QUANTUM HOLOGRAPHY AND NEUROCOMPUTER
ARCHITECTURES

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QUANTUM HOLOGRAPHY AND NEUROCOMPUTER ARCHITECTURES

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The digital computer is extremely effective at producing precise answers to well-defined questions. The nervous system accepts fuzzy, poorly conditioned input, performs a computation that is ill-defined, and produces approximate output. The systems are thus different in essential and fundamentally irreconcilable ways. Our struggles with digital computers have taught us much about how neural computation is not done; unfortunately, they have taught us relatively little about how it is done. Part of the reason for this failure is that a large proportion of neural computation is done in an analog rather than in a digital manner.

Carver A. Mead (1989)

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The actions of digital computers themselves depend vitally upon quantum effects - effects that are not, in my opinion, entirely free of difficulties inherent in the quantum theory. What is this 'vital' quantum dependence? In modern electronic computers, the existence of discrete states is needed (say, coding the digits 0 and 1), so that it becomes a clear-cut matter when the computer is in one of these states and when in another. This is the very essence of the 'digital' nature of computer operation. This discreteness depends ultimately on quantum mechanics.

Roger Penrose (1989)

1. Overview. The development of faster and more efficient computers in recent years has been driven by a seemingly unending thirst for communication, interaction, automation, control issues, information availability, and a yearning for new understanding of the self-organization principles of ourselves and our environment. The challenges of the future force to create and study new concepts of adaptive information processing and to implement for faster communication novel computer architectures based on fundamental quantum theoretical principles.

Until now the increased power has been driven largely by continued refinements to microelectronic fabrication techniques, such as electronic switches (miniaturized transistors) with higher switching speeds and associated integrated circuits (ICs) with increased levels of integration on silicon chips. Although the advancements in the IC hardwiring and packaging functions have been significant, their prospect for continuing at the same steady rates from very large scale integration (VLSI) to ultra large scale integration (ULSI) are being dimmed by physical limitations associated with further miniaturization. Limitations of electronics include:

- o electromagnetic interference at high speed,
- o distorted edge transitions,
- o complexity of metal connections,
- o drive requirements for pins,
- o large peak power levels,
- o impedance matching effects.

Electromagnetic interference arises because the inductances of two current carrying wires are coupled. Sharp edge transitions must be maintained for proper switching, but higher frequencies are attenuated greater than lower frequencies, resulting in sloppy edges at high speeds. The complexity of metal connections on

chips, on circuit boards, and between system components affects interconnection topology and introduces fields and unequal path lengths. This translates to signal skews that are overcome by slowing the system clock rate so that signals overlap sufficiently in time. Large peak power levels are needed to overcome residual capacitances, and impedance matching effects at connections require high currents which result in lower system speeds. Even if much smaller logic gates are produced by utilizing new techniques such as X-ray lithography yielding an order-of-magnitude reduction in feature size, the speed of the IC will be limited by the interconnection delays between transistors. Unlike transistor response times, these time lags are reluctant to scale down with size. As a result there is a factor 10^3 disparity between the speed of the fastest electronic switching components, presently transistors that can switch in 5 ps, and the clock rates of 5 ns of the fastest digital electronic computers.

To ensure further progress, it is prudent not to rely upon continued refinements to hardware and software implementations. In fact, computer architects are turning to the design of parallel processors to continue the drive toward faster and more powerful computers. The massively parallel organization principles which distinguish analog neural systems from the small scale interconnection architectures of special-purpose parallel electronic processors and even more from the von Neumann architecture of standard digital computer hardware are one of the main reasons for the largely emerging interest in neurocomputer science. Since presently some areas of microelectronics are approaching their natural physical limitations, it is necessary to examine other technologies that may offer denser, faster communication between chips or logic gate arrays or even provide alternatives to the gates themselves. If light could be used to transfer data between chips or gates, the interconnection-delay problems of electronics would be avoided and the communication would occur at the speed of light itself.

The system of linear interconnects by which nonlinear processing elements can share information among themselves is the most important component of any parallel computer. Just as photonics is becoming the technology of the future for telecommunication and machine-to-machine interconnects, it also penetrates computer hardware and affects communications within a single computer, especially for processor-to-processor interconnects and board-to-board interconnects of parallel computer architectures, and even more of neurocomputer architectures. Not only does coherent light emitted by a laser have a much higher information capacity than electrical hardwires, but due to their boson character optical photons do not interfere during free-space propagation. Thus optical beams can occupy the same region of space without mutual interaction, allowing data streams to pass through one another without crosstalk and quantum interference, hence allowing multiple signals to travel the same channel in parallel. Indeed, a good lens can image tens of thousands of fully resolved points from one plane to another, each of these parallel channels having a theoretical bandwidth far in excess of 1 THz. Thus a single lens could easily carry all the telephone conversations simultaneously going on in the world at any moment in time. In this way, holographic optical interconnect technology leads to a very high packing density, to a simplified connection complexity,

and to reduced drive requirements.

In summary of these arguments, optical technology includes the following advantages:

- o high connectivity through coherent imaging,
- o no physical contact for interconnects,
- o non-interference of free-space propagating signals,
- o high spatial and temporal bandwidth,
- o no feedback to the power source,
- o inherently low signal dispersion.

High bandwidth is achieved in space because of the non-interference of optical signals, and high bandwidth is achieved in time because propagating wavefronts do not mutually interact. There is no feedback to the power source as in electronics, so that there are no data dependent loads. Finally, inherently low signal dispersion means that the shape of a pulse as it leaves its source is virtually unchanged when it reaches its destination.

At the interchip level of the interconnection hierarchy, two types of holographic optical interconnects are available: free-space and guided-wave optical technology. In the free-space type, a large array of optical signal beams emitted from a light source is distributed by imaging it to a planar array of optical detectors using a holographic optical element (HOE) as a flat light diffracting device. This type of holographic optical interconnect is three-dimensional and provides flexible implementation of wiring schemes which are impossible to fabricate with conventional refractive/reflective optical technology. However, it requires space because the HOE must be located above the arrays in the optical module. In the guided-wave type, an optical signal is transmitted and distributed from a coherent light source to an array of optical detectors via a guided-wave optical medium such as optical fibers or integrated waveguides. This type of holographic optical interconnect is more compact and more mechanically stable than the free-space type. Its compactness is due to the planar layout of the optical elements and the use of small waveguides. Its stability is due to the fact that all optical elements are fixed on a common substrate. A hybrid integration procedure then enables integration of laser diodes and photodiodes at any position on the guided-wave circuit surface.

As an example of massively parallel extrinsic connections between hologram planes implemented by free-space optics, the information throughput in one cycle of the AT & T Bell Laboratories' looped digital optical pipeline processor is higher than that of all the hardwired telephone nets on the whole world together. The processor is based on a family of optical modules which creates by split-and-shift hologratings from a pair of laser beams the array of power supply beams to read chips containing large arrays of self-electro-optic effect devices (SEEDs) of 5 μm square. It operates at 1 million cycles per second, slower than most personal computers, but the most optimistic perspectives predict the implementation of an

all-optical processor operating at several hundred million cycles per second – faster than most supercomputers – within the next five years.

REMARK 1. The SEED processing elements form very-low-energy electro-optic modulators and optical logic gates based on multiple quantum well (MQW) structures that are fabricated by gallium arsenide GaAs-gallium aluminum arsenide GaAlAs technology utilizing molecular beam epitaxy (MBE). Large arrays of a family of SEEDs control the intensity of a beam of 850-nm-wavelength laser light passing through them by making use of the quantum confined Stark effect (QCSE). This quantum phenomenon causes an electrical voltage of a few volts applied normal to the plane of the quantum wells to decrease the material's ability to absorb light at 850 nm wavelength. Thus an electrical signal can be converted to an optical signal carried by a laser beam. In a symmetric SEED (S-SEED) the quantum-well material is grown inside the intrinsic region of a PIN photodiode structure, and two such diodes are connected in series with a dc bias voltage applied. If laser light is incident on one of the diodes, a photocurrent is generated, the other diode acts as an electrical load, and the voltage across the first diode drops. This voltage drop causes an increase in optical absorption via the QCSE, thus generating more carriers and increasing the photocurrent. Positive feedback ensues, and the S-SEED switches into a stable state in which the first diode has a maximum absorption and transmits only a little light, while the second diode has low adsorption and high transmission. If a higher light intensity is applied to the second diode, the S-SEED will flip into the opposite state with the first diode absorbing and the second diode transmitting. The logic state of the device depends only upon the ratio of the input beams, and the state of the device can be read out by a pair of equally powered beams without altering its condition, thus providing time-sequential gain. In this way the S-SEEDs can be used as fully cascadable optical differential logic devices with low switching energies (currently subpicojoule) and the potential speed of a billion operations per second. In fact, they are the first optical logic gates that are competitive with microelectronic processing elements in terms of switch energy and cascability. Arrays of thousands of highly uniform S-SEEDs contained in GaAs chips can be addressed by arrays of laser beams incident normal to the plane of the chip. The beams can set and read out the logic state of the gates and transfer the data to subsequent similar arrays using imaging optics ([81], [129], [130]).

Another example of massively parallel extrinsic connections between hologram planes is the guided-wave optical interconnection technology used in the field of amacronics. Amacronic structures are hybrid analog neural processors formed by layers of optics, electronics, and detector arrays organized in a parallel way similar to the amacrine clustered processing layers in front of the human retina. Like diurnal insects, amacronic sensor technology finally may be able to dynamically trade sensitivity for resolution.

All the holographic optical interconnect technology underlies the fundamental fact that in the quantized theory of the electromagnetic field the bosons (integral-spin particles) present in a beam of coherent light traveling in a well-defined direction are the photons. For light quanta, however, the quantum parallelism occurs

according to which different alternatives at the quantum level are allowed to coexist in quantum complex linear superposition. The key idea of quantum holography is to mathematically model the quantum parallelism by the Kirillov quantization. This procedure allows to identify in a first step the hologram plane with the three-dimensional Heisenberg nilpotent Lie group quotiented by its one-dimensional center, then to restrict in a second step the sesquilinear holographic transform

$$\psi(t')dt' \otimes \varphi(t)dt \mapsto H_1(\psi, \varphi; x, y).dx \wedge dy$$

to the holographic lattices which form two-dimensional pixel arrays inside the hologram plane, and finally to recognize in a third step the hologram plane as a neural plane of local neural networks.

Quantum or photon holography as a part of quantum optics or photonics is the procedure of mathematically modeling the quantum parallelism by the Kirillov quantization. It allows a unified approach to planar optical components of digital optical computers and analog amacronic processors. Based on the beam splitter quantum interference experiment as an elementary building unit, the quantum holographic approach is also applicable to the Soffer optical resonator and the optical processing of synthetic aperture radar (SAR) data which represent particularly important examples of optical neurocomputer architectures.

Utilizing a unified quantum holographic approach to artificial neural network models implemented with coherent optical, optoelectronic, or analog electronic neurocomputer architectures, the paper establishes a new identity for the matching polynomials of complete bichromatic graphs which implement the intrinsic connections between neurons of local networks located in the neural plane. In microoptics and nanotechnology, the quantum theoretical treatment of optical holography is imperative because it involves only small differences of energy and not because atoms coherently excited by short laser pulses may be as large as some transistors of microelectronic ICs and the pathways between them inside the hybrid VLSI neurochips of amacronics. Actually, quantum effects can occur over distances of several meters or even billion of light years for quasars.

Until recently optical computing was looked upon as an alternative technology for performing an old task. Now, a paradigm shift is coming about as a result of the realization that optical computers are fundamentally different from, and in many senses superior to, any electronic computer. Certain optical computers are the only available ones that are intrinsically quantum mechanical processors.

H. John Caulfield and Joseph Shamir (1990)

Not enough has been written about the philosophical problems involved in the application of mathematics, and particularly of group theory, to physics. On the one hand, mathematics is created to solve specific problems arising in physics, and, on the other hand, it provides the very language in which the laws of physics are formulated. One need only think of calculus or of Fourier analysis as examples of this dual relationship.

We are all familiar with the exploitation of symmetry in the solution of a mathematical problem. On the other hand, the very assertion of symmetry is often the most profound formulation of a physical law or the key step in the development of a new theory.

V. Guillemin and S. Sternberg (1984)

2. Introductory Comments. Real-time image analysis and processing, computer vision, automatic target recognition for intelligent robots, remote surveillance, autonomous navigation, sound localization, speech processing and understanding, smart sensors processing, and various other application areas of artificial intelligence (AI) need to process an immense amount of data with very high velocity. The computational power required exceeds by many orders of magnitude the capabilities of sequential digital computers. The Space Station program's Earth Observing System (Eos) polar orbiting platforms, for instance, require to process data rates up to 1.5 gigabits per second. High resolution color images running at frame rates as low as 30 frames per second will require 10^8 bits per second. If one adds some form of autonomous feature identification to the system, the processing requirement will be between 10^{10} and 10^{13} operations per second. As a final example, a human-like speech recognizer must simultaneously perform phonetic, phonemic, syntactic, semantic, and pragmatic analyses of its inputs and match them to $5 \cdot 10^4$ words in real time. These processing throughputs exceed even the most optimistic projections for sequential supercomputers.

The problem of large-volume and high-speed computations can be solved by

- o data compression techniques,
- o parallel data processing.

Since their very beginning, artificial neural networks have been considered as massively parallel computing paradigms. Indeed, neural nets offer the potential of providing massive parallelism, adaption to dynamic data structures, and new algorithmic approaches to problems in image processing, computer vision, speech

recognition, robotic control, knowledge processing, among other application fields of AI. Even the fastest sequential digital electronic computers (including advanced parallel architectures) typically require processing times ranging from many minutes to several hours for non-complex low-level image processing tasks on large image arrays.

The advantages of neural computation are now widely recognized and neural networks form one of the most rapidly expanding areas of contemporary research. In fact, research in neurocomputer science, stimulated by major advancements in neurophysiological studies, neurosynergetic understanding, optoelectronic technology, molecular engineering, and bioelectronic material processing is currently in the midst of a gold rush period, an intense period of rapid discovery and exploitation. Everywhere new veins of gold are being uncovered and mined by thousands of prospectors, most of whom have crossed over into this exciting new research area from a diversity of other disciplines – neurobiology, neurosynergetics, quantum physics, imaging optics, electrical engineering, mathematics, and computer science.

The fundamental characteristics of all known neurocomputer architectures are the linear synaptic interconnections between simple nonlinear processing elements, called neurons, to form a concurrent distributed processing pattern of extensive connectivity. The processing units like the S-SEED logic devices (cf. Remark 1 supra) are arranged as two-dimensional arrays of neurons in the neural plane. Information is stored in the neurocomputer almost exclusively in the interconnection pattern, called neural network. Large scale (LS) collective systems like artificial neural networks exhibit many properties, including robustness, reliability, and fault tolerance, an ability to deal with ill-posed problems and noisy data, which conventional digital computer architectures do not. Adopting the synergetic point of view, neurobiology provides existence theorem on effectiveness of neural network parallel algorithms on appropriate problems.

