**ON THE CONVERGENCE OF FINITE ELEMENT
APPROXIMATIONS OF A RELAXED VARIATIONAL PROBLEM**

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Abstract: The accuracy of finite element approximations to a convex but not strictly convex variational problem is considered. Convergence is proved for a finite element approximation of a particular vector field related to the solution. In a special one dimensional case $O(h)$ convergence is shown for a piecewise linear approximation of the derivative. h denotes the size of each element domain. Numerical results are also presented for this one dimensional case.

§1 Introduction.

We analyze the finite element approximation of minimizers u of the functional

$$(1.1a) \quad J(u) = \int_{\Omega} (\psi(|\nabla u|) + fu)$$

where

$$\psi(0) = 0,$$

$$\psi'(t) = \begin{cases} \mu_1 t, & t < t_1 \\ \mu_1 t_1 = \mu_2 t_2, & t_1 \leq t \leq t_2 \\ \mu_2 t, & t > t_2. \end{cases}$$

t_1, t_2, μ_1, μ_2 are positive constants, $t_1 < t_2$ and $\mu_2 < \mu_1$. We require $\mu_2 < \mu_1$ to ensure ψ is convex. Ω is a bounded open domain and $u = u(x)$ satisfies the condition

$$(1.1b) \quad u = g \text{ on } \partial\Omega.$$

This problem arises as the relaxation of some nonconvex problems.

Background: In Bauman and Phillips [BP] and Gurtin and Temam [GT] (1.1) is the relaxation of an elastostatic antiplane shear problem with nonconvex

energy. In [BP] Ω is a convex ring, u is the axial displacement which is assigned the constant value H on the inner boundary and 0 on the outer boundary, and $f \equiv 0$.

In Goodman, Kohn and Reyna [GKR] (1.1) arises in their study of a problem from optimal design. They take $f = \lambda$, a constant, Ω is the unit square and $g \equiv 0$. In part of the paper a finite difference approximation is obtained by multigrid methods.

Silling also considers the antiplane shear problem with nonconvex energy and its relaxation (1.1) in [S1] and [S2]. He computes finite difference solutions to a related time dependent problem and considers the numerical steady states as approximations to possible solutions.

Under minimal assumptions there exists a solution to (1.1); its uniqueness is, in general, an open question. In [BP] the minimizer is shown to be unique. In all cases the following vector field is uniquely determined.

$$(1.2) \quad \sigma = \begin{cases} \nabla u & \text{if } |\nabla u| < t_1 \text{ on } |\nabla u| > t_2 \\ \frac{\nabla u}{|\nabla u|} & \text{if } t_1 \leq |\nabla u| \leq t_2 \end{cases}$$

Note that the transition region where only $\frac{\nabla u}{|\nabla u|}$ is determined uniquely is generally thought to be small. In the numerical results of [GKR] and [S1] it occupies only about 15% of Ω .

Finite element approximation: Let $0 < h < \frac{1}{2}$ be a parameter. Let $\mathcal{T}_h$ be a partition of Ω into open elements τ_h each with diameter approximately h . Assume, for simplicity, that Ω has a boundary on which the mesh functions fit exactly. (i.e.

$$(1.3) \quad \bar{\Omega} = \bigcup_{\tau_h \in \mathcal{T}_h} \bar{\tau}_h$$

Typically $\partial\Omega$ is polygonal. Let S^h be a finite dimensional subspace of $W^{1,\infty}(\Omega)$ (Space of functions defined on Ω whose first derivative is in $L^\infty(\Omega)$). Restrict g to be a function for which there exists $\chi \in S^h$ such that

$$(1.4) \quad \chi \equiv g \text{ on } \partial\Omega.$$

We will require the family of meshes $\{\mathcal{T}_h\}_{0 < h < \frac{1}{2}}$ to be quasi-uniform; there exists constants ρ_0 and ρ_1 , independent of h , such that for each element domain τ_h : (i) τ_h contains a disk of radius $\rho_0 h$. (ii) $\text{diam}(\tau_h) \leq \rho_1 h$.

Assume that for any $v \in H^1(\Omega)$ there exists $\{\chi_h\}_{0 < h < \frac{1}{2}}$ where $\chi_h \in S^h$ such that

$$(1.5) \quad \lim_{h \rightarrow 0} \|v - \chi_h\|_{1,\Omega} = 0$$

where $H^m(A)$ denotes the usual Sobolev space of functions defined on a bounded open set A with m generalized derivatives in $L^2(A)$ that has norm $\|\cdot\|_{m,A}$ and semi-norms $|\cdot|_{j,A}$ for $j = 0, \dots, m$. If we write $\|\cdot\|_m$ or $|\cdot|_j$ we mean $\|\cdot\|_{m,\Omega}$ or $|\cdot|_{j,\Omega}$ respectively.

Let

$$H_g^1(\Omega) = \{v \in H^1(\Omega) : v = g \text{ on } \partial\Omega\}$$

and consider the reformulation of (1.1). Find $u \in H_g^1(\Omega)$ such that

$$(1.6) \quad J(u) = \min_{v \in H_g^1(\Omega)} J(v)$$

The finite element problem is the following: find $u_h \in S_g^h$ such that

$$(1.7) \quad J(u_h) \leq \min_{\chi \in S_g^h} J(\chi)$$

where

$$S_g^h = \{\chi \in S^h : \chi = g \text{ on } \partial\Omega\}.$$

Also define

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$$

and

$$S_0^h = \{x \in S^h : x = 0 \text{ on } \partial\Omega\}.$$

Finally assume all integrations on the approximating functions are performed exactly.

