**THE YANG-BAXTER EQUATIONS
AND DIFFERENTIAL IDENTITIES**

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THE YANG-BAXTER EQUATIONS AND DIFFERENTIAL IDENTITIES

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Abstract. The solution of the Yang-Baxter equation for integrable systems is shown to be equivalent to the existence of a differential identity. Quantum integration formulas for the calculation of commutators of monodromy matrices are given. Based on the integration formulas and the systematic use of differential identities, the Yang-Baxter equations for the Nonlinear Schrödinger model for the quantum case of both Bosons and Fermions are derived. The case for discrete models is also included. The parallelism between the classical and quantum case and the classical limiting process from the latter to the former are discussed.

1. Introduction. The calculation of Poisson brackets for the scattering data for the KdV equation by CISM (classical inverse scattering method) was first carried out by Zakharov and Faddeev [1]. The calculations depended on a set of differential identities and on the evaluation of certain limits in the sense of distributions. After the formulation of QISM (quantum inverse scattering method) by Faddeev and his collaborators [2], in which the solution of the Yang-Baxter equation is the important step of the problem, CISM was recast in terms of the solution of a classical Yang-Baxter equation [3]. In this paper, we first repeat the calculation of Poisson brackets for monodromy matrices of the NS (nonlinear Schrödinger) model by the use of an integration formula [3] and differential identity, and then generalize the formula to the quantum cases to calculate commutators and obtain the Yang-Baxter equations of the NS model for both Bosons and Fermions by carrying out the integration with the use of corresponding differential identities. We present a simple proof that the Yang-Baxter equation is equivalent to a differential identity. These differential identities can be chosen to be the same as those that have been used to derive Yang-Baxter equations by our generalized formulas. Further, the differential identity is shown to be equivalent to the local Yang-Baxter equation. We do not claim that the establishment of differential identities are always simpler than the algebraic solutions of local Yang-Baxter equations but aim to clarify the relations between them in various contexts. Finally, we show how the classical results are obtained by the classical limit from the quantum ones. The corresponding results for discrete models are similar where the differential identities are replaced by difference identities. We discuss the discrete model of the Heisenberg chain as an illustration.

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2. Poisson Brackets for Classical Systems. In this section we recapitulate the calculation of the Poisson brackets by the use of differential identities. We use the following integration formula for the Poisson bracket of monodromy matrices [3, p. 192]:

$$\begin{aligned}
(2.1) \quad & \{T(x, y|\lambda) \otimes T(x, y|\mu)\} \\
& = \int_y^x \int_y^x dz dz' T(x, z|\lambda) \otimes T(x, z'|\mu) \{U(z, \lambda) \otimes U(z', \mu)\} \\
& \quad T(z, y|\lambda) \otimes T(z', y|\mu)
\end{aligned}$$

where $T(x, y|\lambda)$ is the monodromy matrix defined by

$$\begin{aligned}
(2.2) \quad & \frac{\partial}{\partial x} T(x, y|\lambda) = U(x, \lambda) T(x, y|\lambda) \\
& T(x, x|\lambda) = 1
\end{aligned}$$

For the NS model, the Hamiltonian takes the form

$$(2.3) \quad H = \int dx \left(\frac{\partial u^*(x)}{\partial x} \frac{\partial u(x)}{\partial x} + c u^*(x) u^*(x) u(x) u(x) \right),$$

where c , the coupling constant, is positive for repulsive interaction, and

$$(2.4) \quad U(x, \lambda) = \frac{i}{2} \lambda \sigma_3 + i\sqrt{c} u(x) \sigma_+ - i\sqrt{c} u^*(x) \sigma_-.$$

Here $\sigma_{\pm} = \frac{1}{2} (\sigma_1 \pm i\sigma_2)$, where $\sigma_i (i = 1, 2, 3)$ are the Pauli matrices.

Let F and G be (2×2) matrix valued functionals of u and u^* . The Poisson bracket of F and G is [3, p. 187]

$$\{F \otimes G\} = i \int_{-\infty}^{\infty} \left(\frac{\delta F}{\delta u(x)} \otimes \frac{\delta G}{\delta u^*(x)} - \frac{\delta F}{\delta u^*(x)} \otimes \frac{\delta G}{\delta u(x)} \right) dx.$$

Considering $U(x, \lambda)$ as a functional of $u(z)$ and $u^*(x)$, we have

$$\begin{aligned}
\frac{\delta U(x, \lambda)}{\delta u(z)} &= c\sqrt{c} \delta(x - z) \sigma_+ \\
\frac{\delta U(x, \lambda)}{\delta u^*(z)} &= -c\sqrt{c} \delta(x - z) \sigma_-.
\end{aligned}$$

Hence

$$\begin{aligned}
& \{U(x, \lambda) \otimes U(y, \mu)\} = \\
& = i \int dz \left(\frac{\delta U(x, \lambda)}{\delta u(z)} \otimes \frac{\delta U(y, \mu)}{\delta u^*(z)} - \frac{\delta U(x, \lambda)}{\delta u^*(z)} \otimes \frac{\delta U(y, \mu)}{\delta u(z)} \right) \\
& = ic\delta(x - y)(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+) \\
(2.5) \quad & = \frac{ic}{2} (\sigma_3 \otimes I - I \otimes \sigma_3)P
\end{aligned}$$

where the identity

$$(2.6) \quad (\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+) = \frac{1}{2} (\sigma_3 \otimes I - I \otimes \sigma_3)P$$

has been used. In (2.6) the symbol P stands for the permutation matrix, equal to

$$(2.7) \quad P = \frac{1}{2} (I \otimes I + \sum_{i=1}^3 \sigma_i \otimes \sigma_i)$$

where I is the 2×2 identity matrix. P acts on $V \otimes V$ by $(\xi \otimes \eta)P = \eta \otimes \xi$ and on matrices by $P(A \otimes B) = (B \otimes A)P$. Clearly, $P^2 = I$ and

$$[(A \otimes B)P]_{ij,kl} = (A \otimes B)_{ij,lk},$$

$$[P(A \otimes B)]_{ij,kl} = (A \otimes B)_{ji,kl},$$

Substituting (2.5) into (2.1), we obtain

$$\begin{aligned}
& \{T(x, y|\lambda) \otimes T(x, y|\mu)\} \\
& = \frac{ic}{2} \int_y^x dz \left(T(x, z|\lambda) \sigma_3 T(z, y|\mu) \otimes T(x, z|\mu) T(z, y|\lambda) \right. \\
& \quad \left. - T(x, z|\lambda) T(z, y|\mu) \otimes T(x, z|\mu) \sigma_3 T(z, y|\lambda) \right) P.
\end{aligned}$$

