REDUCED MATRICES AND q-LOG CONCAVITY PROPERTIES OF q-STIRLING NUMBERS

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IMA Preprint Series # 501

February 1989

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Abstract

We prove the q-log-concavity of the q-Stirling numbers of the second kind, which was recently conjectured by Lynne Butler, by suitably extending her injective proof of the analogous property of the q-binomial coefficients. For this we introduce new combinatorial interpretations of Stirling numbers of both kinds in terms of "0-1 tableaux" inspired from a row-reduced echelon matrix representation of restricted growth functions. Other related results, methods, counterexamples and conjectures are discussed.

Key words: Unimodality, log-concavity, Stirling numbers, q-analogues, row-reduced echelon matrices, partitions, restricted growth functions, 0-1 tableaux, weight preserving involutions.

AMS subject classification: Primary 05A15; secondary 11B65.

This research was supported in part by the Institute for Mathematics and its Applications, with funds provided by the National Science Foundation of the United States; this work was also supported by grants A5660 of NSERC (Canada) and EQ1608 of FCAR (Québec).

Introduction

A sequence of real numbers $\{a_k\}_{0 \le k \le n}$ is said to be **unimodal** if there exists m, $0 \le m \le n$, such that

$$(0.1) \qquad a_0 \le a_1 \le \ldots \le a_m \ge a_{m+1} \ge \ldots \ge a_n$$

The sequence is called **log-concave** if

$$(0.2) \qquad a_{k-1}\, a_{k+1} \le a_k^2 \, .$$

for all k, $1 \le k \le n-1$.

Unimodal and log-concave sequences are fairly frequent in combinatorics and in other branches of mathematics, and numerous methods of proof, some of them quite sophisticated, are available (see [Stl] for a recent survey). For example the three following basic combinatorial families of integers are log-concave (in k) and consequently unimodal, for all fixed $n \ge 0$:

- the binomial coefficients $\binom{n}{k}$
- the (signless) Stirling numbers of the first kind $c(n,k)$,
- the Stirling numbers of the second kind $S(n,k)$.

These facts can be established combinatorially, for example, by constructing injective mappings between appropriate sets (see [Sa1]):

$$(0.3) \qquad A_{k-1} \times A_{k+1} \longrightarrow A_k \times A_k$$

Many of the combinatorial sequences $\{a_k\}$ admit q-analogues, that is polynomials $a_k(q)$ in a variable q such that $a_k(1) = a_k$. In particular, we will consider the following q-analogues of the previous three families:

- the well-known q-binomial coefficients $\left[{n \atop k}\right]_q$, defined by

(0.4) $$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q\,[n-k]!_q}$$

where

(0.5) $$[n]!_q = [1]_q\,[2]_q \ldots [n]_q,\ n \geq 1 \ ;\ [0]!_q = 1$$

with

(0.6) $$[n]_q = 1+q+\ldots+q^{n-1} = \frac{1-q^n}{1-q},\ n\geq 1\ ;\ [0]_q = 0$$

. the q-Stirling numbers of the first kind $c_q[n,k]$, (see [Go]), defined inductively by

(0.7) $$\begin{cases} c_q[0,0] = 1, \quad c_q[n,k] = 0 \quad \text{for} \quad k<0 \quad \text{or } k>n, \\ c_q[n,k] = c_q[n-1,k-1] + [n-1]_q\, c_q[n-1,k],\ n\geq 1 \end{cases}$$

. the q-Stirling numbers of the second kind $S_q[n,k]$ (see [Ca, GaR, Go, Mi, WW]; notations vary!) defined inductively by

(0.8) $$\begin{aligned} &S_q[0,0] = 1, \quad S_q[n,k] = 0 \quad \text{for} \quad k<0 \quad \text{or} \quad k>n \\ &S_q[n,k] = S_q[n-1,k-1] + [k]_q\, S_q[n-1,k],\ n\geq 1 \end{aligned}$$

It is then natural to introduce the finer concepts of **q-unimodality** and **q-log-concavity** for polynomials in q. This is done simply by interpreting the inequalities in (0.1) and (0.2) (see [Bu1], [Bu2]) as coefficient-wise inequalities. In other words, for two polynomials f(q) and g(q), we have

(0.9) $$f(q) \leq g(q) \quad \Leftrightarrow \quad g(q) - f(q) \text{ has non-negative coefficients}$$

At the 840th meeting of the American Mathematical Society in East Lansing, Michigan, Lynne Butler presented a beautiful injective proof of the q-log-concavity in k of the q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and announced the conjecture that the sequence of q-Stirling numbers of the second kind $S_q[n,k]$ is q-log-concave in k (see [Bu2]). In the following weeks a number of combinatorialists visiting the Institute for Mathematics and its Applications in Minneapolis settled this conjecture and related open questions concerning q- and p,q-Stirling numbers (see [WW]) of the first and second kinds. Various methods of proof were proposed, in particular mathematical induction [Sa2], the theory of symmetric functions [Stn], and injections. The purpose of this paper is to report on these results and more particularly to present injective proofs, using an appropriate extension of L. Butler's method which is based on involutions on pairs of Ferrers diagrams [Bu2].

For this purpose, we introduce a new combinatorial model for set partitions that is inspired from reduced matrix representations [Le2, Le3] of restricted growth functions. It consists of special 0-1 fillings of Ferrers diagrams that we call *0-1 tableaux*, from which the q and p statistics "number of inversions" and "number of non-inversions" can be easily read (see §2 for more details), thus providing new combinatorial interpretations for the polynomials in p and q $S_{pq}[n,k]$ and $c_{pq}[n,k]$; setting p=1 gives the usual q-analogues of the Stirling numbers.

In §3 we adapt the involutions "arm" and "leg" of [Bu] to pairs of 0-1 tableaux and apply them in §4 to establish various pq-log-concavity results by describing weight preserving injections of the form (0.3). Other results, proofs, counterexamples and conjectures are given.

We first briefly describe in §1 various classes of reduced matrices, closed under matrix multiplication, and some generalizations, that are of interest in combinatorics and that gave rise to the concept of 0-1 tableaux.

Acknowledgments

The author woulk like to acknowledge the warm and efficient hospitality of the Institute for Mathematics and its Applications (IMA) in March and April 1988. This includes the use of their computer facilities, MAPLE in particular. He also thanks Lynne Butler, Bruce Sagan, Dennis Stanton, Dennis White and many other visitors and members of the IMA for fruitful discussions.

§1. Categories of Reduced matrices.

Let $0 \leq k \leq n$ be integers and $\mathbb{K}$ be a field. We will call a $k \times n$ matrix over $\mathbb{K}$ **reduced** if it is row-reduced echelon, of rank k. We conveniently assume the existence of a unique $0 \times n$ reduced matrix for all $n \geq 0$. It is a well known and widely used fact (see e.g. [Kn, NSW]) that reduced $k \times n$ matrices R can be used to codify k-dimensional subspaces W of $\mathbb{K}^n$ (take W = W(R), the row-space of R) and to enumerate them when $\mathbb{K}$ is a finite field GF(q). Indeed we can remove, without loss of information, the pivot columns of a reduced $k \times n$ matrix (the remaining 0's to the left of the leading 1 of each row are also removed) to obtain a Ferrers diagram (in the third quadrant, for the moment) which fit in a k by n-k rectangle and is filled with arbitrary elements of $\mathbb{K}$ (stars, in Fig. 1). When $\mathbb{K}$ = GF(q), the number of such subspaces is given by (0.4) and hence we find

(1.1)
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \in \mathfrak{F}(k,n-k)} q^{|\lambda|}$$

where $\mathfrak{F}(k, n\text{-}k)$ denotes the set of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, with $n-k \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$, and $|\lambda| = \sum_i \lambda_i$.

$$\begin{pmatrix} 1 & * & 0 & * & 0 & * & * & * & * & 0 & * & * \\ 0 & 0 & 1 & * & 0 & * & * & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{pmatrix} \longleftrightarrow \begin{array}{|c|c|c|c|c|c|c|c|} \hline * & * & * & * & * & * & * & * \\ \hline & * & * & * & * & * & * & * \\ \hline & & * & * & * & * & * & * \\ \hline & & & & & & * & * \\ \hline \end{array}$$