Main result: Let

$$(1.8a) \quad E = \{x \in \Omega : |\nabla u| < t_1 \text{ or } |\nabla u| > t_2\}$$

and

$$(1.8b) \quad F = \{x \in \Omega : t_1 \leq |\nabla u| \leq t_2\} = \Omega \setminus E.$$

Note that E and F are identical for solutions of (1.1) since the vector field σ in (1.2) is unique. We show that for any solution u of (1.6) and any approximate solution u_h which satisfies (1.7)

$$(1.9a) \quad \nabla u_h \rightarrow \nabla u \text{ as } h \rightarrow 0 \text{ on } E.$$

$$(1.9b) \quad \psi'(|\nabla u_h|) \rightarrow \psi'(|\nabla u|) \text{ as } h \rightarrow 0 \text{ on } F.$$

and

$$(1.9c) \quad \frac{\nabla u_h}{|\nabla u_h|} \rightarrow \frac{\nabla u}{|\nabla u|} \text{ as } h \rightarrow 0 \text{ on } F.$$

where the convergence is in L^2 norms. (See Theorem 3.1 for the precise statement.) In [GKR] and [S2] numerical approximations of (1.1) are obtained; however, no discussion of the accuracy is given. Our analysis offers some evidence that these results are valid. Our argument is based on techniques developed in Rannacher and Frehse [RF] and Johnson and Thomée [JT].

Outline of the paper: The results (1.9) are proved in §3 while some key auxiliary Theorems are given in §2.

In §4 we consider the approximation of (1.6) by piecewise linear finite elements in one dimension where $\Omega = (a,b)$ and F is a finite set of points. In this simple case (1.6) has a unique solution (it is not clear that (1.7) does however). We show for any solution u_h of (1.7) that $u'_h \rightarrow u'$ at an optimal rate $O(h)$ in L^∞ on element domains that do not contain points in the set F . (See Theorem 4.1.)

In §5 some numerical calculations are presented in the one dimensional problem described above. Both linear and quadratic finite elements are used.

§2 Properties of J :

In this section we state some of the properties of the functional J that will be used in the convergence proofs in later sections.

J is differentiable; for $u \in H_g^1(\Omega)$

$$(2.1) \quad J'(u)[v] = \int_{\Omega} \left[\frac{\psi'(|\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v - fv \right]$$

where $v \in H_0^1(\Omega)$. We will consider the following reformulations of problems (1.6) and (1.7): Find $u \in H_g^1(\Omega)$ such that

$$(2.2) \quad J'(u)[v] = 0 \quad \text{for all } v \in H_0^1(\Omega).$$

Find $u_h \in S_g^h$ such that

$$(2.3) \quad J'(u_h)[\chi] = 0 \quad \text{for all } \chi \in S_0^h.$$

By standard arguments we can show the following: (See Ciarlet [C] for instance).

Proposition 2.1: If $f \in L^2(\Omega)$ then there exists solutions u and u_h of (1.6) and (1.7) respectively. Furthermore u solves (1.6) if and only if it is a solution of (2.2) and u_h solves (1.7) if and only if it is a solution of (2.3).

For $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ define $G = \begin{bmatrix} G_1(\xi) \\ G_2(\xi) \end{bmatrix}$ by

$$G(\xi) = \frac{\psi'(|\xi|)}{|\xi|} \xi$$

If $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ define a 2×2 matrix $\tilde{A}$ with entries

$$[\tilde{A}(\eta)]_{ij} = \frac{\partial G_i}{\partial \eta_j}(\eta)$$

Also define the related 2×2 matrix A whose entries are given by the expression

$$[A(\xi, \eta)]_{ij} = \int_0^1 [\tilde{A}(\eta + t(\xi - \eta))]_{ij} dt.$$

Note $\tilde{A}$ and A are symmetric and

$$\begin{aligned} (2.4) \quad A(\xi, \eta)(\xi - \eta) &= \int_0^1 \tilde{A}(\eta + t(\xi - \eta))(\xi - \eta) dt \\ &= G(\xi) - G(\eta). \end{aligned}$$

The following identity is useful:

$$(2.5) \quad (J'(w) - J'(z))[w - z] = \int_{\Omega} (G(\nabla w) - G(\nabla z)) \cdot \nabla(w - z)$$

$$\begin{aligned}
(J'(w) - J'(z))[w - z] &= \int_{\Omega} \nabla(w - z) \cdot A(\nabla w, \nabla z) \nabla(w - z) \\
&= \int_{\Omega} \int_0^1 \nabla(w - z) \cdot \tilde{A}(\nabla w + t \nabla(w - z)) \nabla(w - z) dt.
\end{aligned}$$

We calculate the eigenvalues of $\tilde{A}$ to show it is positive semi-definite.

Let

$$g(s) = \frac{\psi'(s)}{s}.$$

Then

$$\frac{\partial G_1}{\partial \xi_1}(\xi) = g'(|\xi|) \frac{\xi_1^2}{|\xi|} + g(|\xi|).$$

$$\frac{\partial G_1}{\partial \xi_2}(\xi) = \frac{\partial G_2}{\partial \xi_1}(\xi) = g'(|\xi|) \frac{\xi_1 \xi_2}{|\xi|}, \text{ and}$$

$$\frac{\partial G_2}{\partial \xi_1}(\xi) = g'(|\xi|) \frac{\xi_2^2}{|\xi|} + g(|\xi|).$$

A short calculation shows that the eigenvalues are given by the expressions

$$\lambda_1 = g(|\xi|).$$

$$\lambda_2 = g'(|\xi|)|\xi| + g(|\xi|).$$

Since

$$\begin{aligned} g'(s)s + g(s) &= s \left[\frac{\psi''(s)}{s} - \frac{\psi'(s)}{s^2} \right] + \frac{\psi'(s)}{s} \\ &= \psi''(s) \end{aligned}$$

we have

$$(2.6) \quad \lambda_1 = \psi'(|\xi|)/|\xi|.$$

$$\lambda_2 = \psi''(|\xi|).$$

Since both eigenvalues are nonnegative A is positive semi-definite. Note that $\lambda_1 \geq \lambda_2$.

