ON THE 0-1-MAXIMIZATION OF POSITIVE DEFINITE QUADRATIC FORMS

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Introduction. We are concerned with the complexity of the problem of maximizing a positive definite quadratic form over all 0-1-vectors. This is a problem in *Pseudoboolean Programming*, whose general task may be described in decision form as follows.

PSEUDOBOOLEAN PROGRAMMING.

> Instance: *$n \in \mathbb{N}$; a function $f : \{0,1\}^n \longrightarrow \mathbb{R}$; an integer λ.*
> Question: *Does there exist a vector $x \in \{0,1\}^n$ such that $f(x) \geq \lambda$?*

Obviously, for practical purposes one has to impose some conditions on the function f which imply that f is computable in the underlying model of computation.

A wide variety of problems, including many of great practical importance, can be formulated in terms of pseudoboolean programming [HR68]. This includes many optimization problems on graphs, and also the problems of linear or quadratic integer programming which are known to be NP-complete. Thus, on the one hand, pseudoboolean programming provides a rich framework for optimization, and on the other hand, many of its problems are algorithmically very hard. For these reasons it is of interest to

- identify broad classes of functions that can be maximized over $\{0,1\}^n$ in polynomial time.

Some such classes of quadratic functions are given in [PQ82], [Ba86] and [HS86]. It is also of interest to

- identify narrow classes of functions for which the maximization over $\{0,1\}^n$ is NP-hard.

Hammer and Simeone [HS80] show that such a class is formed by the quadratic functions of the form x^TAx, where A is an upper triangular matrix that has at most one negative entry in each row. We show here that the positive definite quadratic functions also form such a class. Thus we focus on the following problem.

POSDEF-0-1-MAX.

> Instance: *$n \in \mathbb{N}$; a positive definite symmetric integer $n \times n$-matrix B; a positive integer λ.*

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Question: *Does there exist a vector $x \in \{0,1\}^n$ such that $x^T Bx \geq \lambda$?*

Pseudoboolean programming for positive definite quadratic forms has been discussed previously in [Ko80], [Ph84] and [PR87].

Since we plan to discuss the computational complexity of PosDef-0-1-Max and some related problems, we must specify the model of computation that will be used. It is the *binary* or *Turing machine model*, in which the input data is encoded in binary form and the performance of an algorithm is measured in terms of the number of required elementary operations as a function of the input size. In particular, the size of the input of PosDef-0-1-Max is the encoding length of n, $B = (\beta_{i,j})$ and λ in the binary encoding scheme. That is

$$(1 + \lceil \log n \rceil) + \Sigma_{\beta_{i,j} \neq 0} (2 + \lceil \log |\beta_{i,j}| \rceil) + (1 + \lceil \log \lambda \rceil),$$

where the logarithms are to the base 2 and $\lceil x \rceil$ denotes the integer ceiling of x. The input size is defined similarly for the other problems discussed here.

It is easy to see that the problem PosDef-0-1-Max belongs to the class NP. The guessing algorithm simply guesses a vector $x \in \{0,1\}^n$; the checking algorithm computes $x^T Bx$ and compares it with λ. The time required for checking is bounded by a polynomial in the size of the input.

One of the purposes of this note is to show that the problem PosDef-0-1-Max is NP-complete. The other purpose, perhaps even more important, is to explain how methods and results from *Computational Convexity* come into play to establish this result. (We have coined the term *Computational Convexity* to denote computational results and methods that grow out of Convexity Theory.)

What we will actually do is to outline arguments showing that PosDef-0-1-Max is polynomially equivalent to a certain geometric problem, Simplex-Width. The latter is shown elsewhere [GK89a] to be NP-complete, so that establishes the NP-completeness of PosDef-0-1-Max. The equivalence is perhaps surprising, since the flavor of PosDef-0-1-Max is so combinatorial while that of Simplex-Width is so geometrical. However, the polynomial transformations between the two problems are quite direct and have applications beyond the one presented here.

For a polytope P in Euclidean n-space and for a unit vector $u \in \mathbb{R}^n$ the *breadth* $b_u(P)$ of P in direction u is defined as

$$b_u(P) = \max_{x,y \in P} (x-y)^T u,$$

and the *width* $w(P)$ is then defined as

$$w(P) = \min_{\|u\|=1} b_u(P).$$

Simplex-Width is essentially the problem of computing the width of a simplex. More precisely, it is the following decision problem.

SIMPLEX-WIDTH.

Instance: $n \in \mathbb{N}$; *an* n*-simplex* S *given either as the convex hull of* $n+1$ *vectors in* $\mathbb{Z}^n$ *or as the set* $\{x|Ax \leq b\}$*, where* A *is an integer* $(n+1) \times n$*-matrix and* $b \in \mathbb{Z}^{n+1}$*; a positive integer* γ.

