**MULTI-HAMILTONIAN STRUCTURE
OF THE BORN-INFELD EQUATION**

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Abstract

We exhibit the multi-Hamiltonian structure, conservation laws and higher order symmetries for the Born-Infeld equation. A new transformation of this equation to the equations of a Chaplygin gas is presented and explored. The equation is distinguished among two-dimensional hyperbolic systems by its wealth of such structures.

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1. Introduction.

A nonlinear modification of Maxwell's electrodynamics was proposed by Born and Infeld in 1934¹. The simplest example of this system of nonlinear field equations is the quasi-linear second order equation in 1+1 dimensions

$$(1 + \varphi_x^2) \varphi_{tt} - 2 \varphi_t \varphi_x \varphi_{xt} - (1 - \varphi_t^2) \varphi_{xx} = 0, \quad (1.1)$$

which is known as the Born-Infeld equation². It is also the equation governing minimal surfaces in 2+1 dimensional Minkowski space, which is a special case of the Nambu string³. The world-sheet of the string is parametrized by harmonic coordinates, familiar from the theory of minimal surfaces, rather than the light cone gauge⁴. We will also consider the representation of this equation in null coordinates

$$x' = x + t, \quad t' = x - t,$$

in terms of which the Born-Infeld equation can be rewritten as

$$\varphi_x^2 \varphi_{t't'} - 2 (2 + \varphi_t \varphi_x) \varphi_{x't'} + \varphi_t^2 \varphi_{x'x'} = 0. \quad (1.2)$$

In this paper we shall discuss the Hamiltonian structure, symmetries and conservation laws of the Born-Infeld equation. We shall find that it has a remarkably rich structure. The first step is to recast the equation as a first order quasi-linear Hamiltonian system of hydrodynamic type^{5,6}. Remarkably, this can be done in three inequivalent ways, one of which corresponds to a system of isentropic gas dynamics with adiabatic index $\gamma = -1$, corresponding to the pressure-density relation $P = -1/\rho$, which is known as a *Chaplygin gas*⁷. Each of these systems is separable, and therefore the extensive results on Hamiltonian structures, symmetries and conservation laws of Sheftel⁸, and Olver and Nutku⁹ can be used. Even among the separable two-dimensional systems, the Born Infeld system has a much richer algebraic structure than most, in part due to the multiple Hamiltonian reformulations of the equation. We will see that the Born-Infeld equation, admits (at least) six independent Hamiltonian structures, in contrast to the two Hamiltonian structures for a general separable system, and four Hamiltonian structures in the more general polytropic case. Moreover, the diagonalization techniques introduced by Verosky¹⁰ are then applied to show that these system admits first order conserved densities depending on arbitrary functions - which is special to these particular systems.

We assume that the reader is familiar with the basics of Hamiltonian systems of

evolution equations, symmetries and conservation laws, as presented, for instance, in Olver¹¹. In the interests of brevity, we have omitted many of the more complicated computations.

2. Hyperbolic forms of the Born-Infeld Equation.

We begin by showing how the Born-Infeld equation can be rewritten as a first order system of quasi-linear hyperbolic evolution equations in several ways. All of these representations have the form⁹

$$u_t = -D_x \frac{\partial H}{\partial v}, \quad v_t = -D_x \frac{\partial H}{\partial u}, \quad (2.1)$$

where $\mathcal{H}[u,v] = \int H(u,v) dx$ is the Hamiltonian functional, and D_x is the total x -derivative. In vector form, if we let $u(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix}$, then equations (2.1) are in elementary Hamiltonian form¹¹

$$u_t = \mathcal{D}^* E_u[H] \quad (2.2)$$

where E_u denotes the Euler operator, or variational derivative with respect to u . The Hamiltonian operator in (2.2) is the constant coefficient skew-adjoint differential operator

$$\mathcal{D}^* = -\sigma_1 D_x = \begin{pmatrix} 0 & -D_x \\ -D_x & 0 \end{pmatrix}, \quad \text{where } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.3)$$

The induced Poisson bracket on the space of densities is given by the standard formula

$$\{ \mathcal{F}, \mathcal{H} \} = \frac{1}{2} \int E_u[F] \cdot \mathcal{D}^* E_u[H] dx = \frac{1}{2} \int \{ E_v[F] D_x E_u[H] - E_u[F] D_x E_v[H] \} dx.$$

We begin by looking at the equation in the physical variables, (1.1). Since this equation can be derived from a variational principal where the Lagrangian depends only on the gradient of ϕ , we know that it can be expressed as the integrability condition of a first order system⁶. To effect this change, we introduce a new potential ψ , given by

$$\psi_x = \frac{\phi_x}{\sqrt{1 + \phi_x^2 - \phi_t^2}}, \quad \psi_t = \frac{\phi_t}{\sqrt{1 + \phi_x^2 - \phi_t^2}}; \quad (2.4)$$