For artificial neural networks to become ultimately useful, neuromorphic hardware must be developed. The effectiveness of neuromorphic hardware is in direct proportion to the attention it pays to the guiding neurobiological metaphor. Development efforts in the field of sixth generation computers have concentrated on one of two goals: to build

- o efficient hardware that effectively executes software simulations,
- o actual hardware emulators for specific neural network models.

Examples of the first are the Hecht-Nielsen Neurocomputer (HNC) accelerator board for conventional serial personal computers, and the Delta board by Science Applications International Corporation (SAIC). HNC is also pioneering a new computer language, *AXON*, that is designed for programming digital computers to simulate advanced neural networks. An important application of the SAIC neural network software simulation is the detection of explosives in checked airline baggage: the luggage is bathed in low energy (thermal) neutrons and the gamma rays resulting from neutron absorption by atomic elements in the luggage are analyzed. The artificial neural network software then searches for specific combinations of

atomic elements that characterize explosives including dynamites and water gels.

Examples of the second can be viewed hierarchically. On the simplest level, the information is recorded and retrieved from an erasable magneto-optic disk by optical techniques. Higher-level building-blocks are two-dimensional arrays of coherent optical processors ([2], [3], [4], [5], [6], [125], [126], [127], [128], [141], [142], [143], [144], [145]) for the analog implementation of neural network models by holographic optical interconnects, and neural network analog VLSI chips. For instance, the analog silicon models of the orientation-selective retina for pattern recognition ([1], [113], [115]), and the analog electronic cochlea for auditory localization ([90], [113], [114]) belong to this category. The amacronic and the cochlea VLSI neurochips are made with a standard complementary metal oxide semiconductor (CMOS) process ([185]).

Although the implementation of the various neural network models needs to overcome many difficult optical, optoelectronic, and analog electronic design problems, their performance is modest compared with the powerful organizing principles found in biological neural wetware. The visual system of a single human being does more image processing than do the entire world's supply of supercomputers, and the nervous system of even a very simple animal like the common house-fly (*Musca domestica*) contains computing paradigms that are orders of magnitude more effective than are those found in systems made by humans. Unlike conventional computer hardware, neural circuitry is not hardwired or specified as an explicit set of point-to-point connections. Instead it develops under the influence of a genetic specification and epigenetic factors, such as electrical activity, both before and after birth. How this happens is in large part unknown.

Neurobiological development processes are far too complex to hope that a relatively complete understanding of how a perceptual system develops and functions will soon emerge. But we are familiar with complex synthetic systems whose principles of neural organization can be understood without one's knowing in detail how the components work. Furthermore, the same principles can be used to build neurocomputers in any of several different technologies. Presently the most advanced neural network analog CMOS VLSI chips model, to a first approximation, the time-frequency domain processing of two highly spectacular biological neural systems: the active echolocating system of the horseshoe bats (*Rhinolophidae*), and the passive auditory localization system of the barn owl (*Tyto alba*) which both produce complete maps of the auditory space from the time-frequency coding pathways. Continuing evolution, however, of hybrid submicron optoelectronic technology combined with neurocomputer science, the highly promising and exciting new field of studying how computations can be carried out in extensive networks formed by two-dimensional arrays of heavily interconnected simple processing elements, will create advanced neurocomputers within the next decade which will be able to solve problems intractable for even the largest conventional digital computers.

This paper concentrates on a unified quantum holographic approach to massively parallel coherent optical, optoelectronic, and analog electronic neurocomputer architectures. Notice that the borders between physical optics, electronics,

and solid state physics are getting more and more fuzzy. The primary assumption of quantum or photon holography is that the energy transmitted by a beam of coherent radiation is divided into discrete wave packets, or photons, much as an electric current is made up of a flow of electrons. Detailed analysis shows that the arrival times at a photodetector of photons from a classical coherent radiation source such as a laser, exhibit the same Poissonian statistics as does the thermionic emission of electrons from the hot cathode of a vacuum tube. Thus, the photocurrent exhibits fluctuations which resemble the shot noise of the current in the vacuum tube. The quantum noise produced by a photoelectric detector is therefore an intrinsic property of the radiation itself, rather than of the photon detector.

The concept of photon arises from the quantization of the electromagnetic field. The spatial part of Maxwell's equations in vacuo

$$dF = 0 \quad \text{and} \quad d * F = 4\pi j$$

(differential 2-form $F = B + E \wedge dt$ on the Lorentz space $\mathbb{R}^4$, d = exterior derivative, $*$ = Hodge star operator on $\Lambda^2(\mathbb{R}^4)$) and the boundary conditions determine the possible frequencies ν , direction and spatial wavefront profile of the field, collectively referred to as the mode. The field modes are described by wave vectors $\mathbf{k} \neq 0$ in $\mathbb{R}^3$. In spite of warnings, sometimes the suggestion occurs that the Maxwell waves do form the wave functions of the photon. This suggestion, however, is a false idea. Although a single photon can have a definite position at one time, it is impossible to construct a position operator for the photon. Therefore it cannot have definite positions at all times in a specified time interval, i.e., it cannot have a definite trajectory (Bohr's indeterminacy principle; see [139], [178]). Since in the presence of sources, photons can be absorbed or emitted, one cannot introduce a linear Schrödinger evolution equation for a single photon. In fact, the modes of the electromagnetic field must be quantized and photons of energy $h\nu$ then occur as elementary excitations driven by the quantized Maxwell field modes of frequency ν and label $\mathbf{k}$.

In quantum optics or photonics, the quantization of the Maxwell field modes is done by expressing the time dependent part of Maxwell's equations in the form of the equation of motion for a classical harmonic oscillator and then replacing the classical harmonic oscillator by its quantum-mechanical counterpart. In this way, the electromagnetic field is considered as an assemblage of driven harmonic oscillators. As a consequence, the energy of the radiation field is quantized and the quanta are referred to as photons. The state of the field can be expressed in terms of number states $|n_{\mathbf{k}}\rangle$ which are states with $n_{\mathbf{k}}$ quanta occupying the mode $\mathbf{k}$. These number states are eigenstates of the Hamiltonian of the quantum-mechanical harmonic oscillator. However, they are not a realistic description of a coherent radiation field, as emitted by a laser. One formal series of number states, the so called coherent state $|\alpha_{\mathbf{k}}\rangle$, is used to represent the coherent radiation field produced by an ideal source such as an ideal laser operating well above threshold. In the Dirac notation, the Glauber formal series expansion ([133])

$$|\alpha_{\mathbf{k}}\rangle = \exp\left(-\frac{1}{2} |\alpha_{\mathbf{k}}|^2\right) \cdot \sum_{n \geq 0} \frac{(\alpha_{\mathbf{k}})^n}{\sqrt{n!}} |n_{\mathbf{k}}\rangle$$

describes a state $|\alpha_{\mathfrak{f}}\rangle$ where the probability of finding the mode occupied by $n_{\mathfrak{f}}$ photons exhibits a Poissonian distribution about a mean of $|\alpha_{\mathfrak{f}}|^2$ with a width $|\alpha_{\mathfrak{f}}|$. Ignoring the slowly time varying phase diffusion, the coherent state is considered to be a good approximation to the field produced by a coherent radiation source as a laser.

A mathematical description of photons or, more generally, of bosons is given by the Bargmann–Fock model of quantics ([22], [76]). Actually, the Bargmann–Fock model is based on the quantum-field-theory annihilation and creation operators for bosons and therefore on harmonic analysis of the three-dimensional isotropic Heisenberg nilpotent Lie group G . A first important advantage of the Heisenberg group approach is that it allows to define two-time averages, that is to say, mean values of mixed boson states taken at two different times t and $t' = t - x$ of fixed difference $t - t' = x$. A second important advantage is that the Kirillov quantization reveals the planar geometry of the unitary dual of the Heisenberg group G , that is to say, the planarly multilayered structure of the unitary dual manifold. To each flat layer, the Kirillov correspondence associates representations of G of linear Schrödinger type, or representations of G of linear Fraunhofer type, respectively. Because in the quantized theory of the electromagnetic field the bosons present in a beam of coherent light traveling in a well-defined direction are the photons, the Kirillov quantization approach allows to model, among other basic optical phenomena, the quantum parallelism performed by the beam splitter quantum interference experiment and optical holography, the functionality of holographic optical interconnects, optical phase conjugation, three-dimensional planarly multilayered optical devices like display holograms, and spatial light modulators (SLMs).

Display holograms are probably one of the most impressive realizations of the unitary dual manifold of G . Starting with the beam splitter quantum interference experiment as an elementary building unit, the key idea of quantum holography is to identify the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ with G quotiented by its one-dimensional center C_G and to recognize via the Kirillov quantization procedure the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ as a neural plane of local neural networks. As a result, harmonic analysis on the central projection G -slice G/C_G provides filtered backpropagation formulae which are at the base of the holographic reciprocity principle. Moreover, it gives rise to the elementary holograms and the Gabor wavelets which form total families of approximating functions in $L^2(\mathbb{R} \oplus \mathbb{R})$ of decorrelating and correlating code primitives of artificial neural networks. The neural network implemented in the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ explains the robust “associative” optical memory realized by optical holograms by the distributed nature of holographic recordings. Finally, a series of new identities for theta-null values which arises from artificial neural network identities shows that studies in computational mathematics combined with synthetic neurobiology may have an unexpected spin-off in pure mathematics.

Emphasis throughout the paper is placed on the application of quantum holography to neural computer architectures. For the fairly deep details of the Mackey machinery and the Kirillov quantization procedure underlying the harmonic ana-

lysis of the half-line bundle G over the two-sided symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$, the reader is referred to the monograph [151]. Technological details of the implementations are described in the references indicated below and in the references listed in the monograph [50].

The double-slit experiment is a phenomenon which has in it the heart of quantum-mechanics; in reality it contains the only mystery of the theory.

Richard P. Feynman (1918–1988)

In any attempt of a pictorial representation of the behaviour of the photon we would meet with the difficulty: to be obliged to say, on the one hand, that the photon always chooses one of the two ways and, on the other hand, that it behaves as if it had passed both ways.

Niels Bohr (1885–1962)

The unitary Schrödinger evolution is totally deterministic, maintains quantum complex superposition, and acts in a continuous way, but the completely different procedure of forming the squared moduli of quantum amplitudes and only this non-deterministic reduction of the state-vector (or, as it is sometimes graphically described: collapse of the wavefunction) introduces uncertainties and probabilities into quantum theory. It is a probabilistic law which grossly violates quantum complex superposition and is blatantly discontinuous.

Roger Penrose (1989)

3. Quantum Holography. The vertebrate vision system is perhaps the most complex neural assembly known. Although more details are known about vision than about any other neural system, it is by far not yet fully understood. On the deepest level of molecular operation, visual imaging is a quantum process. A solid object is seen because light scattered by the object causes chemical changes in the retinal cells of the eye. The eye is quite a good light detector: Experiments have shown that the vertebrate's retinal rod photoreceptors can respond to the absorption of even a single photon ([13]). In general, however, many photons are

absorbed by the eye without reaching a light sensitive cell. For this reason only a few photons in every hundred that enter the eye are detected. Obviously the chemical changes involved in seeing an object must be reversible. In fact, the cell reverts to its normal state after about one-tenth of a second. It is this short light storage period that limits the sensitivity of the eye for detecting faint objects. Photography can overcome this limitation of the eye by storing the changes in a permanent way on photographic emulsion.

Photographic emulsion consists of individual grains of a silver halide compound, in which the silver atoms are ionized. When a photon is absorbed by the photographic emulsion, an electron is emitted, in the same way as electrons are knocked out of a metallic surface in the photo-electric effect. This electron can now be attracted by a silver ion to form a neutral atom of silver. Left to itself, the neutral silver atom, surrounded by the ionic silver halide compound, is unstable, and will eventually eject the electron and revert back to an ion. However, if before this happens, other photons have produced several other neutral silver atoms nearby, a stable development center consisting of a small number of atoms can be formed. In contrast, each grain of the photographic emulsion contains billions of silver ions. However, when the photographic emulsion is developed, this assembly of neutral silver atoms induces all the remaining silver ions in the grain to be deposited as opaque metallic silver.

Figure 1

Figure 1 shows several photographs of the same person taken at different exposures. In the top left picture about $3 \cdot 10^3$ photons enter the camera. Most of these photons are absorbed without causing permanent change in the photographic emulsion. It is evident that $3 \cdot 10^3$ photons are not enough to generate a recognizable image and the photograph appears like random clusters of light dots. However, when

the exposure increases the number of photons entering the camera increases. The top right picture involves about 10^4 photons and already, although there is no clear image, a blurred impression of an image is beginning to show up within the clusters of light dots. The improvement continues as the number of photons increases, and the final exposure involves more than 3.10^7 photons. In this last picture, the image intensity seems to vary smoothly although it is built up out of individual development centers created by the arrival of individual photons. Although in the lowest exposure photograph the positions of the bright dots, signifying the presence of a development center in that grain emulsion, seem to be random, they are not. Centres are more likely to develop in places where the image will eventually be bright. Thus, even in a photograph, the quantum theoretical, probabilistic nature of light detection can be seen. It is not possible to predict with certainty where any particular photon will land, or in which grain of the photographic emulsion a development center will be produced.

Photographic emulsions are not sensitive to individual photons. Several neutral atoms must be produced in the photographic detector to form a development center. More efficient for imaging applications than photographic plates are the CCD (charge coupled device) detectors. They are formed by a two-dimensional array of photon detectors laid out on a single silicon chip typically comprising $2^9 \times 2^9$ pixels of size $20 \mu\text{m} \times 20 \mu\text{m}$. In fact, there are many formats for CCDs available. The arrival of single photons is detected directly by CCDs by converting them to electron-hole pairs, and then collecting the electrons into a potential well created by a depletion region. The accumulated charge at each position over the detector array then corresponds to the pattern of photons striking the CCD. The charge is read out by clocking the potential so that buckets of electrons transfer from well to well until they reach an integrating capacity and an on-chip preamplifier. An important feature of the device is that each pixel can be separately addressed, making CCDs extremely powerful for imaging applications.

Whereas photography first processes the optical information to form an image which is then recorded on the emulsion of a photographic plate or a CCD detector array, it is also possible to record the raw optical data in a non-localized way on the photographic emulsion or CCD and then place the processing in the future with the viewer. The method for recording the complete raw optical data is called optical holography. It is a two-step processing method which involves the phenomena of

- o scattering,
- o stationary quantum interference,
- o diffraction.

In the first processing step, the holographic image encoding procedure, the beams scattered by the solid object are mixed with the coherent reference beam and the generated stationary quantum interference pattern is recorded as an optical hologram. In the second processing step, the holographic image decoding procedure, the quantum interference pattern is read out.

The quantum interference patterns are not stationary unless the interfering light wavelets are coherent. There are two types of coherence, both of which are required at least to some extent, to get a stationary interference fringe pattern. One is temporal (longitudinal or axial) coherence, which is the requirement that the light wavelets all travel in the same time or same frequency, i.e., monochromatic light. The other is spatial (transverse or lateral) coherence, which is the requirement that the light wavelets are moving together in phase as if they started from a single point in space. The laser produces a high degree of both temporal and spatial coherence.