M $\qquad\qquad$ $\lambda = (8, 7, 6, 2)$

Figure 1.1

It is however less often observed that reduced matrices are closed under matrix multiplication (see e.g. [Le4, exer. 2.19]) and that this multiplication contains all the information about the inclusion of subspaces: if R and T are reduced $k \times n$ and $m \times n$ matrices resp., then $W(R) \subseteq W(T)$ if and only if there exists a (unique) reduced $k \times m$ matrix S such that $R = S \cdot T$. Since the identity matrices are the only reduced square matrices, we see that the reduced matrices constitute, with matrix multiplication, arrows of a triangular category, denoted by $\mathcal{Red}$, according to the following definition [Le2]:

1.1 Definition. A category $\mathcal{C}$ is called **triangular** if the set ob($\mathcal{C}$) of objects of $\mathcal{C}$ is equal to $\mathbb{N} = \{0, 1, 2, \ldots\}$ and the cardinalities $A(k,n) = |\mathcal{C}(k,n)|$ of sets of arrows ($k, n \in \mathbb{N}$) constitute a **triangular family of numbers,** that is

(1.2) $$k > n \quad \Rightarrow \quad A(k,n) = 0 ,$$

(1.3) $$A(n,n) = 1 \quad , \quad \text{for all } n \in \mathbb{N}.$$

Most of the basic combinatorial triangular families of numbers can be represented by triangular categories and many of them occur as subcategories of $\mathcal{Red}$ (i.e. classes of reduced matrices closed under matrix multiplication) or as generalizations of $\mathcal{Red}$. There is usually an induced partial order on each set $\mathcal{C}(n) = \bigcup_{k\geq 0} \mathcal{C}(k,n)$, determined by the factorizations of the arrows (i.e. the reduced matrices) as in the case of subspaces of $\mathbb{K}^n$. Moreover the incidence algebras of these posets and their Möbius functions are closely related to those of the triangular category (see [DoRS, CLL, Le2]).

A first example is the poset $\mathcal{Sse}(n)$ of subsets S of $[n] = \{1, 2, \ldots, n\}$; these can be represented as reduced matrices M with all stars equal to 0 (take S = set of pivot columns of M). If $M = [m_{ij}]$ is of format $k \times n$, then M can also be seen as the *row to column matrix representation* of the unique injective and increasing mapping $f: [k] \longrightarrow [n]$ whose image is S, that is

$$m_{ij} = \chi(\, j = f(i)\,)$$

where χ is the usual truth function. Writing $M = M(f)$ one has also

(1.4) $$M(gf) = M(f)\, M(g)$$

This triangular category of reduced matrices, denoted by $\mathcal{Sse}$, is isomorphic by (1.4) to the category Δ_{face} of injective increasing mappings of finite sets of the form $[n]$, $n \geq 0$. We have

(1.5) $$|\mathcal{Sse}(k,n)| = \binom{n}{k} \quad .$$

Another example, most important for the present paper, comes from the lattice $\mathcal{Par}(n)$ of set partitions of [n]. Given a partition π of [n] into k blocks, written in standard form (as in Fig. 1.2, where n = 12 and k = 4), that is where the blocks are ordered according to their smallest elements, we can define a surjective function $g = g_\pi : [n] \longrightarrow [k]$ by

(1.6) $$g(j) = i \ \text{ iff } j \text{ belongs to the } i^{th} \text{ block of } \pi .$$

This function g is of *restricted growth*, that is satisfies the condition

(1.7) $$g(j) \leq \max_{i<j} g(i) + 1, \quad \text{for all } j \in [n].$$

$$\pi = 1, 2, 4, 7 / 3, 9, 12 / 5, 6, 8, 11 / 10$$

$$g_\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 2 & 1 & 3 & 3 & 1 & 3 & 2 & 4 & 3 & 2 \end{pmatrix}$$

$$C(g) = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$T(\pi) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline & & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline & & & & & & 0 & 0 \\ \hline \end{array}$$

Figure 1.2

The **column to row** $k \times n$ **matrix representation** $C(g) = [c_{ij}]$ of a mapping $g:[n] \rightarrow [k]$ is defined by (see figure 1.2)

(1.8) $$c_{ij} = \chi(\, i = g(j)\,).$$

It is an elementary fact that the mapping g is of restricted growth and surjective if and only if C(g) is reduced and also that

(1.9) $$C(g)\, C(f) = C(gf)$$

Thus we obtain a triangular category, denoted by *Par*, of reduced matrices of the form $C(\pi) = C(g_\pi)$, where π is a partition, such that

(1.10) $$|\mathcal{P}ar\,(k,n)| = S(n,k)$$

and for which the order relation induced on *Par*(n) by the factorizations is the usual *order by refinement*. Reduced matrices of the form $C=C(\pi)$ can be characterized by the following facts:

- C is reduced,
- all entries of C are 0's or 1's.
- there is exactly one 1 in each column.

Now again we can remove, without loss of information, the pivot columns of $C(\pi)$ to obtain a so-called **0-1 tableau** $T(\pi)$ (see fig. 1.2) which represents the given partition π. This is the model that will be used in the following sections.

Other triangular categories of reduced matrices include *Comp*, consisting of compositions of integers, which can alternatively be described as set partitions whose blocks contain only consecutive elements (see fig. 1.3) and *Parge* (from French "partage"), consisting of integer partitions, which can be described as compositions whose parts are decreasing. As triangular categories, we have

(1.11) $$\mathcal{P}arge \subseteq \mathcal{C}omp \subseteq \mathcal{P}ar \subseteq \mathcal{R}ed.$$

Many other triangular families of numbers involving combinatorial objects (e.g. partitions with ordered blocks, permutations, bipartite graphs) can be represented as triangular categories (see [Le2]) often using generalized reduced matrices.

§2. p,q-Stirling numbers

The first combinatorial interpretation of the q-Stirling numbers was given by S. Milne [Mi] in terms of *inversion numbers* of partitions or of restricted growth functions. M. Wachs and D. White [WW] have recently introduced a second statistic on partitions which could be called *non-inversions*. The generating functions in two variables p and q for the joint distribution of these two statistics are then called the *p,q-Stirling numbers* of the second kind and will be denoted by $S_{p,q}[n,k]$. Using the bijection between partitions π and 0-1 tableaux $T(\pi)$ described earlier (see fig.1.2), these statistics are easily defined as follows:

- the inversion number of π, $inv(\pi)$, is equal to the number of 0's below a 1 in $T(\pi)$ (stars in fig.2.1);
- the non-inversion number of π, $nin(\pi)$ is equal to the number of 0's above a 1 in $T(\pi)$ (plusses in fig 2.1).

1	1	+	1	+	+	+	+
	*	+	*	+	1	+	1
		1	*	1	*	1	*
						*	*

Figure 2.1

We then set, for $n \geq k$,

$$S_{p,q}[n,k] = \sum_{\pi \in \mathcal{P}ar(k,n)} p^{nin(\pi)} q^{inv(\pi)}, \tag{2.1}$$

and obtain immediately, from the combinatorial model (try to add a new column), the recurrence

$$S_{p,q}[n,k] = S_{p,q}[n-1, k-1] + [k]_{p,q} S_{p,q}[n-1, k], \quad n \geq 1, \tag{2.2}$$

which, together with the boundary values

$$S_{p,q}[0,0] = 1 \ ; \quad S_{p,q}[n,k] = 0 \ \text{ for } \ k<0 \ \text{ or } \ k>n \tag{2.3}$$

characterize these polynomials. Here the notation $[n]_{p,q}$ is used for the p,q-analogue of n, defined by

$$(2.4) \qquad [0]_{p,q} = 0\,; \;\; [n]_{p,q} = p^{n-1} + p^{n-2}q + \ldots + q^{n-1} = \frac{p^n - q^n}{p-q}, \;\; n \geq 1$$

and can be seen as the p,q-generating function for 0-1 tableaux having one column, of height n.