Inverting these equations for the first derivatives of ϕ we find the same expressions with the roles of ϕ and ψ interchanged. Equation (1.1) is then realized as the integrability condition for the system (2.4). Moreover, its companion equation expressing the integrability conditions for ψ is again (2.4), with ψ replacing ϕ . We shall now formulate these equations in terms of a pair of conservation laws. For this purpose, we introduce the variables

$$r = \phi_x, \quad s = \psi_x.$$

Solving (2.6) for ϕ_t, ψ_t we deduce that the 1-forms

$$\alpha = r dx + s \sqrt{\frac{1+r^2}{1+s^2}} dt = d\phi, \quad \omega = s dx + r \sqrt{\frac{1+s^2}{1+r^2}} dt = d\psi,$$

are exact. The implication that they are closed gives rise to the following pair of quasi-linear evolution equations:

$$\begin{aligned} r_t &= \frac{rs}{\sqrt{(1+r^2)(1+s^2)}} r_x + \sqrt{\frac{1+r^2}{(1+s^2)^3}} s_x, \\ s_t &= \sqrt{\frac{1+s^2}{(1+r^2)^3}} r_x + \frac{rs}{\sqrt{(1+r^2)(1+s^2)}} s_x. \end{aligned} \tag{2.5}$$

We will call the quasi-linear system (2.5) the *physical version* of the Born-Infeld equation. It is not hard to see that it is in the standard Hamiltonian form (2.2), where

$$\tilde{H}^*(r, s) = \sqrt{(1+r^2)(1+s^2)} \tag{2.6}$$

is the Hamiltonian density. We note that there are alternative ways of re-expressing (2.1) as the integrability condition of a first order system such as (2.2), but there is a unique choice of ψ which will result in a Hamiltonian system of equations. (An alternative first order form of the Born-Infeld equation which is not Hamiltonian can be found in Whitham².)

A similar reasoning applies to the Born-Infeld equation, rewritten in null coordinates (1.2). We similarly introduce a new potential χ by the equations

$$\chi_x = - \frac{\phi_x}{\sqrt{1 + \phi_x \phi_t}}, \quad \chi_t = \frac{\phi_t}{\sqrt{1 + \phi_x \phi_t}}. \quad (2.7)$$

As above, the companion equation for χ is identical to (2.7). Define

$$z = \phi_x, \quad w = \chi_x.$$

Note that the 1-forms

$$\alpha = z dx - \left(\frac{1}{z} - \frac{z}{w^2} \right) dt = d\phi \quad \omega = w dx - \left(\frac{1}{w} - \frac{w}{z^2} \right) dt = d\chi,$$

are exact, leading to an alternative system of quasi-linear evolution equations

$$\begin{aligned} z_t &= \left(\frac{1}{z^2} + \frac{1}{w^2} \right) z_x - \frac{2z}{w^3} w_x, \\ w_t &= - \frac{2w}{z^3} z_x + \left(\frac{1}{z^2} + \frac{1}{w^2} \right) w_x, \end{aligned} \quad (2.8)$$

which will be called the *null coordinate version* of the Born-Infeld equation. Again, these are in Hamiltonian form (2.2), with the Hamiltonian density

$$\hat{H}^*(z, w) = \frac{z}{w} + \frac{w}{z}. \quad (2.9)$$

Although these two versions of the Born-Infeld equation can be obtained by a transformation between physical and null coordinates, it is rather remarkable that there is also a transformation of just the dependent variables which maps one to the other:

Theorem 1. Given r, s , with $rs > 1$, define the transformation

$$\begin{aligned} z &= (1 + r^2)^{1/4} (1 + s^2)^{1/4} [(rs + 1)^{1/2} + (rs - 1)^{1/2}], \\ w &= (1 + r^2)^{1/4} (1 + s^2)^{1/4} [(rs + 1)^{1/2} - (rs - 1)^{1/2}]. \end{aligned} \quad (2.10)$$

If (r, s) satisfy the physical version of Born-Infeld (2.5), then (z, w) satisfy the null coordinate version (2.8).

The proof is just a straightforward but lengthy calculation. In the next section we

shall see how the transformation (2.10) can be systematically deduced by referring to the second Hamiltonian structure of (2.5).

We now turn to a remarkable transformation from the Born-Infeld system to a system of quasi-linear equations arising in polytropic gas dynamics.

