**REMARKS ON INERTIAL RADII, PERSISTENT
NORMAL STRESSES, SECONDARY MOTIONS,
AND NON-ELASTIC EXTENSIONAL VISCOSITIES**

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Remarks on inertial radii, persistent normal stresses, secondary motions, and non-elastic extensional viscosities

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In this note I discuss some consequences of the balance of inertia and normal stresses in nearly steady slow motions. I argue that the fluid's elasticity cannot be determined from its extensional viscosity. I give a formula for the extensional viscosity at quadratic order in the stretch rate and show that one and the same intensity factor for the extensional viscosity arises universally as the intensity factor for the non-Newtonian contribution to the vorticity. It is argued that the appearance of intense vortices and relatively large extensional viscosities are linked in nearly steady slow flow.

Second order fluids and Reiner-Rivlin fluids

The stress $\mathbf{T}$ in an incompressible simple fluid can be expressed as

$$\mathbf{T} = -p\mathbf{1} + \mathbf{S} , \quad (1)$$

where the extra stress is determined as a functional of the history of the relative Cauchy strain. The isotropic part of the stress has no constitutive equation; it is determined by dynamics as one of the unknown fields governed by the equations of motion. Most liquids which are isotropic in the rest state fit in this framework. Many models for the constitutive equation relating to $\mathbf{S}$ to deformation have been proposed. Whatever may be the choice for the

functional relating stress to relative strain, it may be approximated by expressions of the form

$$\mathbf{S}^{(n)} \stackrel{\text{def}}{=} \mathbf{S}_1[\mathbf{U}] + \mathbf{S}_2[\mathbf{U}] + \dots + \mathbf{S}_n[\mathbf{U}] , \quad (2)$$

with an error of $O(\|\mathbf{U}\|^{n+1})$, where $\mathbf{U}(\mathbf{x},t)$ is the velocity of a nearly steady slow motion

$$\mathbf{U}(\mathbf{x},t) = \varepsilon \mathbf{v}(\mathbf{x},\tau,\varepsilon) , \quad \tau = \varepsilon t , \quad (3)$$

and ε is a small parameter measuring slowness usually associated with prescribed data,

$$\|\mathbf{v}(\mathbf{x},\tau,0)\| < \infty \quad (4)$$

and t is a slow time, $O(1)$ variations in the time interval $\Delta\tau = \varepsilon\Delta t$ are slow in the time interval Δt ; that is, $\frac{\partial}{\partial t} = \varepsilon \frac{\partial}{\partial \tau}$. Moreover,

$$\mathbf{S}^{(n)} = \varepsilon \mathbf{S}_1[\mathbf{v}] + \varepsilon^2 \mathbf{S}_2[\mathbf{v}] + \dots + \varepsilon^n \mathbf{S}_n[\mathbf{v}] . \quad (5)$$

For slow, steady motions, $\mathbf{S}_n$ is homogeneous of degree n in $\mathbf{U}$. Nearly steady slow motions are equivalent to those which Coleman and Noll [1] call retarded; they are discussed in greater detail in references [2] and [5].

The expansions (2) were introduced by Coleman and Noll [1] and they were justified in the context of their theory of fading memory. These expansions are robust in the sense that they carry over to more general theories [2], which contain most of the constitutive equations presently used by rheologists. The form which such expansions must take is strongly suggested in the (1955) theory of Rivlin and Ericksen [3]. We are interested in the stress $\mathbf{S}^{(2)}$ of grade two where

$$\mathbf{S}_1 = \mu \mathbf{A}_1[\mathbf{U}] ,$$

$$\mathbf{S}_2 = \alpha_1 \mathbf{A}_2[\mathbf{U}] + \alpha_2 \mathbf{A}_1^2[\mathbf{U}] , \quad (6)$$

μ is the zero shear viscosity, α_1 and α_2 are constants, which I call the quadratic constants of $\mathbf{S}^{(2)}$,

$$\mathbf{A}_1[\mathbf{U}] = \nabla \mathbf{U} + \nabla \mathbf{U}^T ,$$

and

$$\mathbf{A}_2[\mathbf{U}] = \frac{\partial \mathbf{A}_1}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{A}_1 + \mathbf{A}_1 \nabla \mathbf{U} + \nabla \mathbf{U}^T \mathbf{A}_1 .$$

