INTEGRABLE AND CHAOTIC POLARIZATION DYNAMICS IN NONLINEAR OPTICAL BEAMS

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Abstract

The problem of two counterpropagating optical laser beams in a nonlinear medium is investigated as a Hamiltonian system. The phase space for travelling-wave solutions is the manifold $\mathbb{C}^2 \times \mathbb{C}^2$, coordinatized by two complex two-component electric field amplitudes, one for each beam. Invariance of the Hamiltonian function under various actions of the rotation group allows for reduction of the phase space to the two-sphere S^2, on which the reduced Hamiltonian system, being two-dimensional, is completely integrable. We determine all the fixed points of the system and describe the bifurcations of the phase portrait which occur as parameters are varied. Among the various orbits in the reduced phase space, those connecting the hyperbolic fixed points are special and correspond to soliton-like and kink-like travelling-wave solutions. We also investigate how chaos, in particular Horseshoe chaos and Arnold diffusion, arises when the system is subjected to certain types of pertubations.

1. Reduction of the problem to the sphere.

We examine two counterpropagating optical laser beams in a lossless, nonlinear, Kerr like, parity invariant homogeneous medium with small nonlinearities, e.g., laser beams in an optical fiber, characterized by a fourth-rank susceptibility tensor $\chi^{(3)}$. Such a configuration can be described in terms of a quasi-monochromatic electric field is given by

$$\mathbf{E}(z, t) = \left[e(z, t)e^{ik_0z} + \bar{e}(z, t)e^{-ik_0z} \right] e^{-i\omega_0 t} + \text{c.c.} \tag{1.1}$$

where the "bar" distinguishes between the electric fields associated with each of the two counterpropagating beams; the two e's are complex, two-component vectors in $\mathbb{C}^2$. Inserting the above prescription (1.1) into Maxwell's equations leads via standard assumptions [1] to the following set of equations on $\mathbb{C}^2 \times \mathbb{C}^2$ (the r's are the preserved magnitudes of the e's, the kappas are intensity parameters, and τ is a scaled space-time characteristic variable):

$$\frac{\partial e_i}{\partial \tau} = \frac{i\bar{\kappa}}{\bar{r}} \chi^{(3)}_{ijkl}[e_j e_k e_l^* + 2e_j \bar{e}_k \bar{e}_l^*], \qquad \frac{\partial \bar{e}_i}{\partial \tau} = \frac{i\kappa}{r} \chi^{(3)}_{ijkl}[\bar{e}_j \bar{e}_k \bar{e}_l^* + 2\bar{e}_j e_k e_l^*], \tag{1.2}$$

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where i,j,k,l = 1,2 and $\chi^{(3)}{}_{ijkl} = \chi^{(3)*}{}_{jilk}$ for lossless media. These equations describe travelling-wave solutions, to which we will restrict ourselves here; the analysis of the more general case when (1.2) assumes the form of a pair of *partial* differential equations, thereby defining an infinite-dimensional dynamical system, will be presented elsewhere. Furthermore, this system is Hamiltonian on the symplectic manifold $\mathbb{C}^2 \times \mathbb{C}^2$; indeed, the Hamiltonian function and the Poisson bracket are defined by

$$H = \tfrac{1}{2}\chi^{(3)}_{ijkl}[e_i{}^*e_je_ke_l{}^* + \bar{e}_i{}^*\bar{e}_j\bar{e}_k\bar{e}_l{}^* + 4e_i{}^*e_j\bar{e}_k\bar{e}_l{}^*],$$
$$\{F, G\} = \frac{i\bar{\kappa}}{\bar{r}}\left[\frac{\partial G}{\partial e^*}\frac{\partial F}{\partial e} - \frac{\partial F}{\partial e^*}\frac{\partial G}{\partial e}\right] + \frac{i\kappa}{r}\left[\frac{\partial G}{\partial \bar{e}^*}\frac{\partial F}{\partial \bar{e}} - \frac{\partial F}{\partial \bar{e}^*}\frac{\partial G}{\partial \bar{e}}\right], \tag{1.3}$$

and, as usual, the evolution of a dynamical quantity f is given by $df/d\tau = \{f, H\}$.

A remarkable feature of this Hamiltonian is that it is invariant under a diagonal action of the group U(1): this permits us to apply the Marsden-Weinstein reduction procedure for Hamiltonian systems with symmetry [2,3] and show that a cascade of Lie-Poisson maps reduces the system to the 2-sphere S^2.