Theorem 2. Define the variables

$$u = -\left(\frac{1}{z^2} + \frac{1}{w^2}\right), \quad v = \frac{zw}{2}. \quad (2.11)$$

Then z, w satisfy the Born-Infeld system (2.8) if and only if u, v satisfy the gas dynamics system

$$u_t + uu_x + v^{-3} v_x = 0, \quad v_t + (uv)_x = 0. \quad (2.12)$$

The proof is again a straightforward calculation. The system (2.12) corresponds to the equations of isentropic, polytropic gas dynamics with adiabatic index $\gamma = -1$, known as a Chaplygin gas⁷. It is distinguished among such quasi-linear hyperbolic systems by the fact that shocks do not form^{12,2}. This system is also in the elementary Hamiltonian form (2.2), with Hamiltonian density

$$H^*(u,v) = \frac{u^2 v}{2} + \frac{1}{2v}. \quad (2.13)$$

We remark that the reduction of a gas dynamics system to a single second order hyperbolic equation, which includes the reduction of a Chaplygin gas to the Born-Infeld equation (1.2), can be found in Garabedian¹³. Note also that the physical version (2.8) can be transformed directly to the gas dynamics version (2.12) by composing the transformations (2.10), (2.11):

$$u = \frac{rs}{\sqrt{(1+r^2)(1+s^2)}}, \quad v = \sqrt{(1+r^2)(1+s^2)}. \quad (2.14)$$

We thus have three distinct ways of reformulating the Born-Infeld equation as a Hamiltonian system of quasi-linear evolution equations of the type (2.2). To keep track of various functions and operators in the different coordinate systems, we will adopt the following convention: in the physical (r,s) -version, these quantities will have a tilde over them, e.g. $\tilde{H}$; in the null (z,w) -version, they will have a hat over them, e.g. $\hat{H}$; in the

gas dynamics (u,v)-coordinates, there will be no distinguishing mark, e.g. H.

3. First order Hamiltonian operators.

We now investigate other first order Hamiltonian structures for the Born-Infeld equation, using the methods in references 6 and 9. First we recall that the most general skew-adjoint first order matrix differential operator has the form

$$\mathcal{B} = M \cdot D_x + D_x \cdot M + Q_x = \begin{pmatrix} 2m D_x + m_x & 2p D_x + p_x + q_x \\ 2p D_x + p_x - q_x & 2n D_x + n_x \end{pmatrix}, \quad (3.1)$$

where

$$M = \begin{pmatrix} m & p \\ p & n \end{pmatrix} \quad \text{is a general symmetric matrix,}$$

$$Q = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix} \quad \text{is a general skew-symmetric matrix,}$$

and where the coefficients m, n, p, q are allowed to depend on the dependent variables. The particular Hamiltonian operator (2.3) corresponds to the choice

$$\mathcal{D}^*: \quad m^* = n^* = q^* = 0, \quad p^* = -\frac{1}{2}. \quad (3.2)$$

In order that the Poisson bracket associated with the operator (3.1) satisfy the Jacobi identity, the coefficients m, n, p, q must satisfy additional first order partial differential equations⁶.

Besides the standard Hamiltonian form (2.2), any polytropic gas dynamics system can be written in two additional, alternative Hamiltonian forms involving first order Hamiltonian operators⁶, making it a tri-Hamiltonian system:

$$u_t = \mathcal{D}_0 E_u(H_2) = \mathcal{D}_1 E_u(H_1) = \mathcal{D}_2 E_u(H_0). \quad (3.3)$$

For the case of adiabatic index $\gamma = -1$, these Hamiltonian operators have the following form:

$$\mathcal{D}_0 = \mathcal{D}^*: \quad m_0 = 0, \quad n_0 = 0, \quad p_0 = -\frac{1}{2}, \quad q_0 = 0, \quad (3.4)$$

$$\mathcal{D}_1: \quad m_1 = \frac{1}{v^3}, \quad n_1 = v, \quad p_1 = -u, \quad q_1 = 2u. \quad (3.5)$$

$$\mathcal{D}_2: \quad m_2 = \frac{u}{v^3}, \quad n_2 = uv, \quad p_2 = -\frac{u^2}{2} - \frac{1}{2v^2}, \quad q_2 = u^2. \quad (3.6)$$

and are mutually compatible¹¹ Hamiltonian operators. We note that these are genuinely distinct Hamiltonian operators, meaning that $\mathcal{D}_2$ is not related to $\mathcal{D}_0$ and $\mathcal{D}_1$ by a well-known recurrence formula¹⁴ generating higher order Hamiltonian operators from any biHamiltonian system. The corresponding Hamiltonian densities placing (2.12) in the tri-Hamiltonian form (3.3) are

$$H_0 = v, \quad H_1 = uv, \quad H_2 = \frac{u^2v}{2} + \frac{1}{2v}, \quad (3.7)$$

which appear in the well-known hierarchy of conserved densities for gas dynamics⁹.

Before proceeding on to the tri-Hamiltonian structure of the other two evolutionary versions of the Born-Infeld equation, it helps to recall how Hamiltonian operator transform under change of variables.