Setting p=1 in (2.1) means ignoring the statistic $nin(\pi)$ and yields the q-Stirling numbers of the second kind $S_q[n,k]$ defined by (0.8). Of course, setting p=q=1 gives the usual Stirling numbers of the second kind.

The main result of this paper is the following theorem which will be proved injectively in §4:

Theorem 2.1 The p,q-Stirling numbers of the second kind are p,q-log concave in n, that is satisfy, for $n \geq 1$ and $2 \leq k \leq n-1$, the inequality

$$(2.5) \qquad S_{p,q}[n, k-1]\; S_{p,q}[n, k+1] \leq_{p,q} (S_{p,q}[n,k])^2$$

where $f \leq_{p,q} g$ is to be interpreted as "g - f is a polynomial in the variables p and q with non-negative coefficients".

For p =1 , this is the conjecture of Lynne Butler; this result, as well as the more general theorem 4.1, has been proven independently by Bruce Sagan [Sa2] using induction.

0	0	0	0	1	0	1	1
1	0	1	0	0	0	0	
0	1	0	1	0	1		
0	0						

Figure 2.2

In order to follow more closely L. Butler's proof of the q-log-concavity of the Gaussian coefficients, we will now adopt the Anglo-Saxon representation of Ferrers diagrams in the fourth quadrant of the plane and give an intrinsic description of 0-1 tableaux (see fig. 2.2). Indeed a 0-1 tableau can be seen as a pair $\varphi = (\lambda, f)$ where $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ is a partition of an integer

$m = |\lambda|$ and $f = (f_{ij})$ is a "filling" of the corresponding Ferrers diagram with 0's and 1's, with exactly one 1 in each column. Let $nin(\varphi)$ (resp. $inv(\varphi)$) denote the number of 0's that are located above (resp. below) a 1 in the 0-1 tableaux φ. We also denote by $\mathfrak{T}$ the set of all 0-1 tableaux and by $\mathfrak{T}(k,r)$ the set of all those $\varphi = (\lambda, f) \in \mathfrak{T}$ such that the number of (non-zero) parts of λ is at most k and that $\lambda_1 = r$ (we insist that the largest part be equal to r), for $k \geq 0$, $r \geq 0$. By convention, there is one 0-1 tableau $\varphi \in \mathfrak{T}^0_k$, with $nin(\varphi) = inv(\varphi) = 0$, for $k \geq 0$ but $\mathfrak{T}(0,r) = 0$, for $r > 0$. It should then be clear that for $r = n-k$, we have

$$S_{p,q}[n,k] = \sum_{\varphi \in \mathfrak{T}(k,n-k)} p^{nin(\varphi)} \; q^{inv(\varphi)} \tag{2.6}$$

Two properties of $S_{p,q}[n,k]$ are immediate consequences of this combinatorial definition: the symmetry in p and q and Gould's formula.

Proposition 2.2 The p,q-Stirling numbers of the second kind $S_{p,q}[n,k]$ are symmetric polynomials in the variables p and q.

Proof. This fact is already obvious from the recurrence (2.2), since $[n]_{p,q}$ is a symmetric polynomial in p and q. An explicit involution Φ on $\mathfrak{T}$ which interchanges the statistics *nin* and *inv* can be described as follows: restricted to single columns, Φ is the obvious involution which yields the symmetry of $[n]_{p,q}$ (see fig. 2.3); for a general 0-1 tableau, apply this involution separately to each of its columns. □

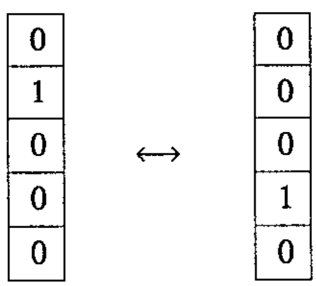


Figure 2.3

In particular we have

$$S_{1,q}[n,k] = S_{q,1}[n,k] \tag{2.7}$$

and we see that the two statistics $nin(\varphi)$ and $inv(\varphi)$ have the same distribution, for $\varphi \in \mathfrak{T}(k,n-k)$.

Proposition 2.3. The polynomial $S_{p,q}[n,k]$ admits the two following equivalent expressions which, for p = 1, correspond to H.W. Gould's [Go] original definition of the q-Stirling numbers of the second kind:

(2.8)
$$S_{p,q}[n,k] = \sum_{\lambda \in \mathfrak{F}(k,n-k)} [\lambda'_1]_{p,q} \cdots [\lambda'_{n-k}]_{p,q},$$

where λ' denotes the conjugate partition of λ,

(2.9)
$$S_{p,q}[n,k] = \sum_{1 \leq j_1 \leq \ldots \leq j_{n-k} \leq k} [j_1]_{p,q} \cdots [j_{n-k}]_{p,q}$$

Proof: These expressions simply describe the construction, column by column, of 0-1 tableaux.

□

Note also that formula (2.9) can be translated in terms of generating functions:

(2.10)
$$\sum_{r \geq 0} S_{p,q}[k+r, k]\, x^r = \frac{1}{1-[1]_{p,q}\, x} \cdots \frac{1}{1-[k]_{p,q}\, x}$$

or of the homogeneous symmetric polynomials h_r:

(2.11)
$$S_{p,q}[n,k] = h_{n-k}([1]_{p,q}, \ldots, [k]_{p,q})$$

We now turn to p,q-Stirling numbers of the first kind. Inspired by Gould's formulas [Go], we set

(2.12)
$$c[n,k] = \sum_{\varphi \in \mathfrak{T}d\,(n-1,\, n-k)} p^{nin(\varphi)}\, q^{inv(\varphi)}$$

where $\mathfrak{T}d\,(h,r)$ denotes the subset of $\mathfrak{T}\,(h,r)$ consisting of 0-1 tableaux with *distinct columns*.

Equivalently we have

$$c[n,k] = \sum_{1 \le j_1 < \ldots < j_{n-k} \le n-1} [j_1]_{p,q} \cdots [j_{n-k}]_{p,q} \tag{2.13}$$

$$\sum_{r \ge 0} c[n, n-r]\, x^r = (1 + [1]_{p,q}\, x) \ldots (1 + [n-1]_{p,q}\, x) \tag{2.14}$$

$$c[n,k] = e_{n-k}([1]_{p,q}, \ldots, [n-1]_{p,q}) \tag{2.15}$$

where e_n denotes the elementary symmetric function of degree n.

It is left as an exercise to the reader to find a bijection between 0-1 tableaux in $\mathcal{T}d$ (n-1,n-k) and permutations of the set {1, . . . ,n} which factorize into k disjoint cycles, and to interpret the statistics *nin* and *inv* accordingly.

§3. The involutions "arm" and "leg"

As a tool for constructing injective proofs, L. Butler has introduced in [Bu2], for $m \geq 1$, weight preserving involutions "arm" $\mathcal{A} = \mathcal{A}_m$ and "leg" $\mathcal{L} = \mathcal{L}_m$ on the set $\mathfrak{F} \times \mathfrak{F}$ of pairs (λ,μ) of Ferrers diagrams, the weight being defined as

(3.1) $$w(\lambda,\mu) = q^{|\lambda| + |\mu|}$$

The purpose of this section is to extend these to weight preserving involutions of the set $\mathfrak{T} \times \mathfrak{T}$ of pairs (φ,ψ) of 0-1 tableaux, with weight

(3.2) $$w(\varphi,\psi) = p^{nin(\varphi) + nin(\psi)} \; q^{inv(\varphi) + inv(\psi)}$$

We will use an alternate description of L. Butler's involutions which has a Gessel-Viennot lattice path flavor [GeV].