Figure 2

The basic experimental set-up to generate optical holograms is a modified version of the archetypical double-slit quantum interference experiment by which Thomas Young in 1803 conclusively verified the wave character of monochromatic light, and G.P. Thomson, the son of J.J. Thomson who first demonstrated that electrons behave like particles, and also Davisson and Germer in 1927 conclusively revealed that electrons also behave like waves: downstream, the primary wave is divided by a beam splitter into two coherent wavelets which travel different paths before recombination and detection; see Figure 2. If in the beam splitter quantum interference experiment the two photon routes possible in the linear Mach-Zehnder interferometer are exactly equal in length, there is 100% probability that the photon reaches the detector A and a 0% probability that it reaches the other detector B. In other words: the photon is certain to strike the detector A. If an absorbing screen is placed in the way either of the two interferometer routes, then it becomes equally probable that the photon reaches A or B; but when both paths are open and of the same lengths only detector A can be reached. Blocking off one of the two routes allows B to be reached. Therefore the photon must have actually traveled both routes at once. If a scattering object is placed in the way either of the

two interferometer routes, a stationary quantum interference pattern is generated in the hologram plane due to the incremental measuring principle of the linear Mach–Zehnder interferometer. The organization of random clusters to stationary quantum interference fringe patterns in the hologram plane can be visualized by the photon-counting image acquisition system (PIAS–TI) which is capable to detect even single photons like the vertebrate’s retinal rod and to obtain images at a level of darkness not obtainable even by ultra-high sensitivity video cameras.

THEOREM 1. *The holographic image encoding procedure is formed by a complex linear superposition of beam splitter quantum interference experiments. Conversely, the beam splitter quantum interference experiment performed by a linear Mach–Zehnder interferometer forms a degenerate holographic image encoding procedure.*

From this result it follows that the basic quantum phenomenon of optical holography is the quantum parallelism according to which different alternatives at the photon level are allowed to coexist in quantum complex linear superposition. The great thirty-year dialogue between Bohr and Einstein ([186], [187], [188], [189], [190], [191], [192]) concerning the issues of the beam splitter quantum interference experiment demonstrates the fundamental importance of the basic holographic image encoding procedure (see Theorem 7 infra).

Die kodierte Form der Amplituden – und Phasenverteilung trägt die Bezeichnung Hologramm. Im Grunde genommen stellt das Hologramm ein Interferenzmuster dar, das durch die Überlagerung der vom Objekt gestreuten Wellen mit der Referenzwelle zustande kommt. Die Funktion der Referenzwelle kann man sich auch so verdeutlichen, daß durch sie eine Lichtwelle im Raum “eingefroren” wird. Es addieren sich die Amplituden unter Berücksichtigung ihrer Phasenbeziehungen und nicht die Intensitäten.

Jurij I. Ostrovskij (1988)

The various descriptions, Doppler filtering, aperture synthesis, holography, and cross-correlation, diverse as they are when described physically, become identically when formulated mathematically.

Emmett N. Leith (1978)

4. The Holographic Image Encoding Procedure. The fundamental fact about optical holograms is that they are light diffracting elements. As planar diffractive optical elements (DOEs) they are of central importance for the implementation of neurocomputer architectures by holographic optical interconnects. The starting point of the holographic principle is the important fact that all detectors are phase-blind at the frequency range of the visible light. To overcome the phase-blindness, optical holography encodes in the writing step the phase information of the optical signals by a geometric encoding procedure. The word “holography” comes from the Greek, *holos* meaning entire, complete, *graphein* meaning to write, to record. Subsequently, optical holographic technique decodes in the readout step the phase information by light diffraction. The holographic reciprocity principle mathematically describes the decoding procedure.

In order to get mathematical insight into the geometric encoding procedure of optical holography by sesquilinearization of the multiplexed signal energy, let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of complex-valued $\mathcal{C}^\infty$ wavelet packet amplitude densities on the real line $\mathbb{R}$ rapidly decreasing at infinity. Consider $\mathcal{S}(\mathbb{R})$ as a dense vector subspace of the complex Hilbert space $L^2(\mathbb{R})$ of square integrable complex-valued densities with respect to Lebesgue measure dt of $\mathbb{R}$ under its natural isometric embedding. Endow $\mathcal{S}(\mathbb{R})$ with the standard scalar product $\langle \cdot | \cdot \rangle$ and the associated total signal energy norm $\| \cdot \|_2$. In optical holography, a square-law detector encodes in a massively parallel way the optical path length difference $x \in \mathbb{R}$ and the phase difference $y \in \mathbb{R}$ of two coherent signals having the same center frequency $\nu \neq 0$. Assuming that the writing complex-valued wavelet packet amplitude densities $\psi(t')dt'$ and $\varphi(t)dt$ with respect to Lebesgue measure $dt' = dt$ belong to the space $\mathcal{S}(\mathbb{R})$, the coordinates (x, y) of the stationary quantum interference pattern are simultaneously recorded in the hologram plane $\mathbb{R} \oplus \mathbb{R}$ by the coherent two-wavelet mixing $\psi(t')dt' \otimes \varphi(t)dt$. In view of the phase-conjugate cross terms or interference terms $\langle \psi | \varphi \rangle$ and $\langle \bar{\psi} | \bar{\varphi} \rangle$ of the total signal energy distribution identity or signal intensity relation

$$\|v\psi + w\varphi\|_2^2 = \langle v\psi + w\varphi | v\psi + w\varphi \rangle$$

with complex weights $v, w \in \mathbb{C}$, the sesquilinear extension to $\mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R})$ of the mapping

$$\psi \otimes \varphi \mapsto H_\nu(\psi, \varphi; x, y) = \int_{\mathbb{R}} \psi(t-x) \bar{\varphi}(t) e^{2\pi i \nu y t} dt \quad (\nu \neq 0)$$

describes by quantum complex linear superposition of the phased two-time average the first step of the angle image encoding procedure of optical holography. In this first processing step, each object to be globally stored by the coherent object signal beam is encoded in the hologram prior to its recording by mixing (or multiplexing or heterodyning) an unfocused linearly polarized coherent non-object-bearing reference signal beam having a particular angle between its wave vector and the normal vector of the hologram plane $\mathbb{R} \oplus \mathbb{R}$. Therefore the sesquilinear mapping defined by the

$$\psi(t')dt' \otimes \varphi(t)dt \mapsto H_1(\psi, \varphi; x, y).dx \wedge dy$$

is called the holographic transform of the complex-valued writing wavelet packet amplitude densities ([159], [160], [161]). It should be observed that unlike sequential data processing, the holographic transform of mixed wavelet packet amplitude densities as indicated above does not treat time as a sequencer but as an expresser of information similarly to biological neural systems where time is used throughout as one of the fundamental representational coordinates. Moreover, it should be noticed that the coordinates (x, y) of the stationary quantum interference pattern are independent of the distance between the object to be globally recorded and the square-law detector located in the hologram plane. It follows by quantum theoretical state-vector reduction which is a nonlinear procedure:

THEOREM 2. *Let ψ and φ be wavelets in $\mathcal{S}(\mathbb{R})$ of unit energy $\|\psi\|_2 = \|\varphi\|_2 = 1$. Then the phased two-time average*

$$|H_1(\psi, \varphi; x, y)|$$

which records the stationary quantum interference pattern generated by the coherent two-wavelet mixing $\psi(t')dt' \otimes \varphi(t)dt$ provides the probability of detecting a photon within a unit area attached to the point (x, y) of the hologram plane $\mathbb{R} \oplus \mathbb{R}$.

The method of optical holography or coherent wavefront reconstruction applies to all waves: to electron waves, X-rays, light waves, acoustic waves, and seismic waves, providing the wavelets are coherent enough to form the required stationary quantum interference patterns in the hologram plane ([158]). Therefore a laser is not really needed for optical holography; it is merely the use of solid, three-dimensional objects that calls for light wavelets whose coherence length exceeds the path differences due to the unevenness of such objects. Dennis Gabor used in 1947 a filtered mercury arc lamp to get temporal coherence and a pinhole to get spatial coherence in the first optical hologram. The main reason for the discouraging quality of his optical holograms was that no light source existed at that time with the combined intensity and coherence that was needed. When laser light became available, the quantum interference experiments by Emmett N. Leith and Juris Upatnieks at the University of Michigan in 1962 resulted in excellent optical display holograms that astonished the scientific community.

In radar analysis, the mapping $(x, y) \mapsto H_\nu(\psi, \varphi; x, y)$ is called the narrowband cross-ambiguity function ([15], [46], [92], [117], [121], [124], [150], [151]) and is used to characterize the resolution performance of radar signals. In the following it will be convenient to define the narrowband auto-ambiguity function by $H_\nu(\psi; \cdot, \cdot) := H_\nu(\psi, \psi; \cdot, \cdot)$. The mapping

$$\psi(t)dt \mapsto H_1(\psi; x, y).dx \wedge dy$$

which describes the self-interference of photons is called the holographic trace transform. In view of Theorems 1 and 2 *supra*, the holographic trace transform models the classical beam-splitter quantum interference experiment.

REMARK 2. The only examples of strictly convex objects for which the scattering amplitude density has been analyzed fairly completely for high frequencies ($|\nu| \rightarrow \infty$) are the compact spheres of the Euclidean space $\mathbb{R}^3$. According to the synergetic point of view, however, optical holography does not attempt to mathematically predict the scattering amplitude densities but geometrically encodes and decodes the scattering amplitude densities and their phases as an experimental result utilizing coherent reference beams.

REMARK 3. A vital element of optical neurocomputer architectures is the medium for optical hologram recording because it plays the role of an optical holographic associative memory. An associative memory has the basic capability of storing a number of associated information patterns (u, v) , so that subsequent presentation of one pattern u recalls its paired pattern v . This is an inherently parallel procedure, and the attractiveness of optical implementations of holographic associative memories has been recognized for some time. Electro-optical photorefractive crystals (PRCs) are known to form reusable optical holographic storage materials that can be infinitely recycled and do not require additional processing. The crystals of the sillenite family, bismuth silicon oxide $\text{Bi}_{12}\text{SiO}_{20}$ (BSO), bismuth titanium oxide $\text{Bi}_{12}\text{TiO}_{20}$ (BTO), and bismuth germanium oxide $\text{Bi}_{12}\text{GeO}_{20}$ (BGO) exhibit the highest sensitivity to light among presently known PRCs ([176]). Optical holograms are recorded inside PRCs directly by illuminating the crystal with laser light. The light induces a charge redistribution inside the crystal ([47], [48]) and in a certain characteristic time interval a dynamic equilibrium between distributions of the recording light intensity and internal electric charge is established. The electric charge induces an internal electrostatic field that changes the refractive index of the crystal by the electro-optical (Pockels or Kerr) effect and forms a volume holographic optical element (VHOE). As the interference pattern undergoes changes, a new charge distribution is formed, hence a new optical hologram is recorded. This charge distribution again comes to a dynamic equilibrium with the recording quantum interference pattern. If the period during which the interference pattern changes is sufficiently long, the electro-optical crystal rerecords an optical hologram. Hence the electro-optical PRCs can adapt itself to varying external conditions, such as occasional temperature-induced changes of the phase difference between the writing object signal beam and reference signal beam, or mechanical instabilities. This is an extremely important feature because it allows more reliable storage of scattering objects by almost-real-time quantum holography.

Research in the area of real-time quantum holography in electro-optical PRCs needs to focus on materials in order to achieve a faster speed of photo-response (< 1 msec), greater photorefractive sensitivity, control over image decay, and reduced fanning. Molecular engineering recently developed the highly interesting and promising organic crystals. As an alternative, bioelectronics or molecular electronics

are offering photochemically sensitive materials like biopolymers of the chlorophyll-protein complex and the retinal-protein complex for real-time holographic recording. It has been discovered that specifically bacteriorhodopsin which belongs to the retinal-protein complex and which can be extracted from the purple membrane of *Halobacterium halobium* is a very attractive recording material for real-time optical signal processing. Depending on the preparation procedure, these materials have a very wide range of photoresponse time running from 100 sec down to 10 psec, and an extremely high spatial resolution limited by the dimensions of the molecules. However, research in this area is still in the early stage of development and for the present the studies are far from the practical implementation of potential biological neurocomputers. Nevertheless, investigations of the simplest optical processors and of associative memory elements based on biopolymers are being intensively developed in various laboratories all over the world so that it is expected that on the basis of purple membranes of halobacteria an optical memory with a capacity of 10^9 bits/cm² will be created in near future ([147]).

REMARK 4. According to the rules of quantum theory, any two states whatever, irrespective of how different from one another they might be, can coexist in the any quantum complex linear superposition. This is the deep and decisive reason for the fact that high-resolution radar imagery of the terrain and optical holographic imaging are closely related concepts. In fact, airborne and spaceborne SAR imaging systems are active remote sensing systems which illuminate the terrain with electromagnetic energy at relatively long wavelengths (radar L-band center wavelength $\lambda = 23$ cm, C-band center wavelength $\lambda = 5.7$ cm, X-band center wavelength $\lambda = 3.1$ cm) as the platform moves with respect to the ground being mapped. SAR imaging systems coherently detect the signals returning from the terrain (called radar return) in order to store them in amplitude density including phase until all of the returns are collected. Simultaneous amplitude density and phase recording is performed by multiplexing or heterodyning the radar returns with a reference signal in order to generate microwave holograms ([53], [91]). The signals that are collected and coherently superposed in SAR systems by small antennas are not already focused as is the case in real aperture systems like radar altimeters. Because extensive processing is required to form the SAR image from the radar return, optical signal processing techniques have been applied to the collection and processing of SAR data. Chronologically, the coherent optical systems developed at the University of Michigan regarding applications to the processing of SAR data form the oldest branch in the family tree of optical computing ([30], [31], [59], [88], [92]). SAR coherent imaging systems can be regarded as optical neurocomputers which implement a Doppler filter bank by a relatively static reflection pattern of the architecture mirror ([24], [86]). The two-dimensional quantum parallelism inherent to the optical data processing approach is in large part responsible for the success of SAR coherent imaging.