Performing the integration with the help of the differential identity

$$\begin{aligned}
& \frac{\partial}{\partial z} \left(T(x, z|\lambda) T(z, y|\mu) \otimes T(x, z|\mu) T(z, y|\lambda) \right) \\
(2.8) \quad &= -\frac{i}{2} (\lambda - \mu) \left(T(x, z|\lambda) \sigma_3 T(z, y|\mu) \otimes T(x, z|\mu) T(z, y|\lambda) \right. \\
&\quad \left. - T(x, z|\lambda) T(z, y|\mu) \otimes T(x, z|\mu) \sigma_3 T(z, y|\lambda) \right),
\end{aligned}$$

we finally obtain [3]

$$\{T(x, y|\lambda) \otimes T(x, y|\mu)\} = [r(\lambda, \mu), T(x, y|\lambda) \otimes T(x, y|\mu)]$$

where

$$(2.9) \quad r(\lambda, \mu) = \frac{-cP}{\lambda - \mu} .$$

3. The Quantum Case. In §2 we showed that the classical Yang-Baxter equation for the monodromy matrix can be obtained from the differential equation for the monodromy matrix by the use of a certain differential identity. This is also true for the quantum case. We demonstrate this point for the NS model for Bosons. The Hamiltonian in this case is again (2.3), but u, u^* now are operators acting on Fock space and satisfy the commutation relations:

$$\begin{aligned}
(3.1) \quad & [u(x), u^*(y)] = \delta(x - y) \\
& [u(x), u(y)] = [u^*(x), u^*(y)] = 0
\end{aligned}$$

The monodromy matrix $T(x, y|\lambda)$ satisfies the forward and backward differential equations

$$\begin{aligned}
& \frac{\partial}{\partial x} T(x, y|\lambda) = : U(x, \lambda) T(x, y|\lambda) : \\
(3.2) \quad & \frac{\partial}{\partial y} T(x, y|\lambda) = - : T(x, y|\lambda) U(y, \lambda) :
\end{aligned}$$

$$T(x, x|\lambda) = 1$$

where $U(x, \lambda)$ is given by (2.4).

The notation $:A:$ means that an operator A , depending on the creation and annihilation operators u and u^* , is to be written in *normal order*, meaning that the destruction operators u are to appear all the ways to the right. For example, u^*u^2 is in normal order, while uu^*u is not. The reasons for writing expressions in normal order are subtle. (In particular, we found that the order in which the operators appear in an expression can have an effect on the asymptotic behavior of its expectations as $x \rightarrow \pm\infty$.)

The following commutation relations hold [4] [5] ($y < x$).

$$\begin{aligned}
(3.3) \quad & [u(z), T(x, y|\lambda)] = [u^*(z), T(x, y|\lambda)] = 0 \quad z \notin [y, x] \\
& [u(x), T(x, y|\lambda)] = -i\frac{\sqrt{c}}{2} \sigma_- T(x, y|\lambda) \\
& [u(y), T(x, y|\lambda)] = -i\frac{\sqrt{c}}{2} T(x, y|\lambda) \sigma_- \\
& [u^*(x), T(x, y|\lambda)] = -i\frac{\sqrt{c}}{2} \sigma_+ T(x, y|\lambda) \\
& [u^*(y), T(x, y|\lambda)] = -i\frac{\sqrt{c}}{2} T(x, y|\lambda) \sigma_+ \\
& [u(z), T(x, y, \lambda)] = -i\sqrt{c} T(x, z|\lambda) \sigma_- T(z, y|\lambda) \quad y < z < x
\end{aligned}$$

With the help of (3.3), Eqs. (3.2) can be rewritten in *standard order*

$$\frac{\partial}{\partial x} T(x, y|\lambda) = V(x, \lambda) T(x, y|\lambda)$$

$$\frac{\partial}{\partial y} T(x, y|\lambda) = -T(x, y|\lambda) V(y, \lambda),$$

where

$$(3.4) \quad V(x, \lambda) = U(x, \lambda) + \frac{c}{2} \sigma_+ \sigma_-$$

We now generalize (2.1) to the quantum case. First, we define

$$[T(x, y|\lambda) \otimes T(x, y|\mu)]$$

to be a matrix with elements

$$(3.5) \quad [T(x, y|\lambda) \otimes T(x, y|\mu)]_{ij,kl} = [T_{ik}(x, y|\lambda), T_{jl}(x, y|\mu)].$$

It is easy to see that

$$(3.6) \quad [T(x, y|\lambda) \otimes T(x, y|\mu)] = T(x, y|\lambda) \otimes T(x, y|\mu) - PT(x, y|\mu) \otimes T(x, y|\lambda)P$$

We state our generalized formula as:

THEOREM 3.1.

$$(3.7) \quad [T(x, y|\lambda) \otimes T(x, y|\mu)] = \int_y^x \int_y^x dz \, dz' T(x, z|\lambda) \otimes T(x, z'|\mu) [V(z, \lambda) \otimes V(z', \mu)] P \, T(z', y|\mu) \otimes T(z, y|\lambda) P,$$

where

$$\frac{\partial}{\partial x} T(x, y|\lambda) = : U(x, \lambda) T(x, y|\lambda) : = V(x, \lambda) T(x, y|\lambda).$$

