w > 0$. Hence w satisfies the variational inequality

$$\begin{aligned} -\Delta w + 3\frac{h'}{h}w_x &\geq -g, \\ w &\geq 0, \end{aligned} \quad (3.13)$$

$$w[-\Delta w + 3\frac{h'}{h}w_x + g] = 0$$

a.e. in Ω_1 and $w_x > 0$ in Ω_1^+ . Note that $w \in C_{\text{loc}}^{1,1}(\Omega_1)$ by general regularity results for variational inequalities(cf. [6, p.44]).

For any $r \in (0, r_0)$, let $B_r = B_r(P)$ and $B_{r/2} = B_{r/2}(P)$. Let ζ be a smooth function such that $0 \leq \zeta \leq 1$, $\zeta = 1$ on ∂B_r and $\zeta = 0$ in $B_{r/2}$. Following the method used in [8], we consider the function

$$v(x, y) = -bw_x(x, y) + cw_y(x, y) + w(x, y) - d\zeta(x, y),$$

where b, c and d are constants with $b > 0$, $|c|$ small and $d > 0$. Since

$$\begin{aligned} -\Delta w &= -3\frac{h'}{h}w_x - g, \\ \Delta w_x &= 3\frac{h'}{h}w_{xx} + 3\frac{h''}{h}w_x - 3\frac{h'^2}{h^2}w_x - a \sin x, \\ -\Delta w_y &= -3\frac{h'}{h}w_{xy} + (1+a)\theta' \end{aligned}$$

in $B_r \cap \Omega_1^+$, we obtain

$$\begin{aligned} -\Delta v + 3\frac{h'}{h}v_x &= 3bw_x\left(\frac{h''}{h} - \frac{h'^2}{h^2}\right) - ba \sin x \\ &\quad + c(1+a)\theta' - g + d(\Delta\zeta - 3\frac{h'}{h}\zeta_x) \end{aligned} \quad (3.14)$$

in $B_r \cap \Omega_1^+$. Since $a \sin x \geq \delta$ in B_0 for some positive constant δ and there is a positive constant M such that

$$\|w\|_{C^{1,1}(B_0)} \leq M,$$

one can choose r so small that the first two terms on the right-hand side of (3.14) is less than or equal to $-b\delta/2$. Hence we obtain

$$-\Delta v + 3\frac{h'}{h}v_x \leq 0 \quad \text{in } B_r \cap \Omega_1^+, \quad (3.15)$$

if b is sufficiently large and $|c|$ and d are sufficiently small. For the boundary conditions, we have

$$\begin{aligned} v &= -d\zeta \leq 0 \quad \text{on } S_1, \\ v &\leq cw_y + d/2 - d \leq 0 \quad \text{on } \partial B_r \cap \Omega_1^+ \cap \{w \leq d/2\}, \end{aligned}$$

if $|c|$ is sufficiently small, and

$$v \leq -b \inf_{\Gamma} w_x + d/2 + \sup w - d \leq 0 \quad \text{on } \Gamma \equiv \partial B_r \cap \Omega_1^+ \cap \{w > d/2\},$$

if b is sufficiently large. Therefore, it follows from (3.15) and the maximum principle that $v \leq 0$ in $B_r \cap \Omega_1^+$. In particular, we have

$$-bw_x + cw_y \leq 0 \quad \text{in } B_{r/2}. \quad (3.16)$$

and the theorem follows. $\square$

Remark 3.4. From (3.16) and the regularity theorems for the free boundary of variational inequality(cf. [6, Chap. 2]) it follows that $S_1 \setminus T$ is in $C_{\text{loc}}^{m+1,\alpha}$ if $g \in C^{m,\alpha}$, where $m \geq 1$ and $0 < \alpha < 1$. Notice that the inhomogeneous term $-g$ in (3.13) being strictly uniformly negative is needed to get C^1 regularity.