Theorem 1. *For isotropic media, the Hamiltonian system (1.2) is reducible to a 2-dimensional system on S^2 with one additional quadrature.*

We give only a sketch of the proof and make some comments; for details see ref. [4]. The first step of the reduction consists in restricting the initial phase space to $S^3 \times S^3$; this is done by rewriting the system in terms of bilinear forms in the electric field amplitudes,

$$(e, \bar{e}) \longrightarrow (u = e^\dagger \vec{\sigma} e,\ \bar{u} = \bar{e}^\dagger \vec{\sigma} \bar{e}), \quad W = \vec{\sigma}_{ij}\chi^{(3)}_{ijkl}\vec{\sigma}_{kl} = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3). \tag{1.4}$$

This map is sometimes refered to as the Stokes map. It proves convenient to introduce the parameters λ_i; here, we will consider isotropic media for which $\lambda_3 = \lambda_1$ and we also mention that there always exists a choice of coordinate frame such that the tensor W is diagonal as in (1.4). The next step consists in noting that the norms of the u's (the r's) are Casimir (or distinguished) functions for the reduced Lie-Poisson bracket in these variables; they are in fact invariant under the action of the group SO(2). This invariance induces a Hopf map which brings down the phase space to a product of two so-called Poincaré spheres: $S^3 \times S^3 \to S^2 \times S^2$, which may be coordinatized by spherical angles:

$$(u, \bar{u}) \longrightarrow (\theta, \phi, \bar{\theta}, \bar{\phi}). \tag{1.5}$$

The Poincaré sphere provides a very convenient way of describing the polarization states of a beam; points on the poles represent the two opposite circularly polarized states, those on the equator correspond to linearly polarized states, and all other points describe elliptically polarized states [4]. Note that going once around the Poincaré sphere, parallel to the equatorial plane, yields a phase advance by π of the original polarization state. Going twice around the Poincaré sphere along a latitude however restores an initial state; this is a remnant of the spinorial quality of the two-component complex electric field amplitudes. Furthermore, the Stokes map eliminates an inessential (non-measurable) absolute phase in the complex amplitude description. The change of variables $\mathbf{C}^2 \times \mathbf{C}^2 \to S^2 \times S^2$ is analogous, in the study of the motion of a rigid body, to the reduction from the Cayley-Klein parameters to the body angular momentum variables.

When $\lambda_3 = \lambda_1$ is assumed, a further SO(2) symmetry exists, leading to the conservation of the quantity σ defined below which is physically interpretable as the net angular momentum (per unit length) of the beams in the direction of propagation. This symmetry implies a further reduction of $S^2 \times S^2$ to S^2, coordinatized by angles ψ and α which are defined through the following formulae:

$$\alpha = \phi - \bar{\phi}, \qquad \beta = \phi + \bar{\phi}, \qquad \sigma = \kappa\cos\theta + \bar{\kappa}\cos\bar{\theta},$$
$$\omega(\psi) = \kappa\cos\theta - \bar{\kappa}\cos\bar{\theta} = \omega_0 + R\cos\psi. \tag{1.6a}$$

In the definition of ω, the values of the constants ω_0 and R depend upon the magnitude of σ as follows:

$$\sigma \geq |\bar{\kappa}| - |\kappa|: \qquad \omega_0 = |\kappa| - |\bar{\kappa}|, \qquad R = |\kappa| + |\bar{\kappa}| - \sigma;$$
$$|\kappa| - |\bar{\kappa}| \leq \sigma \leq |\bar{\kappa}| - |\kappa|: \qquad \omega_0 = -\sigma, \qquad R = 2|\kappa|; \tag{1.6b}$$
$$\sigma \leq |\kappa| - |\bar{\kappa}|: \qquad \omega_0 = |\bar{\kappa}| - |\kappa|, \qquad R = |\kappa| + |\bar{\kappa}| + \sigma.$$