Question: *Is* $w^2(S) \leq \gamma$?

Before proceeding, let us answer two questions that might occur to the reader. First, since it is generally a major computational task to pass from one sort of representation of a polytope to the other ([Mc70], [Dy83], [Sw85], [Se87]), why don't we distinguish between the two ways of representing a simplex? The reason is that this passage is easy in the case of simplices. We can, in polynomial time, pass back and forth between the two sorts of representation, and since we are concerned only with establishing polynomial-time computability or NP-hardness there is no need to distinguish between the two sorts.

A second question is why we consider $w^2(S)$ rather than the width $w(S)$ itself. That is because $w(S)$ may be irrational and hence not computable in our binary model. However, $w^2(S)$ is always rational and of size bounded by a polynomial in the size of the input. In fact, candidates for a strip of minimum width that contains S are only those for which the union of the two parallel bounding hyperplanes contains all vertices of S. This makes it easy to compute the square of the width of any such strip, and also to show that SIMPLEX-WIDTH belongs to the class NP.

For further background material on various aspects of computational geometry, convexity theory, complexity theory, and mathematical programming that are touched here, we mention [AHU74], [Ed87], [Eg69], [GJ79], [GLS88] and [PS85].

The Transformation. Let us say that two decision problems PROB1 and PROB2 are *equal* if their sets of instances coincide and an instance is a "yes"-instance for PROB1 if and only if it is a "yes"-instance for PROB2. We say that PROB1 is *transformable* to PROB2 if there is a polynomial-time algorthm that converts an instance of PROB1 into an instance of PROB2 such that these are both "yes" instances or both "no" instances. And the two problems are *polynomially equivalent* if each is transformable to the other.

When B is positive definite, the functional x^TBx is convex and hence its maximum on any polytope is attained at some vertex of the polytope. From this it follows that POSDEF-0-1-MAX is equal to the problem

CUBEMAX.

Instance: $n \in \mathbb{N}$; *a positive definite symmetric integer* $n \times n$*-matrix* B*; a positive integer* λ.

Question: *Does there exist a vector* $x \in [0,1]^n$ *such that* $x^TBx \geq \lambda$?

CUBEMAX is the decision version of the problem of maximizing x^TBx on the standard unit cube $C = [0,1]^n = \sum_{i=1}^n [0,1]e_i$, where e_i is the ith standard unit vector.

Let us now apply LDU-factorization to express B as the product LDU of a lower triangular matrix L, a diagonal matrix D and an upper triangular matrix U. This is done essentially by Gaussian elimination and thus works in polynomial time. Since B is symmetric and positive definite, $L = U^T$ and all elements of D are positive. Thus, with $V = \sqrt{D}U$, where $\sqrt{D}$ denotes the matrix obtained from D by taking the square root of all entries, we have $B = V^T V$. It is due to the $\sqrt{D}$-step that V cannot be computed in our binary model. However, it turns out that a suitable approximation will do and that CUBEMAX is polynomially equivalent to the following problem.

PARMAX.

> Instance: $n \in \mathbb{N}$; *linearly independent vectors* $v_1, \ldots, v_n \in \mathbb{Z}^n$; *a positive integer* λ.
>
> Question: *Does there exist a vector* $x \in \sum_{i=1}^{n} [0,1] v_i$ *such that* $x^T x \geq \lambda$?

PARMAX is just the decision version of the problem of finding a point of the parallelotope $\sum_{i=1}^{n}[0,1]v_i$ that is farthest from the origin. It is a special case of the following problem.

ZONMAX.

> Instance: $m, n \in \mathbb{N}$; *vectors* $w_1, \ldots, w_m$ *of* $\mathbb{Z}^n$ *a positive integer* λ.
>
> Question: *Does there exist a vector* $x \in \sum_{i=1}^{m}[0,1]w_i$ *such that* $x^T x \geq \lambda$?

In ZONMAX, the vectors w_i are not required to be linearly independent and the sum $\sum_{i=1}^{m}[0,1]w_i$ is the sort of geometric figure known as a *zonotope* – the Minkowski (or vector) sum of finitely many line segments.

Zonotopes play an important role in several contexts, partly because of their close relationship to arrangements of hyperplanes (see e.g. [Ed87]). However, in connection with POSDEF-0-1-MAX and SIMPLEX-WIDTH we are concerned with the special zonotopes in $\mathbb{R}^n$ that are generated by $n+1$ line segments $[0,1]w_i$ such that the w_i's sum up to 0. In fact, it turns out that PARMAX is equal to the following problem.

ZEROSUM-(N+1)-ZONMAX.