REMARK 5. Since the advent of optical holography there has been a strong interest in replacing lenses and other crucial parts of optical systems by high performance HOEs. In particular, optical SAR data processing systems may be re-

alized by optical heads which include high-performance hololenses. Many HOEs are fabricated by recording the stationary quantum interference pattern between two mixing laser beams. The use of digital computer-generated hologram (CGH) techniques, however, avoids the technological difficulties involved in the interferometric HOE fabrication. Moreover, one benefit that digital CGHs can offer that is not available with optical holography is the ability to deal with objects that exist only mathematically. Finally, high quality digital CGHs to implement holographic optical interconnects of high circuit density may be fabricated with the same technology used in the manufacture of CMOS VLSI circuit chips ([77], [79], [80], [85], [87], [123], [174], [175]). The geometrical CGH encoding computations for specific HOE pattern parameters are performed with a standard computer aided design (CAD) station. Upon completion of the HOE pattern database generation and conversion of the pattern by a subroutine to the required formatted data, a digital computer controlled output device such as a Perkin-Elmer electron-beam high-resolution micro-lithographic system then writes the desired geometric pattern on photoresist, which is subsequently processed to produce the finished transmissive or reflective holographic element. It is at this intermediate level of lower throughput requirements where sequential processors play a role in vision and image processing. Alternately, digital CGHs may be realized by writing the appropriate geometrical pattern on a SLM. In any case, digital CGHs are at the base of a technology transfer from microelectronics to microoptics or amacronics and form a bridge between digital computer and optical neurocomputer architectures. Since atoms coherently excited by short laser pulses (Rydberg atoms) may be as large as some transistors of microelectronic ICs and the pathways between them inside the CMOS VLSI chips ([8], [198], [199]), the quantum theoretical treatment of optical holography is of particular importance for amacronics (see Section 10 *infra*) and nanotechnology.

Investigators in one field may very well never have been aware that the Heisenberg group had been found in some field not seemingly related to theirs. Another factor certainly contributory to its relative obscurity is that what I call “the Heisenberg group” is not in fact one object, but a collection of similar objects, rather like a functor, or a scheme in algebraic geometry, or even a combination of several overlapping functors. Thus one has to look with a certain pair of spectacles in order to see the topics as being united via a single common phenomenon.

Roger Howe (1980)

Perhaps the most rewarding aspect of analog computation is the extent to which elementary computational primitives are a direct consequence of fundamental laws of physics.

Carver A. Mead (1989)

I would like to express my belief that the holographic concept of Gabor is as fundamental as the general relativity theorem of Einstein, and it has to be explored further for a better understanding of nature in which we live.

Pál Greguss (1970)

Man sollte alles so einfach wie möglich machen, aber nicht einfacher.

Albert Einstein (1879–1955)

5. The Kirillov Quantization. Let G denote the multiplicative group of all unipotent real matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} := (x, y, z)$$

with unit element $(0, 0, 0)$. Then G is a simply connected two-step nilpotent Lie group, diffeomorphic to the differential manifold $(\mathbb{R} \oplus \mathbb{R}) \times \mathbb{R}$, with one-dimensional center $C_G = \{(0, 0, z) | z \in \mathbb{R}\}$. The polarized presentation

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

and the equivalent isotropic presentation

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1))$$

show that G is a realization of the three-dimensional Heisenberg group ([22], [54], [117], [151]). The Haar measure of G is Lebesgue measure $dx \otimes dy \otimes dz$ of the underlying differential manifold $\mathbb{R}^3$. The Lie algebra $\mathfrak{g} = T_{(0,0,0)}(G)$ of G is formed by the upper triangular matrices $\{(x, y, z) - (0, 0, 0) | x, y, z \in \mathbb{R}\}$. In terms of the canonical basis $\{P, Q, Z\}$ of $\mathfrak{g}$ which is given by the matrices

$$P := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the Heisenberg commutation relations read as follows:

$$[P, Q] = PQ - QP = Z, \quad [P, Z] = 0, \quad [Q, Z] = 0.$$

Thus the center $\mathfrak{c} = \mathbb{R}Z$ of the Heisenberg Lie algebra $\mathfrak{g}$ is one-dimensional and satisfies $\exp(\mathfrak{c}) = C_G$. Obviously

$$\text{ad}_{\mathfrak{g}} \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -b & a & 0 \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$. The adjoint action $(x, y, z) \mapsto \text{Ad}_G(x, y, z)$ of G on $\mathfrak{g}$ linearizes the action of G on itself by inner automorphisms and is therefore defined by conjugation:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & a & c + bx - ay \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

With respect to the basis $\{P, Q, Z\}$ of $\mathfrak{g}$ it follows

$$\text{Ad}_G(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y & x & 1 \end{pmatrix} \quad ((x, y, z) \in G).$$

Consequently the identity

$$\text{Ad}_G \circ \exp = \exp \circ \text{ad}_{\mathfrak{g}}$$

holds as usual on $\mathfrak{g}$. If $\{P^*, Q^*, Z^*\}$ denotes the dual basis of $\{P, Q, Z\}$, the coadjoint action $(x, y, z) \mapsto \text{CoAd}_G(x, y, z)$ of G in the dual vector space $\mathfrak{g}^* = T_{(0,0,0)}^*(G)$ of $\mathfrak{g}$ is given by the formula

$$\text{CoAd}_G(x, y, z) = \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix} \quad ((x, y, z) \in G).$$

Hence

$$\text{CoAd}_G(x, y, z)(\xi P^* + \eta Q^* + \nu Z^*) = (\xi + \nu y)P^* + (\eta - \nu x)Q^* + \nu Z^*,$$

where the triple (ξ, η, ν) denotes real coordinates. From this identity the Kirillov coadjoint orbit picture of G becomes apparent: For each center frequency $\nu \neq 0$ the orbit of the point $(0, 0, \nu)$ under the CoAd_G -action of G is the affine plane $\mathcal{O}_{\nu}$ in $\mathfrak{g}^*$ through the point νZ^* parallel to the plane spanned by $\{P^*, Q^*\}$ through the origin of $\mathfrak{g}^*$. For $\nu = 0$ the points $(\xi, \eta, 0)$ are zero-dimensional coadjoint orbits $\mathcal{O}_{(\xi, \eta)}$ of G located in the plane spanned by $\{P^*, Q^*\}$ through the origin of $\mathfrak{g}^*$. Notice that the symplectic plane $\mathcal{O}_{\nu}(\nu \neq 0)$ carries the canonical differential 2-form

$$\omega_{\mathcal{O}_{\nu}} = \nu d\xi \wedge d\eta$$

which endows $\mathcal{O}_\nu$ with a well-defined orientation. The point-orbit $\mathcal{O}_{(\xi,\eta)}((\xi,\eta) \in \mathbb{R} \oplus \mathbb{R})$ can be identified with the Dirac measure $\varepsilon_{(\xi,\eta)}$ located at the point (ξ,η) of the “singular” plane $\nu = 0$.

In terms of the Heisenberg nilpotent Lie group G , the Kirillov quantization procedure means the choice of an irreducible unitary linear representation U of G acting in a complex Hilbert space $\mathcal{H}$ and the coadjoint orbit $\mathcal{O}_U$ associated with the isomorphy class of U . Recall that U is a continuous homomorphism of G into the unitary group $\mathbf{U}(\mathcal{H})$ of $\mathcal{H}$, i.e., $U : G \rightarrow \mathbf{U}(\mathcal{H})$ is a mapping such that

$$\begin{aligned} U((x_1, y_1, z_1) \cdot (x_2, y_2, z_2)) &= U(x_1, y_1, z_1) \circ U(x_2, y_2, z_2), \\ U(0, 0, 0) &= id_{\mathcal{H}}, \end{aligned}$$

and such that the mapping $G \times \mathcal{H} \ni ((x, y, z), \psi) \mapsto U(x, y, z)\psi \in \mathcal{H}$ is continuous. Irreducibility means that $(U, \mathcal{H})$ is not obtained as a direct sum of two nontrivial linear subrepresentations of G . Equivalently, there exists no proper closed vector subspace $\neq \{0\}$ of $\mathcal{H}$ invariant under the action of G by U in $\mathcal{H}$.

According to the Stone–von Neumann–Mackey theorem ([151]) the Kirillov quantization problem has a solution unique up to unitary isomorphy: For any given center frequency $\nu \neq 0$, the central character

$$\chi_\nu : C_G \ni (0, 0, z) \mapsto e^{2\pi i \nu z}$$

determines up to a unitary isomorphism a unique infinite-dimensional irreducible unitary linear representation U_ν of G in the standard Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ which acts on the vector subspace $\mathcal{S}(\mathbb{R})$ according to the rule

$$U_\nu(x, y, z)\psi(t) = e^{2\pi i \nu(z + yt)}\psi(t - x) \quad (t \in \mathbb{R}).$$

Thus for all elements $(x, y, z) \in G$, the transition amplitude $U_\nu(x, y, z)\psi \in \mathcal{S}(\mathbb{R})$ is obtained by time shifting and phasor multiplication with respect to the frequency $\nu \neq 0$ of the wavelet $\psi \in \mathcal{S}(\mathbb{R})$.

Linear representations of G that are unitarily isomorphic to one of the representations $(U_\nu, L^2(\mathbb{R}))(\nu \neq 0)$ of G are called linear representations of Schrödinger type. The Kirillov correspondence assigns to the coadjoint orbit $(\mathcal{O}_\nu, \omega_{\mathcal{O}_\nu}) \in \mathfrak{g}^*/CoAd_G(G)$ ($\nu \neq 0$) of the point $(0, 0, \nu)$ in $\mathfrak{g}^*$ the isomorphy class of $(U_\nu, L^2(\mathbb{R}))$. Notice that this isomorphy class contains the Bargmann–Fock model of quantics describing bosons by annihilation and creation operators (cf. Section 1 *supra*), and also the linear lattice representation of G (see Section 11 *infra*). Each element of the isomorphy class, i.e., each linear representation of Schrödinger type of G realizes $(U_\nu, L^2(\mathbb{R}))$ by quantum complex linear superposition.

Notice that the complex vector space of $\mathcal{C}^\infty$ -vectors for the linear representation U_ν ($\nu \neq 0$) of G acting on $\mathcal{H} = L^2(\mathbb{R})$ is given by the Schwartz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$, and that the differentiated form of U_ν reads

$$\begin{aligned} U_\nu(P) &= \frac{d}{ds} U_\nu(\exp(sP))_{s=0} = -\frac{\partial}{\partial t}, \\ U_\nu(Q) &= \frac{d}{ds} U_\nu(\exp(sQ))_{s=0} = 2\pi i \nu t, \\ U_\nu(Z) &= \frac{d}{ds} U_\nu(\exp(sZ))_{s=0} = 2\pi i \nu. \end{aligned}$$

The linear operators $\{-\partial/\partial t, 2\pi i\nu t\}$ determine a representation of the Schrödinger operators by skew-symmetric operators. In particular, these operators satisfy the Heisenberg commutation relation ([22], [151])

$$[P, Q] = PQ - QP = 2\pi i\nu \cdot \text{id} \quad (\nu \in \mathbb{R}, \quad \nu \neq 0).$$

Shortly after Werner Heisenberg introduced the commutation relations in quantics, Hermann Weyl discovered in 1928 that they could be interpreted as the structure relations for the real Heisenberg Lie algebra $\mathfrak{g}$. The commutation relations combined with the Parseval–Plancherel theorem and the Cauchy–Schwarz inequality provide the Heisenberg inequality ([14])

$$\int_{\mathbb{R}} t^2 |\psi(t)|^2 dt \cdot \int_{\mathbb{R}} s^2 |\mathcal{F}\psi(s)|^2 ds \geq \frac{1}{16\pi^2} \left(\int_{\mathbb{R}} |\psi(t)|^2 dt \right)^2$$

which expresses the local/global duality between the wavelet $\psi \in \mathcal{S}(\mathbb{R})$ and its Fourier transform $\mathcal{F}\psi \in \mathcal{S}(\mathbb{R})$. A standard density argument shows that the Heisenberg inequality extends to all elements of the complex Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. It implies the classical Heisenberg Uncertainty Principle

$$\Delta U_\nu(P) \cdot \Delta U_\nu(Q) \geq \pi |\nu| \quad (\nu \in \mathbb{R}, \quad \nu \neq 0)$$

in terms of the standard root-mean-square deviations of the operators $U_\nu(P)$ and $U_\nu(Q)$ acting in $\mathcal{H} = L^2(\mathbb{R})$. Expressed in terms of the canonical basis $\{P, Q, Z\}$ of the Heisenberg Lie algebra $\mathfrak{g}$, the classical Uncertainty Principle takes the form of the Robertson relation ([148])

$$\Delta U_\nu(P) \cdot \Delta U_\nu(Q) \geq \frac{1}{2} |U_\nu(Z)| \quad (\nu \in \mathbb{R}, \quad \nu \neq 0).$$

The Uncertainty Principle has been one of the key relationships in quantum theory for over sixty years. In his Chicago lectures of spring 1929, Werner Heisenberg regarded this inequality as the precise mathematical expression of the Uncertainty Principle within the formalism of quantics ([67]). Moreover, it has been recognized as one of the fundamental results in signal processing ([12], [38], [138], [194]). Nevertheless, in the context of quantum holography it is very important to appreciate that the Heisenberg Uncertainty Principle does have a number of serious weaknesses. These are particularly related to the fact that the standard deviations $\Delta U_\nu(P)$ and $\Delta U_\nu(Q)$ which are defined by the square root of the expectation values only give very general information about the spreads of the probability density functions and are insensitive to the fine structure of quantum interference patterns ([139], [178], [179]). The structure of the Heisenberg group G , however, includes the Poisson summation formula and is therefore rich enough to getting around this inadequacy of the Heisenberg Uncertainty Principle. Indeed, an application of the linear lattice representation δ_1 of G to the interfering wavelet packet amplitude densities enables to rigorously establish the quantum parallelism according to which different

alternatives at the photon level are allowed to coexist in quantum complex linear superposition (see Theorem 7 infra).

Let $\overline{U}_\nu$ denote the contragradient representation of U_ν so that

$$\overline{U}_\nu(x, y, z) = {}^t U_\nu((x, y, z)^{-1})$$

holds for all elements $(x, y, z) \in G$. Obviously

$$U_\nu|_{C_G} = \chi_\nu, \quad \overline{U}_\nu|_{C_G} = \chi_{-\nu} \quad (\nu \in \mathbb{R}, \quad \nu \neq 0).$$

In terms of neural network theory, $\overline{U}_\nu$ is the feedback or backprojection representation of G associated to U_ν ($\nu \neq 0$). The flatness of the coadjoint orbits $(\mathcal{O}_\nu, \omega_{\mathcal{O}_\nu}) \in \mathfrak{g}^*/CoAd_G(G)$ and $(\mathcal{O}_{-\nu}, \omega_{\mathcal{O}_{-\nu}}) \in \mathfrak{g}^*/CoAd_G(G)$ ($\nu \neq 0$) in the dual vector space $\mathfrak{g}^*$ of the Heisenberg Lie algebra $\mathfrak{g}$ associated by the Kirillov correspondence with the isomorphy classes of U_ν and $\overline{U}_\nu$, respectively, is equivalent to the square integrability modulo C_G of U_ν and $\overline{U}_\nu$. If G/C_G is endowed with the differential 2-form

$$\omega_1 = dx \wedge dy,$$

induced by the form $\omega_{\mathcal{O}_1}$ on $\mathcal{O}_1$, the central projection and backprojection G -slice theorem follows:

THEOREM 3. *The holographic transform is the coefficient form of the linear Schrödinger representation U_1 of the polarized Heisenberg group G projected along the center C_G onto $(G/C_G, \omega_1)$. Thus the Kirillov quantization identities*

$$\left\{ \begin{array}{l} H_1(\psi', \varphi'; x, y).dx \wedge dy = \langle U_1(x, y, 0)\psi' | \varphi' \rangle .\omega_1 \\ -\overline{H}_1(\psi, \varphi; x, y).dx \wedge dy = -\langle \overline{U}_1(x, y, 0)\overline{\psi} | \overline{\varphi} \rangle .\omega_1 \end{array} \right.$$

hold.