Remark 3.5. The proof of Theorem 3.3 cannot be applied at the points $P \in S_1 \cap T$, since we do not know whether u satisfies an elliptic equation in a neighborhood of P .

In the sequel, we let $\pi_y(A)$ be the projection of A onto the y -axis.

THEOREM 3.6. $\rho_1(y)$ is continuous.

Proof. In view of Theorem 3.3, it remains to prove the continuity of ρ_1 only at points $y \in \pi_y(S_1 \cap T)$. Suppose for contradiction that ρ_1 is not continuous at y_0 for some $y_0 \in \pi_y(S_1 \cap T)$. Since ρ_1 is upper semi-continuous, there is a sequence $y_n \rightarrow y_0$ such that

$$x_0 \equiv \lim_{n \rightarrow \infty} \rho_1(y_n) < \rho_1(y_0) = \rho(y_0).$$

Set $x_n = \rho_1(y_n)$, $P_0 = (x_0, y_0)$ and $P_n = (x_n, y_n)$. Let r_0 be a positive number such that $B = B_{r_0}(P_0) \subset \Omega_1$. Then there are positive constants δ and M such that

$$a \sin x \geq \delta \quad \text{in } B, \quad \|w\|_{C^{1,1}(B)} \leq M.$$

Without loss of generality we may assume that $P_n \in B_{r_0/2}(P_0), \forall n \geq 1$. From the proof of Theorem 3.3 there exist positive constants ω and r depending only on δ and M such that $u > 0$ in $A_\omega(P_n) \cap B_r(P_n)$ for all n , where $A_\omega(P_n)$ is the cone with axis parallel to x-axis, vertex at P_n and opening angle ω . This is a contradiction, since $u = 0$ on $\{0 \leq x \leq \rho(y_0), y = y_0\}$ by Lemma 2.10. Hence the theorem follows. $\square$

COROLLARY 3.7. For any solution (u, γ) of Problem(P_0),

$$\gamma = I_{\Omega_1^+} \quad \text{a.e. in } \Omega_1.$$

Proof. Let (x_0, y_0) be a point in Ω_1 with $x_0 < \rho_1(y_0)$. From Theorem 3.6 it follows that there exists a positive number r such that the ball $B = B_r(x_0, y_0)$ is contained in the set $\{u = 0\}$. By Lemma 3.2, we obtain that $\gamma = 0$ a.e. in $\tilde{B}$, where $\tilde{B}$ is defined as in Lemma 3.2. Since the point (x_0, y_0) is arbitrary and the measure of the set $\{(x, y); x = \rho_1(y)\}$ is zero, the corollary follows. $\square$

PROPOSITION 3.8. The set $\{y; \rho_1(y) = \rho(y)\}$ has no interior points.

Proof. Suppose that there exist y_0 and $\delta > 0$ such that

$$\rho_1(y) = \rho(y), \forall y \in (y_0 - \delta, y_0 + \delta).$$

Let $P = (\rho(y_0), y_0)$ and $B = B_\delta(P)$. Recall the notations n , u^r and u^l in the proof of Lemma 2.10. Then by the Hopf boundary lemma we have

$$\frac{\partial u^r}{\partial n} > 0 \quad \text{on } B \cap T.$$

On the other hand, for any $\zeta \in C_0^\infty(B)$ with $\zeta \geq 0$, since $u = \gamma = 0$ in $B \cap \Omega_1$, we have

$$\int_{B \cap \Omega_2} h^3 \nabla u \cdot \nabla \zeta \geq \int_{B \cap \Omega_2} (a \sin x) \zeta. \quad (3.17)$$

The left-hand side of (3.17) can be computed as

$$\int_{B \cap \Omega_2} h^3 \nabla u \cdot \nabla \zeta = \int_{B \cap \Omega_2} (a \sin x) \zeta - \int_{B \cap T} h^3 \frac{\partial u^r}{\partial n} \zeta.$$

Hence we obtain from (3.17) that $\partial u^r / \partial n \leq 0$ on $B \cap T$, a contradiction. $\square$