Lemma 3.^{14,15} Let $u = \varphi(z)$ be a change of variables, and let J denote the Jacobian matrix of φ . Let $\mathcal{D}$ denote a Hamiltonian operator in the u coordinates, and $\hat{\mathcal{D}}$ the corresponding Hamiltonian operator in the z coordinates. Then these two operators are related by the change of variables formula

$$\mathcal{D} = J \cdot \hat{\mathcal{D}} \cdot J^T. \quad (3.8)$$

Thus, for Hamiltonian operators of the form (3.1), we find the corresponding coefficient matrices have the change of variables formulae

$$M = J \cdot \hat{M} \cdot J^T, \quad Q_x = J \cdot \hat{Q}_x \cdot J^T + J \cdot \hat{M} \cdot J_x^T - J_x \cdot \hat{M} \cdot J^T. \quad (3.9)$$

Dubrovin and Novikov⁵ have pointed out that the Poisson brackets defined by Hamiltonian operators for equations of hydrodynamic type give rise to Riemannian metrics with vanishing torsion and curvature. The metric corresponding to an operator of the form (3.1) is given by

$$ds^2 = \frac{n du^2 - 2p du dv + m dv^2}{mn - p^2} . \quad (3.10)$$

Since the metric (3.10) is flat we know that a (possibly complex) change of variables $u = \varphi(z)$ will bring it to the canonical form $ds^2 = 2 dz dw$, determining the maximal analytic extension of the metric, and which corresponds to the elementary Hamiltonian operator (2.3). Remarkably, the transformations (2.11), (2.14) are precisely the ones needed to place the metrics determined by the Hamiltonian operators $\mathcal{D}_1, \mathcal{D}_2$ in canonical form.

Proposition 4. Under the transformations (2.8), (2.14), the Hamiltonian operators and densities for the gas dynamics system (2.12) are mapped to the following Hamiltonian operators and densities for the physical and null versions of the Born-Infeld equation:

Null coordinate version:

Hamiltonian operators:

$$\hat{\mathcal{D}}_0: \quad \hat{m}_0 = z w^{-3} (z^{-2} - w^{-2})^{-2}, \quad \hat{n}_0 = w z^{-3} (z^{-2} - w^{-2})^{-2},$$

$$\hat{p}_0 = -\frac{1}{2} (z^{-2} - w^{-2})^{-1}, \quad \hat{q}_0 = (z^{-2} - w^{-2})^{-1}.$$

$$\hat{\mathcal{D}}_1 = -2 \mathcal{D}^*: \quad \hat{m}_1 = 0, \quad \hat{n}_1 = 0, \quad \hat{p}_1 = 1, \quad \hat{q}_1 = 0,$$

$$\hat{\mathcal{D}}_2: \quad \hat{m}_2 = -\frac{z}{w^3}, \quad \hat{n}_2 = -\frac{w}{z^3}, \quad \hat{p}_2 = -\frac{1}{2z^2} + \frac{1}{2w^2}, \quad \hat{q}_2 = \frac{1}{z^2} - \frac{1}{w^2}.$$

Hamiltonian densities:

$$\hat{H}_0(z, w) = \frac{z w}{2},$$

$$\hat{H}_1(z, w) = -\frac{z}{2w} - \frac{w}{2z},$$

$$\hat{H}_2(z, w) = \frac{w}{4z^3} + \frac{3}{2zw} + \frac{z}{4w^3}.$$

Physical version:

Hamiltonian operators:

$$\tilde{m}_0 = \frac{-2 r s (1 + r^2)^2}{(r^2 - s^2)^2}, \quad \tilde{n}_0 = \frac{-2 r s (1 + s^2)^2}{(r^2 - s^2)^2},$$

$\tilde{\mathcal{D}}_0$:

$$\tilde{p}_0 = \frac{(r^2 + s^2) (1 + r^2) (1 + s^2)}{(r^2 - s^2)^2}, \quad \tilde{q}_0 = \frac{(r^2 + s^2 + 2r^2 s^2)}{(r^2 - s^2)}.$$

$$\tilde{m}_1 = \frac{(1 + r^2)^{3/2} (r^2 + s^2 + 2r^2 s^2)}{(1 + s^2)^{1/2} (r^2 - s^2)^2},$$

$$\tilde{n}_1 = \frac{(1 + s^2)^{3/2} (r^2 + s^2 + 2r^2 s^2)}{(1 + r^2)^{1/2} (r^2 - s^2)^2},$$

$\tilde{\mathcal{D}}_1$:

$$\tilde{p}_1 = \frac{(1 + r^2)^{1/2} (1 + s^2)^{1/2}}{(r^2 - s^2)^2} r s (2 + r^2 + s^2),$$

$$\tilde{q}_1 = \frac{2 r s}{r^2 - s^2} (1 + r^2)^{1/2} (1 + s^2)^{1/2},$$

$$\tilde{\mathcal{D}}_2 = \mathcal{D}^*: \quad \tilde{m}_2 = 0, \quad \tilde{n}_2 = 0, \quad \tilde{p}_2 = -\frac{1}{2}, \quad \tilde{q}_2 = 0,$$