Proof. The $(\alpha\beta, \gamma\delta)$ component of the right hand side of (3.7) is

$$\int_y^x \int_y^x dz \, dz' T_{\alpha\rho}(x, z|\lambda) T_{\beta\sigma}(x, z'|\mu) (V_{\rho\vartheta}(z, \lambda) V_{\sigma\phi}(z', \mu) - V_{\sigma\phi}(z', \mu) V_{\rho\vartheta}(z, \lambda)) T_{\phi\delta}(z', y|\mu) T_{\vartheta\gamma}(z, y|\lambda),$$

where, as usual, repeated indices mean summation. Using the forward and backward equations in component form

$$\begin{aligned} V_{\sigma\phi}(z', \mu) T_{\phi\delta}(z', y|\mu) &= \frac{\partial}{\partial z'} T_{\sigma\delta}(z', y|\mu) \\ -T_{\beta\sigma}(x, z'|\mu) V_{\sigma\phi}(z', \mu) &= \frac{\partial}{\partial z'} T_{\beta\phi}(x, z'|\mu), \end{aligned}$$

we can write this as

$$\int_y^x dz T_{\alpha\rho}(x, z|\lambda) \left[\int_y^x dz' \frac{\partial}{\partial z'} (T_{\beta\sigma}(x, z'|\mu) V_{\rho\vartheta}(z, \lambda) T_{\sigma\delta}(z', y|\mu)) \right] T_{\vartheta\gamma}(z, y|\lambda).$$

Integrating with respect to z' and using the fact that $T_{\beta\sigma}(x, x|\mu) = \delta_{\beta\sigma}$, for example, we get

$$\begin{aligned} \int_y^x dz \left[T_{\alpha\rho}(x, z|\lambda) V_{\rho\vartheta}(z, \lambda) T_{\beta\delta}(x, y|\mu) T_{\vartheta\delta}(z, y|\lambda) \right. \\ \left. - T_{\alpha\rho}(x, z|\lambda) T_{\beta\delta}(x, y|\mu) V_{\rho\vartheta}(z, \lambda) T_{\vartheta\gamma}(z, y|\lambda) \right]. \end{aligned}$$

Similarly, by the use of

$$\begin{aligned} T_{\alpha\rho}(x, z|x) V_{\rho\vartheta}(z, \lambda) &= -\frac{\partial}{\partial z} T_{\alpha\vartheta}(x, z|\lambda) \\ V_{\rho\vartheta}(z, \lambda) T_{\vartheta\gamma}(z, y|\lambda) &= \frac{\partial}{\partial z} T_{\rho\gamma}(z, y|\lambda), \end{aligned}$$

the integration of z is carried out. Finally, we obtain

$$-T_{\beta\delta}(x, y|\mu)T_{\alpha\gamma}(x, y|\lambda) + T_{\alpha\gamma}(x, y|\lambda) T_{\beta\delta}(x, y|\mu),$$

which is just $[T(x, y|\lambda) \otimes T(x, y|\mu)]_{\alpha\beta, \gamma\delta}$; by definition, thus, the theorem is proved.

By a straightforward calculation, we have

$$\begin{aligned} [V(z, \lambda) \otimes V(z', \mu)] &= [U(z, \lambda) \otimes U(z', \mu)] \\ &= c(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+)\delta(z - z'). \end{aligned}$$

Substituting this into Theorem 3.1 and using (2.6), we obtain

$$\begin{aligned} &[T(x, y|\lambda) \otimes T(x, y|\mu)] \\ &= \frac{c}{2} \int_y^x dz T(x, z|\lambda) \otimes T(x, z|\mu) (\sigma_3 \otimes I - I \otimes \sigma_3) T(z, y|\mu) \otimes T(z, y|\lambda) P. \end{aligned}$$

Since $[T(x, z|\mu), T(z, y|\mu)] = 0$, the right hand side can be written as

$$\begin{aligned} &\frac{c}{2} \int_y^x dz \left(T(x, z|\lambda) \sigma_3 T(z, y|\mu) \otimes T(x, z|\mu) T(z, y|\lambda) \right. \\ &\quad \left. - T(x, z|\lambda) T(z, y|\mu) \otimes T(x, z|\mu) \sigma_3 T(z, y|\lambda) \right) P \end{aligned}$$

Because of (3.4), the differential identity (2.8) remains valid in this case for Bosons. So the integral is calculated using (2.8). In this way, we obtain

$$\begin{aligned} &[T(x, y|\lambda) \otimes T(x, y|\mu)] \\ (3.8) \quad &= \frac{ic}{\lambda - \mu} \left(T(x, y|\mu) \otimes T(x, y|\lambda) - T(x, y|\lambda) \otimes T(x, y|\mu) \right) P \end{aligned}$$

Using (3.2), equation (3.8) may be rewritten in the form

$$(3.9) \quad R(\lambda, \mu) T(x, y|\lambda) \otimes T(x, y|\mu) = T(x, y|\mu) \otimes T(x, y|\lambda) R(\lambda, \mu)$$

where

$$R(\lambda, \mu) = \frac{-ic}{\lambda - \mu - ic} I + \frac{\lambda - \mu}{\lambda - \mu - ic} P$$

This is the result of [2] [4].