> Instance: $n \in \mathbb{N}$; *linearly independent vectors* $v_1, \ldots, v_n \in \mathbb{Z}^n$; *a positive integer* λ.
>
> Question: *With* $v_0 = -\sum_{i=1}^{n} v_i$ *does there exist a vector* $x \in \sum_{i=0}^{n}[0,1]v_i$ *such that* $x^T x \geq \lambda$?

Let us now study the problem for a fixed such zonotope

$$Z = \sum_{i=0}^{n} [0,1] v_i, \quad \text{with} \quad \sum_{i=0}^{n} v_i = 0.$$

The vertices of Z, which are the points of interest in maximizing $x^T x$, are of the form $\sum_{i \in I} v_i$ with $I \subset \{0, \ldots, n\}$. Let us fix I for a moment, and suppose that $\sum_{i \in I} v_i$ is a vertex of Z. If we normalize our vectors and set

$$\hat{v} = \frac{v}{\| v \|}, \quad \hat{v}_i = \frac{v_i}{\| v_i \|} \quad \text{for} \quad i = 0, \ldots, n,$$

then of course

$$\| v \| = v^T \hat{v} = \sum_{i \in I} \| v_i \| \, \hat{v}_i^T \hat{v}.$$

A crucial step in our transformation is now the geometric interpretation of the real number $\| v_i \| \, \hat{v}_i^T \hat{v}$. Let Q_i denote any $(n-1)$-dimensional polytope in $\mathbb{R}^n$ whose affine hull is orthogonal to $\hat{v}_i$ and whose $(n-1)$-dimensional volume is $\| v_i \|$. Let L denote the linear subspace orthogonal to $\hat{v}$. Then $\| v_i \| \, \hat{v}_i^T \hat{v}$ is the $(n-1)$-dimensional signed volume of the orthogonal projection of Q_i onto L.

Now, can we unify this interpretation to be able to deal with all projections simultaneously, and thus with $\sum_{i \in I} \| v_i \| \, \hat{v}_i^T \hat{v}$; i.e., can we construct a polytope which is related to all the Q_i's? The answer (at least from the theoretical point of view) is contained in Minkowski's classical theorem [Mi03] on the existence of polytopes with prescribed facet normals and facet volumes.

Minkowski's Theorem. *Let $z_0, \ldots, z_m$ be (mutually different and spanning) unit vectors of $\mathbb{R}^n$, and let $\nu_0, \ldots, \nu_m$ be positive reals such that $\sum_{i=0}^m \nu_i z_i = 0$. Then there is a polytope P, unique up to translation, that has $z_0, \ldots, z_m$ as the outer normals and $\nu_0, \ldots, \nu_m$ as the $(n-1)$-volumes of its facets.*

Application of this theorem with $m = n$, $z_i = v_i$, $\nu_i = \| v_i \|$ $(i = 0, \ldots, n)$ yields a simplex S with facets F_i (this is of course a special choice for the polytopes Q_i) such that v_i is the outer normal of F_i and $\| \nu_i \|$ is the $(n-1)$-volume of F_i. This result is not directly useful for our purpose since Minkowski's theorem is not algorithmic. However, it turns out that it is possible to design a polynomial-time algorithm which determines S approximately. The tools for that are Brunn-Minkowski theory and the ellipsoid algorithm.

The last crucial step in our transformation uses an observation employed in [Eg69] to prove a result of Steinhagen [St20], [St22]. With the aid of the projection interpretation of $\| v \|$ one can show by means of an elementary dissection argument that

$$\| v \| \, b_{\hat{v}}(S) = nV(S),$$

where $V(S)$ is the volume of S. This relation indicates that maximizing the norm over vertices of Z is equivalent to minimizing the breadth of S over all directions $\hat{v}$ associated with subsets I of $\{0, \ldots, n\}$. But as indicated earlier, this minimum is the width of S. Clearly the last transformation can be reversed (the reverse direction does not require use of Minkowski's Theorem) and we conclude that PosDef-0-1-Max is polynomially equivalent to Simplex-Width.

Concluding Remarks. The previous section shows that the the problems PosDef-0-1-Max and Simplex-Width are related by means of a two-way polynomial-time transformation. In [GK89a], the NP-completeness of Simplex-Width is proved by a transformation from Partition, so it follows that PosDef-0-1-Max is also NP-complete. The transformation between PosDef-0-1-Max and Simplex-Width also makes it possible to transfer heuristics and approximative algorithms from either problem to the other [GK89b]. Moreover, the auxiliary problems ZonMax and ParMax are of interest in their own right and are also NP-complete. For ZonMax, this is established in a different manner (transformation from Not-All-Equal 3Sat) by J. Bodlaender and J. van Leeuwen (private communication).