Hamiltonian densities:

$$\tilde{H}_0(r, s) = \sqrt{(1 + r^2)(1 + s^2)},$$

$$\tilde{H}_1(r, s) = r s,$$

$$\tilde{H}_2(r, s) = \frac{r^2 s^2 + 1}{2 \sqrt{(1 + r^2)(1 + s^2)}}.$$

4. Recursion operators and conserved densities.

According to Magri's Theorem¹⁶, any compatible biHamiltonian system has an associated recursion operator. The Hamiltonian operators $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2$, are mutually compatible⁶, and so there are three recursion operators for the gas dynamics system:

$$\mathcal{R}_1 = \mathcal{D}_1 \cdot \mathcal{D}_0^{-1}, \quad \mathcal{R}_2 = \mathcal{D}_2 \cdot \mathcal{D}_0^{-1}, \quad \mathcal{R}_3 = \mathcal{D}_2 \cdot \mathcal{D}_1^{-1}, \quad (4.1)$$

although there is a trivial relation between them:

$$\mathcal{R}_2 = \mathcal{R}_3 \cdot \mathcal{R}_1.$$

Similar recursion operators can be constructed for the other two versions of the Born-Infeld equation. Now, a curious phenomenon happens when we apply the recursion operator to the hierarchy where the Born-Infeld Hamiltonian lies. We find that the hierarchy of Hamiltonian flows are corresponding conserved densities terminates after just two steps,

$$\mathcal{R}_1: \quad H_0 \rightarrow H_1 \rightarrow H_2 \rightarrow 0,$$

because the second Hamiltonian H_2 is a distinguished functional (Casimir) for the Hamiltonian structure determined by $\mathcal{D}_1$. Therefore the hierarchy guaranteed by Magri's Theorem degenerates into just three independent Hamiltonians, and we have a nontrivial example of a biHamiltonian system which does not satisfy one of the technical hypotheses of Magri's Theorem that the hierarchy of Hamiltonians be independent functionals^{11,15}. However, the second recursion operator $\mathcal{R}_3$ does generate further members of the gas dynamics hierarchy of conserved densities⁸. (We remark that in reference 9 we failed to remark that this property of the hierarchy of flows generated by the recursion operator can occasionally degenerate; the equation after (4.3) there should read

$$\mathcal{R}_1(Q_n) = (n\gamma - n - 1) Q_{n+1}, \quad \mathcal{R}_2(Q_n) = \frac{n}{2} (n\gamma + \gamma - n - 3) Q_{n+2},$$

$$\mathcal{R}_1(\tilde{Q}_n) = (n\gamma - n + 1) \tilde{Q}_{n+1}, \quad \mathcal{R}_2(\tilde{Q}_n) = \frac{n+1}{2} (n\gamma - n + 1) \tilde{Q}_{n+2},$$

leading to degeneracies if γ has one of the forms $1 \pm \frac{1}{n}$, $1 \pm \frac{2}{n}$, for some integer n .)

Another interesting anomaly happens for the physical version of Born-Infeld. Here, from the point of view of reference 6, the most natural recursion operator would be

the operator

$$\mathcal{R}^* = \tilde{\mathcal{D}}_1 \cdot \tilde{\mathcal{D}}_2^{-1} = \tilde{\mathcal{D}}_1 \cdot \mathcal{D}^{*-1}$$

Again, this recursion operator does not produce a hierarchy of symmetries and conserved Hamiltonian densities. In fact, as the reader can check, the recursion operator repeats after two steps:

$$\mathcal{R}^*: H_0 \rightarrow H_1 \rightarrow H_0 \rightarrow H_1 \rightarrow H_0 \rightarrow \dots$$

resulting in an infinite loop, and again the functionals produced by Magri's theorem are not independent. (At first glance, this result does not seem to reconcile with the gas dynamics version under the transformation (2.14). However, we note that since the recursion operator involves the inverse of the Hamiltonian operator $\mathcal{D}^*$, we can add in any element of its kernel at each step. Thus, the explanation is that we have just chosen different elements of $\ker \mathcal{D}_2$ to add in.)