These three choices ensure reduction to a smooth manifold, generically. A special situation arises, however, for which the reduced space will be singular, namely when the σ is equal to the difference of the magnitudes of the κ's; in that case, one of the poles, say P, will be a singular point and the reduced phase space may be viewed as $S^2 \setminus P$. In addition, if $\sigma = 0$, then both poles (N and S) are singular and the phase space can be viewed as $S^2 \setminus \{N, S\}$. That causes no difficulty for analyzing the motion; in fact, interesting dynamical bifurcations take place in these singular cases, as we will see below. On S^2, the Hamiltonian function, the induced Lie-Poisson bracket, and the equations of motion are

$$H = \tfrac{1}{2}\lambda_1\left[r^2 + \bar{r}^2 + (r\bar{r}/\kappa\bar{\kappa})[\Gamma\omega^2 + \Delta\sigma^2 + 2E\sigma\omega + f(\psi)\bar{f}(\psi)\cos\alpha\right],$$
$$\{F, G\} = \frac{2\kappa\bar{\kappa}}{r\bar{r}R\sin\psi}\left[\frac{\partial F}{\partial\psi}\frac{\partial G}{\partial\alpha} - \frac{\partial G}{\partial\psi}\frac{\partial F}{\partial\alpha} + 2R\sin\psi\left\{\frac{\partial F}{\partial\beta}\frac{\partial G}{\partial\sigma} - \frac{\partial G}{\partial\beta}\frac{\partial F}{\partial\sigma}\right\}\right],$$
$$\partial\psi/\partial\tau = -\lambda_1 f(\psi)\bar{f}(\psi)\sin\alpha/R\sin\psi, \tag{1.7}$$
$$\partial\alpha/\partial\tau = 2\lambda_1(\Gamma\omega + E\sigma) + \lambda_1[(\sigma - \omega)f(\psi)/\bar{f}(\psi) - (\sigma + \omega)\bar{f}(\psi)/f(\psi)]\cos\alpha,$$
$$\partial\beta/\partial\tau = 2\lambda_1(\Delta\sigma + E\omega) - \lambda_1[(\sigma - \omega)f(\psi)/\bar{f}(\psi) + (\sigma + \omega)\bar{f}(\psi)/f(\psi)]\cos\alpha,$$

where ω is given in (1.6b) and

$$\Gamma = \tfrac{1}{4}(L-1)\rho_+ - L, \qquad \Delta = \tfrac{1}{4}(L-1)\rho_+ + L, \qquad E = \tfrac{1}{4}(L-1)\rho_-, \qquad L = \lambda_2/\lambda_1,$$

$$\rho_\pm = (\bar{r}\bar{\kappa}/r\kappa \pm \bar{r}\kappa/r\bar{\kappa}), \qquad f(\psi) = \sqrt{4\kappa^2 - (\sigma+\omega)^2}, \qquad \bar{f}(\psi) = \sqrt{4\bar{\kappa}^2 - (\sigma-\omega)^2}.$$

Note that the right-hand side of the equation for β depends only on the variables ψ and α. This is why the above system can be considered as a system on the sphere for the latter variables (with Lie-Poisson bracket given by the first two terms of that given above); β can be recovered from the solutions of this system by performing a quadrature. Since the reduced system is defined on a two-dimensional manifold and derivable from a Hamiltonian function, there immediately follows the

Corollary 1. *The system (1.2) is completely integrable.*

We now formulate a theorem to the effect that the above reduction procedure causes no loss of information, in the sense that the (travelling-wave) solutions of our initial system (1.2) can be recovered up to an overall phase determined from the initial conditions by using the solutions of (1.7). In other words, once the Cauchy data has been specified, the system is forced to evolve on a two-dimensional sphere, and the reduction procedure (unlike a projection) preserves the information needed to reconstruct the solution in the original phase space.

Theorem 2. *The solution manifold of the Hamiltonian system (1.2) is completely determined by that of the reduced Hamiltonian system on S^2.*

This statement may be proved by going backwards through the proof of Theorem 1. First, by integrating for β, and inverting (1.5) we construct the solution to the system on the product $S^2 \times S^2$. Then, inverting (1.5) permits us to find the solutions defined on $S^3 \times S^3$. Next, we invert the transformation (1.4) and use the fact that the preserved intensities of the beams define an immersion into the initial phase space $\mathbb{C}^2 \times \mathbb{C}^2$; this defines the solution of the initial system, up to a phase, for each field. Finally, these phases are reconstructed by substituting in (1.2): this yields equations for these phases, which we integrate to get their space-time profiles.