For the NS model of Fermions [5], the Hamiltonian takes the form

$$(3.10) \quad H = \int dx \left(\frac{\partial u_i^*}{\partial x} \frac{\partial u_i}{\partial x} + c : u_i^* u_i u_j^* u_j : \right)$$

Here we again consider only the case of repulsive interaction $c > 0$. The dummy indices i, j are now summed over $1, 2$; u_i, u_i^* act on Fock space and satisfy the anticommutation relations

$$(3.11) \quad \begin{aligned} [u_i(x), u_j^*(y)]_+ &= \delta_{ij} \delta(x - y) \\ [u_i(x), u_j(y)]_+ &= [u_i^*(x), u_j^*(y)]_+ = 0 \end{aligned}$$

The monodromy matrix $T(x, y|\lambda)$ is a 3×3 matrix defined by

$$(3.12) \quad \frac{\partial}{\partial x} T(x, y|\lambda) =: U(x, \lambda) T(x, y|\lambda) :$$

$$T(x, x|\lambda) = 1$$

where

$$(3.13) \quad U(x, \lambda) = \frac{i\lambda}{2} J + i\sqrt{c} u_j(x) e_{j3} - i\sqrt{c} u_j^*(x) e_{3j},$$

$J = \text{diag}(1, 1, -1)$, and e_{jk} is a 3×3 matrix defined by $(e_{jk})_{lm} = \delta_{jl} \delta_{km}$.

The following relations hold [6] ($y < x$) for $j = 1, 2$

$$\begin{aligned} u_j(z)T(x, y|\lambda) &= J T(x, y|\lambda)J u_j(z) & z \notin [x, y] \\ T(x, y|\lambda)u_j^*(z) &= u_j^*(z) J T(x, y|\lambda)J & z \notin [x, y] \end{aligned}$$

$$(3.14) \quad \begin{aligned} u_j(x)T(x, y|\lambda) &= J T(x, y|\lambda)J u_j(x) - \frac{i\sqrt{c}}{2} e_{3j}T(x, y|\lambda) \\ u_j(y)T(x, y|\lambda) &= J T(x, y|\lambda)J u_j(y) - \frac{i\sqrt{c}}{2} J T(x, y|\lambda)J e_{3j} \\ T(x, y|\lambda)u_j^*(x) &= u_j^*(x) J T(x, y|\lambda)J + \frac{i\sqrt{c}}{2} J T(x, y|\lambda)J e_{j3} \\ T(x, y|\lambda)u_j^*(y) &= u_j^*(y)J T(x, y|\lambda)J + \frac{i\sqrt{c}}{2} T(x, y|\lambda)e_{j3} \end{aligned}$$

From (3.12) and (3.14), it follows

$$\frac{\partial}{\partial x} T(x, y|\lambda) = V(x, \lambda) T(x, y|\lambda)$$

$$\frac{\partial}{\partial y} T(x, y|\lambda) = -T(x, y|\lambda) V(y, \lambda)$$

where

$$(3.15) \quad V(x, \lambda) = U(x, \lambda) - \frac{c}{2} e_{j3} e_{3j}$$

In dealing with problems involving fermions, it is necessary to introduce *supermatrices* [5]. For a supermatrix we assign parity $p(i)$ of i th row (column) = 0 when the parity is even, and $p(i) = 1$ when the parity is odd. The super tensor product of two supermatrices A and B is defined by

$$(A \otimes_s B)_{ij,kl} = A_{ik} B_{jl} (-1)^{p(j)(p(i)+p(k))}.$$

We define $[A \otimes_s B]$ to be the matrix with elements given by

$$[A \otimes_s B]_{ij,kl} = A_{ik} B_{jl} (-1)^{p(j)(p(i)+p(k))} - B_{jl} A_{ik} (-1)^{p(l)(p(i)+p(k))},$$

It is easy to see that

$$(3.16) \quad [A \otimes_s B] = A \otimes_s B - P_s B \otimes_s A P_s,$$

where $P_s = \sum e_{ij} \otimes e_{ji} (-1)^{p(i)p(j)}$ is the super permutation operator. In analogy with Theorem 3.1, we have:

THEOREM 3.2.

$$(3.17) \quad [T(x, y|\lambda) \otimes_s T(x, y|\mu)] =$$

$$\int_y^x \int_y^x dz dz' T(x, z|\lambda) \otimes_s T(x, z'|\mu) [U(z, \lambda) \otimes_s U(z', \mu)] P_s T(z', y|\mu) \otimes_s T(z, y|\lambda) P_s$$

Proof. The proof is similar to that of Theorem 2.1 with consideration of the additional signs due to the parity.

We now return to the NS model for Fermions, described by the Hamiltonian (3.10). For the present problem the parity should be chosen as

$$p(i) = \begin{cases} 0 & i = 1, 2 \\ 1 & i = 3 \end{cases}$$

It follows from a straightforward calculation that

$$\begin{aligned} [V(z, \lambda) \otimes_s V(z', \mu)] &= [U(z, \lambda) \otimes_s U(z', \mu)] \\ &= c(e_{3j} \otimes_s e_{j3} + e_{j3} \otimes_s e_{3j})\delta(z - z') \end{aligned}$$

Substituting this into (3.17) and taking into account the identity

$$(e_{3j} \otimes_s e_{j3} + e_{j3} \otimes_s e_{3j})P_s = \frac{1}{2} (I \otimes_s J - J \otimes_s I),$$

we obtain

$$\begin{aligned} & [T(x, y|\lambda) \otimes_s T(x, y|\mu)] \\ (3.18) \quad & = \frac{c}{2} \int_y^x dz T(x, z|\lambda) \otimes_s T(x, z|\mu) (I \otimes_s J - J \otimes_s I) T(z, y|\mu) \otimes_s T(z, y|\lambda) P_s. \end{aligned}$$

For non-overlapping intervals (x, z) and (z, y) ($x < z < y$), $[T(x, z|\mu) \otimes_s T(z, y|\mu)] = 0$ so (3.18) becomes

$$\begin{aligned} (3.19) \quad & \frac{c}{2} \int_y^x dz \left(T(x, z|\lambda) T(z, y|\mu) \otimes_s T(x, z|\mu) J T(z, y|\lambda) \right. \\ & \left. - T(x, z|\lambda) J T(z, y|\mu) \otimes_s T(x, z|\mu) T(z, y|\lambda) \right) P_s. \end{aligned}$$