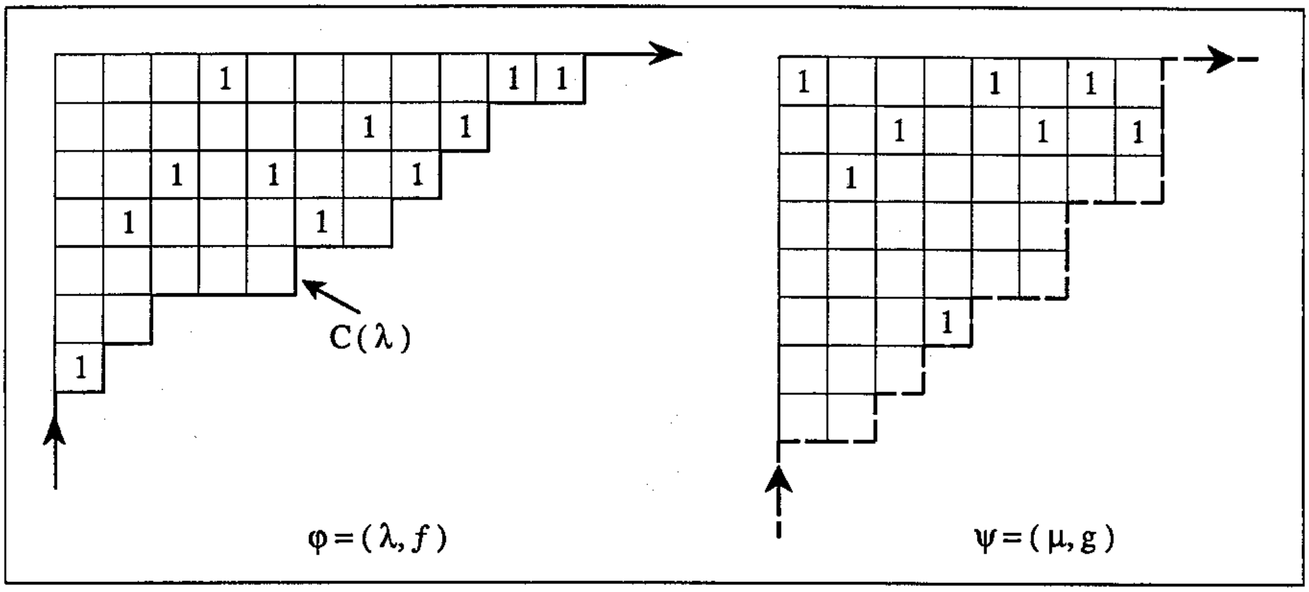


Figure 3.1

Let $\varphi = (\lambda,f)$ and $\psi = (\mu,g)$ be 0-1 tableaux and let $m \geq 1$. The construction of $(\tilde{\varphi},\tilde{\psi}) = \mathcal{A}_m(\varphi,\psi)$ is illustrated in the figures 3.1, 3.2 and 3.3, in the case m=2. With any partition λ, we associate the infinite lattice path $c(\lambda)$ in the plane that goes upward along the negative y-axis until it reaches the Ferrers diagram of λ, follows the south-west boundary of the diagram to the x-axis and then goes right to infinity on the x-axis (see fig. 3.1).

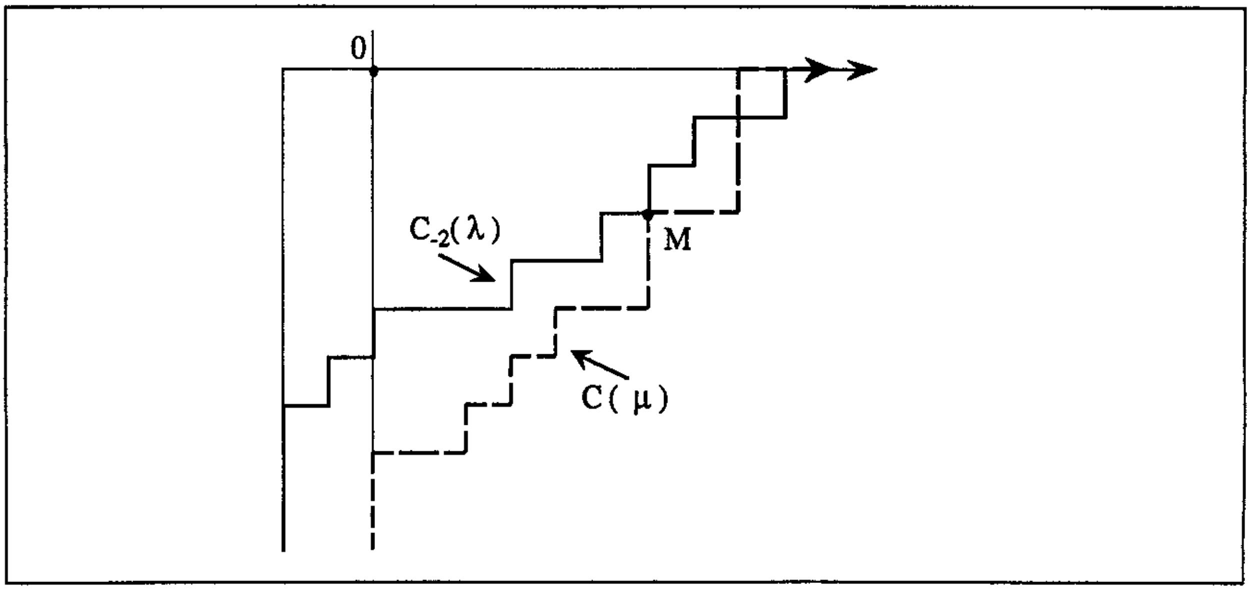


Figure 3.2

To define $(\tilde{\varphi}, \tilde{\psi}) = \mathcal{A}_m(\varphi,\psi)$, we first shift the path $c(\lambda)$ m steps to the left and superimpose the resulting path, denoted by $c_{-m}(\lambda)$, over $c(\mu)$ (see fig. 3.2). These two paths must intersect; let M denote the first point of intersection. The involution $\mathcal{A}_m$ then interchanges the two sections of paths that start at M, giving a pair of new shapes $(\tilde{\lambda}, \tilde{\mu})$; to obtain the new fillings, we simply interchange the columns of φ and ψ that are to the right of the point M (see fig. 3.3). This produces the new pair $(\tilde{\varphi}, \tilde{\psi}) = \mathcal{A}_m(\varphi,\psi)$.

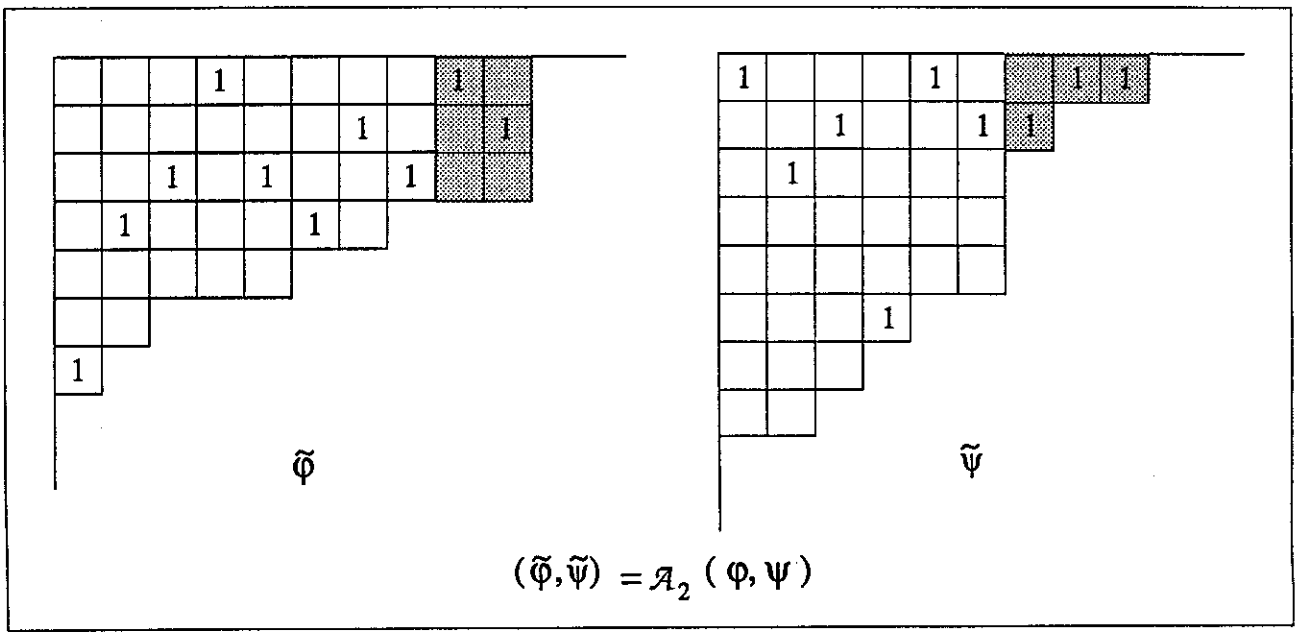


Figure 3.3

Proposition 3.1 The endofunction $\mathcal{A}_m$ described above on $\mathcal{T} \times \mathcal{T}$ is involutive and weight preserving. Moreover we have