PROPOSITION 3.9. *The closure of $\pi_y(S_1)$ is $[0, 1]$.*

Proof. Suppose that there is y_0 such that $y_0 \in (0, 1) \setminus A$, where $A = \overline{\pi_y(S_1)}$. Then there exists a positive constant δ such that $J \equiv (y_0 - \delta, y_0 + \delta) \subset (0, 1) \setminus A$. Hence we have $\rho_1(y) = 0, \forall y \in J$. Let $B = B_\delta(0, y_0)$. For any $\zeta \in C_0^\infty(B)$, since the function $u \pm \zeta$ restricted to Ω is in K_0 and $u > 0$ in $B \cap \Omega$, we have

$$\int_{B \cap \Omega_1} h^3 \nabla u \cdot \nabla \zeta = \int_{B \cap \Omega_1} g \zeta_x,$$

or

$$\int_{B \cap \Omega_1} (a \sin x) \zeta - \int_{B \cap \partial_0 \Omega} h^3 u_x \zeta = \int_{B \cap \Omega_1} (a \sin x) \zeta - \int_{B \cap \partial_0 \Omega} g \zeta.$$

Hence

$$u_x = \frac{1 - \theta}{(1 + a)^2} \quad \text{on } B \cap \partial_0 \Omega. \quad (3.18)$$

From (2.7), (3.18) and Theorem 2.3 it follows that

$$\frac{1 - \theta(y)}{(1 + a)^2} \leq \frac{1 - \tilde{\theta}(y)}{(1 + a)^2}, \forall y \in J.$$

That is we have $\theta(y) \geq \tilde{\theta}(y)$ for all $y \in J$. This contradicts to the assumption that $\theta < \tilde{\theta}$ in $(0, 1)$. Therefore, the set $(0, 1) \setminus A$ is empty. Since A is closed, the proposition follows. $\square$

Remark 3.10. If one can show that $S_1 \cap T = \emptyset$ and S_2 is smooth, then by Theorem 3.3 u will be a classical solution of Problem(P_c) for any solution (u, γ) of Problem(P_0). Note that if $S_1 \cap T \neq \emptyset$ then S_1 cannot be smooth near any point $P \in S_1 \cap T$. Indeed, otherwise we have $\partial u / \partial \nu < 0$ at P by the Hopf boundary point lemma. But from (0.5) and the fact $g = 0$ on T it follows that $\partial u / \partial \nu = 0$ at P , a contradiction. Note also that if one can prove that S_2 is a y-graph, then S_2 will be locally real analytic by the same reasoning as in Theorem 2.2.

§4. Uniqueness. Let (u_1, γ_1) and (u_2, γ_2) be two solutions of Problem(P_0). We shall prove that $u_1 = u_2$ and $\gamma_1 = \gamma_2$ using the method of Carrillo and Chipot(cf. [5]). Let ρ_i be the function defined in Lemma 2.10 corresponding to the solution $(u_i, \gamma_i), i = 1, 2$, and let

$$u_0 = \min(u_1, u_2), \quad \gamma_0 = \min(\gamma_1, \gamma_2), \quad \rho_0 = \min(\rho_1, \rho_2), \\ D_i = \{y \in (0, 1); \rho_i(y) > \rho_0(y)\}, i = 1, 2.$$

LEMMA 4.1. *For any $\zeta \in H^1(\Omega) \cap C^0(\bar{\Omega})$ such that $\zeta \geq 0$ in Ω , we have*

$$\int_{\Omega} h^3 \nabla(u_1 - u_0) \cdot \nabla \zeta - \int_{\Omega_1} (\gamma_1 - \gamma_0) g \zeta_x \leq \int_{D_2} (g \zeta)(\rho_1(y), y) dy, \quad (4.1)$$

$$\int_{\Omega} h^3 \nabla(u_2 - u_0) \cdot \nabla \zeta - \int_{\Omega_1} (\gamma_2 - \gamma_0) g \zeta_x \leq \int_{D_1} (g \zeta)(\rho_2(y), y) dy. \quad (4.2)$$