2. *Fixed points, bifurcations, and special solutions.*

The reduced system (1.7) for ψ and α on S^2 exhibits several interesting bifurcation sequences. To appreciate these, one begins by determining the set of fixed points of the system, as well as their existence conditions (or definition domains in the parameter space) and stability conditions; these fixed points are physically interpretable as steady state solutions of the system. Bifurcations are then observed to occur upon crossing certain critical hypersurfaces in the parameter space where fixed points are created or destroyed, or where their stability changes. For our problem, the parameter space is of dimension five (for instance, L, σ, the κ's, and the ratio of the intensities are sufficient to span the parameter space) and therefore one might expect rather complicated bifurcation sequences. We present here a few special cases which, although they do not provide a complete picture (see ref. [4] for a more detailed description of the various bifurcation sequences of both the one-beam and the two-beam problem), are the most interesting as far as dynamics is concerned. The generic phase portrait, i.e., that which occupies the major portion of the parameter space, corresponds to two stable fixed points, opposite to each other, migrating on the great circle described by $\sin\psi = 0$; thus all solutions of the system are periodic in the generic case. A more structured phase portrait is obtained, e.g., either when $\sigma = 0$ and the κ's have different magnitudes, or when the κ's are equal but $\sigma \neq 0$; this situation is depicted in Figure 1, below. When $|\Gamma|$ is sufficiently large, the portrait consists of a center as well as a triplet formed by a saddle point with two centers enclosed within a pair of homoclinic loops. These loops collapse as the parameter Γ passes through some critical values $\Gamma_{\pm}$, at which point the portrait undergoes a pitchfork bifurcation, resulting in two stable centers on the equator with a family of periodic orbits.

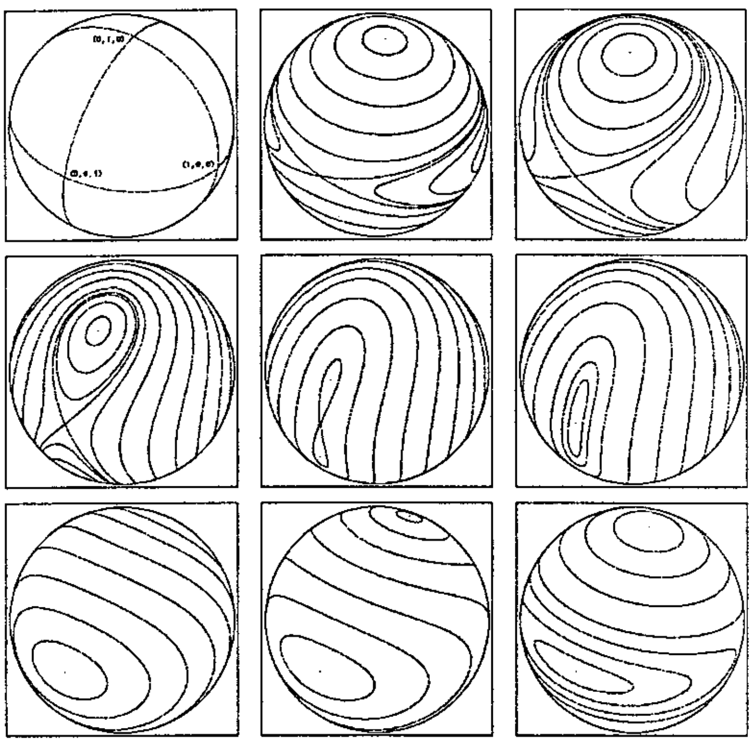

Figure 1. The phase portrait and its pitchfork bifurcations for $\sigma = 0$ and $|\kappa|/|\kappa| \neq 1$, or $\sigma \neq 0$ and $|\kappa|/|\kappa| = 1$.

The second case of interest occurs for the case of a mirror symmetry between the two beams, when the kappas have equal magnitudes and $\sigma = 0$. This symmetric situation gives rise to what we term the *Butterfly* bifurcation, which is presented pictorially below in Figure 2. The pictures show how the phase space portrait changes as the parameter Γ is varied. Two asymptotic bifurcations take place when $|\Gamma| \to \infty$, as the poles which are stable fixed points (centers) have homoclinic orbits encircling them which merge to form a circle of unstable fixed points on the equator. Two other bifurcations occcur at $|\Gamma| = 1$ when the two homoclinic orbits collapse to a half great-circle of fixed points which then opens up into a pair of heteroclinic orbits connecting unstable (saddle) fixed points at the poles while the saddle point on the equator changes into a center. This case may be obtained from the preceding case by taking an appropriate limit. In this limit, the two centers within the homoclinic loops are located exactly at the poles independently of the value of Γ; the poles are therefore always fixed points so that the pitchfork bifurcation cannot take place.