$$(3.3) \qquad \mathcal{A}_m(\mathcal{T}(h,r) \times \mathcal{T}(k,s)) \subseteq \mathcal{T}(h, s+m) \times \mathcal{T}(k, r-m)$$

whenever $r \geq s+m$ and $h \leq k$.

Proof. The first part of the proposition should be evident. Now if $(\varphi,\psi) \in \mathcal{T}(h,r) \times \mathcal{T}(r,s)$ we have $\lambda_1 = r$ and $\mu_1 = s$ so that if $r \geq s+m$, then the point of intersection M will appear before the x-axis. This means that an "arm" interchange will effectively take place and that $\tilde{\lambda}_1 = \mu_1+m$ and $\tilde{\mu}_1 = \lambda_1 - m$. Concerning the legs, we will have $\tilde{\lambda}'_1 = \lambda'_1 \leq h$ and either $\tilde{\mu}'_1 = \mu'_1 \leq k$ or $\tilde{\mu}'_1 \leq \lambda'_1 \leq h \leq k$, when M lies on the y-axis. In any case (3.3) holds. □

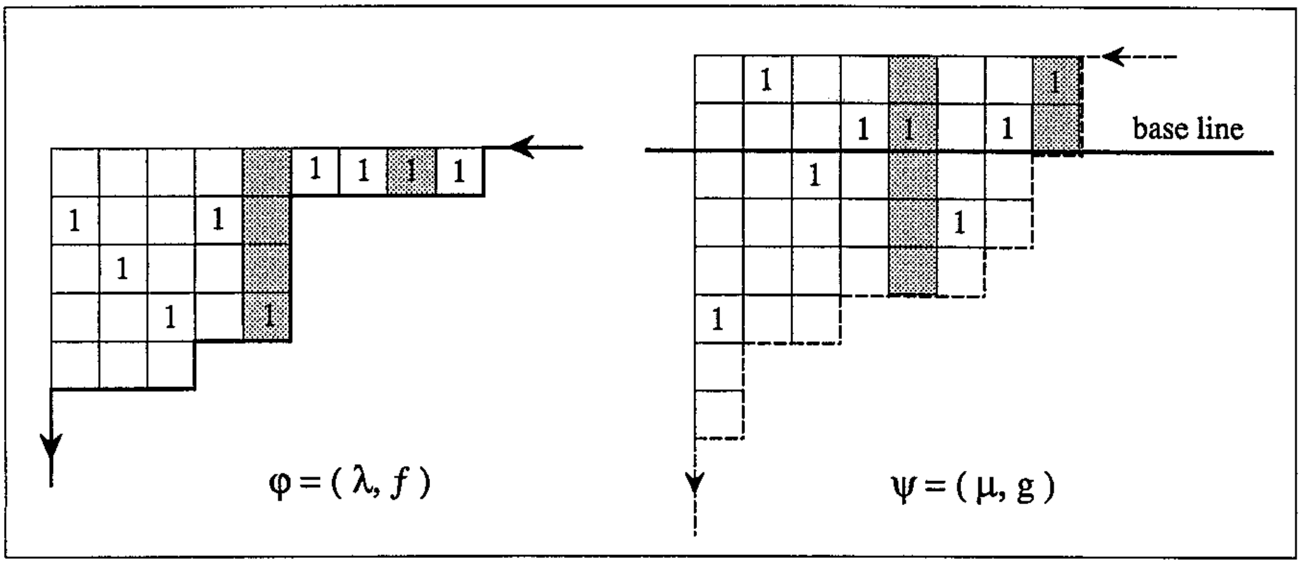


Figure 3.4

The construction of the "leg" involution $\mathcal{L}_m$ is similar, but crucial changes must be made in Butler's construction. It will be illustrated, again with m=2, by the figures 3.4 - 3.9. Let $(\varphi = (\lambda,f)$, $\psi = (\mu,g))$ be a pair of 0-1 tableaux. We introduce a horizontal line, called the **base line,** which is m steps below the top of the Ferrers diagram of μ (see fig. 3.4). The section of ψ which lies below the baseline will be called the **sub-tableau** of ψ. The involution $\mathcal{L}_m$ will somehow interchange a number of columns of φ with the corresponding columns in the sub-tableau of ψ. We naturally want to preserve the property that there is exactly one 1 in each column.

Assume for the moment that φ and ψ each contain only one column. There are two cases:

Case 1. *There is no 1 above the base line .* Then φ and the sub-tableau of ψ are simply interchanged by $\mathcal{L}_m$ (see fig. 3.5).

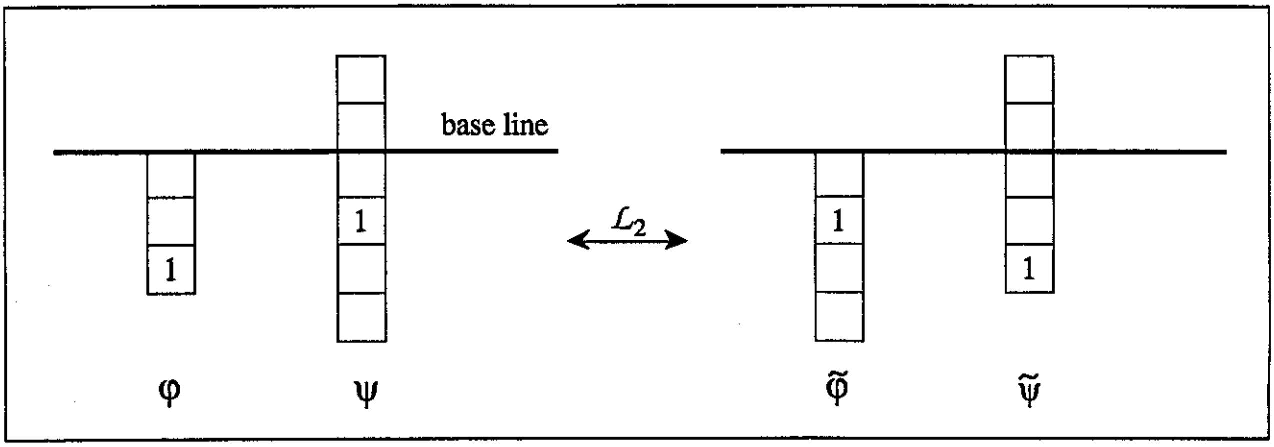


Figure 3.5

Case 2. *There is a 1 above the base line in ψ.* We then introduce a second horizontal line, called the **cut line,** h steps below the baseline, where

$$h = \min (\lambda'_1 , \mu'_1 - m)$$

(see fig. 3.6 & 3.7). Here again λ' denotes the conjugate partition of λ so that λ'_1 is the size of the first (and only) column of φ. Two subcases can occur:

Subcase 2.1 *There is no 1 below the cut line .* In this case, $\mathcal{L}_m$ simply moves from left to right, or from right to left, the cells that lie below the cut-line (see fig. 3.6).