Since $\psi''(|\xi|) = 0$ if $t_1 \leq |\xi| \leq t_2$ strict monotonicity is not guaranteed. We note a one dimensional example from [GKR] which has many solutions.

Consider

$$J(u) = \int_0^1 \psi(|u'|)$$

and look for functions that satisfy the boundary conditions

$$u(0) = 0 \quad , \quad u(1) = \frac{1}{2}(t_1 + t_2).$$

Any continuous piecewise linear function u that satisfies the boundary conditions and has derivative that satisfies

$$t_1 \leq u' \leq t_2$$

is a solution.

In the following propositions we show that the vector field (1.2) is uniquely defined for both the true and approximate solutions.

Proposition 2.2: Let u be any solution of (1.6) and define E and F as in (1.8). If v is another solution then

$$(2.7) \quad \nabla u = \nabla v \quad \text{a.e. on } E,$$

$$(2.8) \quad t_1 \leq |\nabla v| \leq t_2 \quad \text{a.e. on } F, \text{ and}$$

$$(2.9) \quad \frac{\nabla u}{|\nabla u|} = \frac{\nabla v}{|\nabla v|} \quad \text{a.e. on } F.$$

Proof:

We first note the following useful identity. Let ξ and η be two nonzero vectors in $\mathbb{R}^2$ and θ the angle between them.

$$(2.10) \quad \left[\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right] \cdot (\xi - \eta) = |\xi| + |\eta| - \frac{\xi \cdot \eta}{|\eta|} - \frac{\xi \cdot \eta}{|\xi|}$$

$$\begin{aligned}
\left(\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right) \cdot (\xi - \eta) &= (|\xi| + |\eta|) - \frac{\xi \cdot \eta}{|\xi| |\eta|} (|\xi| + |\eta|) \\
&= (|\xi| + |\eta|)(1 - \cos \theta)
\end{aligned}$$

From the Euler-Lagrange equations, (2.2), we have

$$(G(\nabla u), \nabla e) = (G(\nabla v), \nabla e)$$

where u and v are solutions, $e = u - v$, and

$$(w, z) = \int_{\Omega} wz.$$

Then

$$0 = (G(\nabla u) - G(\nabla v), \nabla e)$$

$$= \int_{\Omega} \left[\int_0^1 \tilde{A}(\nabla u + t \nabla e) \nabla e \cdot \nabla e \, dt \right]$$

Since $\tilde{A}(\cdot)$ is symmetric positive semi-definite and $\lambda_2 = \psi''(|\cdot|)$ is its smallest eigenvalue we have

$$\int_0^1 \psi''(\nabla u + t \nabla e) dt |\nabla e|^2 = 0 \quad \text{a.e. } \Omega.$$

At points where $|\nabla u| < t_1$ or $|\nabla u| > t_2$ or $|\nabla v| < t_1$ or $|\nabla v| > t_2$ we have

$$\int_0^1 \psi''(\nabla u + t \nabla v) dt > 0$$

so $\nabla e = 0$ and thus $\nabla u = \nabla v$. This proves (2.7) and (2.8).

If $|\nabla u|, |\nabla v| \in [t_1, t_2]$ then since the integrand is nonnegative

$$G(\nabla u) \cdot \nabla e = G(\nabla v) \cdot \nabla e.$$

But $\psi'(|\nabla u|) = \psi'(|\nabla v|)$ so

$$\left[\frac{\nabla u}{|\nabla u|} - \frac{\nabla v}{|\nabla v|} \right] \cdot \nabla(u - v) = 0.$$

By (2.10) we have

$$(1 - \cos \theta(\nabla u, \nabla v)) = 0$$

where $\theta(\nabla u, \nabla v)$ is the angle between ∇u and ∇v . (2.9) now follows. ■

By the same argument we can prove the following:

Proposition 2.3: Let u_h be any solution of (1.7) and set

$$F_h = \{x \in \Omega : t_1 \leq |\nabla u_h| \leq t_2\}$$

$$E_h = \Omega \setminus F_h.$$

If v_h is another solution then

$$(2.11) \quad \nabla u_h = \nabla v_h \text{ a.e. on } E_h,$$

$$(2.12) \quad t_1 \leq |\nabla v_h| \leq t_2 \text{ a.e. on } F_h, \text{ and}$$

$$(2.13) \quad \frac{\nabla u_h}{|\nabla u_h|} = \frac{\nabla v_h}{|\nabla v_h|} \text{ a.e. on } F_h.$$

§3 Convergence in two dimensions:

We state and prove our main Theorem. Note that a similar result holds in one dimension.