The irreducibility combined with the unitarity of the linear Schrödinger representation U_1 of G implies that the commutant of U_1 is isomorphic to the compact torus group $\mathbb{T}$. Hence from the central projection G -slice theorem it follows:

COROLLARY. *The holographic trace transform $\psi(t)dt \mapsto H_1(\psi; x, y).dx \wedge dy$ extends to a mapping of $L^2(\mathbb{R})$ into $L^2(\mathbb{R} \oplus \mathbb{R})$ such that the identity*

$$H_1(\psi; x, y).dx \wedge dy = H_1(\varphi; x, y).dx \wedge dy$$

implies $\psi(t)dt = c\varphi(t)dt$ where $c \in \mathbb{T}$ denotes a constant phase factor.

The free choice of the phase factor $c \in \mathbb{T}$ reflects the fact that only the phase difference is of physical importance. Therefore the holographic image encoding

procedure needs the mixing of a coherent reference signal beam by a linear Mach-Zehnder interferometer to incrementally record the phase of the object signal beam in the hologram plane $\mathbb{R} \oplus \mathbb{R}$.

Holograms are recordings of the intensities of interference patterns said to be formed with a “subject wave” and a “reference wave.” Yet in the mathematical representation of the interference pattern intensity there is nothing except arbitrary notation to distinguish one wave from the other. We find the hologram transmittance of a plane hologram to be symmetric in the complex amplitudes of the two forming waves.

Robert J. Collier, Christoph B. Burkhardt,
and Lawrence H. Lin (1971)

The cross-correlation viewpoint, however, better than any other, renders understandable the well known all-range-focusing capability of the synthetic aperture radar system, implied in our holographic viewpoint. Since the form of the recorded signal, as manifested in the quadratic phase factor, is a function of range, it is apparent that each range element must be processed differently, for example, by correlation with a reference function which is different for each range. Since the pulsing provides resolution in range, we can store the data from each range separately and process them differently, so that each range is cross-correlated with the reference function proper for that range. Thus, the synthetic antenna is in effect focused simultaneously at all ranges, a most remarkable feat when viewed in terms of the capabilities of conventional antennas.

Emmett N. Leith (1978)

6. Metaplectic Covariance. Another important advantage of the Kirillov quantization approach lies in the fact that the hidden symmetries of the holographic

transform

$$\psi(t')dt' \otimes \varphi(t)dt \mapsto H_1(\psi, \varphi; x, y).dx \wedge dy$$

can be expressed by the group of automorphisms of the Heisenberg nilpotent Lie group G keeping the center C_G pointwise fixed. This group, the metaplectic group $\mathbf{Mp}(1, \mathbb{R})$, forms a twofold cover of the symplectic group $\mathbf{Sp}(1, \mathbb{R}) = \mathbf{SL}(2, \mathbb{R})$. Its natural action on the hologram plane $\mathbb{R} \oplus \mathbb{R}$ preserves the center frequencies $\nu \neq 0$ ([151], [164], [184]). Its action on the complex Hilbert space $L^2(\mathbb{R})$ is performed by the metaplectic representation σ . The representation σ of $\mathbf{Mp}(1, \mathbb{R})$ is a projective unitary linear representation of $\mathbf{Sp}(1, \mathbb{R})$ in $L^2(\mathbb{R})$ and satisfies the metaplectic covariance condition of the Kirillov quantization

$$\sigma(g)^{-1} \circ U_\nu(x, y, 0) \circ \sigma(g) = U_\nu((g^{-1}(x, y), 0)) \quad ((x, y) \in \mathbb{R} \oplus \mathbb{R})$$

for all $g \in \mathbf{Sp}(1, \mathbb{R})$. It follows

THEOREM 4. *The holographic transform satisfies the metaplectic covariance identity*

$$H_1(\sigma(g)\psi, \sigma(g)\varphi; x, y).dx \wedge dy = H_1(\psi, \varphi; g^{-1}(x, y)).dx \wedge dy$$

for all complex-valued wavelet packet amplitude densities $\psi(t')dt'$ and $\varphi(t)dt$ belonging to $\mathcal{S}(\mathbb{R})$ and all elements $g \in \mathbf{Sp}(1, \mathbb{R})$.

Notice that the action of $\mathbf{Sp}(1, \mathbb{R})$ in $L^2(\mathbb{R})$ by the metaplectic representation σ includes the dilations by real scaling factors $a \neq 0$, and the one-dimensional Fourier transform $\mathcal{F}$, both being of importance for the Gabor wavelet transform. Indeed, for the diagonal matrix

$$g_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

in $\mathbf{Sp}(1, \mathbb{R})$, the scaling (or zooming) identity

$$\sigma(g_a)\psi(t)dt = |a|^{-1/2}\psi(a^{-1}t)dt$$

follows, and similarly for the Weyl matrix

$$g_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of $\mathbf{Sp}(1, \mathbb{R})$ the relation

$$\sigma(g_0)\psi(t)dt = \mathcal{F}\psi(t)dt$$

holds for all $\psi \in \mathcal{S}(\mathbb{R})$.

A Fourier transform hologram is an optical hologram which records the stationary quantum interference pattern of two coherent wavelets whose complex-valued wavelet packet amplitude densities at the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ are the Fourier transforms of both the object and the reference wavelet. It follows as a special case of the metaplectic covariance identity of the holographic transform:

COROLLARY 1. For the complex-valued wavelet packet amplitude densities $\psi(t')dt'$ and $\varphi(t)dt$ in $\mathcal{S}(\mathbb{R})$ the 90° rotation identity

$$H_1(\mathcal{F}\psi, \mathcal{F}\varphi; x, y).dx \wedge dy = H_1(\psi, \varphi; -y, x).dx \wedge dy$$

holds.

If $\psi^-(t)dt = -\psi(-t)dt$ denotes the complex-valued wavelet packet amplitude density of the time-reversed optical signal, the Fourier inversion theorem yields the identity

$$\sigma(g_0^2)\psi(t)dt = \psi^-(t)dt.$$

Thus the hologram plane rotated through 180° corresponds to the time-reversed writing signals.

It should be observed that the Weyl matrix g_0 , the diagonal matrices $g_a (a \neq 0)$, and the unipotent matrices

$$g^u = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$$

for $u \in \mathbb{R}$ are generators of the group $\mathbf{Sp}(1, \mathbb{R})$. In fact they give rise to an Iwasawa decomposition

$$\mathbf{Sp}(1, \mathbb{R}) = \mathbf{KAN}$$

into compact, diagonal, and unipotent subgroups. As a paraxial ray-transfer matrix, $g^u \in \mathbf{Sp}(1, \mathbb{R})$ defines a thin cylindrical lens of focal length $f = -\frac{1}{u}$ and $\sigma(g^u)$ defines the chirp modulation operator

$$\sigma(g^u)\psi(t)dt = e^{-i(u/2)t^2}\psi(t)dt$$

of chirp rate $u \neq 0$. For $u < 0$ the chirp modulation operator $\sigma(g^u) \in \mathbf{U}(L^2(\mathbb{R}))$ defines an up chirp amplitude density modulation and for $u > 0$ a down chirp amplitude density modulation.

COROLLARY 2. The chirp amplitude density modulation $\sigma(g^u)$ of chirp rate $u \neq 0$ can be corrected by a thin cylindrical lens of focal length $f = \frac{1}{u}$.

Finally, for the drift-length transfer matrix

$$g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

the identity

$$\mathcal{F}(\sigma(g)\psi)(t)dt = e^{-i(\pi/4)\text{sign } u}\sigma(g^{-u})\mathcal{F}\psi(t)dt$$

follows where $\text{sign } u = u/|u|$. The phasor occurring in this formula arises by the Maslov index which is responsible for the fact that $\mathbf{Mp}(1, \mathbb{R}) = \widetilde{\mathbf{Sp}}(1, \mathbb{R})$ forms a twofold cover of $\mathbf{Sp}(1, \mathbb{R})$. Since optical holography is phase sensitive, it is exactly this sudden change in phase (Gouy effect) of

$$\pi/2 = \pi/4 - (-\pi/4)$$

which makes it not appropriate to place the hologram plane in the focal planes of the beam expanding lenses of the basic interferometric set-up.

It should be observed that the construction of the metaplectic representation σ of $\mathbf{Mp}(1, \mathbb{R})$ is completely analogous to the construction of the spin representations of the orthogonal groups (symmetric tensors taking place of anti-symmetric tensors).

Example 1. Let $T > 0$ be given and denote by

$$\psi_T(t) = \begin{cases} 1 & |t| \leq \frac{1}{2} T, \\ 0 & |t| > \frac{1}{2} T, \end{cases}$$

the rectangular pulse of duration T . In terms of the triangular pulse

$$\Lambda(t) = \begin{cases} 1 - |t| & |t| \leq 1, \\ 0 & |t| > 1, \end{cases}$$

and the cardinal sine mother wavelet

$$\text{sinc } x = \begin{cases} \frac{\sin \pi x}{\pi x} & x \neq 0, \\ 1 & x = 0, \end{cases}$$

the holographic trace transform of $\psi_T(t)dt$ takes the form

$$H_1(\psi_T; x, y).dx \wedge dy = \Lambda\left(\frac{x}{T}\right) \text{sinc } y(T - |x|).dx \wedge dy.$$

An application of Theorem 4 *supra* shows that the chirp pulse density $\sigma(g^u)\psi_T(t)dt$ of duration T and chirp rate $u \neq 0$ admits the holographic trace transform

$$H_1(\sigma(g^u)\psi_T; x, y).dx \wedge dy = \Lambda\left(\frac{x}{T}\right) \text{sinc } (y - ux)(T - |x|).dx \wedge dy.$$

Satellite altimetry uses the ranging capability of radar sensors to measure the surface topographic profile. An example of an advanced-type system is the SEASAT altimeter which was put into orbit in June 1978. The satellite orbital altitude was 790 km and the platform velocity (ground track) $v=6.6$ km/sec. SEASAT was in operation for a total of 105 days. During that time, the altimeter provided profiles of the ocean surface with an accuracy of a fraction of a meter. In the altimetry mode, SEASAT operated at a center frequency of 13.5 GHz. The stable local oscillator (STALO) generated a sequence of 12.5 nanosec pulses at a 250 MHz center frequency which has been applied to the chirp generator. The SEASAT chirp generator was a surface acoustic wave (SAW) device fabricated on a lithium tantalate substrate. The resulting chirp modulated pulse had a linearly decreasing frequency with an 80 MHz bandwidth and a pulse duration $T = 3.2 \mu\text{sec}$. The pulse repetition frequency (PRF) was 1020 Hz. During the transmit mode, the chirp pulse at 250 MHz has been upconverted to 3375 MHz, amplified to a 1 W level, and multiplied by 4 to 13.5

GHz. This also multiplied the bandwidth by 4 in order to achieve the desired 320 MHz bandwidth and height measurement accuracy of 0.47 m. In the receive mode the chirp pulse have been upconverted to 3250 MHz, amplified to 0.1 W, multiplied by 4 to 13.0 GHz, and used for mixing with the received echo signal.

Example 2. In SAR remote sensing systems (see Remark 4 *supra*), a target at distance r_0 with velocity v relative to the moving platform induces a relativistic chirp amplitude density modulation $\sigma(g^u)$ of the received echo signal of chirp rate

$$u = \frac{4\pi v^2}{\lambda r_0}$$

where $\lambda = c/|\nu|$ denotes the center wavelength of the coherent radar. The dependence of the chirp rate u of the range r_0 is called the range-azimuth coupling.

In the SAR optical signal processor, let λ' denote the wavelength of the coherent light scanning the holographic film of transport velocity v' . If

$$v_0 = v/v', \quad \lambda_0 = \lambda/\lambda'$$

are the relative SAR platform velocity and the relative radar wavelength, respectively, then the radar return focus at distance

$$f = \frac{\lambda_0 r_0}{4\pi v_0^2}$$

from the hologram plane. It follows that the relativistic effect of the platform motion generates an axial astigmatism. To compensate the linear range variation of the focal length f , a wide-screen equalizer is introduced in the hologram plane. Such an equalizer takes the form of a conical lens or a tilted cylindrical lens ([88], [91], [92]) which are components of a correcting anamorphic optical system (cf. Corollary 2 of Theorem 4 *supra*). The recent developments of SLMs have supplied an attractive replacement for the holographic film as an input medium. Moreover, two-dimensional optical data processors using laser diode illumination, acousto-optic (AO) cell data input and a CCD detector array for the output have been designed. For each realization of the SAR data processor, however, it is important to notice that the spatial resolution of SAR imaging systems is independent of the range r_0 to the target and the velocity v of the radar platform.

In the imaging mode, SEASAT SAR operated at a center frequency of 1275 MHz (L-band, $\lambda = 23.5$ cm) with pulse duration $T = 34 \mu\text{sec}$ and PRF selections of 1464, 1537, 1580, and 1647 Hz admitting a spatial resolution of 25 m. The depression angle ranged between 67° to 73° and produced an image-swath width of 100 km. The antenna was a 10.74 m by 2.16 m phased array system deployed after orbit insertion. The microwave holographic data for each 100 km wide image-swath have been optically processed to produce four film strips each of which covered a width of 25 km and a length of several thousand kilometers.

The first Shuttle imaging radar (SIR-A) experiment was launched on the second flight of Columbia in November 1981. The satellite orbital altitude was 250 km and the image-swath width 50 km in order to cover a surface area of about 10 million km². The SAR antenna of 9.44 m by 2.09 m radiating area was stowed inside the Shuttle cargo bay and operated when the Shuttle was in an inverted attitude. As in the SEASAT SAR, the transmitted pulse was a chirp pulse of 1275 MHz center frequency admitting a spatial resolution of 38 m. The image data were recorded as holographic film on board the Shuttle. The data film was developed and then processed at the laboratory by coherent laser light to generate the original image film at a scale of 1 : 500.000.

A second Shuttle imaging radar (SIR-B) experiment was conducted in October 1984. For SIR-B the SAR antenna was modified, however, to permit the depression angle to be changed during the mission within a range of 30° to 75°. The center wavelength λ was the same as in the earlier missions. The satellite orbital altitude was 225 km and the spatial resolution improved to 25 m.

Figure 3

As an example, Figure 3 shows a SAR image of the Lakshmi region of the planet Venus. It has been generated by the Soviet Union VENERA 15 and 16 orbiters through the cloud-shrouding atmosphere of Venus which is impenetrable for visible light. NASA flew a SAR around Venus in 1990 for the Magellan mission. The radar is operating at a center frequency of 2385 MHz and provides a resolution down to 250 m; see Figures 4 and 5. By the late 1990s the Cassini spacecraft may be put in an orbit around Saturn and image its moon, Titan, at L-band and K-band on flybys.