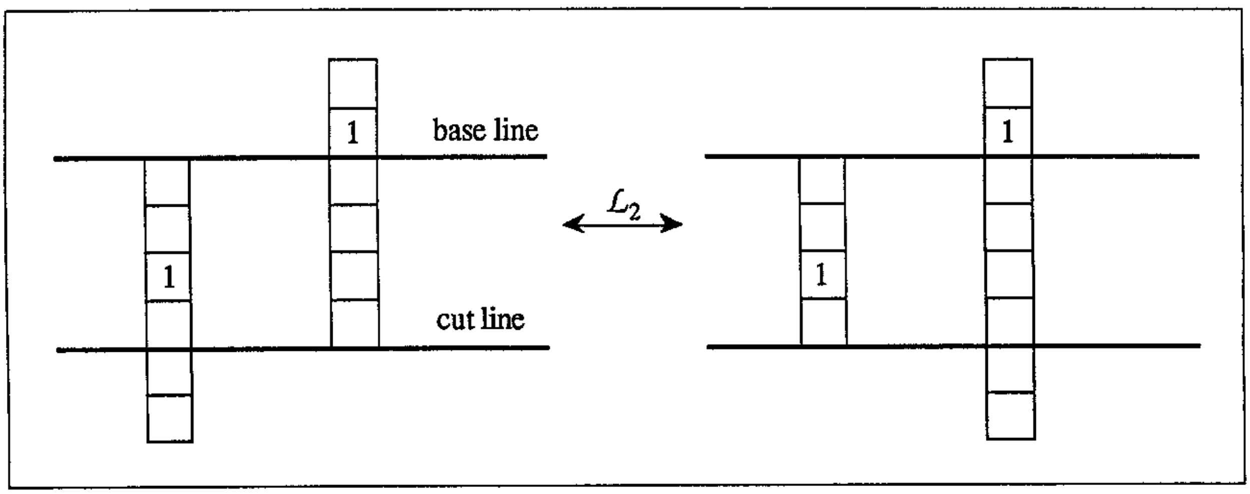


Figure 3.6

Subcase 2.2 *There is a 1 below the cut line,* necessarily in φ (fig 3.7). Such a pair of columns is called **untouchable** and is left fixed by $\mathcal{L}_m$. Note that in this case, $\lambda'_1 > \mu'_1 - m$.

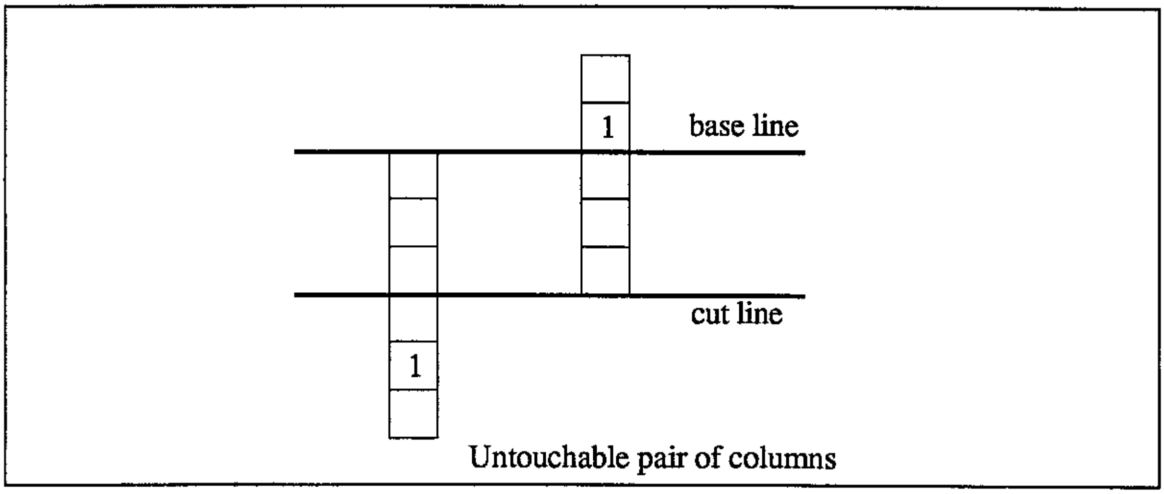


Figure 3.7

Now in the case of a general pair (φ, ψ) we first locate the untouchable pairs of corresponding columns, if there are any (5^{th} and 8^{th} columns in fig. 3.4). Let d(λ) denote the path associated with the Ferrers diagram of λ obtained from c(λ) simply by reversing its direction. We then superpose the paths d(λ) and $d_{-m}(\mu)$, obtained by shifting d(μ) m steps upwards. This gives a first intersection point M but more interesting in our case is the first intersection point P to the left of which lies no untouchable pair of columns (see fig. 3.8).

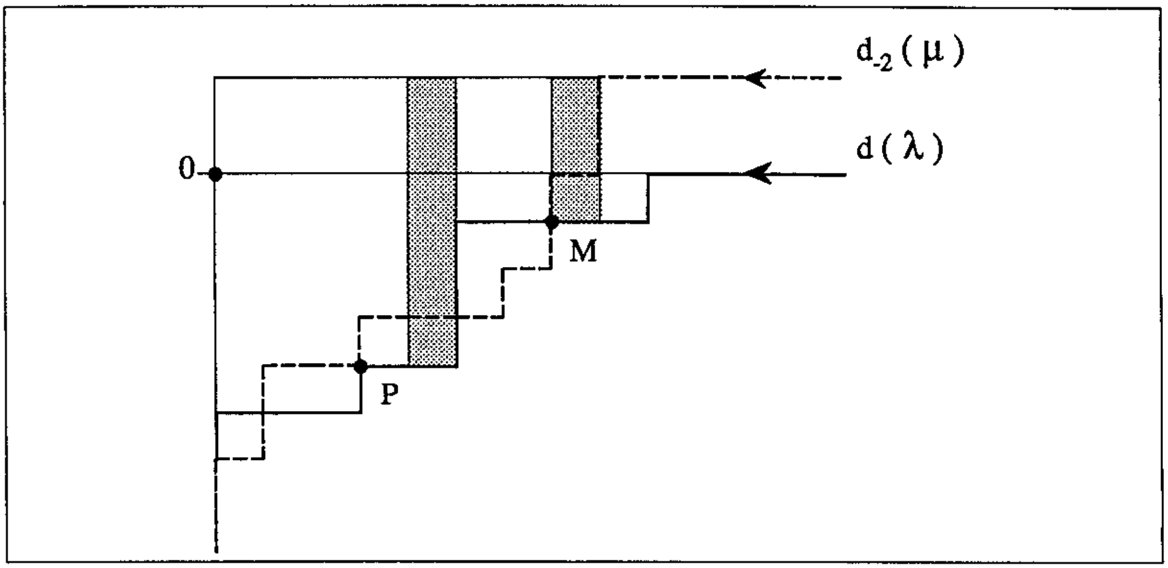
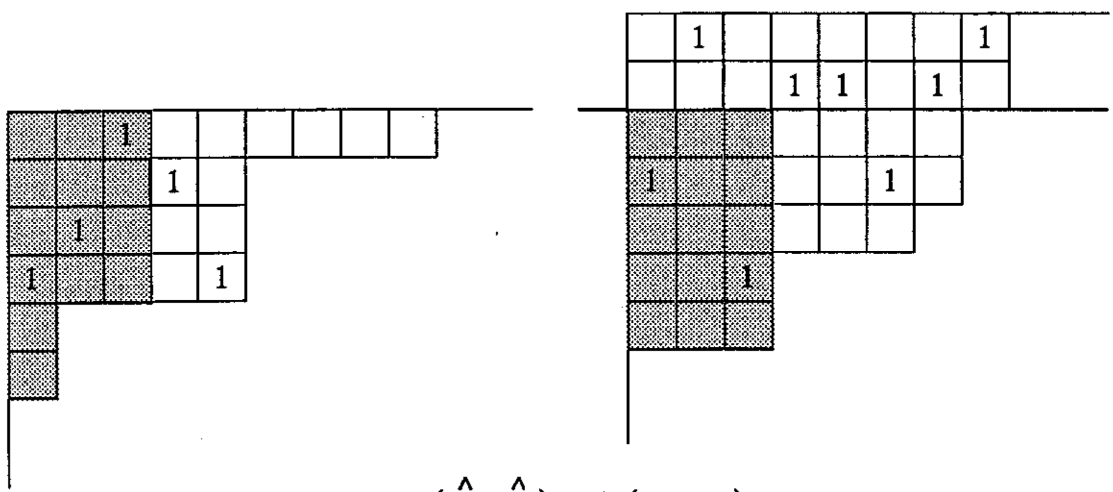