Proof. Let ϵ be an arbitrary small positive number. Since $\phi = u_1 \pm \min(u_1 - u_0, \epsilon\zeta) \in K_0$, we have

$$\begin{aligned} & \int_{\Omega} h^3 \nabla u_1 \cdot \nabla [\min(u_1 - u_0, \epsilon\zeta)] \\ &= \int_{\Omega_1} \gamma_1 g[\min(u_1 - u_0, \epsilon\zeta)]_x + \int_{\Omega_2} (a \sin x) [\min(u_1 - u_0, \epsilon\zeta)]. \end{aligned} \quad (4.3)$$

On the other hand, $\phi = u_2 + \min(u_1 - u_0, \epsilon\zeta) \in K_0$ implies that

$$\begin{aligned} & \int_{\Omega} h^3 \nabla u_2 \cdot \nabla [\min(u_1 - u_0, \epsilon\zeta)] \\ & \geq \int_{\Omega_1} \gamma_2 g[\min(u_1 - u_0, \epsilon\zeta)]_x + \int_{\Omega_2} (a \sin x) [\min(u_1 - u_0, \epsilon\zeta)]. \end{aligned} \quad (4.4)$$

Subtracting (4.4) from (4.3), we get

$$\int_{\Omega} h^3 \nabla(u_1 - u_2) \cdot \nabla [\min(u_1 - u_0, \epsilon\zeta)] \leq \int_{\Omega_1} (\gamma_1 - \gamma_2) g[\min(u_1 - u_0, \epsilon\zeta)]_x. \quad (4.5)$$

We write $u_1 - u_2 = (u_1 - u_0) + (u_0 - u_2)$ and $\gamma_1 - \gamma_2 = (\gamma_1 - \gamma_0) + (\gamma_0 - \gamma_2)$. Then from (4.5) and the identities $(u_0 - u_2)(u_1 - u_0) = 0$ and $(\gamma_0 - \gamma_2)(u_1 - u_0) = 0$ it follows that

$$\int_{\Omega} h^3 \nabla(u_1 - u_0) \cdot \nabla [\min(u_1 - u_0, \epsilon\zeta)] \leq \int_{\Omega_1} (\gamma_1 - \gamma_0) g[\min(u_1 - u_0, \epsilon\zeta)]_x. \quad (4.6)$$

Since

$$\int_{\Omega_1} (\gamma_1 - \gamma_0) g[\min(u_1 - u_0, \epsilon\zeta)]_x = \epsilon \int_{\Omega_1} (\gamma_1 - \gamma_0) g\zeta_x - \epsilon \int_{\Omega_1} (\gamma_1 - \gamma_0) g[(\zeta - \frac{u_1 - u_0}{\epsilon})^+]_x$$

and

$$\begin{aligned} & \int_{\Omega} h^3 \nabla(u_1 - u_0) \cdot \nabla [\min(u_1 - u_0, \epsilon\zeta)] \\ &= \epsilon \int_{\Omega \cap \{u_1 - u_0 > \epsilon\zeta\}} h^3 \nabla(u_1 - u_0) \cdot \nabla \zeta + \int_{\Omega \cap \{u_1 - u_0 \leq \epsilon\zeta\}} h^3 |\nabla(u_1 - u_0)|^2, \end{aligned}$$

we obtain from (4.6)

$$\int_{\Omega \cap \{u_1 - u_0 > \epsilon\zeta\}} h^3 \nabla(u_1 - u_0) \cdot \nabla \zeta \leq \int_{\Omega_1} (\gamma_1 - \gamma_0) g\zeta_x - \int_{\Omega_1} (\gamma_1 - \gamma_0) g[(\zeta - \frac{u_1 - u_0}{\epsilon})^+]_x. \quad (4.7)$$

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