Finally, we want to point out some applications that involve the problems considered here. We have already mentioned the importance of zonotopes in computational geometry, due in part to their relationship with arrangements of hyperplanes. However, they are useful in other ways as well. For example, in [MS85] they play a role in finding the minimum of areas of a polytope's orthogonal projections.

The problem of computing the width of a general polytope is of interest in connection with certain problems in robotics and in the sensitivity analysis of linear programming [GK89a]. And in connection with computer vision, there has been some interest in Minkowski's theorem (see [Li85]) – that is, in constructing polytopes from their facet normals and facet areas. Extensions of our algorithmic approach lead to further applications in this area.

REFERENCES

[AHU74] A.V. AHO, J.E. HOPCROFT AND J.D. ULLMAN, *The Design and Analysis of Computer Algorithms*, Addison-Wesley, Reading, Mass., 1974.

[Ba86] F. BARAHONA, *A solvable case for quadratic 0-1 programming*, Discrete Appl. Math., 13 (1986), pp. 23–26.

[Dy83] M.E. DYER, *The complexity of vertex enumeration methods*, Math. of Operations Res., 8 (1983), pp. 381–402.

[Ed87] H. EDELSBRUNNER, *Algorithms in Computational Geometry*, Springer, New York etc., 1987.

[Eg69] H.G. EGGLESTON, *Convexity*, Cambridge Univ. Press, Cambridge, 1958, 1969.

[GJ79] M.R. GAREY AND D.S. JOHNSON, *Computers and Intractability*, Freeman, San Francisco, 1979.

[GK89a] P. GRITZMANN AND V. KLEE, *Inner and outer j-radii of polytopes in finite dimensional ℓ^p-spaces: computational aspects*, in preparation.

[GK89b] P. GRITZMANN AND V. KLEE, *On the error of polynomial computations of width and diameter*, in preparation.

[GLS88] M. GRÖTSCHEL, L. LOVÁSZ AND A. SCHRIJVER, *Geometric algorithms and Combinatorial Optimization*, Springer, Berlin, 1988.

[HR68] P.L. HAMMER AND S. RUDEANU, *Boolean Methods in Operations Research and Related Areas*, Springer, New York, 1968.

[HS80] P.L. HAMMER AND B. SIMMEONE, *Quasimonotone boolean functions and bistellar graphs*, Ann. Discrete Math., 8 (1980), pp. 107–119.

[HS86] P. HANSEN AND B. SIMEONE, *Unimodular functions*, Discrete Appl. Math., 14 (1986), pp. 269–281.

[Ko80] H. KONNO, *Maximization of a convex quadratic function over a hypercube*, J. Oper. Res. Soc. Japan, 23 (1980), pp. 171–189.

[Li85] J.J. LITTLE, *Extended Gaussian images, mixed volumes, and shape reconstruction*, Proc. 1st Conf. on Computational Geometry, Baltimore (1985), pp. 15–23.

[MS85] M. McKENNA AND R. SEIDEL, *Finding the optimal shadow of a convex polytope*, Proc. Symp. Comp. Geom. (1985), pp. 24–28.

[Mc70] P. McMULLEN, *The maximum number of faces of a convex polytope*, Mathematika, 17 (1970), pp. 179–184.

[Mi03] H. MINKOWSKI, *Volumen und Oberfläche*, Math. Ann., 57 (1903), pp. 447–495.

[PR87] P.M. PARDALOS AND J.B. ROSEN, *Constrained Global Optimization*, Lecture Notes in Comp. Sci. vol. 268, Springer-Verlag, Berlin, 1987.

[Ph84] D.T. PHAM, *Algorithmes de calcul du maximum de formes quadratiques sur la boule unité de la norme du maximum*, Numer. Math., 45 (1984), pp. 377–401.

[PQ82] J.C. PICARD AND M. QUEYRANNE, *Selected applications of min cut in networks*, INFOR, 20 (1982), pp. 395–422.

[PS85] F.P. PREPARATA AND M.I. SHAMOS, *Computational Geometry*, Springer, New York etc., 1985.

[Se87] R. SEIDEL, *Output-Size Sensitive Algorithms for Constructive Problems in Computational Geometry*, Springer Ph.D. Theses, Department of Computer Science, Cornell University, Ithaca, N.Y., 1987.

[St20] P. STEINHAGEN, *Beiträge zur Theorie der konvexen Köper*, Diss., Hamburg, 1920.

[St22] P. STEINHAGEN, *Über die größte Kugel in einer konvexen Punktmenge*, Abh. Math. Sem. Univ. Hamburg, 1 (1922), pp. 15–22.

[Sw85] G. SWART, *Finding the convex hull facet by facet*, J. of Algorithms, 6 (1985), pp. 17–38.

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