Making use of the differential identity

$$\begin{aligned} (3.20) \quad & \frac{\partial}{\partial z} \left(T(x, z|\lambda) T(z, y|\mu) \otimes_s T(x, z|\mu) T(z, y|\lambda) \right) \\ & = \frac{i}{2} (\lambda - \mu) \left(T(x, z|\lambda) T(z, y|\mu) \otimes_s T(x, z|\mu) J T(z, y|\lambda) \right. \\ & \quad \left. - T(x, z|\lambda) J T(z, y|\mu) \otimes_s T(x, z|\mu) T(z, y|\lambda) \right), \end{aligned}$$

((3.20) follows from (3.15)), the integral of (3.19) is reduced to

$$\frac{-c}{\lambda - \mu} \left(T(x, y|\mu) \otimes_s T(x, y|\lambda) - T(x, y|\lambda) \otimes_s T(x, y|\mu) \right) P_s$$

Thus, we have shown that

$$(3.21) \quad [T(x, y|\lambda) \otimes_s T(x, y|\mu)] = \frac{-ic}{\lambda - \mu} \left(T(x, y|\mu) \otimes_s T(x, y|\lambda) - T(x, y|\lambda) \otimes_s T(x, y|\mu) \right) P_s.$$

According to (3.16) the left hand side of (3.21) is

$$T(x, y|\lambda) \otimes_s T(x, y|\mu) - P_s T(x, y|\mu) \otimes_s T(x, y|\lambda) P_s$$

Therefore, it is easy to see that (3.17) contains the following Yang-Baxter equation

$$R(\lambda, \mu) T(x, y|\lambda) \otimes_s T(x, y|\mu) = T(x, y|\mu) \otimes_s T(x, y|\lambda) R(\lambda, \mu)$$

where

$$(3.22) \quad R(\lambda, \mu) = \frac{ic}{\lambda - \mu + ic} I + \frac{\lambda - \mu}{\lambda - \mu + ic} P_s$$

Here I denotes the 3×3 identity matrix. This is the result obtained in [5].

We point out that both theorem 3.1 and 3.2 are valid in general for the quantum case, but that Theorem 3.1 is useful for the Boson case, while Theorem 3.2 is useful for the Fermion case. Further, Theorem 3.1 can be considered as a particular case of Theorem 3.2 when $p(i) = 0$ for all i .

For a discrete model on a 1 dimensional lattice, define

$$(3.23) \quad \begin{aligned} T(i, j|\lambda) &= L_{i-1}(\lambda) \dots L_j(\lambda) \quad j \leq i-1 \\ T(i, i|\lambda) &= 1, \end{aligned}$$

where $L_j(\lambda)$ is the associated linear operator of the given model. If we define the difference operator Δ_i by $\Delta_i f(i) = f(i+1) - f(i)$, then the forward and backward difference equations are respectively

$$(3.24) \quad \begin{aligned} \Delta_i T(i, j|\lambda) &= U_i(\lambda) T(i, j|\lambda) \\ \Delta_j T(i, j|\lambda) &= -T(i, j+1) U_j(\lambda), \\ \text{where } U_i(\lambda) &= L_i(\lambda) - 1 \end{aligned}$$

The analogs of Theorems 3.1 and 3.2 are

We obtain

$$\begin{aligned}
(RHS)_{\alpha\beta,\gamma\delta} = & \sum_{k=j}^{i-1} \left(T_{\alpha\rho}(i, k+1|\lambda) L_k(\lambda)_{\rho\vartheta} T_{\beta\delta}(i, j|\mu) T_{\vartheta\gamma}(k, j|\lambda) (-1)^{p(\beta)(p(\alpha)+p(\vartheta))+p(\delta)(p(\vartheta)+p(\gamma))} \right. \\
& \left. - T_{\alpha\rho}(i, k+1|\lambda) T_{\beta\delta}(i, j|\mu) L_k(\lambda)_{\rho\vartheta} T_{\vartheta\gamma}(k, j|\lambda) (-1)^{p(\delta)(p(\gamma)+p(\rho))+p(\beta)(p(\alpha)+p(\rho))} \right)
\end{aligned}$$

Repeating the process for the summation on k , we arrive at

$$\begin{aligned}
(RHS)_{\alpha\beta,\gamma\delta} &= T_{\alpha\gamma}(i, j|\lambda) T_{\beta\delta}(i, j|\mu) (-1)^{p(\beta)(p(\alpha)+p(\gamma))} - T_{\beta\gamma}(i, j|\mu) T_{\alpha\gamma}(i, j|\lambda) (-1)^{p(\delta)(p(\alpha))} \\
&= [T(i, j|\lambda) \otimes T(i, j|\mu)]_{\alpha\beta,\gamma\delta} \\
&= (LHS)_{\alpha\beta,\gamma\delta}
\end{aligned}$$

As an example we consider the Heisenberg spin chain, for which $L_k(\lambda)$ is a 2×2 matrix given by [2]

$$L_k(\lambda) = \lambda I + i \sigma_\alpha S_k^\alpha$$

where S_k^α are spin operators satisfying the commutation relations

$$[S_k^\alpha, S_{k'}^\beta] = i \delta_{kk'} \epsilon_{\alpha\beta\gamma} S_k^\gamma.$$

By a straightforward calculation, we obtain

$$(3.29) \quad [L_k(\lambda) \otimes L_{k'}(\mu)] = -i \delta_{kk'} \epsilon_{\alpha\beta\gamma} S_k^\gamma \sigma^\alpha \otimes \sigma^\beta$$

Substituting (3.29) into (3.25), we have

$$\begin{aligned}
(3.30) \quad & [T(i, j|\lambda) \otimes T(i, j|\mu)] \\
&= -i \sum_{k=j}^{i-1} T(i, k+1|\lambda) \otimes T(i, k+1|\mu) \epsilon_{\alpha\beta\gamma} \sigma^\alpha \otimes \sigma^\beta P S_k^\gamma T(k, j|\mu) \otimes T(k, j|\lambda) P.
\end{aligned}$$