The stress $\mathbf{S} = \mathbf{f}(\mathbf{A}_1)$ in a nonlinear viscous fluid may be represented, without loss of generality, in the form

$$\mathbf{S} = \phi_1(\text{II}, \text{III}) \mathbf{A}_1 + \phi_2(\text{II}, \text{III}) \mathbf{A}_1^2 , \quad (7)$$

where ϕ_1 and ϕ_2 are functions of the second and third invariants of $\mathbf{A}_1$. The expression (7) is called a Reiner-Rivlin fluid. It is not presently used in rheology because it has a zero first normal stress in shear flow. On the other hand, every constitutive equation can be represented by Eq. (7) in steady motions which are purely extensional since the history of such motions can be expressed in terms of $\mathbf{A}_1$ alone. $\mathbf{S}^{(2)}$ is also of the form (7) because in pure extension $\mathbf{A}_2 = \mathbf{A}_1^2$.

It is not uncommon for rheologists to say that the extensional viscosity is associated with the elasticity, or the memory of a fluid. This type of thinking is flawed because we have just seen that the extensional viscosity is described uniquely, rigorously, and completely by an equation (7) for a purely viscous fluid without memory. Since $\mathbf{A}_1$ is the same for all past time in steady extension, the

fluid has nothing different than its present state to remember. The elasticity of the fluid cannot be detected. To illustrate this with an elastic fluid, we may think of microstructural models of elastic dumbbells or long flexible molecules in which high extensional viscosities are associated with stretching of the model "molecules" in the flow, an elastic response in the sense that we have stored energy which is not manifest (because it remains constant) during a steady extensional flow. On the other hand, enormous extensional viscosities can be achieved in fluids with long, slender, inextensible and completely non-elastic particles. It is certainly possible to obtain the extensional functions of any simple fluid by devising an appropriate purely viscous fluid.

The stress $\mathbf{S}^{(2)}$ has been widely used in rheological studies because it is one of the simpler nonlinear constitutive equations and, in fact, is the form which is taken by every constitutive equation when the motion is nearly steady and slow.

Quadratic constants

Tables of $3\alpha_1+2\alpha_2$, obtained from rod climbing measurements, can be found in the paper of Joseph, Beavers, Cers, Dewald, Hoger, and Than [4].

The zero shear viscosity μ and the quadratic constants α_1 and α_2 are related to moments of kernels appearing in the integral expansions of the stress (see, for example, Joseph and Beavers, [5]) and to limiting values of the first and second normal stress. The zero shear viscosity μ is given by

$$\mu = \int_0^{\infty} G(s)ds = \lim_{\kappa \rightarrow \infty} \frac{T_{12}(\kappa)}{\kappa}, \quad (8)$$

the first quadratic constant is given by

$$\alpha_1 = -\int_0^\infty sG(s)ds = -\lim_{\kappa \rightarrow 0} \frac{N_1(\kappa^2)}{2\kappa^2} \quad (9)$$

and the second quadratic constant is given by [5]

$$\alpha_2 = \int_0^\infty \int_0^\infty \gamma(s_1, s_2) ds_1 ds_2 = \lim_{\kappa \rightarrow 0} \frac{N_1(\kappa^2) + N_2(\kappa^2)}{\kappa^2} = \phi_2(0,0) \quad (10)$$

where $\phi_2(II,III)$ appears in (7). These relations imply that μ is positive, α_1 is negative and the sign of α_2 is indeterminate. In polymeric liquids so far studied, N_2 is small and negative and α_2 is positive.

The parameter α_1 is associated with the extra tension $N_1 = T_{11} - T_{22}$ along streamlines in shear flow. At the same time $-\alpha_1$ may be thought to be a kind of relaxation time for the fluid, associated with the memory or elasticity. It could be said that N_1 is associated with elasticity. On the other hand, α_2 is present already in the Reiner-Rivlin fluid, which is instantaneously viscous and is not associated with memory. We are going to show that any fluid with a large positive first normal stress difference and small α_2 will give rise to a **negative** extensional viscosity. This type of elasticity aids rather than resists stretching.

Balance of inertia and normal stresses. Something for nothing.

I am going to discuss two features of flow in the range of validity of second order theory which are not represented in the rheological literature. The first is the characteristic signature of dynamics of flow of second order fluids, the balance of inertia and normal stress at the quadratic level. The second is the

persistence of normal stress in regions of vanishing dimension which I call “something for nothing.”