All three versions of the Born-Infeld equation are examples of *separable systems*^{8,9}, meaning that the Hamiltonian density H in the representation (2.2) satisfies

$$\frac{H_{uu}}{H_{vv}} = \frac{\lambda(u)}{\mu(v)}. \quad (4.2)$$

For the three versions of Born-Infeld, the separation coefficients are given by

Gas dynamics	(2.12)	$\lambda(u) = 1,$	$\mu(v) = v^{-4},$	
Null version	(2.8)	$\hat{\lambda}(z) = z^{-4},$	$\hat{\mu}(w) = w^{-4},$	(4.3)
Physical version	(2.5)	$\tilde{\lambda}(r) = (1 + r^2)^{-2},$	$\tilde{\mu}(s) = (1 + s^2)^{-2},$	

It is standard that the zeroth order conserved densities for such a system can be found by solving a separable linear wave equation.

Proposition 5. A function $F(u,v)$ is a conserved density of a separable Hamiltonian system (2.2), (4.2), if and only if it is a solution to the linear wave equation

$$\frac{F_{uu}}{\lambda(u)} = \frac{F_{vv}}{\mu(v)}. \quad (4.4)$$

Any Hamiltonian system (2.2) admits the conserved densities $1, u, v, uv$. In the separable case, there are four fundamental hierarchies of solutions to the wave equation (4.4), each of the form

$$H_n(u, v) = \sum_{i=0}^n F_i(u) \cdot G_{n-i}(v), \quad (4.5)$$

where the functions F_i and G_i are generated by the recursion relations

$$\frac{\partial^2 F_i}{\partial u^2} = \lambda(u) F_{i-1}, \quad F_i(0) = F'_i(0) = 0,$$

$$\frac{\partial^2 G_i}{\partial v^2} = \mu(v) G_{i-1}, \quad G_i(0) = G'_i(0) = 0.$$

The hierarchies depend on the initial selection of $H_0 = F_0 \cdot G_0$:

$$\begin{aligned} H_0^{(1)} &= 1, & F_0^{(1)} &= G_0^{(1)} = 1, \\ H_0^{(2)} &= u, & F_0^{(2)} &= u, \quad G_0^{(2)} = 1, \\ H_0^{(3)} &= v, & F_0^{(3)} &= 1, \quad G_0^{(3)} = v, \\ H_0^{(4)} &= uv, & F_0^{(4)} &= u, \quad G_0^{(4)} = v. \end{aligned}$$

Our transformations do not respect this hierarchical structure of the conserved densities. For instance, (2.11) maps the first and fourth null Born-Infeld hierarchies to combinations of all four gas dynamics hierarchies, so that, up to multiple,

$$\begin{aligned} \hat{H}_{2j}^{(1)} &\rightarrow H_j^{(1)}, & \hat{H}_{2j}^{(4)} &\rightarrow H_j^{(3)}, \\ \hat{H}_{2j+1}^{(1)} &\rightarrow H_j^{(2)}, & \hat{H}_{2j+1}^{(4)} &\rightarrow H_j^{(4)}. \end{aligned}$$

On the other hand, the second and third hierarchies get mapped to algebraic conserved densities for the gas dynamics version (2.12). For example, the conserved density $\hat{H}_0^{(2)} = z$, is mapped to the conserved density

$$\sqrt{v - uv^2} + \sqrt{-v - uv^2},$$

which does not show up in any of the standard gas dynamics hierarchies. The hierarchies

in the physical r, s variables are no longer rational functions, and we shall not write them down explicitly. They do not correspond to any of the hierarchies in the other variables (with isolated exceptions), and provide yet other non-polynomial conserved densities for the gas dynamics system (2.12).

5. Higher order Hamiltonian structures

In Olver and Nutku⁹ it was shown how any separable Hamiltonian system has a second Hamiltonian structure involving a complicated third order matrix differential operator. The resulting recursion operator recovers results on symmetries and conservation laws due to Sheftel⁸. For the Born-Infeld equation, each of the versions is separable, and so we are lead to three distinct third order Hamiltonian structures. This is probably quite special to these particular systems, but we have no proof of this fact. In particular, it would be interesting to see whether any of the polytropic gas dynamics systems have additional Hamiltonian structures.

Theorem 6. Consider a separable Hamiltonian system (2.2), where the Hamiltonian density satisfies (4.2). Define the matrix variables

$$U_x = \begin{pmatrix} u_x & \mu(v)v_x \\ v_x & \lambda(u)u_x \end{pmatrix} \quad \text{and} \quad V_x = \begin{pmatrix} \lambda(u)u_x & \mu(v)v_x \\ v_x & u_x \end{pmatrix}. \quad (5.1)$$

Then the system can be written in biHamiltonian form

$$u_t = \mathcal{D}^* E_u(H) = \mathcal{E} E_u(H^*), \quad (5.2)$$

using the third order matrix differential operator

$$\mathcal{E} = D_x \cdot V_x^{-1} \cdot D_x \cdot U_x^{-1} \cdot \sigma_1 \cdot D_x = D_x \cdot V_x^{-1} \cdot D_x \cdot \sigma_1 \cdot V_x^T \cdot D_x. \quad (5.3)$$

In particular, $\mathcal{E}$ is Hamiltonian, and compatible with $\mathcal{D}^*$.