From the algebraic identity

$$\epsilon_{\alpha\beta\gamma} \sigma^\alpha \otimes \sigma^\beta P = i(\sigma^\gamma \otimes I - I \otimes \sigma^\gamma)$$

we obtain

$$(3.31) \quad \epsilon_{\alpha\beta\gamma}\sigma^\alpha \otimes \sigma^\beta S_k^\gamma P = L_k(\lambda) \otimes I - I \otimes L_k(\lambda).$$

Using (3.31) and the fact that $L_k(\lambda)$ commutes with $T(i, k+1|\mu)$ and $T(k, j|\mu)$, the right hand side of (3.30) can be written as

$$(3.32) \quad \begin{aligned} & -i \sum_{k=j}^{i-1} \left(T(i, k+1|\lambda) L_k(\lambda) T(k, j|\mu) \otimes T(i, k+1|\mu) T(k, j|\lambda) \right. \\ & \quad \left. - T(i, k+1|\lambda) T(k, j|\mu) \otimes T(i, k+1|\mu) L_k(\lambda) T(k, j|\lambda) \right) P \\ & = -i \sum_{k=j}^{i-1} \left(T(i, k|\lambda) T(k, j|\mu) \otimes T(i, k+1|\mu) T(k, j|\lambda) \right. \\ & \quad \left. - T(i, k+1|\lambda) T(k, j|\mu) \otimes T(i, k+1|\mu) T(k+1, j|\lambda) \right) P \end{aligned}$$

The next step is to use difference identity

$$\begin{aligned} & \Delta_k \left(T(i, k|\lambda) T(k, j|\mu) \otimes T(i, k|\mu) T(k, j|\lambda) \right) \\ & = \left((\lambda - \mu) (T(i, k|\lambda) T(k, j|\mu) \otimes T(i, k+1|\mu) T(k, j|\lambda) - \right. \\ & \quad \left. - T(i, k+1|\lambda) T(k, j|\mu) \otimes T(i, k+1|\mu) T(k+1, j|\lambda) \right) \end{aligned}$$

to carry out the summation. Finally we obtain

$$(3.33) \quad \begin{aligned} & [T(i, j|\lambda) \otimes T(k, j|\mu)] = \\ & \frac{-i}{\lambda - \mu} \left(T(i, j|\mu) \otimes T(i, j|\lambda) - T(i, j|\lambda) \otimes T(i, j|\mu) \right) P \end{aligned}$$

which is equivalent to

$$R(\lambda, \mu) T(i, j|\lambda) \otimes T(i, j|\mu) = T(i, j|\mu) \otimes T(i, j|\lambda) R(\lambda, \mu)$$

with

$$R(\lambda, \mu) = \frac{i}{\lambda - \mu + i} I + \frac{\lambda - \mu}{\lambda - \mu + i} P$$

This is the result obtained in [2].

4. Concluding Remarks. In the preceding section, we derived the Yang-Baxter equations on the basis of integration (summation) formulas and with the help of certain differential (difference) identities. In fact, we may show that the Yang-Baxter equation is equivalent to a differential identity:

THEOREM 4.1. *The Yang-Baxter equation*

$$(4.1) \quad R(\lambda, \mu)T(x, y|\lambda) \otimes T(x, y|\mu) = T(x, y|\mu) \otimes T(x, y|\lambda)R(\lambda, \mu)$$

is equivalent to the differential identity

$$(4.2) \quad \frac{\partial}{\partial z} \left(T(x, z|\mu) \otimes T(x, z|\lambda)R(\lambda, \mu)T(z, y|\lambda) \otimes T(z, y|\mu) \right) = 0$$

Proof. Integrating this identity over the interval $y \leq z \leq x$, we get the Yang-Baxter equation. Conversely, if the Yang-Baxter equation holds for all x and y , then it certainly holds for $y = z$. Multiplying the Yang-Baxter equation for x, z on the right by $T(z, y|\lambda) \otimes T(z, y|\mu)$, we get

$$\begin{aligned} & R(\lambda, \mu) \left(T(x, z|\lambda) \otimes T(x, z|\mu) \right) \left(T(z, y|\lambda) \otimes T(z, y|\mu) \right) \\ &= T(x, z|\mu) \otimes T(x, z|\lambda)R(\lambda, \mu)T(z, y|\lambda) \otimes T(z, y|\mu). \end{aligned}$$

Since $T(x, z|\mu)$ and $T(z, y|\lambda)$ depend on field operators on non-overlapping intervals, and since the field operators commute on non-overlapping intervals, $(T(x, z|\lambda) \otimes T(x, z|\mu))(T(z, y|\lambda) \otimes T(z, y|\mu)) = T(x, y|\lambda) \otimes T(x, y|\mu)$. Therefore the left side above is independent of z , and the differentiation of this identity with respect to z proves the other half of Theorem 4.1.

The differential identity (4.2) for the NS model for Bosons can be written

$$(4.3) \quad \begin{aligned} & \frac{\partial}{\partial z} \left(P \ T(x, z|\mu) \otimes T(x, z|\lambda)P \ T(z, y|\lambda) \otimes T(z, y|\mu) \right) \\ &= \frac{ic}{\lambda - \mu} P \frac{\partial}{\partial z} \left(T(x, z|\mu)T(z, y|\lambda) \otimes T(x, z|\lambda)T(z, y|\mu) \right) \end{aligned}$$

On the other hand, we have [4]

$$\begin{aligned} \frac{\partial}{\partial z} T(z, y|\lambda) \otimes T(z, y|\mu) &= : \Gamma(z|\lambda, \mu)T(z, y|\lambda) \otimes T(z, y|\mu) : \\ \frac{\partial}{\partial z} T(x, z|\mu) \otimes T(x, z|\lambda) &= - : T(x, z|\mu) \otimes T(x, z|\lambda)\Gamma(z|\mu, \lambda) : \end{aligned}$$

where

$$\Gamma(z|\lambda, \mu) = U(z, \lambda) \otimes I + I \otimes U(z, \mu) + c\sigma_+ \otimes \sigma_-.$$

The term $c\sigma_+ \otimes \sigma_-$ is the quantum correction due to the noncommutivity of the field operators.