Theorem 3.1: Let u be any solution of (1.6) and u_h be any solution of (1.7) then

$$(3.1) \quad \int_E w(d) |\nabla e_h|^2 + \int_{E_h \cap F} |\psi'(|\nabla u|) - \psi'(|\nabla u_h|)|^2 \\ + \int_F (1 - \cos \theta_h) \leq C |u - \chi|_{1,\Omega}^2 \quad \text{for all } \chi \in S_g^h.$$

θ_h is the angle between ∇u and ∇u_h ; $e_h = u - u_h$; C depends only on μ_1 , μ_2 , t_1 , and t_2 ; and

$$w(d) = \frac{d}{t_2 + d}$$

where

$$d = \begin{cases} |\nabla u| - t_2 & \text{if } |\nabla u| > t_2 \\ t_1 - |\nabla u| & \text{if } |\nabla u| < t_1 \end{cases}$$

Proof:

Since $u, u_h \in H_g^1(\Omega)$ it follows $e_h \in H_0^1(\Omega)$. From (2.1) and (2.2) we have

$$(3.2) \quad (J'(u) - J'(u_h))[\chi] = 0 \quad \text{for all } \chi \in S_0^h$$

We apply Nitsche's trick to begin the error estimate: Let $\Lambda = \Lambda(\nabla u, \nabla u_h)$ then

$$\begin{aligned} (\nabla e_h, \Lambda \nabla e_h) &= (G(\nabla u) - G(\nabla u_h), \nabla e_h) \\ &= (J'(u) - J'(u_h))[e_h]. \end{aligned}$$

Let $z_h, \chi \in S_g^h$. By (3.2) the above becomes

$$\begin{aligned} (\nabla e_h, \Lambda \nabla e_h) &= (J'(u) - J'(u_h))[(u - z_h) - (z_h - u_h)] \\ &= (J'(u) - J'(u_h))[(u - z_h) - (z_h - \chi)] \\ &= (J'(u) - J'(u_h))[u - \chi] \\ &= (\nabla e_h, \Lambda \nabla (u - \chi)) \end{aligned}$$

By the Schwarz inequality we have

$$(3.3) \quad (\nabla e_h, \Lambda \nabla e_h) \leq (\nabla(u - \chi), \Lambda \nabla(u - \chi)).$$

Setting

$$\eta(t) = \nabla u_h + t \nabla e_h$$

we estimate the right-hand-side (RHS).

$$\begin{aligned} \text{RHS} &= \int_{\Omega} \int_0^1 (\nabla(u - \chi) \cdot \tilde{A}(\eta(t)) \nabla(u - \chi)) dt \\ &\leq \int_{\Omega} \left[\int_0^1 \frac{\psi'(|\eta(t)|)}{|\eta(t)|} dt \right] |\nabla(u - \chi)|^2 \end{aligned}$$

Since

$$\frac{\psi'(s)}{s} \leq \mu_1$$

we conclude

$$(3.4) \quad \text{RHS} \leq \mu_1 |u - \chi|_{1,\Omega}^2$$

On the left-hand-side (LHS) we have

$$\begin{aligned} (3.5) \quad \text{LHS} &= \int_E \nabla e_h \cdot A \nabla e_h + \int_F (G(\nabla u) - G(\nabla u_h)) \cdot \nabla e_h \\ &\geq \int_E \left[\int_0^1 \psi''(\eta(t)) dt \right] |\nabla e_h|^2 \\ &\quad + \int_F (G(\nabla u) - G(\nabla u_h)) \cdot \nabla e_h. \end{aligned}$$

Let

$$\gamma(x) = \int_0^1 \psi''(\eta(t)) dt.$$

We claim for now that

$$(3.6) \quad \gamma(x) \geq C_0 w(d)$$

if $x \in E$ where C_0 depends only on μ_1 and μ_2 . Thus (3.5) becomes

$$(3.7) \quad \begin{aligned} \text{LHS} &\geq C_0 \int_E w(d) |\nabla e_h|^2 + \int_F (G(\nabla u) - G(\nabla u_h)) \cdot \nabla e_h \\ &= I + L \end{aligned}$$

On F $\psi'(|\nabla u|)$ is a positive constant; $\mu_1 t_1 = \mu_2 t_2$. Thus, letting

$$\beta_h = \frac{\psi'(|\nabla u_h|)}{\psi'(|\nabla u|)},$$

$$\begin{aligned}
(3.8) \quad L &= \psi'(|\nabla u|) \int_F \left[\frac{\nabla u}{|\nabla u|} - \beta_h \frac{\nabla u_h}{|\nabla u_h|} \right] \cdot \nabla(u - u_h) \\
&= \psi'(|\nabla u|) \int_F [(|\nabla u| + \beta_h |\nabla u_h|) - (\beta_h |\nabla u| + |\nabla u_h|) \cos \theta_h] \\
&= \psi'(|\nabla u|) \int_F [(\beta_h |\nabla u| + |\nabla u_h|)(1 - \cos \theta_h) \\
&\quad + (1 - \beta_h)(|\nabla u| - |\nabla u_h|)] \\
&= \psi'(|\nabla u|) \int_F [(\beta_h |\nabla u| + |\nabla u_h|)(1 - \cos \theta_h)] \\
&\quad + \int_F [(\psi'(|\nabla u|) - \psi'(|\nabla u_h|))(|\nabla u| - |\nabla u_h|)].
\end{aligned}$$

Define

$$h(t) = \frac{1}{\mu_1} \psi'(t) - t \quad \text{for } t \geq 0.$$

$h(0) = 0$ and h is monotone decreasing. Thus if $t < s$ then

$$h(t) \geq h(s)$$

or

$$\frac{1}{\mu_1} (\psi'(s) - \psi'(t)) \leq s - t.$$

Multiplying by the nonnegative quantity $\psi'(s) - \psi'(t)$ we have

$$(3.9) \quad \frac{1}{\mu_1} (\psi'(s) - \psi'(t))^2 \leq (s - t)(\psi'(s) - \psi'(t)).$$