Let us consider flows which perturb rest with steady flow proportional to a small parameter writing

$$\mathbf{U} = \varepsilon \mathbf{U}_1 + \varepsilon^2 \mathbf{U} + \text{higher order terms} . \quad (11)$$

The perturbation equations for slow steady flow are well known. At first order, we get Stokes flow of Newtonian fluid. In the limit $\varepsilon \rightarrow 0$, there is no difference between Newtonian and non-Newtonian fluids. At second order we get

$$\mu \nabla^2 \mathbf{U}_2 - \nabla p_2 = \rho \mathbf{U}_1 \nabla \mathbf{U}_1 - \text{div}(\alpha_1 \mathbf{B} + \alpha_2 \mathbf{A}^2) \quad (12)$$

The inhomogeneous terms are the force of inertia

$$\rho \mathbf{U}_1 \nabla \mathbf{U}_1$$

and the divergence of the normal stress

$$\text{div}(\alpha_1 \mathbf{B} + \alpha_2 \mathbf{A}^2)$$

where $\mathbf{A}[\mathbf{U}_1]$ and $\mathbf{B}[\mathbf{U}_1, \mathbf{U}_1]$ are evaluated on $\mathbf{U}_1$. The same equation (12), but with $\alpha_1 = 0$, governs the flow of the Reiner-Rivlin fluid (7). Suppose now that $\mathbf{U}_1$ scales with U and that $\mathbf{U}_1$ has an order one variation in a distance L . Then inertia scales like $\rho U^2/L$ and normal stresses like $\alpha U^2/L^3$ where α is representative of an appropriate linear combination of α_1 and α_2 . The two effects are equally important at distances of the order

$$L = \sqrt{\alpha/\rho}$$

In many problems there is a critical distance, of order L , with inertia dominant for larger distances and normal stresses dominant for smaller distances. We could call the critical distance an inertial radius. This type of balance appears in many problems; for example, in the flow outside a rotating sphere or between a cone and plate [6], outside a rotating wavy rod [8], or in the flow associated with rod climbing [4], [5], among others. There is an inertial radius for secondary motions which in rotating flows takes form as a dividing surface for eddies with a different direction of circulation. There is also an inertial radius for the pressure. Suppose that $U=r\omega(r)$ where $\omega(r)$ is the angular velocity of rotation in the first order Newtonian problem. Inertia

$$\rho r^2 \omega^2$$

goes to zero quadratically with r , but the normal stresses

$$\propto \frac{U^2}{r^2} \sim \alpha \omega^2$$

do not go to zero, no matter how small is the distance r . This last astonishing result (something for nothing) appears to be a valid description of normal stress effects in rod climbing, at “pressure holes” (where $\omega=\kappa d$, the shear rate times the hole diameter) and elsewhere, though its effects at the smallest distance are probably masked by other small dynamical effects like surface tension, which become relatively large in this limit.

The concept of an inertial radius or critical distance dividing fields into regions dominated by inertia and regions dominated by normal stress requires a competition between normal stresses and inertia. Sometimes people refer to this as a competition between elasticity and inertia, but since α_2 is involved, this choice of words may be misleading. Examples of competing effects are

abundant. In steady flows around rotating bodies, it appears that inertia drives eddies with one sense, and normal stresses drive them in the opposite sense. The inertia of fluid will cause it to sink near a rotating rod and normal stresses will make it rise. The effect of normal stresses is to make the extrudate from a pipe swell, while inertia makes it contract. Inertia produces an error of one sign at pressure hole and the normal stresses produce an error of the other sign.

It is not obligatory that normal stresses and inertia must compete. Counter examples can be found. For example, when a rod is rotated in a two-layer system, with the viscoelastic liquid above, this liquid will climb down the rod. Analysis shows that the effect of gravity on the rotating heavy fluid below is to depress the free surface into a “bathtub vortex” with greater depression near the rod. The normal stresses push the viscoelastic fluid up into the air and down into the heavy fluid. Hence, inertia and normal stresses are in conflict at the air-liquid surface and in concert at the two-fluid interface. A casual inspection of the equations of motion might suggest that normal stresses and inertia do not need to act in opposition in general, in spite of so many examples in which they are opposed. Perhaps it is possible to find a general condition in which opposing effects are more than a rule of thumb.

Extensional viscosity and secondary motions

We can show that fluids which have large extensional viscosities at low stretching are likely to give rise to intense secondary motions; both are proportional to $\alpha_1 + \alpha_2$.