Figure 4

Figure 5

REMARK 6. The Heisenberg group G carries a sub-Riemannian metric and a sub-Laplacian ([173]). On the fibre $T_{(x,y,z)}^*(G)$ with base point $(x, y, z) \in G$ of the cotangent bundle $T^*(G)$ of G , the associated bundled quadratic form Q is given by

$$Q_{(x,y,z)}(\xi, \eta, \nu) = (\xi + \nu y)^2 + (\eta - \nu x)^2.$$

Thus $Q_{(x,y,z)}$ is a parabolic quadratic form with one-dimensional null-space spanned by the vector $(-y, x, 1) \in \mathcal{O}_1$, $(\mathcal{O}_1, \omega_{\mathcal{O}_1}) \in \mathfrak{g}^*/CoAd_G(G)$. The sub-Laplacian $\square_G$ of G forms a sub-elliptic linear differential operator given by

$$\square_G = \left(\frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right)^2 + \left(\frac{\partial}{\partial y} - x \frac{\partial}{\partial z} \right)^2.$$

The Heisenberg helix is the analog of a geodesic for the sub-Riemannian geometry of G defined by the sub-elliptic bundled quadratic form Q on $T^*(G)$. This fact corresponds to the “expansion theorem” discovered by Dennis Gabor in 1965 which says that information attached to an optical signal pattern is not carried by “rays”, but by a certain “tube of rays” the cross-section of which is proportional to the square of the center wavelength λ of the optical signal ([55]).

I have no doubt that my latest publication is my luckiest find yet. I have also much perfected the experimental method, and I can now produce really pretty reconstructions from apparently hopelessly muddled diffraction patterns.

Dennis Gabor (1900–1979) to Max Born (1882–1970) on 15th June 1948

The coherence of laser light finds a spectacular application in the making of holograms. A typical hologram looks like a gray piece of plastic with no evident image on it. However, the hologram actually has a microscopically fine and highly complex pattern of lines and spaces. Now illuminate the developed hologram by the same laser system, except that the object has been removed. The pattern on the hologram converts the pure laser beam into a precise replica of the pattern of ordinary light that would be obtained if the object were still there. In this way, the hologram acts as a window. Each eye looking

at the illuminated hologram sees a different point of view, thus creating a three-dimensional image by an illusion of depth and solidity. By changing one's vantage point, it is possible to see behind things and around corners, just as if one was looking at the real object through an ordinary window. The image has a realistic three-dimensional appearance. Holography resulted in a whole new concept in the development of imaging systems and technology.

Enders A. Robinson (1989)

7. The Holographic Image Decoding Procedure. In the following, the symplectic homogeneous G -manifolds

$$\begin{aligned}(\mathcal{O}_1, \omega_{\mathcal{O}_1}) &\in \mathfrak{g}^*/\text{CoAd}_G(G), \\ (\mathcal{O}_{-1}, \omega_{-1}) &\in \mathfrak{g}^*/\text{CoAd}_G(G),\end{aligned}$$

and the central projection G -slice G/C_G will be identified with the two-sided symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$. Then the Heisenberg nilpotent Lie group G is a half-line bundle over the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ the two sides of which carry the canonical differential 2-forms $\omega_1 = dx \wedge dy$ and $\omega_{-1} = -dx \wedge dy = dy \wedge dx$, respectively. In view of the square integrability modulo C_G of the irreducible unitary linear representations U_1 and $\overline{U}_1$ of G , an application of Schur's lemma provides the biorthogonality relations ([118], [135], [151], [167])

$$\iint_{\mathbb{R} \oplus \mathbb{R}} H_1(\psi', \varphi'; x, y) \overline{H}_1(\psi, \varphi; x, y) dx dy = \langle \psi' \otimes \varphi | \psi \otimes \varphi' \rangle$$

for the complex valued wavelet packet amplitude densities $\psi'(t)dt$, $\varphi'(t)dt$, $\psi(t)dt$, $\varphi(t)dt$ in $\mathcal{S}(\mathbb{R})$. Therefore the dyads

$$\begin{cases} E(\psi', .; x, y) : \varphi' \mapsto H_1(\psi', \varphi'; x, y) \overline{U}_1(x, y, 0) \overline{\psi}' \\ \overline{E}(\psi, .; x, y) : \varphi \mapsto \overline{H}_1(\psi, \varphi; x, y) U_1(x, y, 0) \psi \end{cases} \quad ((x, y) \in \mathbb{R} \oplus \mathbb{R})$$

which embed $\psi' \in \mathcal{S}(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R})$, respectively, into the Hilbert Schmidt (HS) operator-valued differential 2-forms acting on $L^2(\mathbb{R})$, define a U_1 -system $(E(., .; x, y))_{(x, y) \in \mathbb{R} \oplus \mathbb{R}}$, and a $\overline{U}_1$ -system $(\overline{E}(., .; x, y))_{(x, y) \in \mathbb{R} \oplus \mathbb{R}}$ of coherent states based on the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ ([119], [133]). Observe that these coherent state systems provide a quantum theoretical description of nonspreading wavelet packets ([55], [133], [159]) and therefore of the Gabor tubes of rays (see Remark 6 supra).

THEOREM 5. For all complex-valued writing wavelet packet amplitude densities $\psi'(t)dt$, $\varphi'(t)dt$, $\psi(t)dt$, $\varphi(t)dt$ in $\mathcal{S}(\mathbb{R})$ the gain equations

$$\iint_{\mathbb{R} \oplus \mathbb{R}} E(\psi', \varphi'; x, y) dx dy = \|\psi'\|_2 \overline{\varphi'},$$

$$\iint_{\mathbb{R} \oplus \mathbb{R}} \overline{E}(\psi, \varphi; x, y) dx dy = \|\psi\|_2 \varphi$$

hold.

REMARK 7. Similar inversion formulas can be established for the affine coherent states defined by the wavelet transform and the irreducible unitary linear representations of the non-unimodular affine Lie group

$$G_+ = \{(\alpha, \beta) | \alpha > 0, \beta \in \mathbb{R}\}$$

of the real line $\mathbb{R}$ (“ $\alpha t + \beta$ group”). The affine wavelets are particularly useful code primitives for voice decomposition ([63]).

The non-abelian solvable Lie group G_+ has the presentation

$$(\alpha_1, \beta_1).(\alpha_2, \beta_2) = (\alpha_1 \alpha_2, \alpha_1 \beta_2 + \beta_1).$$

Of course, G_+ may also be represented as the group of real matrices

$$\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \quad (\alpha > 0, \beta \in \mathbb{R})$$

under matrix multiplication. The left Haar measure of G_+ is given by $\frac{d\alpha \otimes d\beta}{\alpha^2}$ and the right Haar measure by $\frac{d\alpha \otimes d\beta}{\alpha}$. Apart from the trivial one-point coadjoint

orbits located on the real line $\mathbb{R}$, the affine group G_+ of $\mathbb{R}$ admits exactly two non-trivial coadjoint orbits, the open upper half-plane $\mathcal{O}_+$ and the open lower half-plane $\mathcal{O}_-$. It follows from the Kirillov coadjoint orbit picture of G_+ that every irreducible unitary linear representation of G_+ of dimension > 1 is unitarily isomorphic to either U or its contragredient representation $\overline{U}$, where U can be realized on the complex Hilbert space $L^2(\mathbb{R})$ by the assignment

$$U(\alpha, \beta)\psi(t) = e^{i\beta e^t} \psi(t + \log \alpha) \quad (t \in \mathbb{R}),$$

and $\overline{U}$ by the action

$$\overline{U}(\alpha, \beta)\psi(t) = e^{-i\beta e^t} \psi(t + \log \alpha) \quad (t \in \mathbb{R})$$

on $\psi \in \mathcal{S}(\mathbb{R})$. More convenient are the realizations on $L^2(\mathbb{R}_+)$ given by

$$U(\alpha, \beta)\psi(t) = e^{i\beta t} \sqrt{\alpha} \psi(\alpha t) \quad (t > 0),$$

and on $L^2(\mathbb{R}_-)$ given by

$$\overline{U}(\alpha, \beta)\psi(t) = e^{-i\beta t} \sqrt{\alpha} \psi(\alpha t) \quad (t < 0).$$

Notice that the irreducible unitary linear representations U and $\overline{U}$ of G_+ are square integrable and that their coefficient functions form the wideband ambiguity functions ([46], [121]).

REMARK 8. One of the most dramatic deployments of computer technology in radiological diagnostic imaging is the development of computer-aided tomography (CT). In this case and, more recently, in magnetic resonance imaging (MRI) systems, the computational capability made possible by the advent of high speed computers has been an absolutely essential ingredient in the process of image formation. Similarly to holography, the raw data provided by the physical imaging system in CT or MRI is in an encoded form which bears no discernible resemblance to the two-dimensional array of information comprising an image that can be visually perceived. CT generates an image of a cross-sectional slice of the body lying perpendicular to the long axis of the patient being examined. Unlike optical holography, in which the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ is transversal to the direction of laser irradiation, CT is based upon the measurement of the attenuation of X-ray beams lying entirely within the plane of the section being imaged. Turning from optical holography to CT ([120]), the preceding identities give rise by an application of the spectral theory of the irreducible reductive dual pair $(\widetilde{\mathbf{Sp}}(1, \mathbb{R}), \widetilde{\mathbf{O}}(n, \mathbb{R}))$ inside $\widetilde{\mathbf{Sp}}(n, \mathbb{R})$ ([76], [153]), to the singular value decomposition of the Radon transform $\mathcal{R}: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R} \times \mathbf{S}_{n-1})$ acting on functions $f \in \mathcal{S}(\mathbb{R}^n)$ according to

$$\mathcal{R}f(r, \omega) = \int_{\mathbb{R}^n} f(x) \varepsilon_{(r - \langle \omega | x \rangle)} dx$$

(ε_p = Dirac measure located at the point $p \in \mathbb{R}$). It follows that the inversion problem for the Radon transform $\mathcal{R}$ which underlies CT, MRI, and tomographic reconstruction for geophysical applications is ill-posed. Neurocomputers, however, seem to be more appropriate to solve ill-posed problems than conventional digital computers.

As a special case we obtain from Theorem 5 supra the following result which describes the readout procedure of optical holograms, i.e., the retrieval of geometrically encoded information by adaptive resonance. It is important to appreciate that the energy normalization is a nonlinear procedure.

COROLLARY. *Let $\varphi \in \mathcal{S}(\mathbb{R})$ and assume that $\psi \in \mathcal{S}(\mathbb{R})$ satisfies the normalization condition $\|\psi\|_2 = 1$. If $\mathcal{F}$ denotes the Fourier transform acting on $\mathcal{S}(\mathbb{R})$ then the filtered backpropagation formulac of degenerate coherent four-wavelet mixing*

$$\left\{ \begin{array}{l} \iint_{\mathbb{R} \oplus \mathbb{R}} H_1(\psi, \varphi; x, y) e^{-2\pi i y t} \overline{\psi}(t - x) dx dy = \overline{\varphi}(t) \\ \iint_{\mathbb{R} \oplus \mathbb{R}} H_1(\psi, \varphi; -y, x) e^{-2\pi i y t} \mathcal{F} \overline{\psi}(t - x) dx dy = \mathcal{F} \overline{\varphi}(t) \end{array} \right. \quad (t \in \mathbb{R})$$

hold.

The preceding reproducing diffraction integrals prove the fundamental law of optical holography or holographic reciprocity principle which governs the angle image decoding procedure of optical holograms: The complex-valued wavelet packet

amplitude density including the magnitude and the phase of the conjugate object signal recorded in the hologram can be read out simultaneously by illuminating the hologram with the conjugate to the original reference signal beam. The conjugate beam which becomes unfocused by a beam expander provides the illuminating wavelets with their proper phase factors for adaptive resonance. Thus the geometric encoding procedure of optical holography is able to overcome the phase-blindness of the detectors at the frequency range of the visible light: the holographic decoding procedure reconstructs the complete wavefront creating a real pseudoscopic image of the object. The reconstruction of a Fourier transform hologram establishes the 90° rotation identity of Corollary 1 of Theorem 4 *supra*.

A complex-valued wavefront recorded in a planar optical hologram is effectively stored for future reconstruction by an application of the fundamental law of optical holography. Holographic interferometry is concerned with the formation and interpretation of the stationary quantum interference patterns which are created when a coherent wavelet, generated at some earlier time and stored in an optical hologram, is later reconstructed according to the holographic reciprocity principle, and caused to interfere with a phase-related comparison wavelet. It is the storage or time delay aspect which gives the holographic procedure a unique advantage over conventional optical interferometry. It permits diffusely reflecting or scattering objects which are subjected to stress to be interferometrically compared with their non-deformed state. Actually, the holographic interferometry has become one of the most important applications of the fundamental law of optical holography.

After the quantum of energy has already gone through the double slit screen, a last-instant free choice on our part gives at will a double-slit interference record or a one-slit-beam count. Does this result mean that present choice influences past dynamics, in contravention of every formulation of causality? Or does it mean, calculate pedantically and don't ask questions? Neither; the lesson presents itself rather as this, that the past has no existence except as it is recorded in the present.

John A. Wheeler (1978)

I argue that the very structure of all quantum theories suggests a revision of the classical notion of space and time. I will present evidence that two copies of space-time, rather than one, are the proper arena for all quantum processes. At the heart of this observation lies the very well known fact

that every set of equations and formulae in quantum theory, from which all the transition amplitudes are determined, may always be written in two equivalent forms, differing by complex conjugation. We obtain one set from the other by reversing the sign of the imaginary unit i .

Iwo Bialynicki-Birula (1986)

8. Optical Wavefront Conjugation. The pair of reproducing diffraction integrals of the Corollary of Theorem 5 supra describing the holographic filter bank is also at the basis of optical wavefront conjugation by means of real-time quantum holography ([47], [48]) in electro-optical PRCs. Two of the beams are referred to as pump beams and are arranged such that they are co-linear and counter-propagating and overlap both spatially and temporally in the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$. The third beam, commonly called the probe beam, can interfere with each of the pump beams to generate transient phase gratings within the electro-optical PRC. These gratings arise because the refractive index of the PRC changes in response to the intensity of laser light: as the pump beams interfere with each other, the regions of constructive and destructive interference cause a corresponding modulation of the refractive index. The pump beams then entering the PRC can be deflected by the induced gratings to produce the fourth, wave-front conjugate beam. The wavefront conjugate beam propagates back along the path of the probe beam with a wave vector opposite to the wave vector of the probe beam.