Using this result in (3.8) we have

$$\begin{aligned} L \geq \psi'(|\nabla u|) \int_F (\beta_h |\nabla u| + |\nabla u_h|)(1 - \cos \theta_h) \\ + \frac{1}{\mu_1} \int_F (\psi'(|\nabla u|) - \psi'(|\nabla u_h|))^2. \end{aligned}$$

Returning to (3.7) and recalling (3.4) this estimate becomes

$$\begin{aligned} (3.10) \quad C \int_E w(d) |\nabla e_h|^2 + \frac{1}{\mu_1} \int_F (\psi'(|\nabla u|) - \psi'(|\nabla u_h|))^2 \\ + \psi'(|\nabla u|) \int_F (\beta_h |\nabla u| + |\nabla u_h|)(1 - \cos \theta_h) \\ \leq \mu_1 |u - \chi|_{1,\Omega}^2. \end{aligned}$$

Since the third term on the LHS is nonnegative we conclude the following:

$$(3.11) \quad \int_F (\psi'(|\nabla u|) - \psi'(|\nabla u_h|))^2 \leq C |u - \chi|_{1,\Omega}^2.$$

$$(3.12) \quad \int_E w(d) |\nabla e_h|^2 \leq C |u - \chi|_{1,\Omega}^2.$$

C depends only on μ_1 and μ_2 . From (3.10) we have

$$\psi'(|\nabla u|) \int_F (\beta_h |\nabla u| + |\nabla u_h|)(1 - \cos \theta_h) \leq \mu_1 |u - \chi|_{1,\Omega}^2$$

or

$$\begin{aligned} & \int_F \psi'(|\nabla u|)(|\nabla u| + |\nabla u_h|)(1 - \cos \theta_h) \\ & \leq \mu_1 |u - \chi|_{1,\Omega}^2 + \int_F (\psi'(|\nabla u_h|) - \psi'(|\nabla u|)) |\nabla u| (1 - \cos \theta_h) \\ & \leq \mu_1 |u - \chi|_{1,\Omega}^2 + C_\epsilon \int_F |\nabla u|^2 (\psi'(|\nabla u_h|) - \psi'(|\nabla u|))^2 \\ & \quad + \epsilon \int_F (1 - \cos \theta_h)^2 \end{aligned}$$

Since $\psi'(|\nabla u|) \geq \mu_1 t_1$, $t_1 \leq |\nabla u| \leq t_2$, and $(1 - \cos \theta_h) \leq 2$ this inequality becomes

$$\mu_1 t_1^2 \int_F (1 - \cos \theta_h) \leq C_\epsilon |u - \chi|_{1,\Omega}^2 + 2\epsilon \int_F (1 - \cos \theta_h).$$

Picking ϵ sufficiently small depending only on μ_1 and t_1 we conclude

$$(3.13) \quad \int_F (1 - \cos \theta_h) \leq C |u - x|_{1,\Omega}^2$$

Combining (3.11), (3.12) and (3.13) we have proved (3.1).

Returning to the verification of (3.6) recall for $s \geq 0$

$$\psi''(s) = \begin{cases} \mu_1 & \text{if } 0 \leq s < t_1 \\ 0 & \text{if } t_1 < s < t_2 \\ \mu_2 & \text{if } t_2 < s \end{cases}$$

ψ'' is undefined at t_1 and t_2 , and $\mu_2 < \mu_1$. We consider several cases depending on locations of v_u and v_{u_h} in $\mathbb{R}^2$ when evaluated at a point $x \in E$.

First suppose $|v_u| > t_2$ and $|v_{u_h}| < t_2$. (See Figure 3.1.) Let L be the line connecting v_u to v_{u_h} , $D(r)$ the disk with radius r and center at the origin. Then $v_{u_h} \in D(t_2)$ and

$$\begin{aligned} \gamma(x) &\geq \mu_2 \frac{\text{Total length of } L \text{ that lies outside } D(t_2)}{\text{Total length of } L} \\ &\geq \mu_2 \frac{d}{\text{diam}(D(t_2 + d))} \\ &\geq \mu_2 \frac{d}{2(t_2 + d)} \end{aligned}$$

so (3.6) holds in this case. If $v_{u_h} \in D(t_2)$ and $\delta = |v_{u_h}| - t_2$ we have (See Figure 3.2).

$$\begin{aligned}
 \gamma(x) &\geq \mu_2 \frac{d + \delta}{\text{diam}(D(t_2 + d + \delta))} \\
 &\geq \mu_2 \frac{d + \delta}{2((t_2 + d) + \delta)}
 \end{aligned}$$

Since the function $\frac{a+x}{b+x} \geq \frac{a}{b}$ for $x \geq 0$ if $a \leq b$ we conclude (3.6) holds.

Now suppose $\forall u \in D(t_1)$. If $\forall u_h \in D(t_2) \setminus D(t_1)$ then

$$\gamma(x) \geq \mu_1 \frac{d}{\text{diam}(D(t_2))} \geq \mu_1 \frac{d}{2t_2} \geq \frac{\mu_1}{2} \frac{d}{t_2 + d}.$$

Finally if $\forall u_h \notin D(t_2)$ then

$$\begin{aligned}
 \gamma(x) &\geq \frac{\mu_1 d + \mu_2 \delta}{\text{diam}(D(t_2 + \delta))} \\
 &\geq C \frac{d + \delta}{t_2 + \delta}
 \end{aligned}$$

As argued previously we can again conclude (3.6) holds. This finishes the proof of the claim (3.6) and the Theorem.