Figure 3.8

The involution $\mathcal{L}_m$ then interchanges the two sections of paths that originate at P and the new fillings are obtained by applying to successive pairs of columns left of the point P, the rules described earlier in cases 1 and 2.1. This gives $(\hat{\varphi}, \hat{\psi}) = \mathcal{L}_m(\varphi,\psi)$ (see Fig. 3.9)



$(\hat{\varphi}, \hat{\psi}) = \mathcal{L}_2(\varphi, \psi)$

Figure 3.9

Proposition 3.2 . The endofunction $\mathcal{L}_m$ on $\mathfrak{T} \times \mathfrak{T}$ is involutive and weight preserving. Moreover, we have

$$\text{(3.4)} \qquad \mathcal{L}_m\,(\mathfrak{T}(h,r) \times \mathfrak{T}(k,s)) \subseteq \mathfrak{T}(k\text{-}m, r) \times \mathfrak{T}(h\text{+}m,s)$$

whenever $h+m \leq k$ and $r \geq s$.

Proof. Only the second part requires explanation. Let J be the column of the first (from left to right) untouchable pair of columns, if there is any, and $J = \infty$ otherwise. Then as observed before, the path $d(\lambda)$ must lie strictly below $d_{-m}(\mu)$ at the column J. Now if $\lambda'_1 < \mu'_1 - m$, then the point of intersection P must lie to the right of the first column and a "leg" interchange will effectively take place. In that case we will have

$$\text{(3.5)} \qquad \hat{\lambda}'_1 = \mu'_1 - m \leq k - m \ \text{ and } \ \hat{\mu}'_1 = \lambda'_1 + m \leq h + m.$$

If no leg interchange takes place then necessarily $\lambda'_1 \geq \mu'_1$ -m and we have

(3.6) $$\hat{\lambda}'_1 = \lambda'_1 \leq h \leq k - m \quad \text{and} \quad \hat{\mu}'_1 = \mu'_1 \leq \lambda'_1 + m \leq h + m.$$

When $\lambda_1 = r \geq s = \mu_1$, then $\mathcal{L}_m$ could cause an arm interchange only if P=M was on the x-axis and $\lambda_1 > \mu_{m+1}$. But this is impossible since this situation creates untouchable pairs of columns. Hence $\hat{\lambda}_1 = \lambda_1 = r$ and $\hat{\mu}_1 = \mu_1 = s$. In any case we see that (3.4) holds. □

There is a variant of the arm involution which is adapted to 0-1 tableaux with distinct columns, denoted, for m≥1, by

(3.7) $$\mathcal{Ad}_m\colon\ \mathfrak{T}d \times \mathfrak{T}d \longrightarrow \mathfrak{T}d \times \mathfrak{T}d$$

It is defined as follows, for $(\varphi,\psi) \in \mathfrak{T}d \times \mathfrak{T}d$, with $\varphi = (\lambda,f)$ and $\psi = (\mu,g)$: let J be the first column for which $\lambda'_{J+m-1} > \mu'_J$, else, if no such column exists, let $J = \infty$; recall that λ' denotes the conjugate partition of λ and that λ'_j is set equal to 0, for $j \geq \lambda_1$. In terms of the superposed paths $c_{-m}(\lambda)$ and $c(\mu)$, we look for the first vertical segment in their intersection, if there is any. The function $\mathcal{Ad}_m$ then transposes the columns to the right of this segment, that is the columns J+m, J+m+1, . . . of φ are interchanged with the columns J, J+1, . . . of ψ. If $J = \infty$, then $\mathcal{Ad}_m$ leaves the pair fixed.

The proof of the following statement is similar to that of proposition 3.1.

Proposition 3.3. The endofunction $\mathcal{Ad}_m$ described above on $\mathfrak{T}d \times \mathfrak{T}d$ is involutive and weight preserving. Moreover, we have

(3.8) $$\mathcal{Ad}_m(\mathfrak{T}d\,(h,r) \times \mathfrak{T}d\,(h,s)) \subseteq \mathfrak{T}d\,(h, s+m) \times \mathfrak{T}d\,(h, r-m)$$

whenever $r \geq s+m$.

□

§ 4 Results and conjectures

In this section we present a summary of results (with proofs), counterexamples, and conjectures, dealing with log-concavity and unimodality of q- and p,q-analogues of Stirling numbers. Recall that inequality between polynomials means coefficient-wise inequality.

Theorem 4.1 For $n \geq 1$, $2 \leq k \leq \ell \leq n-1$, we have

(4.1) $$S_{p,q}[n, k-1]\ S_{p,q}[n, \ell+1] \ \leq_{p,q}\ S_{p,q}[n, k]\ S_{p,q}[n, \ell]$$

Proof: There is a weight preserving injection

(4.2) $$\mathfrak{T}(k-1, n-k+1) \times \mathfrak{T}(\ell+1, n-\ell-1) \rightarrowtail \mathfrak{T}(k, n-k) \times \mathfrak{T}(\ell, n-\ell)$$

which can be described as the composite $\mathcal{L}_m \circ \mathcal{A}_m$, where $m = \ell - k + 1$. Indeed, since $\mathcal{A}_m$ and $\mathcal{L}_m$ are both weight preserving bijections, their restriction to any subset will be injective and weight preserving, and so will their composite. Moreover it follows from (3.3) and (3.4), since $k \leq \ell$ and $m = \ell - k + 1$, that

(4.3) $$\mathcal{A}_m(\mathfrak{T}(k-1, n-k+1) \times \mathfrak{T}(\ell+1, n-\ell-1) \subseteq \mathfrak{T}(k-1, n-k) \times \mathfrak{T}(\ell+1, n-\ell)$$

and

(4.4) $$\mathcal{L}_m(\mathfrak{T}(k-1, n-k) \times \mathfrak{T}(\ell+1, n-\ell) \subseteq \mathfrak{T}(k, n-k) \times \mathfrak{T}(\ell, n-\ell)$$

This establishes (4.2) and hence (4.1), by virtue of (2.6). □

This theorem was proved independently by Bruce Sagan using induction [Sa2]. In fact he proves a slightly more general result, required for the induction hypothesis, where a factor of $p^i q^j$, with i and j in some range, stands in the left hand side of (4.1). Of course, the case $k = \ell$ is exactly the p,q-log concavity result (Theorem 2.1).

Sagan was the first to prove the analogous result for the Stirling numbers of the first kind (see (4.5) below). He gave two proofs, by induction and also (for $k = \ell$) by observing that his injection g_n in [Sa1] preserves weights (by contrast the injection h_n of [Sa1] does not preserve weights and cannot be used for q-Stirling numbers of the second kind).

We will give two other proofs of theorem 4.2, one using the arm involution adapted to tableaux with distinct columns, $\mathcal{Ad}_m$, and the other, suggested by D. Stanton [Stn], using the Jacobi-Trudi identity for the Schur symmetric functions.

Theorem 4.2 For $n \geq 1$, $2 \leq k \leq \ell \leq n - 1$, we have

(4.5) $$c_{p,q}[n, k-1]\ c_{p,q}[n, \ell + 1] \ \leq_{p,q} \ c_{p,q}[n, k]\ c_{p,q}[n, \ell]$$

Proof 1: A direct application of (3.8), with $m = \ell - k + 1$, $h = n - 1$, $r = n - k + 1$, $s = n - \ell - 1$, gives the required weight preserving injection, bearing in mind (2.12).