In the case of gas dynamics the matrix variables coincide:

$$U_x = V_x = \begin{pmatrix} u_x & v^{-4} v_x \\ v_x & u_x \end{pmatrix},$$

and the corresponding Hamiltonian operator (5.3) is

$$\mathcal{E}_0 = D_x \cdot U_x^{-1} \cdot D_x \cdot U_x^{-1} \cdot \sigma_1 \cdot D_x \quad (5.4)$$

is Hamiltonian, and compatible with $\mathcal{D}_0 = \mathcal{D}^*$. The second Hamiltonian in (5.2) turns out to be

$$H^* = H_2^{(3)} = \frac{u^4 v}{24} + \frac{u^2}{2v} + \frac{1}{24v^3}$$

which appears in the second hierarchy (4.5) of conserved densities. The corresponding recursion operator is the square of the simple recursion operator

$$\mathcal{R} = D_x \cdot U_x^{-1} \cdot \quad (5.5)$$

so that

$$\mathcal{E}_0 \cdot \mathcal{D}_0^{-1} = D_x \cdot U_x^{-1} \cdot D_x \cdot U_x^{-1} = \mathcal{R}^2.$$

Similarly, we have a third order recursion operator in the null variables (z, w) . We define the matrix variables

$$Z_x = \begin{pmatrix} z_x & w^{-4} w_x \\ w_x & z^{-4} z_x \end{pmatrix} \quad \text{and} \quad W_x = \begin{pmatrix} z^{-4} z_x & w^{-4} w_x \\ w_x & z_x \end{pmatrix},$$

and the operator is

$$\hat{\mathcal{E}}_1 = D_x \cdot W_x^{-1} \cdot D_x \cdot Z_x^{-1} \cdot \sigma_1 \cdot D_x = D_x \cdot W_x^{-1} \cdot D_x \cdot \sigma_1 \cdot W_x^{-T} \cdot D_x.$$

is Hamiltonian. Moreover, the Hamiltonian operators $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{D}}_1 = -2 \mathcal{D}^*$ are compatible, and therefore form a Hamiltonian pair. The null Born-Infeld equation (2.8) can be written as a biHamiltonian system

$$z_t = \hat{\mathcal{D}}_1 E_z[\hat{H}] = \hat{\mathcal{E}}_1 E_z[\hat{H}^*],$$

where the Hamiltonian is a multiple of the Hamiltonian $\hat{H}_4^{(2)}$ in the hierarchies (4.5):

$$\hat{H}^*(z, w) = 2 \hat{H}_4^{(2)}(z, w) = \frac{w}{12z^3} + \frac{1}{2zw} + \frac{z}{12w^3}.$$

Note that the transformation (2.11) cannot map these two higher order Hamiltonian operators to each other, since the corresponding biHamiltonian structures do not match up, nor are the compatibility relations preserved. Indeed, a long calculation proves that the gas dynamics recursion operator coming from the Born-Infeld recursion operator (5.3) under the transformation (2.12) is the operator

$$\hat{\mathcal{R}}_1 = \hat{\mathcal{E}}_1 \hat{\mathcal{D}}_1^{-1} \rightarrow -2 \mathcal{R}_1 \mathcal{R}^2,$$

where $\mathcal{R}$ is the gas dynamics recursion operator given by (5.5), and $\mathcal{R}_1$ is the recursion operator (4.1) coming from Nutku's Hamiltonian structures for gas dynamics. Therefore, the operator

$$\mathcal{E}_1 = -2 \mathcal{R}_1 \mathcal{E}_0 \mathcal{R}_1$$

is another third order Hamiltonian operator for the equations (2.12), which is compatible with the first order Hamiltonian operator $\mathcal{D}_1$, but not with either $\mathcal{D}_0$ or $\mathcal{D}_2$.

Finally, there is yet another third order Hamiltonian operator coming from the physical version of Born-Infeld. The operator takes the form

$$\tilde{\mathcal{E}}_2 = D_x \cdot S_x^{-1} \cdot D_x \cdot R_x^{-1} \cdot \sigma_1 \cdot D_x = D_x \cdot S_x^{-1} \cdot D_x \cdot \sigma_1 \cdot S_x^{-T} \cdot D_x,$$

where

$$R_x = \begin{pmatrix} r_x & (1+s^2)^{-2} s_x \\ s_x & (1+r^2)^{-2} r_x \end{pmatrix} \quad \text{and} \quad S_x = \begin{pmatrix} (1+r^2)^{-2} r_x & (1+s^2)^{-2} s_x \\ s_x & r_x \end{pmatrix}.$$

This Hamiltonian operator is compatible with $\tilde{\mathcal{D}}_2 = \mathcal{D}^*$, and so, when transformed back to the other coordinate systems, provides yet another Hamiltonian structure for the Born-Infeld equation.