Recall that neurocomputers consist of weighted linear interconnections between arrays of simple nonlinear processing units, the neurons. Information is stored in the neurocomputer almost exclusively in the interconnection pattern. Learning dynamics are used to evolve the interconnection strength pattern as a succession of small perturbations. Because degenerate four-wavelet-mixing wavefront conjugate mirrors, as described above, provide retroreflection and optical tracking novelty filters, Theorem 5 is at the basis of neural network models implemented by local neural networks of reconfigurable holographic optical interconnect patterns in optical neurocomputer architectures ([2], [3], [4], [5], [6], [77], [79], [80], [81], [123], [125], [126], [127], [128], [129], [130], [141], [142], [143], [144], [145]). In the long term, real-time holography in PRCs appears to be the most appealing reconfigurable optical interconnection technique. If the holographic associative memory has net gain comparable with the losses in the resonator cavity, the output will converge to a real image of the globally stored object: the expanded conjugate reference signal beam acts as an optical scanner for readout of the associate information. In case of a linear resonator memory, gain is supplied by the wavefront conjugate mirror which provides regenerative feedback, whereas in case of a loop resonator memory, gain is supplied by an externally pumped electro-optical PRC.

The Soffer optical resonator forms an implementation of an optical neurocomputer architecture which includes two degenerate four-wavelet-mixing wave-front

conjugate mirrors. For more details, the reader is referred to Section 15 *infra*.

Geometric quantization provides the structure for the geometric realizations of the irreducible unitary representations of the groups involved in physics.

Norman E. Hurt (1983)

9. Radial Isotropy. The vast majority of optical systems are designed to operate over a field of view that is radially isotropic. If the processing elements of an optical system are constrained to be radially symmetric, it is only necessary to optimize the performance over a radial slice of the field of view. The system is then guaranteed to have the same performance over any radial slice of the field of view. The advantages to optimizing over a radial slice as compared with the full field of view are speed and cost. Each additional field point used in the automatic design routine such as CODE V increases the computation time, and, therefore, the expense ([65]).

A complex-valued writing wavelet packet amplitude density $\psi(t)dt$ in $\mathcal{S}(\mathbb{R})$ is called radially isotropic if its holographic trace transform $H_1(\psi; x, y).dx \wedge dy$ is a radial differential 2-form on the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$, i.e., if $H_1(\psi; x, y).dx \wedge dy$ is invariant under the natural action of the orthogonal group $O(2, \mathbb{R})$ in $\mathbb{R} \oplus \mathbb{R}$.

THEOREM 6. *The complex-valued wavelet packet amplitude density $\psi(t)dt$ on $\mathbb{R}$ is radially isotropic if and only if $\psi \in \mathcal{S}(\mathbb{R})$ admits the form of Hermite–Gaussian eigenmodes*

$$\psi = c_n H_n$$

where $c_n \in \mathbb{C}$ is a constant and $H_n(t) = e^{-t^2/2} h_n(t)$ is the Hermite function of degree $n \geq 0$.

The proof follows by Kirillov quantization: There is a complete classification of the irreducible unitary linear representations of the diamond solvable Lie group $\mathbb{T} \times G$ having U_ν as their restrictions to G . The Kirillov corresponding coadjoint orbits in the dual of the non-exponential diamond Lie algebra are paraboloids of revolution ([152]).

It follows from the preceding theorem that a quantum mechanical harmonic oscillator is equivalent to an assembly of bosons each having one polarization state. Notice that the Hermite–Gaussian eigenmodes $(H_n)_{n \geq 0}$ are crucial for the phenomenon of daydreaming in optical resonator neurocomputers ([5], [159]).

COROLLARY 1. *The elementary holograms $(H_1(H_m, H_n; \cdot, \cdot))_{m \geq 0, n \geq 0}$ form a Hilbert basis of the complex Hilbert space $L^2(\mathbb{R} \oplus \mathbb{R})$.*

The orthogonality of the elementary holograms $(H_1(H_m, H_n; \cdot, \cdot))_{m \geq 0, n \geq 0}$ in the complex Hilbert space $L^2(\mathbb{R} \oplus \mathbb{R})$ implies that in the Shannon sense the mutually

information of the code coefficients is zero. Thus the code coefficients are non-redundant, they form a statistically independent ensemble, and image coding in terms of the decorrelating family of code primitives $(H_1(H_m, H_n; \cdot, \cdot))_{m \geq 0, n \geq 0}$ is optimally efficient. Let the element $\psi \in L^2(\mathbb{R} \oplus \mathbb{R})$ admit the expansion

$$\psi = \sum_{m \geq 0, n \geq 0} c_{m,n} H_1(H_m, H_n; \cdot, \cdot)$$

with complex coefficients given by

$$c_{m,n} = \langle \psi | H_1(H_m, H_n; \cdot, \cdot) \rangle \quad (m \geq 0, n \geq 0).$$

It follows

$$\|\psi\|_2^2 = \sum_{m \geq 0, n \geq 0} |c_{m,n}|^2$$

and by switching to the time-asymmetric state-vector reduction procedure of quantics, the probability that $\psi \neq 0$ in $L^2(\mathbb{R} \oplus \mathbb{R})$ collapses to the elementary hologram $H_1(H_m, H_n; \cdot, \cdot)$ is given by the ratio

$$|c_{m,n}|^2 / \|\psi\|_2^2 \quad (m \geq 0, n \geq 0).$$

The non-deterministic collapse of the wavefunction $\psi \in L^2(\mathbb{R} \oplus \mathbb{R})$ represents the nonlinear aspect of the Kirillov quantization procedure because it violates the quantum complex superposition principle. It is complementary to the linear aspects of quantum holography.

COROLLARY 2. *The quantum mechanical mode competition in recognizing $\psi \neq 0$ in $L^2(\mathbb{R} \oplus \mathbb{R})$ is determined by the probabilities $(|c_{m,n}|^2 / \|\psi\|_2^2)_{m \geq 0, n \geq 0}$.*

Amacronics is a name coined for layered structures of processing electronics, binary microoptics, and detector arrays, with applications in imaging systems with processing right at the focal plane. Amacronic structures are based on lessons that we learned from Mother Nature. Human beings live quite happily with a data transfer rate of a few kilocycles, massively parallel yes, but not very fast. All imaging systems suffer from the Von Neumann bottleneck in electro-optics (in computer systems all the processing functions go through a single CPU, the central processor unit; it slows down the overall system). Electro-optical systems are similar; all the optical

information goes through a detector array at the focal plane which is the bottleneck. We are developing layered structures of optics and electronics in a parallel form (a processing unit per pixel), somewhat like what happens in front of the retina of your eye where you have similar amacrine clustered processing layers. The word “amacrine” comes from the Greek a macros meaning short range. The idea is to couple dynamically clusters of detector arrays. With binary optics we may be able to build systems with peripheral vision much more motion-sensitive than on-axis foveal view, or systems tuned for edge detection or noise reduction.

Wilfrid B. Veldkamp (1989)

Binary optics: The optics technology of the 1990s.

Wilfrid B. Veldkamp (1990)

10. Amacronics: The Microoptics Layer. The one-dimensional unitary linear representations of the polarized Heisenberg group G are given by the assignment

$$U_{(\xi,\eta)}(x, y, z)\psi(t) = e^{2\pi i(\xi x + \eta y)}\psi(t) \quad (t \in \mathbb{R}).$$

These representations which are, of course, irreducible are called representations of linear Fraunhofer type of G . Under the Kirillov correspondence they admit one-point coadjoint orbits $\{\mathcal{O}_{(\xi,\eta)} = \varepsilon_{(\xi,\eta)} | (\xi, \eta) \in \mathbb{R} \oplus \mathbb{R}\}$ in the “singular” plane $\nu = 0$ spanned by $\{P^*, Q^*\}$ in $\mathfrak{g}^* = T_{(0,0,0)}^*(G)$ which form a set of Plancherel measure zero. The Plancherel measure π_G of G is uniquely determined by the Haar measure $dx \otimes dy \otimes dz$ and concentrated on $\mathbb{R} - \{0\}$. It is given by

$$\pi_G = |\nu| \, d\nu.$$

The character formula of G ([151]) provides the radial fanin/fanout distribution on the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$:

$$\varepsilon_{(0,0)} = \int_{\mathbb{R}} \text{Tr}_{G/C} U_\nu d\pi_G(\nu)$$

The tempered distributions $Tr_{G/C} U_\nu$ ($\nu \neq 0$) follow from the trace identity for the linear Schrödinger representation U_1 of the Heisenberg group G

$$Tr_{G/C} U_1 = \sum_{n \geq 0} H_1(H_n; \cdot, \cdot)$$

by time scaling $t \mapsto \sqrt{|\nu|} t$. Projection of the Kirillov corresponding paraboloids of revolution in the dual of the diamond Lie algebra along the axis allows to create microscopic multi-level surface relief patterns of high quality diffractive HOEs ([174], [175]); see Figure 6. The focal length of the diffractive microlenses in the HOE arrays is given by

$$f = |\nu|,$$

and therefore inversely proportional to the center wavelength λ . The plane $\nu = 0$ in $\mathfrak{g}^*$ forms therefore the focal plane layer of the amacrine structure. The quantization of the continuous phase profile into discrete phase levels is performed by using the VLSI ion-etching technology.

Figure 6

Both digital and analog optical computing requires sufficiently powerful and bright sources of radiation characterized by a small size and a highly efficient transformation of pumping energy into coherent radiation output. Diode or injection lasers provide the best choice in terms of power consumption and size. The coherence length of their radiation output is sufficient for optical computing purposes.

In the AT & T Bell Laboratories' looped digital optical pipeline processor the array of power supply beams to read the optical logic devices is created from a pair

of laser beams by a HOE component. This holograting of Dammann type ([80], [172], [183]) is a multi-beam splitting DOE the pattern of which is computed to generate a uniform 4×8 array of wavelets from one incident laser beam. The two 850 nm diode lasers can therefore be used to generate two interleaved 4×8 arrays so that one array illuminates all the upper S-SEED diodes and the other illuminates all the lower ones. There are two advantages in this scheme: two lasers can supply twice as much power as one laser, and, by pulsing one of the pair of lasers, all of the devices in the array can be preset into the same logic state. The two beams from the laser pair are combined at a “knife-edge”. The two counterpropagating beams are focused to two adjacent spots, one of which is reflected, the other transmitted. This pair of spots is imaged via the holograting onto the device array.

Presently the most advanced implementation of two-dimensional matrix-addressable arrays of laser diodes for free-space holographic optical interconnect patterns and photonic switching in optical computers is formed by a hybrid optoelectronic chip recently developed by the AT & T Bell Laboratories in collaboration with Bellcore. It includes more than 2 millions of electrically pumped vertical-cavity surface-emitting lasers (VCSELs) arranged on a GaAs substrate of size less than 1 cm^2 . The active area of the micro-lasers emitting infrared laser light of about $\lambda = 8500 \text{ \AA}$ wavelength consists of thin indium gallium arsenide InGaAs layers sandwiched between more than 600 successive molecular beam epitaxial (MBE) or epi layers of GaAs and aluminum arsenide AlAs. Each of the cylindrical microlasers has cross-section of about $5 \text{ }\mu\text{m}$ and has been etched by a photolithographic process. The lengths of the microlasers are about $5.5 \text{ }\mu\text{m}$, and greater than 99.5 % of that is passive material. All the laser diodes are individually addressable, independently of the other ones by a current of about 1 mA and therefore are particularly suitable for performing the angle image encoding and decoding procedures of optical holograms. In practice, a simple 4×4 matrix-addressable surface-emitting laser (SEL) array (MASELA) was used to address an optical hologram containing 16 distinct images, each microlaser reading out a separate image. The two-dimensional MASELA is a technology to which conventional edge-emitting diode lasers have no practical counterpart. It is the aim of the present development in amacronic sensor technology to integrate the optical source chip into a hybrid VLSI neurochip.

The Heisenberg group is a natural setting for defining and analyzing certain continuous and discrete concepts arising from the Fourier transform and associated with nonstationary image representation.

Richard Tolimieri (1990)

Now it is together, blinking happily.

Alan Huang (1990)

Barry R. Masters (1990)

11. Hololattices and Holofractals. The implementation of two-dimensional pixel arrays by holographic optical interconnect patterns ([77], [79], [80], [81], [183]) and analog VLSI wavefront arrays ([1]) suggests to look at the restrictions of the sesquilinear holographic transform

$$\psi(t')dt' \otimes \varphi(t)dt \mapsto H_1(\psi, \varphi; x, y).dx \wedge dy$$

to lattices located inside the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ ([22], [155]). The quadratic lattice $\mathbb{Z} \oplus \mathbb{Z}$ embedded in the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ may be considered as the projection onto G/C_G of the 3-cubic (uniform) lattice

$$L_0 := \{(\mu, \mu', \zeta) | \mu \in \mathbb{Z}, \mu' \in \mathbb{Z}, \zeta \in \mathbb{Z}\}$$

and the normal subgroup

$$L := \mathbb{Z} \oplus \mathbb{Z} \oplus C_G$$

inside the three-dimensional Heisenberg nilpotent Lie group G along its center C_G . Form the compact Heisenberg nilmanifold $L_0 \backslash G$ associated to G which is a circle bundle over the two-dimensional compact torus $\mathbb{T}^2$. An application of the Weil–Zak isomorphism

$$w_1 : \psi \mapsto ((x, y, z) \mapsto e^{2\pi iz} \sum_{m \in \mathbb{Z}} e^{2\pi imy} \psi(m - x)) \quad (\psi \in \mathcal{S}(\mathbb{R}))$$

allows to realize the linear Schrödinger representation U_1 of G as the linear lattice representation

$$\delta_1 = \text{Ind}_L^G(\chi_1)$$

of G ([57], [149], [150], [151]). Thus, δ_1 is a representation of G of linear Schrödinger type which reveals to be of extreme importance in quantum holography.

REMARK 9. The projection of the Weil–Zak isomorphism w_1 onto the first coordinate axis gives rise to the periodization map

$$p : \psi \mapsto (x \mapsto w_1(\psi)(x, 0, 0)) \quad (\psi \in \mathcal{S}(\mathbb{R})).$$

Combined with the one-dimensional Fourier transform $\mathcal{F} = \sigma(g_0)$, the periodization map p gives rise to the Poisson summation formula for the elements of the space $\mathcal{S}(\mathbb{R})$ and therefore to the Whittaker–Shannon–Nyquist–Kotel’nikov sampling theorem which allows the reconstruction of a band-limited signal from its uniformly distributed samples utilizing translates of the cardinal sine mother wavelet (cf. Example 1 supra). Interleaving in the Cross Interleave Reed–Solomon Code (CIRC) is

employed to redistribute data symbols in the bit stream prior to recording so that consecutive words are never adjacent. Recording in a non-localized way guards against the very likely occurrence of burst errors. Upon de-interleaving during the CIRC decoding procedure, the shuffled words are placed back in their original and rightful position in the stream, and the errors are distributed in time.

REMARK 10. For the affine Lie group G_+ of the real line $\mathbb{R}$ (see Remark 7 supra), however, there exists no analog of the linear lattice representation δ_1 of G . Therefore, there is no summation formula of Poisson type for G_+ .

From the isomorphy performed by w_1 between the linear Schrödinger representation U_1 and the linear lattice representation δ_1 of G follows the identity

$$H_1(\psi, \varphi; x, y).dx \wedge dy = \langle \delta_1(x, y, 0)w_1(\psi)|w_1(\varphi) \rangle .dx \wedge dy$$

inside the pixel $]-1/2, +1/2[\times]-1/2, +1/2[$ in the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$.