2. (Suggested by D. Stanton). Let λ denote the conjugate partition of $(n - k, n - \ell)$, i.e. $\lambda = (2^{n-\ell}\ 1^{\ell-k})$. Then the dual form of the Jacobi-Trudi identity (see [Ma], Ch.1, 3.5) expands the Schur function $\mathcal{S}_\lambda(x)$, in the variables $x = (x_1, \ldots, x_{n-1})$, in terms of the elementary symmetric functions $e_j(x)$ as follows:

(4.6) $$\mathcal{S}_\lambda(x) = e_{n-k}(x)\ e_{n-\ell}(x) - e_{n-\ell-1}(x)\ e_{n-k+1}(x)$$

After the substitution $x_i \longmapsto [i]_{p,q}$, for $i = 1, \ldots, n - 1$, we get, by virtue of (2.15),

(4.7) $$c_{p,q}[n, k]\ c_{p,q}[n, \ell] - c_{p,q}[n, k - 1]\ c_{p,q}[n, \ell + 1] = \mathcal{S}_\lambda([1]_{p,q}, \ldots, [n - 1]_{p,q})$$
$$\geq_{p,q} 0$$

since the Schur functions and the $[i]_{p,q}$ have positive coefficients.

□

The next natural question to raise is that of the p,q-unimodality in k. However this turns out to be false, even for $p = 1$; indeed, contrarily to the case of the q-binomial coefficients (see [Bu1]), L. Butler and myself observed that the q-Stirling numbers of the first and of the second kind fail to be q-unimodal in k, in general. For $S_q[n, k]$, this first occurs at $n = 9$, where between $k = 4$ and 5, some coefficients increase and some decrease. For $c_q[n, k]$, this first occurs at $n = 5$, between $k = 2$ and 3.

It is also natural to consider the sequences $S_{p,q}[n, k]$ and $c_{p,q}[n, k]$, $n \geq k$, for a fixed k. These sequences are easily seen to be p,q-increasing and hence p,q-unimodal. The question of p,q-log-concavity in n is solved differently for $S_{p,q}[n, k]$ and $c_{p,q}[n, k]$. In the first case, the following more general formula holds.

Theorem 4.3 For $k \geq 1$, $k + 1 \leq \ell \leq n$, we have

$$S_{p,q}[\ell - 1, k] \; S_{p,q}[n + 1, k] \;\leq_{p,q}\; S_{p,q}[\ell, k] \; S_{p,q}[n, k] \tag{4.8}$$

Proof: This follows simply from proposition 3.1, that is, with $m = n - \ell + 1$,

$$\mathcal{A}_m(\mathcal{T}(k, n + 1 - k) \times \mathcal{T}(k, \ell - 1 - k) \subseteq \mathcal{T}(k, n - k) \times \mathcal{T}(k, \ell - k) \tag{4.9}$$

□

This result was first proved by D. Stanton. Similarly to the proof of theorem 4.2, he used the formula (2.11) and the Jacobi-Trudi identity expressing the Schur function in terms of the homogeneous symmetric function. Stanton's method confirms the fact that the arm involution is closely related to the Gessel-Viennot combinatorial proof of the Jacobi-Trudi identity. This method however doesn't seem to be applicable to theorem 4.1, where both the arm and the leg involutions are needed.

Surprisingly, the analogue of (4.8) for $c_{p,q}[n, k]$ is false, even for $\ell = n$ and $p = q = 1$. Hence the p,q-Stirling numbers of the first kind are not p,q- nor q-log-concave in n. The degrees do not even fit. Numerical evidence however incite us to formulate the following:

Conjecture 4.4 For $1 \leq k$, $r + 1 \leq n$,

$$c_{p,q}[n - 1, k] \; c_{p,q}[n + 1, k] \;\leq_{p,q}\; (p + q) \; (c_{p,q}[n, k])^2 \tag{4.10}$$

The most interesting questions about the q-Stirling numbers that are still outstanding are the *internal* unimodality and log-concavity, that is unimodality and log-concavity of the sequence of coefficients of each of these polynomials in q.

There is a general conjecture by Garsia and Remmel that the generating polynomial $R_k(A, q)$ in q, of placements of k rooks in a Ferrers board A according to the number of inversion, is unimodal (see [GaR], p. 250). This covers the polynomials $S_q[n, k]$ and $c_q[n, k]$; however in this case numerical evidence point out to the stronger property of log-concavity. Note, in contrast, that the q-binomial polynomials, for example, $\left[{4 \atop 2}\right] = 1 + q + 2q^2 + q^3 + q^4$, are not log-concave.

Conjecture 4.5 The polynomial $S_q[n, k]$ is log-concave, for any $1 \leq k \leq n$.

This conjecture was first stated by M. Wachs and D. White [W W] who tested it for $n \leq 20$ on the computer. They observed, moreover, that the sequence of coefficients of $S_{p,q}[n, k]$ as a polynomial in q is not p-log-concave, in general.

Conjecture 4.6 The polynomial $c_q[n, k]$ is log-concave, for any $1 \leq k \leq n$.

References

[Bu1] L.M. Butler, *A unimodality result in the enumeration of subgroups of a finite abelian group*, Proc. Amer. Math. Soc. 101 (1987), 771-775.

[Bu2] L. M. Butler, *The q-log-concavity of q-binomial coefficients*, preprint.

[Ca] L. Carlitz, *On abelian fields*, Trans. Amer. Math. Soc., 35 (1933), 122-136.

[CLL] M. Content, F. Lemay, P. Leroux, *Catégories de Möbius et fonctorialités: un cadre général pour l'inversion de Möbius.* J. Combin. Th. A. 28 (1980), 169-190.

[DoRS] P. Doubilet, G.-C. Rota, & R.P. Stanley, *The idea of generating function,* in Proc. 6th Berkeley Symposium, vol. 2 (1973), 267-318, Un. of California Press.

[GaR] A.M. Garsia & J.B. Remmel, *q-counting Rook Configurations and a Formula of Frobenius.* J. Combin. Th. A, 41 (1986), 246-275.

[GeV] I. Gessel & G.X. Viennot, *Binomial determinants, paths, and hook length formulae,* Adv. in Math. 58 (1988), 300-321.

[Go] H.W. Gould, *The q-Stirling numbers of first and second kinds*, Duke Math. J. 28 (1961), 281-289.

[Kn] D. Knuth, *Subspaces, subsets and partitions*, J. Combin. Th. 10 (1971), 178-180.

[Le1] P. Leroux, *Les catégories de Möbius*, Cahiers de top. et géom. diff. 16 (1975), 280-282.

[Le2] P. Leroux, *Catégories triangulaires: exemples, applications et problèmes*, Rapport de recherche, Université du Québec à Montréal, juillet 1980, 72p.

[Le3] P. Leroux, *Reduced Matrices and Combinatorics*, Contributed Communication, SIAM conference on Applied Linear Algebra, April 26-29, 1982, Raleigh, North Carolina.

[Le4] P. Leroux, *Algèbre linéaire, une approche matricielle*, Modulo Editeur, Montréal, 1983, 500 p. ISBN 2-89113-201-7.

[Ma] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, 1979.

[Mi] S.C. Milne, *Restricted Growth Functions, Rank Row Matchings of Partition Lattices, and q-Stirling Numbers*, Adv. in Math. 43 (1982), 173-196.

[NSW] A. Nijenhuis, A.E. Solow, & H.S. Wilf, *Bijective Methods in the Theory of Finite Vector Spaces*, J. Combin. Th. A, 37 (1984), 80-84.

[Sa1] B. Sagan, *Inductive and injective proofs of log concavity results*, Discrete Math. 68 (1988), 281-292.

[Sa2] B. Sagan, *Inductive Proofs of q-Log Concavity*, Preprint.

[Stl] R.P. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics and geometry*, Annals of the New York Acad. of Sc., to appear.

[Stn] D. Stanton, Personal communication.

[WW] M. Wachs & D. White, *p,q-Stirling Numbers and Set Partition Statistics*, Preprint.