■

$$\nabla u \in D(t_2)$$

$$\nabla u_h \in D(t_2)$$

Figure 3.1

$$\nabla u \in D(t_2)$$

$$\nabla u_h \in D(t_2)$$

Figure 3.2

$$\nabla u \in D(t_1)$$

$$\nabla u_h \in D(t_2)$$

Figure 3.3

§4 A Special one-dimensional case:

We prove in this section that if the set F consists of only a finite number of points and piecewise linear elements are used in the approximation then u'_h converges to u' . u'_h is the derivative of any solution of (1.7) and u' is derivative of the solution to (1.6) which is unique in this case. (This follows from Proposition 2.2)

We specify S^h precisely. Let $\Omega = (a, b)$ and partition it into intervals of uniform length h separated by nodes x_j :

$$a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$$

where $x_j = jh$. Let $I_j = (x_{j-1}, x_j)$ and define a basis for S_h , $\{\varphi_0, \dots, \varphi_N\}$

$$\varphi_i(x) = 0 \quad \text{if } x \notin \bar{I}_i \cup \bar{I}_{i+1}$$

and

$$\varphi'_i = \begin{cases} \frac{x - x_{i-1}}{h} & \text{on } \bar{I}_i \\ \frac{x_{i+1} - x}{h} & \text{on } \bar{I}_{i+1}. \end{cases}$$

These are the usual "hat" functions.

Also define $\tilde{\psi}'$ to be the odd extension of ψ' so $\tilde{\psi}'(t) = \psi'(t)$ if $t \geq 0$ and $\tilde{\psi}'(t) = -\psi'(-t)$ if $t < 0$. Let

$$p = \psi'(|u'|) \frac{u'}{|u'|} = \tilde{\psi}'(u').$$

$$p_i = \frac{1}{h} \int_{I_i} \psi'(|u'_h|) \frac{u'_h}{|u'_h|} = \tilde{\psi}'(u'_h)$$

and

$$f_i = \int_a^b f \varphi_i$$

where $i = 0, \dots, N$ and the second equation holds since $\psi'(|u'_h|) \frac{u'_h}{|u'_h|}$ is a constant on I_i .

Theorem 4.1:

Suppose the set F for the solution of (1.6) consists of only a finite number of points and u has a continuous first derivative except at these points. For $h < \min\{h_0, 1/2\}$

$$(4.1) \quad |p_i - \frac{1}{h} \int_{I_i} p| \leq Ch \quad i = 1, \dots, N.$$

C and $h_0 > 0$ depend on μ_1, μ_2 and $\max_{x \in [a, b]} |p(x)|$.

Remark 4.1:

We observe that if f is of one sign then the solution of (1.7) will have only one point in F . Thus solutions can be constructed that satisfy the conditions of the Theorem.

Remark 4.2:

The estimate (4.1) proves u'_h converges to u' at an optimal rate on any interval I_1 that does not contain a point of F if $u'' \in L^\infty([a,b])$. Suppose $I_1 \subset E_j$ where $j = 1$ or 2 then

$$\max_{x \in I_1} \left| \frac{1}{\mu_j h} \int_{I_1} p - u' \right| \leq Ch \|u''\|_{L^\infty(I_1)}$$

so by the triangle inequality

$$\max_{x \in I_1} \left| \frac{1}{\mu_j} p_i - u' \right| \leq Ch.$$

Thus for h sufficiently small $u'_h = \frac{1}{\mu_j} p_i$.

Proof of Theorem 4.1:

From (1.6), (1.7) and the fact that $\varphi_1 \in H^1(a,b)$

$$-\int_a^b f \varphi_1 = \int_a^b \psi'(|u'_h|) \frac{u'_h}{|u'_h|} \varphi'_1 = \int_a^b \psi'(|u'|) \frac{u'}{|u'|} \varphi'_1.$$

From the definition of φ'_i this becomes

$$(4.2) \quad p_i - p_{i+1} = \frac{1}{h} \int_{I_i} p - \frac{1}{h} \int_{I_{i+1}} p \quad \text{for } i = 1, \dots, N-1.$$

Let δ be a constant to be evaluated later and set

$$(4.3a) \quad p_1 = \frac{1}{h} \int_{I_1} p + \delta.$$

Then we must have

$$(4.3b) \quad p_i = \frac{1}{h} \int_{I_i} p + \delta$$

for $i = 2, \dots, N$.

Let $A \subset (a, b)$ be an interval where on all of A either $|u'| > t_2 + \beta$ or $|u'| < t_1 - \beta$ where $\beta > 0$. Since there are only a finite number of points in $F \setminus A$ and β exist. Without loss of generality suppose $|u'| > t_2 + \beta$.

From Theorem 3.1 we have

$$\|u' - u'_h\|_{L^2(A)} \leq C \|u' - \chi'\|_{L^2(a, b)}$$

For all $\chi \in S_g^h$ where C depends on β . Dividing by μ_2 this becomes

$$\|p - A_h p - \frac{1}{\mu_2} \delta\|_{L^2(A)} \leq C \|u - \chi\|_{L^2(a, b)}.$$

where $A_h p$ is a piecewise constant function;

$$A_h p|_{I_i} = \frac{1}{h} \int_{I_i} p \quad i=1, \dots, N.$$

This becomes

$$|\delta| \leq C(\|p - A_h p\|_{L^2(A)} + \|u' - \chi'\|_{L^2(a,b)}).$$

Since the RHS $\rightarrow 0$ as $h \rightarrow 0$ δ can be made as close to zero as desired by taking h sufficiently small.