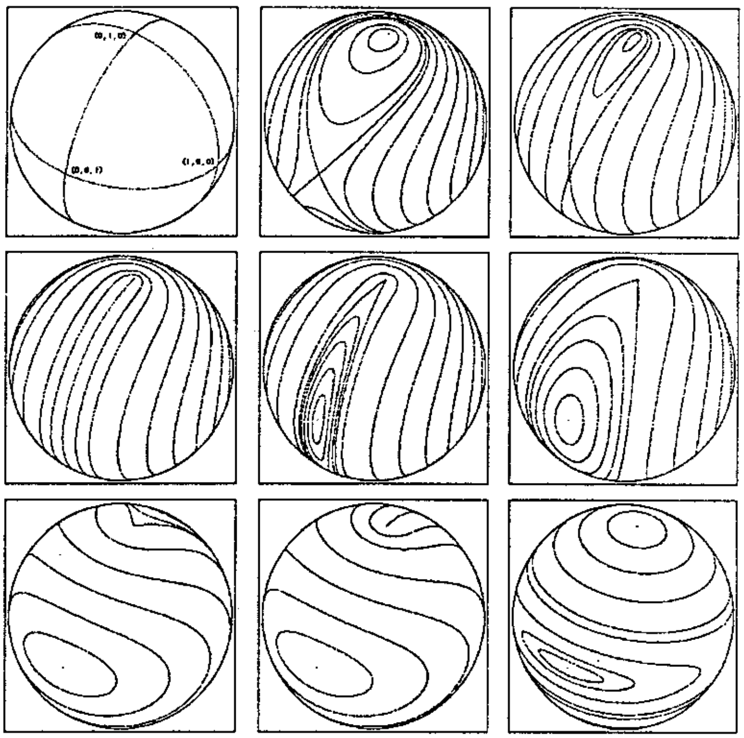

Figure 2. The *Butterfly* bifurcation happens when $\sigma = 0$ and $|\bar{\kappa}|/|\kappa| = 1$. It is degenerate and both poles are singular points on S^2.

The third case that we are presenting here is for σ equal to the *difference* of the magnitudes of the kappas. The previous case is the limit of this one when both kappas have the same magnitude, which forces σ to vanish. In this third case, only one of the poles is a singular point of the reduced phase sphere. In Figure 3, below, the parameters are such that the singular point is the north pole. The phase portrait is characterized by the occurence of a bifurcation which we have termed the *Teardrop* bifurcation. As the pictures show, when $|\Gamma|$ is sufficiently large, the phase portrait consists only of periodic orbits. The bifurcation occurs when $|\Gamma|$ falls below a certain critical value. As this value is attained, the north pole develops a singularity and a single homoclinic loop is created, encircling the top fixed point of center type and meeting with finite angle at the singularity; note that this contrasts with the usual phenomenon, for smooth manifolds, where homoclinic loops come in pairs (one also observes that the Euler index jumps from 2 to 1, which is another indication that the reduced phase space possesses a singular point). This loop then proceeds to stretch, migrates under the sphere and eventually contracts back to a single point where the bifurcation undoes itself as the singularity at the north pole disappears and yields, once again, a single family of periodic orbits.