In summary, then, we have found that the Born-Infeld equation, in any of its evolutionary forms (2.5), (2.8) or (2.12), possesses six distinct Hamiltonian structures: three of them are first order, given by the operators $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$, and three are third order, given by the operators $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$. Whether there are yet more Hamiltonian structures, not trivially related to these, remains an open question!

6. Diagonalization and Higher Order Conservation Laws.

As shown by Olver and Nutku⁹, for a generalized gas dynamics Hamiltonian system, there is an additional hierarchy of higher order conservation laws generalizing Verosky's rational first order conserved density¹⁷,

$$\hat{H}_1[u, v] = \frac{v_x}{u_x^2 - v^{\gamma-3} v_x^2}. \quad (6.1)$$

The case of a Chaplygin gas, $\gamma = -1$, is distinguished in that it admits an infinite collection of distinct first order conserved densities (i.e. they do not differ by a divergence). The easiest way to see this is to apply a diagonalization technique due to Verosky¹⁰ and Tsarev¹⁸.

Definition 8. A first order quasi-linear system is said to be in *diagonal form* if it has the form

$$p_t = A(p, q) p_x, \quad q_t = B(p, q) q_x. \quad (6.2)$$

We remark that the existence of a diagonal form for a quasi-linear first order system is related to the existence of Riemann invariants¹⁸.

Proposition 9. For the Chaplygin system (2.12), the transformation

$$p = u + \frac{1}{v}, \quad q = u - \frac{1}{v}, \quad (6.3)$$

places it in the diagonal form¹⁹

$$p_t = -q p_x, \quad q_t = -p q_x. \quad (6.4)$$

Theorem 10.¹⁰ A two-dimensional diagonal quasi-linear system (6.2) has a first order conservation law $D_t T + D_x X = 0$, with conserved density and flux of the form

$$T = \frac{F(p, q)}{p_x} + \frac{G(p, q)}{q_x}, \quad X = \frac{A F}{p_x} + \frac{B G}{q_x}. \quad (6.5)$$

if and only if F and G satisfy the system of differential equations

$$(A - B) G_p = 2 G B_p, \quad (B - A) F_q = 2 F A_q, \quad F A_p + G B_q = 0. \quad (6.6)$$

For the special case $A = -q$, $B = -p$, corresponding to the Born-Infeld equation, the third equation in (6.6) is vacuous, and so there are solutions

$$F(p,q) = \frac{\alpha(p)}{(p-q)^2}, \quad G(p,q) = \frac{\beta(q)}{(p-q)^2}, \quad (6.7)$$

depending on arbitrary functions $\alpha(p)$, $\beta(q)$. There are similar expressions for the other gas dynamics systems with $\gamma \neq -1$, but there the third equation in (6.4) is not vacuous, and this restricts the corresponding functions to satisfy $\alpha = -\beta$, and so both coefficients must be constant! Thus the Born-Infeld case is very special.

In terms of the gas dynamics variables, the conserved densities have the form

$$T[u,v] = \frac{v^4 \alpha(u + v^{-1})}{v^2 u_x - v_x} + \frac{v^4 \beta(u - v^{-1})}{v^2 u_x + v_x}.$$

Note that the case $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$ reproduces the conserved density (6.1) when $\gamma = -1$. Under the transformation (2.11), these go over into the following conserved densities for the Born-Infeld equation:

$$T[z, w] = \frac{z^4 w^4 \tilde{\alpha}(z^{-1} - w^{-1})}{w^2 z_x - z^2 w_x} + \frac{z^4 w^4 \tilde{\beta}(z^{-1} + w^{-1})}{w^2 z_x + z^2 w_x},$$

where

$$\tilde{\alpha}(s) = \frac{\alpha(-s^2)}{8s}, \quad \tilde{\beta}(s) = \frac{\beta(-s^2)}{8s}.$$

For the particular choices $\tilde{\alpha}(s) = 1$, $\tilde{\beta}(s) = \pm 1$, i.e. $\alpha(s) = 8\sqrt{-s}$, $\beta(s) = 8\sqrt{-s}$, we obtain the conserved densities

$$\frac{z^6 w^6 w_x}{w^4 z_x^2 - z^4 w_x^2}, \quad \frac{z^6 w^6 z_x}{w^4 z_x^2 - z^4 w_x^2},$$

which are more like the first order densities discovered in Verosky¹⁷; see also Olver and Nutku⁹. It is interesting that the transformation (2.12) does *not* map the Verosky-type densities to each other.

It can be shown that the third order evolution equations corresponding to these two

densities are each biHamiltonian systems, and hence the recursion operators lead to two further hierarchies of higher order conserved densities.

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