Let $\psi^* = \mu_1 t_1 = \mu_2 t_2$ and let $y_1, \dots, y_q$ be the points where $|p| = \psi^*$. Thus

$$F = \{y_1, \dots, y_q\}.$$

Let $j_1(h), \dots, j_q(h)$ be the indices where

$$y_k \in I_{j_k(h)}$$

for $k = 1, \dots, q$. We're assuming h is sufficiently small so each $I_{j_k(h)}$ contains only one point of F . To satisfy the boundary conditions we must have

$$(4.4) \quad \int_a^b u'_h = u(b) - u(a)$$

since

$$u(b) - u(a) = \int_a^b u' = \int_a^b \frac{1}{\mu(p)} p$$

and

$$\int_a^b u'_h = \sum_{i=1}^N \frac{1}{\mu(p_i)} h p_i$$

where $\mu(s) = \mu_1$ if $|s| < \psi^*$, $\mu(s) = \mu_2$ if $|s| > \psi^*$ and $\mu_1 \leq \mu(s) \leq \mu_2$ if $|s| = \psi^*$. The set on which $|p| = \psi^*$ is of measure zero so the value of $\mu(p)$ on that set has no effect on the integration. If $|p_i| = \psi^*$ we define $\mu(p_i) = \psi^* / |u'_h|$ where u_h is the specific solution of (1.7) we are analyzing. So (4.4) becomes

$$\begin{aligned} h \sum_{i=1}^N \frac{p_i}{\mu(p_i)} &= \int_a^b \frac{1}{\mu(p)} p \\ &= \sum_{i \in J_1} \frac{1}{\mu_1} \int_{I_i} p + \sum_{i \in J_2} \frac{1}{\mu_2} \int_{I_i} p \\ &\quad + \sum_{i=j_1(h), \dots, j_q(h)} \frac{1}{\mu(p)} p \end{aligned}$$

where

$$J_1 = \{k : \sup_{x \in I_k} |p(x)| < \psi^*\}$$

and

$$J_2 = \{k : \inf_{x \in I_k} |p(x)| > \psi^*\}$$

Define

$$\begin{aligned} H(\delta) = & \sum_{i \in J_1} \left[\frac{1}{\mu_1} \int_{I_i} p - \frac{hp_i}{\mu(p_i)} \right] \\ & + \sum_{i \in J_2} \left[\frac{1}{\mu_2} \int_{I_i} p - \frac{hp_i}{\mu(p_i)} \right] \\ & + \sum_{i=j_1(h), \dots, j_q(h)} \left[\int_{I_i} \frac{p}{\mu(p)} - \frac{hp_i}{\mu(p_i)} \right]. \end{aligned}$$

Using (4.3) this becomes

$$\begin{aligned} (4.5) \quad H(\delta) = & \sum_{i \in J_1} \left[\frac{1}{\mu_1} \int_{I_i} p - \frac{1}{\mu(p_i)} \int_{I_i} p \right] \\ & + \sum_{i \in J_2} \left[\frac{1}{\mu_2} \int_{I_i} p - \frac{1}{\mu(p_i)} \int_{I_i} p \right] \\ & + \sum_{i=j_1(h), \dots, j_q(h)} \left[\int_{I_i} \frac{p}{\mu(p)} - \frac{1}{\mu(p_i)} \int_{I_i} p \right] \end{aligned}$$

$$- h\delta \left[\sum_{i=1}^N \frac{1}{\mu(p_i)} \right].$$

or

$$H(\delta) = T_1(\delta) + T_2(\delta) + D(\delta) - w(h, \delta).$$

where T_1 , T_2 , D , and w depend on δ since $p_i = p_i(\delta)$.

Consider the function

$$t(s) = \frac{s_0}{\mu(s_0 + s)}$$

for $s \in [-\psi^*, \psi^*]$. Recall $0 < \mu_2 \leq \mu \leq \mu_1$. If $s_0 < 0$ then $\mu(s_0 + s)$ is monotone increasing so t is also. If $s_0 \geq 0$ $\mu(s_0 + s)$ is monotone decreasing so t is monotone increasing. Taking $s_0 = \frac{1}{h} \int_{I_1} p$ in (4.5)

we conclude that T_1 , T_2 and D are monotone decreasing in δ on $[-\psi^*, \psi^*]$. Note also that

$$T_1(0) = T_2(0) = 0$$

and D is $O(h)$;

$$\begin{aligned}
 (4.6) \quad |D(\delta)| &= \left| \sum_{i=j_1(h), \dots, j_q(h)} \int_{I_i} \left[\frac{p}{\mu(p)} - \frac{p}{\mu(p_i)} \right] \right| \\
 &\leq \sum_{i=j_1(h), \dots, j_q(h)} \int_{I_i} \left[\left| \frac{p}{\mu(p)} \right| + \left| \frac{p}{\mu(p_i)} \right| \right] \\
 &\leq \frac{2}{\mu_2} \left(\max_{x \in (a, b)} |p| \right) qh.
 \end{aligned}$$

Finally observe that

$$\frac{1}{\mu_1} \leq h \sum_{i=1}^N \frac{1}{\mu(p_i)} \leq \frac{1}{\mu_2}.$$

$$(i.e. \quad \frac{1}{\mu_1} \leq \frac{w(\delta, h)}{\delta} \leq \frac{1}{\mu_2}).$$

We search for δ_h so

$$(4.8) \quad H(\delta_h) = 0.$$

Since there exists at least one solution of (1.7) this equation has a solution. From (4.6) and (4.7) we can rewrite (4.8) as the fixed point equation

$$\delta_h = G(\delta_h).$$

where G is monotone decreasing and $|G(0)| \leq Ch$ for a constant C . From these considerations we must have $\delta_h \in [-Ch, Ch]$. this finishes the proof.