Using (2.6), equation (4.3) may be reduced to (2.8).

In general, if we write the forward differential equation for the tensor product of two monodromy matrices in normal order

$$(4.4) \quad \frac{\partial}{\partial z} \left(T(z, y|\lambda) \otimes T(z, y|\mu) \right) = : \Gamma(z|\lambda, \mu) T(z, y|\lambda) \otimes T(z, y|\mu) :,$$

then the backward differential equation in normal order is

$$(4.5) \quad \frac{\partial}{\partial z} \left(T(x, z|\lambda) \otimes T(x, z|\mu) \right) = - : T(x, z|\lambda) \otimes T(x, z|\mu) \Gamma(z|\lambda, \mu) : .$$

We have the following theorem which was proved in [4] by another method.

THEOREM 4.2. *The Yang-Baxter equation is equivalent to*

$$(4.6) \quad R(\lambda, \mu) \Gamma(z|\lambda, \mu) = \Gamma(z|\mu, \lambda) R(\lambda, \mu)$$

Proof. From (4.2), (4.4), we have

$$\begin{aligned} & \frac{\partial}{\partial z} \left(T(x, z|\mu) \otimes T(x, z|\lambda) R(\lambda, \mu) T(z, y|\lambda) \otimes T(z, y|\mu) \right) \\ &= : T(x, z|\mu) \otimes T(x, z|\lambda) \left(-\Gamma(z|\mu, \lambda) R(\lambda, \mu) + R(\lambda, \mu) \Gamma(z|\lambda, \mu) \right) T(z, y|\lambda) \otimes T(z, y|\mu) : \\ &= 0. \end{aligned}$$

The theorem follows from the conclusion of Theorem 4.1.

Similarly, if we write the differential equations for the tensor product of monodromy matrices in standard order

$$(4.7) \quad \begin{aligned} & \frac{\partial}{\partial z} \left(T(z, y|\lambda) \otimes T(z, y|\mu) \right) = \wedge(z|\lambda, \mu) T(z, y|\lambda) \otimes T(z, y|\mu) \\ & \frac{\partial}{\partial z} \left(T(x, z|\lambda) \otimes T(x, z|\mu) \right) = -T(x, z|\lambda) \otimes T(x, z|\mu) \wedge(z|\lambda, \mu), \end{aligned}$$

then the Yang-Baxter equation is equivalent to

$$(4.8) \quad R(\lambda, \mu) \wedge(z|x, \mu) = \wedge(z|\mu, \lambda) R(\lambda, \mu)$$

Analogously, for supermatrices with assigned parities, we have

THEOREM 4.3. *The Yang-Baxter equation*

$$(4.9) \quad R(\lambda, \mu) T(x, y|\lambda) \otimes_s T(x, y|\mu) = T(x, y|\mu) \otimes_s T(x, y|\lambda) R(\lambda, \mu)$$

is equivalent to the differential identity

$$(4.10) \quad \frac{\partial}{\partial z} \left(T(x, z|\mu) \otimes_s T(x, z|\lambda) R(\lambda, \mu) T(z, y|\lambda) \otimes_s T(z, y|\mu) \right) = 0;$$

and the latter is equivalent to

$$(4.11) \quad R(\lambda, \mu) \Gamma(z|\lambda, \mu) = \Gamma(z|\mu, \lambda) R(\lambda, \mu)$$

$$(4.12) \quad \text{or} \quad R(\lambda, \mu) \wedge (z|\lambda, \mu) = \wedge(z|\mu, \lambda) R(\lambda, \mu)$$

provided that

$$(4.13) \quad \begin{aligned} \frac{\partial}{\partial z} \left(T(z, y|\lambda) \otimes_s T(z, y|\mu) \right) &=: \Gamma(z|\lambda, \mu) T(z, y|\lambda) \otimes_s T(z, y|\mu) : \\ &= \wedge(z|\lambda, \mu) T(z, y|\lambda) \otimes_s T(z, y|\mu). \end{aligned}$$

On account of (3.22), the differential identity (4.10) for the NS model of Fermions with repulsive interaction can be written as

$$(4.14) \quad \begin{aligned} &\frac{\partial}{\partial z} \left(P_s T(x, z|\mu) \otimes_s T(x, z|\lambda) P_s T(z, y|\lambda) \otimes_s T(z, y|\mu) \right) \\ &= \frac{-ic}{\lambda - \mu} P_s \frac{\partial}{\partial z} \left(T(x, z|\mu) T(z, y|\lambda) \otimes_s T(x, z|\lambda) T(z, y|\mu) \right) \end{aligned}$$

From [6]

$$\begin{aligned} &\frac{\partial}{\partial z} \left(T(z, y|\lambda) \otimes_s T(z, y|\mu) \right) \\ &=: (U(z, \lambda) \otimes_s I + I \otimes_s U(z, \mu) + ce_{j3} \otimes_s e_{3j}) T(z, y|\lambda) \otimes_s T(z, y|\mu) : \\ &\frac{\partial}{\partial z} \left(T(x, z|\mu) \otimes_s T(x, z|\lambda) \right) \\ &= - : T(x, z|\mu) \otimes_s T(x, z|\lambda) (U(z, \mu) \otimes_s I + I \otimes_s U(z, \lambda) + ce_{j3} \otimes_s e_{3j}) : \end{aligned}$$

the differentiation on the left side of (4.14) can be completed. Then, (4.14) turns out to be just the differential identity (3.20).