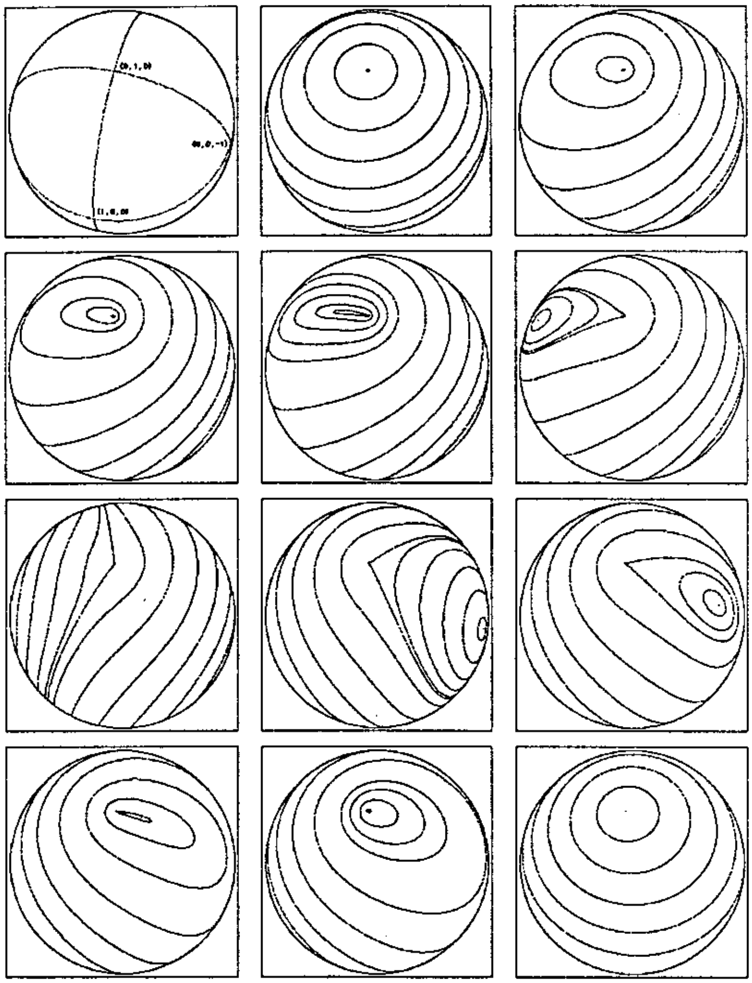

Figure 3. The *Teardrop* bifurcation occurs when $\sigma = |\kappa| - |\bar{\kappa}|$. A single homoclinic loop is connected to a singular point at the pole.

We now give a few examples of special travelling-wave solutions. These solutions are best vizualized on the Poincaré sphere. Thus, we integrate the system on the relevant homoclinic or heteroclinic orbits in S^2 and then lift back to $S^3 \times S^3$. We illustrate the **u**-part of the solutions on the pictures in Figure 4, below (recall that **u** takes its values on a S^2 surface embedded in S^3 because its magnitude is preserved), where the curves represent the time-trajectory of the polarization vector. Consider first the heteroclinic orbits between the north and south poles in figure 2, above. As shown on the left-hand picture in Figure 4, as $t \to -\infty$, the beam is circularly polarized. As time increases, the solution curve spirals out of the north pole (its latitude decreases) and thus the beam is characterized by an elliptical polarization. At $t = 0$, the curve crosses the equator (the beam is then linearly polarized) and subsequently proceeds to spiral again, in a symmetrical manner, towards the south pole, which it reaches asymptotically as $t \to \infty$, so that circular polarization is recovered, but in the opposite sense. This solution is reminiscent of a kink. Other types of special solutions can also be constructed. For instance, the homoclinic orbits in Figure 2 give rise to a continuous family of kinks with linearly polarized asymptotic states (middle picture below); soliton-like solutions arise, as special cases among these, when the asymptotic states coincide (right-hand picture below). Notice the existence of a winding index, taking any integer value, indicating the number of complete turns accomplished about the circular polarization state, so that each of these solutions is characterized by this index as well as its shift in azimuthal angle; soliton-like solutions are thus those solutions with null shift.

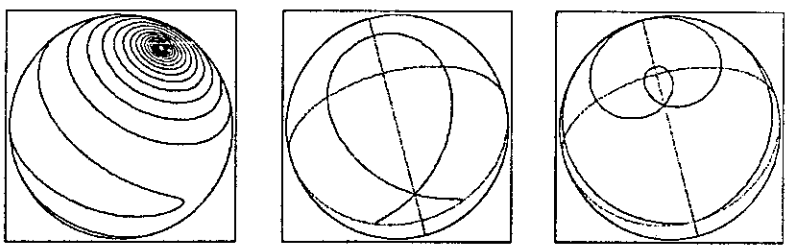

Figure 4. Some kink-like and soliton-like solutions **u** on the Poincaré sphere.