■

§5 One dimensional numerical results:

We constructed a finite element solution to (1.1) for the case studied in §4 where F consists of a finite number of points. We approximated u by piecewise linear elements and continuous quadratic elements on uniform meshes. The main question is the following: How does the lack of smoothness of u' affect the accuracy of u'_h away from the jump discontinuities in u' ? What is the pollution from these jumps?

Let

$$\psi'(s) = \begin{cases} 2s & s \in [0, 1/2) \\ 1 & s \in [1/2, 3/2] \\ 2s/3 & s \in (3/2, \infty) \end{cases}$$

and

$$f(x) = -8x$$

Then we can solve for u ;

$$u'(x) = \begin{cases} 3 - 6x^2 & x \in [0, 1/2) \\ 1 - 2x^2 & x \in [1/2, \sqrt{3}/2] \\ 3 - 6x^2 & x \in (\sqrt{3}/2, 1] \end{cases}$$

and

$$u(x) = \int_0^x u'(s) ds.$$

Note that

$$E_1 = (1/2, \sqrt{3}/2).$$

$$E_2 = [0, 1/2] \cup (\sqrt{3}/2, 1]$$

and

$$F = \{\frac{1}{2}, \frac{\sqrt{3}}{2}\}.$$

Also

$$p = \frac{\psi'(|u'|)}{|u'|} u' = 2 - 4x^2.$$

We found four different approximations. In the first we took S^h to have piecewise linear elements. Since u'_h is constant on each element the integration was done exactly. In the second and third we used approximation functions that were quadratic on each element and continuous on $[a, b]$. In one test we used a three point Gaussian quadrature rule for integration. This was exact on intervals in E_1 or E_2 . It could be inaccurate on intervals where $|u'_h| \in (t_1, t_2)$. As a comparison we used

Simpson's rule with 50 points on each interval; this was exact on intervals in E_1 or E_2 and more accurate on the other intervals.

For the final approximation we solved (1.6) by first approximating

$$P(x) = \int_0^x p(s)ds$$

in the space S^h of continuous piecewise quadratics and then "inverting" $\tilde{\psi}'$ to obtain the function $u'_h = (\tilde{\psi}')^{-1}(P)$ which was not in S^h but was a good approximation to u . The main problem to be solved was

$$-P'' = F \text{ on } (0,1)$$

$$P(0) = 0, \quad P(1) = \int_0^1 p(s)ds.$$

We assumed both boundary conditions were given.

In each case we found the approximation using a simple method.

Initially a guess $u_h^{(0)}$ is formed. Each $u_h^{(n)}$ is computed by partially solving the problem: Find $u_h^{(n)} \in S_g^h$ so that

$$(5.1) \quad \left[\frac{\psi'(|u_h^{(n-1)}'|)}{|u_h^{(n-1)}'|} u_h^{(n)'} \cdot x' \right] = (f, x)$$

for all $x \in S_0^h$. This partial approximation is computed by taking $K(n)$ iterations of the SOR method with initial guess $u_h^{(n-1)}$. We took $K(n) = n$ for the quadratics and $K(n) = \sqrt{n}$ for the piecewise linears.

We stopped the process when

$$\|U^n - U^{n-1}\|_\infty \leq 10^{-10}$$

where $u_n(x) = \sum_{i=0}^N U_i^n \varphi_i(x)$ and $\|\eta\|_\infty = \max_{0, \dots, N} |\eta_i|$. In all cases the system of equations

$$\left[\frac{\psi'(|u_h^{n'}|)}{|u_h^{n'}|} u_h^{n'}, \varphi_i' \right] = (f, \varphi_i) \quad i = \dots,$$

had a residual vector which had magnitude less than 10^{-7} in $\|\cdot\|_\infty$. All computations were done on a SUN 3/50 in double precision arithmetic. (Accuracy $\sim 10^{-14}$.)

To evaluate the accuracy of the approximation we calculated the differences between the first derivatives true and approximate solution at the nodes $x_i, i = 0, \dots, N$. (Since the difference is discontinuous at the x_i right and left hand limits were evaluated.) Since the main interest is the pollution effect the intervals containing the two points in F were left out. Tables 4.1-4.4 display the maximum of the absolute value of these errors.

Table 4.1 displays the results when S^h has piecewise linear elements. The convergence rate is nearly $O(h)$.

Tables 4.2 and 4.3 have results when S^h has continuous quadratics. Table 4.2 has the results when the three point quadrature rule is used and

Table 4.3 has these when the Simpson rule was used. The rate of convergence for both is erratic; the Simpson rule does give slightly better results. Comparing to Table 4.1 observe that the errors using piecewise linears are ~100 times bigger than those found when using the quadratics.

Table 4.4 displays the results of solving for P and "inverting" $\tilde{\psi}'$ to find u' . This is clearly the best method.

Table 4.1: Piecewise Linears

<u>N</u>	<u>Error</u>
11	5.3 e - 1
21	2.8 e - 1
41	1.5 e - 1
81	7.4 e - 2
161	3.7 e - 2

Table 4.2: Quadratics with 3 pt. Gauss Rule

<u>N</u>	<u>Error</u>
11	1.2 e - 2
21	9.1 e - 3
41	9.5 e - 4
81	2.3 e - 3
161	5.0 e - 4

Table 4.3: Quadratics with Simpson's Rule

<u>N</u>	<u>Error</u>
11	1.1 e - 2
21	8.6 e - 3
41	9.3 e - 4
81	2.1 e - 3
161	3.5 e - 4

Table 4.4: Evaluation of P

<u>N</u>	<u>Error</u>
11	8.3 e - 3
21	2.3 e - 3
41	5.9 e - 4
81	1.5 e - 4
161	3.9 e - 5

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