At first order we find that the vorticity $\zeta_1 = \text{curl } \mathbf{u}_1$ is a harmonic vector

$$\nabla^2 \zeta_1 = 0$$

At second order we get

$$\rho \frac{\partial \zeta_2}{\partial t} + \rho \operatorname{curl}(\zeta_1 \wedge \mathbf{u}_1) = \mu \nabla^2 \zeta_2 + (\alpha_1 + \alpha_2) \operatorname{curl} \operatorname{div} \mathbf{A}^2 \quad (13)$$

where $\mathbf{A} = \mathbf{A}[\mathbf{u}_1]$ and the Giesekus [6] relation $\operatorname{div} \mathbf{B} = \operatorname{div} \mathbf{A}^2 + \operatorname{grad} \phi$ has been used. Equation (13) shows that no couples arise from normal stresses at second order when

$$\alpha_1 + \alpha_2 = \lim_{\kappa \rightarrow 0} \left(\frac{1}{2} N_1 + N_2 \right) / \kappa^2 \quad (14)$$

vanishes. In polymeric liquids so far studied N_2 is small and negative. For these liquids $\alpha_1 + \alpha_2$ is positive.

In the case in which inertia and normal stresses compete, we may conclude that the inertial radius for secondary flows is given by

$$L = O \left(\frac{\alpha_1 + \alpha_2}{\rho} \right)^{1/2}$$

At distances greater than the inertial radius for secondary motions, inertia will drive secondary motions; at lesser distances secondary motions are driven by normal stresses.

The motion known as simple extension in the direction $\mathbf{e}_1$ is given by

$$[\mathbf{A}_1] = \begin{pmatrix} 2\dot{s} & 0 & 0 \\ 0 & -\dot{s} & 0 \\ 0 & 0 & -\dot{s} \end{pmatrix},$$

$$[\mathbf{A}_1^2] = \begin{pmatrix} 4\dot{s}^2 & 0 & 0 \\ 0 & \dot{s}^2 & 0 \\ 0 & 0 & \dot{s}^2 \end{pmatrix},$$

$$A_2 = A_1^2,$$

where $\dot{s}$ is the rate of stretching, $dU_1/dx_1 = \dot{s}$. The stress differences in extension

$$T_{11} - T_{22} = T_{11} - T_{33} = S_{11} - S_{22} = S_{11} - S_{33}$$

are evaluated on $S^{(2)} = \mu A_1 + (\alpha_1 + \alpha_2) A_1^2$ as

$$T_{11} - T_{22} = 3[\dot{s}\mu + (\alpha_1 + \alpha_2) \dot{s}^2] = 3\dot{s} [\mu + (\alpha_1 + \alpha_2) \dot{s}] \quad (15)$$

where $\mu + (\alpha_1 + \alpha_2)\dot{s}$ is the extensional viscosity at low stretching. This relation, expressed in terms of normal stress differences as in (14), was first given by Zahorski [9] and later in the form (15) in [4]. Equation (15) is the only model independent formula for the extensional viscosity known to me. It applies to all steady extensional motions (small $\dot{s}$ with errors $O(\dot{s}^3)$).

Second order fluids with $\alpha_1 = 0$ are second order Reiner-Rivlin fluids. For these the extensional viscosity is larger than for any second order fluid with the same α_2 , because α_1 is negative. The fact that the intensity factor $\alpha_1 + \alpha_2$ appears both in (15) and (13) leads to a theorem: **The extensional viscosity is greater and the vortical motions are more intense in the slow steady motions of a Reiner-Rivlin fluid ($\alpha_1=0$) than in any elastic fluid ($\alpha_1<0$) with the same α_2 .** Many intense secondary flows, like those with strong vortices in the flow through contractions, should then arise from Reiner-Rivlin fluids. The larger the value of α_2 , the more intense the motion.

These observations about the relation of extensional viscosity to secondary motions seem to offer a theoretical foundation for recent numerical

results presented by Debbaut and Crochet [7]. They show that "...by including the dependence of the viscosity function upon the third as well as the second invariant in generalized Newtonian and viscoelastic fluids, it is possible to isolate the effects of extensional viscosity and first normal stress difference in axisymmetric flows. It is shown that a high Trouton's ratio leads to vortex enhancement in an abrupt 4:1 circular contraction and to drag increase for the flow round a sphere moving along the axis of a circular tube. For low values of elasticity, it is shown that the first normal stress difference has the opposite effect." We may interpret their observation as related to the fact that for a fixed value of α_2 the intensity of vortex activity and the Trouton viscosity are greatest when $|\alpha_1|$ is smallest.

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