3. Generation of chaotic polarization.

Perturbing a completely integrable dynamical system which possesses either homoclinic or heteroclinic orbits may, when certain global properties of the dynamics are persistent, yield chaotic behavior. This is indeed the case for our system and the existence of complex dynamics may be analytically demonstrated for certain classes of perturbations. These perturbations are physically relevant in applied fields such as communications using fiber optics technology and polarization switching; for instance, perturbations may be created during the manufacturing of fibers, as when twists occur upon winding them onto spools. Specifically, we report the existence of *Smale horseshoe* chaos for periodic perturbations preserving the reduced phase space S^2, as well as for perturbations breaking the invariance of σ, implying that the phase space for the system can only be reduced to $S^2 \times S^2$. Other symmetry breaking perturbations imply the existence of *Arnold diffusion*. The method we use to demonstrate the existence of chaos is known as the *Melnikov* procedure. This method relies on showing that transverse intersection occurs between the stable and unstable manifolds of a given hyperbolic point; this is done by calculating a so-called Melnikov integral function, which is defined as a signed measure of the distance separating the stable and unstable manifolds of this fixed point. When that function possesses simple zeroes, in the two-dimensional case, the Poincaré-Birkhoff-Smale theorem then implies that the Poincaré map of first return possesses a horseshoe construction (in higher dimensions, Arnold diffusion is implied); the underlying mechanism for generating this type of chaos being that the Poincaré map induces both a stretching and folding of the phase points initially nearby the hyperbolic point. See refs [5, 6] for details about the Melnikov method and recent generalizations of it.

The types of perturbations we consider are characterized by (small) deformations of the matrix W (see formula [4]), corresponding to spatially periodic impurities in the nonlinear medium, namely:

$$W = \mathrm{diag}\{\lambda_1, \lambda_2 + \varepsilon\cos[\nu(\tau - \tau_0)], \lambda_1\}, \tag{3.1a}$$

$$W = \mathrm{diag}\{\lambda_1 + \varepsilon, \lambda_2, \lambda_1\} \tag{3.1b}$$

$$W = \mathrm{diag}\{\lambda_1 + \varepsilon\cos[\nu(\tau - \tau_0)], \lambda_2, \lambda_1\}, \tag{3.1c}$$

where ε is small. We examine the consequences of these perturbations on the dynamics of the system near the heteroclinic orbits appearing in the Butterfly bifurcation (see the middle row in Figure 2).

Perturbations of type (3.1a) preserve S^2 as the phase space of the full system. In addition, the Melnikov function is the usual one, i.e., it takes the form of the line integral of the Poisson bracket between the unperturbed (H^0) and perturbed (H^1) Hamiltonian functions ($H \equiv H^0 + \varepsilon H^1$) along the heteroclinic or homoclinic orbit:

$$M(\tau_0) = \int_{\mathbb{R}} \{H^0, H^1\}[\omega(\tau + \tau_0), \alpha(\tau + \tau_0)]d\tau, \tag{3.2a}$$

where H^0 is defined in (1.7) and H^1 is given by

$$H^1 = \tfrac{1}{8}\varepsilon[(r/\kappa)^2(\sigma + \omega)^2 + (\bar{r}/\bar{\kappa})^2(\sigma - \omega)^2 + 4(r\bar{r}/\kappa\bar{\kappa})(\sigma^2 - \omega^2)]\cos(\nu\tau). \tag{3.2b}$$

The Melnikov function integral is shown to be proportional to $\sin(\nu\tau_0)$; because of the very complicated form of the objects involved in the computations, we present a case for which the ratios of the r's and the κ's imply more simple expressions (recall that the heteroclinic orbits in the Butterfly bifurcation exist for $\sigma \equiv 0$):

$$M(\tau_0) = \frac{3rv^2\pi}{8\lambda_1^2\sin^2\alpha_0}\,\mathrm{csch}[v\pi/4\lambda_1 r\sin\alpha_0]\sin(v\tau_0), \qquad \bar{r}/r = 1 = -\bar{\kappa}/\kappa, \tag{3.2c}$$

where α_0 is defined by $\cos(\alpha_0) = -(1 + L)/2$. Existence of simple zeroes for $M(\tau)$ on S^2 yields horseshoe chaos. The type of physical behavior implied by the horseshoe is random, or intermittent, switching between two states of polarization (here the two circular states); this phenomenon is identifiable with binary symbolic shifts.