For the discrete model on a lattice the corresponding theorem is

THEOREM 4.4. *The Yang-Baxter equation*

$$(4.15) \quad R(\lambda, \mu)T(i, j|\lambda) \otimes T(i, j|\mu) = T(i, j|\mu) \otimes T(i, j|\lambda) R(\lambda, \mu)$$

is equivalent to the difference identity

$$(4.16) \quad \Delta_k \left(T(i, k|\mu) \otimes T(i, k|\lambda) R(\lambda, \mu) T(k, j|\lambda) \otimes T(k, j|\mu) \right) = 0$$

In the classical theory of inverse scattering transform the following theorem is true

THEOREM 4.5. *The classical Yang-Baxter equation*

$$(4.17) \quad \{T(x, y|\lambda) \otimes T(x, y|\mu)\} = [\Gamma(\lambda, \mu), T(x, y|\lambda) \otimes T(x, y|\mu)]$$

is equivalent to

$$(4.18) \quad \frac{\partial}{\partial z} \left(\{T(x, z|\lambda) \otimes T(x, z|\mu)\} T(z, y|\lambda) \otimes T(z, y|\mu) \right. \\ \left. + T(x, z|\lambda) \otimes T(x, z|\mu) r(\lambda, \mu) T(z, y|\lambda) \otimes T(z, y|\mu) \right) = 0$$

Proof. Integrating (4.18) from y to x we obtain (4.17). To prove the converse, set y equal to z in (4.17), multiply on the right by $T(z, y|\lambda) \otimes T(z, y|\mu)$ and differentiate with respect to z ; this yields (4.18).

If the monodromy matrix $T(x, y|\lambda)$ is a functional of $u(x)$ and $u^*(x)$, as in the case of the NS model, and the Poisson bracket can be put in the form (cf. [3])

$$= \{T(x, y|\lambda) \otimes T(x, y|\mu)\} \\ = i \int_y^x \left(\frac{\delta T(x, y|\lambda)}{\delta u(z)} \otimes \frac{\delta T(x, y|\mu)}{\delta u^*(z)} - \frac{\delta T(x, y|\lambda)}{\delta u^*(z)} \otimes \frac{\delta T(x, y|\mu)}{\delta u(z)} \right) dz,$$

then

$$\frac{\partial}{\partial z} \left(\{T(x, z|\lambda) \otimes T(x, z|\mu)\} T(z, y|\lambda) \otimes T(z, y|\mu) \right) \\ = -i \left(\frac{\delta T(x, y|\lambda)}{\delta u(z)} \otimes \frac{\delta T(x, y|\mu)}{\delta u^*(z)} - \frac{\delta T(x, y|\lambda)}{\delta u^*(z)} \otimes \frac{\delta T(x, y|\mu)}{\delta u(z)} \right).$$

This follows from the fact that

$$\frac{\delta T(z, y|\lambda)}{\delta u(w)} = 0 \quad \text{when } w \notin [y, z]$$

Therefore, (4.18) takes the form

$$(4.19) \quad \begin{aligned} & \frac{\partial}{\partial z} \left(T(x, y|\lambda) \otimes T(x, y|\mu) r(\lambda, \mu) T(z, y|\lambda) \otimes T(z, y|\mu) \right) \\ &= i \left(\frac{\delta T(x, y|\lambda)}{\delta u(z)} \otimes \frac{\delta T(x, y|\mu)}{\delta u^*(z)} - \frac{\delta T(x, y|\lambda)}{\delta u^*(z)} \otimes \frac{\delta T(x, y|\mu)}{\delta u(z)} \right) \end{aligned}$$

For the NS model with repulsive interaction we recall

$$r(\lambda, \mu) = -\frac{c}{\lambda - \mu} P$$

$$\frac{\delta T(x, y|\lambda)}{\delta u(z)} = T(x, z|\lambda) \sigma_+ T(z, y|\lambda)$$

$$\frac{\delta T(x, y|\lambda)}{\delta u^*(z)} = T(x, z|\lambda) \sigma_- T(z, y|\lambda)$$

Then (4.19) is just the same differential identity (2.8).

Finally, we mention that all formulae obtained in the classical theory can be derived from those in quantum theory by taking the classical limit $\hbar \rightarrow 0$,

$$(4.20) \quad \frac{-1}{i\hbar} [T(x, y|\lambda) \otimes T(x, y|\mu)] \rightarrow \{T(x, y|\lambda) \otimes T(x, y|\mu)\}$$

For example (2.1) can be obtained from Theorem 3.1 in such a way. Likewise, by taking the limit $\hbar \rightarrow 0$

$$(4.21) \quad \frac{-1}{i\hbar} [T(x, y|\lambda) \otimes_s T(x, y|\mu)] \rightarrow \{T(x, y|\lambda) \otimes_s T(x, y|\mu)\}$$

we obtain from (3.17)

$$(4.22) \quad \begin{aligned} & \{T(x, y|\lambda) \otimes_s T(x, y|\mu)\} \\ &= \int_y^x \int_y^x dz \, dz' T(x, z|\lambda) \otimes_s T(x, z'|\mu) \{U(z, \lambda) \otimes_s U(z', \mu)\} \\ & \quad T(z, y|\lambda) \otimes_s T(z', y|\mu) \end{aligned}$$

where

$$\{T(x, y|\lambda) \otimes_s T(x, y|\mu)\}_{ij,kl}$$

$$= i \int_y^x dz \left(\frac{\delta T_{ik}(x, y|\lambda)}{\delta u(z)} \frac{\delta T_{jl}(x, y|\mu)}{\delta u^*(z)} (-1)^{p(j)(p(i)+p(k))} \right. \\ \left. - \frac{\delta T_{jl}(x, y|\mu)}{\delta u(z)} \frac{\delta T_{ik}(x, y|\lambda)}{\delta u^*(z)} (-1)^{p(l)(p(i)+p(k))} \right).$$

This equation can also be derived directly by taking into account of the parity of monodromy matrices in classical theory.

The differential identity (4.18) equivalent to the classical Yang-Baxter equation (4.17) can be obtained from the differential identity (4.10) equivalent to the quantum Yang-Baxter equation (4.9) by putting [2]

$$(4.23) \quad R(\lambda, \mu) = P(1 + i\hbar r(\lambda, \mu))$$

and taking the limit according to (4.20).

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