Perturbations of type (3.1b) lift the phase space of the system to $S^2 \times S^2$. For this second case, the form of the perturbed system falls within category III studied in ref. [6] and the Melnikov function can be written as

$$M(\beta_0) = -\int_{\mathbb{R}} \frac{\partial H^1}{\partial \beta}[\omega, \alpha, \sigma, \beta + \beta_0]dt, \tag{3.3a}$$

where H^1 is given by (the f's are the same as in (1.7)):

$$H^1 = \frac{1}{16}\{(\kappa f/r)^2[1 + \cos(\alpha + \beta)] + (\overline{\kappa f}/\bar{r})^2[1 + \cos(\alpha - \beta)] + (4r\bar{r}/\kappa\bar{\kappa})(\cos\alpha + \cos\beta)\}. \tag{3.3b}$$

This integral is easily shown to be proportional to $\sin(\beta_0)$; choosing the same special ratios for the r's and the κ's, we get (α_0 is as before)

$$M(\beta_0) = \frac{-r^2(2 - \cos\alpha_0)\sin(\beta_0)}{\lambda_1 \sin\alpha_0}. \tag{3.3c}$$

As in the previous case, existence of simple zeroes implies horseshoe chaos; the distinction being that, here, the geometry of the stable and unstable manifolds are toroidal objects embedded in $S^2 \times S^2$. For both of the above cases, the phase space is partitioned into stochastic dynamical layers separated by invariant tori, or KAM surfaces which form impenetrable barriers for the polarization: the polarization vectors wander within these tori.

Type (3.1c) perturbations, in contrast to the first two cases already considered, yield Arnold diffusion; the phase space of the perturbed system is the five-dimensional manifold $S^2 \times S^2 \times \mathbb{R}$ and it can no longer be partitioned into disconnected chaotic regions: the stochasticity domain forms what is called an Arnold's web and solutions diffuse among the invariant tori. The system again belongs to category III of ref. [6]; however the relevant Melnikov function is now a two-component vector function given by:

$$\begin{aligned} M_1(\tau_0, \beta_0) &= \int_{\mathbb{R}} \left[\frac{\partial H^0}{\partial \omega}\frac{\partial H^1}{\partial \alpha} - \frac{\partial H^0}{\partial \alpha}\frac{\partial H^1}{\partial \omega} - \frac{\partial H^0}{\partial \sigma}\frac{\partial H^1}{\partial \beta}\right]d\tau + \frac{\partial H^0}{\partial \sigma}\int_{\mathbb{R}} \frac{\partial H^1}{\partial \beta}\,d\tau, \\ M_2(\tau_0, \beta_0) &= -\int_{\mathbb{R}} \frac{\partial H^1}{\partial \beta}\,d\tau, \end{aligned} \tag{3.4a}$$

Integrating these (for the same ratios of thr r's and κ's), we find that M_1 and M_2 have the form

$$M_1(\tau_0, \beta_0) = \frac{3\nu^2 r\pi[L - \cos\alpha_0\cos\beta_0]}{16\lambda_1^2\sin^2\alpha_0}\operatorname{csch}[\nu\pi/4\lambda_1 r\sin\alpha_0]\sin(\nu\tau_0),$$

$$M_2(\tau_0, \beta_0) = \frac{-\nu\pi[1 - \frac{1}{2}\cos\alpha_0]\sin\beta_0}{4\lambda_1^2\sin^2\alpha_0}\operatorname{csch}[\nu\pi/4\lambda_1 r\sin\alpha_0]\cos(\nu\tau_0). \tag{3.4b}$$

Here, we have two families of simples zeroes; the existence of these is the necessary criterion for the occurence of Arnold diffusion. Physically, this diffusion means that polarization can be transferred back and forth among the nonlinear modes of the system in an erratic manner. More details can be found in ref. [4]; the determination of the form of the Melnikov function for various types of perturbed systems can be found in ref. [6].

5. *Conclusions.*

In this letter, we reported some new properties and aspects of polarizations dynamics resulting from the application of methods of Hamiltonian dynamics to the two-beam problem. In particular, we presented a few bifurcations with degeneracies and demonstrated analytically that chaos occurs, under certain classes of periodic perturbation, in the form of Smale horseshoes or Arnold diffusion. A detailed analysis of similar questions for the one-beam problem will be presented elsewhere.

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