As a first application, the preceding identity allows to decide in a mathematically rigorous way the Bohr–Einstein dialogue ([186], [187], [188], [189], [190], [191], [192]) in favor of quantum theory. Thus the linear lattice representation δ_1 of G allows to overcome the inadequacy of the classical Heisenberg Uncertainty Principle in describing by standard root-mean-square deviations the beam-splitter quantum interference experiment and its application to the holographic image encoding procedure. In fact, Bohr’s intuitive argument cannot be rigorously based on any of the known uncertainty relations. The comprehensive structure of the Heisenberg nilpotent Lie group G , however, is ideally suited for the purposes of quantum holography: An application of the preceding identity establishes the quantum parallelism in a mathematically conclusive way. In fact it proves.

THEOREM 7. *The holographic image encoding procedure implemented by a linear Mach–Zehnder interferometer generates an optical hologram if and only if quantum parallelism holds between the reference and the object wavelets.*

Quantum parallelism according to which any two states of correlated photons must be considered as taking place simultaneously in quantum complex linear superposition, irrespective of how far from one another they might be, is a consequence of the Einstein–Podolsky–Rosen (EPR) type non-locality of quantics. It has been verified by sophisticated and highly accurate laser experiments the results of all of which are in excellent agreement with the quantum theoretical predictions ([7], [8], [9], [10], [11]). The quantum interference pattern has been observed even when the time interval between the arrivals of individual photons was around 30.000 times longer than the time for an individual photon to pass through the linear Mach–Zehnder interferometer ([134], [139]). Therefore, in the context of quantum holography the parallelism of firing neurons (cf. [122]) located at different columns of the visual cortex which has been recently observed at the Max Planck Institute for Brain Research in Frankfurt am Main is highly remarkable.

A second application is the computation of hologratings of Dammann type ([80], [183]) which are based on planar fabrication techniques, such as photolithography

and reactive ion etching ([185]), now standard in VLSI electronic technology. They act as multi-beam splitting DOE components in the AT & T Bell Laboratories' looped digital optical pipeline processor.

A third important consequence is the Parseval–Plancherel type pixel identity

$$\sum_{(\mu, \mu') \in \mathbb{Z} \oplus \mathbb{Z}} H_1(\psi; \mu, \mu') \overline{H_1(\varphi; \mu, \mu')} = \sum_{(\mu, \mu') \in \mathbb{Z} \oplus \mathbb{Z}} |H_1(\psi, \varphi; \mu, \mu')|^2$$

which holds for all complex-valued writing wavelet packet amplitude densities $\psi(t)dt$, $\varphi(t)dt$ in the space $\mathcal{S}(\mathbb{R})$. If the Hermite–Gaussian eigenfunctions H_m and H_n ($m \geq n \geq 0$) are inserted for ψ and φ , respectively, the radial symmetry of the terms of the left-hand side implies by a trace argument the following result ([154], [155], [160]).

Figure 7

Figure 8

THEOREM 8. *The non-oriented lattices of two-dimensional pixel arrays in the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ have the crystallographic dihedral groups D_k ($k \in \{1, 2, 3, 4, 6\}$) of order $2k$ as their groups of symmetry.*

An application of the representations of linear Fraunhofer type of G yields the following

COROLLARY. *The diffraction patterns of the non-oriented lattices located in the hologram plane are the reciprocal lattices.*

Snowflake fractals, i.e., self-similar planar von Koch curves admitting locally the symmetry groups D_k ($k \in \{1, 2, 3, 4, 6\}$) are called holographic fractals or

holofractals, for short. The validity of the preceding results can be experimentally demonstrated by means of optical holofractals readout by the representations of G of linear Fraunhofer type ([160], [180], [181]). For the case $k = 3$ of triadic holofractals having Hausdorff dimension $\log 4 / \log 3 = 1.2619\dots$, a line segment of unit length serves as the initiator and an equilateral triangle becomes the generator; see Figures 7, 8, and 9. Readout of randomly distorted holofractals by the representations of G of linear Fraunhofer type generates radial speckle patterns ([170], [171]).

Figure 9

An application of the Weil–Zak isomorphism w_1 to the readout formulae of the Corollary of Theorem 5 *supra* shows that the scanout of the two-dimensional pixel arrays of the holographic lattices (or hololattices, for short) may be performed by

a time-multiplexing procedure.

REMARK 11. It is a highly remarkable observation of neurophysiology that the presynaptic vesicular grids of the mammalian brain are hexagonal hololattices. The thickness of the presynaptic membrane by which the synaptic vesicles emit their specific neurotransmitter substances is about $50 \text{ \AA}$ whereas the uncertainty of the position of a synaptic vesicle due to the Heisenberg Uncertainty Principle is about $50 \text{ \AA}$ per millisecond ([40], [34]). Of course, this observation and its consequences are also interesting from the philosophical point of view ([41], [193]).

REMARK 12. The hololattices are at the basis of the detour phase method ([156], [162]) of writing digital CGHs of sampled images by use of the fast Fourier transform (FFT) algorithm. The height and the displacement of a single aperture centered at the sampling points of the hololattice are used to encode the complex-valued wavelet packet amplitude density including the phase of the wavefront. Thus the actual encoding of detour phase CGHs is performed without the explicit use of a reference wavelet. The hololattice corresponding to the crystallographic group D_6 of twelfold symmetry offers substantial computational efficiency and a significant reduction of required data storage compared with rectangular sampling: the hexagonal FFT is 25% more efficient than the most efficient rectangular FFT algorithm. The scanout of the wavefront is achieved when the CGH is illuminated with a plane wave and focused with a Fourier-transforming lens.

REMARK 13. The compact disks (CDs) may be regarded as one-dimensional digital CGHs that may be scanned out by the holographic optical read-head of a CD digital audio player. The scanning laser beam which is focused on the surface of the CD is focused on its return path on a quadrant detector located near the laser diode chip. The detector converts the arrays of minute optical holograms which are coherently encoded by mixing the scanning beam with the beam scattered by the pits into a sequence of electric pulses. Thus the massive amount of information arising by scanning the simple interference patterns of pits and lands has to serially pumped off the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ and then to fed into the bit-stream chip or the multi stage noise shaping (MASH) IC of the CD player's microelectronic circuitry. It is the focal plane of the collimating lens which forms the optoelectronic von Neumann bottleneck of the hybrid device. Erasable magneto-optic technology uses laser light both to record and to read data. A blank disk has all its magnetic domains oriented north-pole-down. To record information, a burst of a few nanoseconds of high-intensity light from an infrared laser heats a spot about $1 \mu\text{m}$ across in one magnetic layer of the disk. The coercive force required to change the magnetic orientation of all the domains in the spot from north-pole-down to north-pole-up falls to almost zero as the temperature of the spot increases to 150°C , and the bias magnetic field created by a coil flips the magnetic field. The data are read by a lower-powered beam from the same laser, whose polarization depends by the Kerr magneto-optic effect on whether the magnetic orientation of the spot is north-pole-up or north-pole-down. Optoelectronic ICs in the magneto-optic write-read-head senses the polarization, and the magnetic orientation is interpreted as a digital 1 or

0. The magneto-optic switching technology suggests to consider the spin variables of an erasable CD as a one-dimensional artificial neural network.

The single most important principle in the analysis of electrical circuits is the principle of linear superposition. For an arbitrary network containing resistors and voltage sources, we can find the solution for the network (the voltage at every node and the current through every resistor) by the following procedure. We find the solution for the network in response to each voltage source individually, with all other voltage sources reduced to zero. The full solution, including the effects of all voltage sources, is just the sum of the solutions for the individual voltage sources. In addition to linearity of the component characteristics, there must be a well-defined reference value for voltages (ground), to which all node potentials revert when all sources are reduced to zero. This principle applies to circuits containing current sources as well as to those containing voltage sources. It applies even if the sources are functions of time. It also applies to circuits containing capacitors, provided that any initial charge on a capacitor is treated as though it were a voltage source in series with the capacitor.

Carver A. Mead (1989)

12. Amacronics: The Processing Electronics Layer. Recall from the theory of electrical networks that a simple closed path in the plane $\mathbb{R} \oplus \mathbb{R}$ is called a mesh. A mesh is called hexagonal if it has the dihedral group D_6 as its symmetry group (see Theorem 8 supra). Let us assume that the processing electronics layer is implemented by a linear network of local resistive circuits and that the voltage is constant around the perimeter of each concentric hexagonal mesh about the driven node. Consider the n th concentric hexagonal mesh where all of its $6n$ nodes have the same voltage V_n . On its perimeter there are 6 vertices, and the remaining $6(n - 1)$ nodes lie along the edges. Each of the 6 vertex nodes makes 3 outside interconnections, while each of the $6(n - 1)$ edge nodes makes 2 outside interconnections. Thus the n th hexagonal mesh connects to the $(n + 1)$ th concentric hexagonal mesh through $(12n + 6)$ parallel resistors. Each of the resistors has

resistance R and conductance R_0^{-1} . Therefore the impedance connecting the n th mesh to the $(n+1)$ th concentric mesh is $R/(12n+6)$. Similarly the impedance connecting the n th mesh to the $(n-1)$ th concentric mesh is $R/(12n-6)$. Along the n th mesh there are $6n$ conductances to ground, making a net admittance to ground of $6nR_0^{-1}$. According to Kirchhoff's current law, the current flowing into the n th mesh from its neighbours balances with the current flowing out of the n th mesh to ground. It follows the forward recursion

$$\frac{V_{n+1} - V_n}{\frac{R}{12n+6}} + \frac{V_{n-1} - V_n}{\frac{R}{12n-6}} = 6nR_0^{-1}V_n \quad (n \geq 1).$$

Introducing the parameter $a = RR_0^{-1}$, Kirchhoff's current law takes the more convenient form ([49], [113])

$$(2n+1)V_{n+1} - n(a+4)V_n + (2n-1)V_{n-1} = 0 \quad (n \geq 1).$$

It describes the voltage on a given hexagonal mesh in terms of the voltages on the two smaller concentric hexagonal meshes. For any number $w \in \mathbb{C}$ the identity

$$\sum_{n \geq 1} (w^n(2n+1)V_{n+1} - w^n n(4+a)V_n + w^n(2n-1)V_{n-1}) = 0$$

follows. For the power series $G(w) = \sum_{n \geq 1} V_n w^n$, the relations

$$\begin{aligned} \sum_{n \geq 1} w^{n+1} V_{n+1} &= G(w) - wV_1, \\ \sum_{n \geq 1} w^{n-1} V_{n-1} &= G(w) + V_0 \end{aligned}$$

yield the inhomogeneous linear differential equation of the first order

$$2 \left(w^2 - \left(2 + \frac{a}{2} \right) w + 1 \right) \frac{dG}{dw} + \left(w - \frac{1}{2} \right) G = V_1 - wV_0.$$

The linearity of the ordinary differential equations reflects the linearity of the network. Decompose the quadratic factor

$$w^2 - \left(2 + \frac{a}{2} \right) w + 1 = (w - r_+)(w - r_-)$$

into linear factors. Then the voltage of the first hexagonal mesh takes the form

$$V_1 = \left(1 - \frac{E(r_-^2)}{K(r_-^2)} \right) r_+ V_0$$

in terms of the voltage V_0 at the center of the network and the complete elliptic integrals of the first and second kind $E(r_-^2)$ and $K(r_-^2)$, respectively, evaluated

at the parameter value r_-^2 . The arithmetic-geometric mean algorithm presents an efficient tool to compute V_1 in terms of V_0 and then to apply the forward recursion to compute all the voltages V_n ($n \geq 1$).

Although a cell's response function is in general nonlinear, visual neurophysiologists have found that for many cells, a linear summation approximation is appropriate.

Ralph Linsker (1988)

13. Gabor Wavelets Attached to a Hololattice. In biological vision, the center-surround receptive field profiles of the retinal neurons ([33], [35], [36], [37], [39], [109]) and the cells of the lateral geniculate nucleus are far from forming an orthogonal family in $L^2(\mathbb{R} \oplus \mathbb{R})$. Therefore the resulting neural representation remains highly correlated. Theorem 5 supra suggests to implement a matching filter bank by an adaptive artificial neural network model which is based for $(y, y') \in \mathbb{R} \oplus \mathbb{R}$ on the central projection and backprojection G -slice orbits

$$G_{(y, y')}^{(\mu, \mu')} : (x, x') \mapsto (U_1(x, y, 0) \otimes \bar{U}_1(x', y', 0)(H_0 \otimes H_0))(\mu, \mu') \quad ((\mu, \mu') \in \mathbb{Z} \oplus \mathbb{Z})$$

of the Gaussian mode $H_0 \otimes H_0$ in $L^2(\mathbb{R} \oplus \mathbb{R})$. The irreducibility of the linear Schrödinger representation U_1 of G combined with the Weil–Zak isomorphism w_1 implies:

THEOREM 9. *The approximating family of Gabor wavelets*

$$\left\{ G_{(y, y')}^{(\mu, \mu')} \mid (\mu, \mu') \in \mathbb{Z} \oplus \mathbb{Z}, \quad (y, y') \in \mathbb{R} \oplus \mathbb{R} \right\}$$

attached to the hololattice $\mathbb{Z} \oplus \mathbb{Z}$ inside the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ is total in the complex Hilbert space $L^2(\mathbb{R} \oplus \mathbb{R})$.

Notice that the Gabor wavelets form a non-orthogonal family in the Hilbert space $L^2(\mathbb{R} \oplus \mathbb{R})$. L^2 expansions in terms of Gabor wavelets offer high code compression rates appropriate for image processing purposes ([35]). Early stages of biological visual systems pay for keeping $m = n = 0$ by the non-orthogonality of the center-surround receptive field profiles. The family of Gabor wavelets give excellent fits in the chi-squared statistical sense to the correlating simple cell field profiles empirically studied in the cat striate cortex ([82], [83], [109]). The retina ([39]), an outpost of the central nervous system, and the lateral geniculate nucleus, however, act as decorrelators of the incoming signals. At the level of the mammalian visual cortex, the introduction of orientation selectivity through localized wave modulation combined with quadrature phase relations among paired cells results in a decorrelated neural representation with optimal image compression performance by the Hilbert basis of $L^2(\mathbb{R} \oplus \mathbb{R})$ of elementary holograms $(H_1(H_m, H_n; \cdot, \cdot))_{m \geq 0, n \geq 0}$. Signal preprocessing and processing in the auditory parts of the cortex follow similar basic lines.

I saw my first hologram at the Ontario Science Centre in Canada, and have been obsessed with holography ever since.

Sunny Bains (1987)

The resolution can indeed be very good, since the effective aperture for the system is not the aperture of the object-bearing system but is instead the aperture of the other branch of the interferometer.

Emmett N. Leith (1986)

14. Optical Display Holograms and Superresolution Imaging. Optical phase conjugation by degenerate four-wavelet-mixing requires a coherent light source. Due to the degrading effects of coherent artifact noise which results from the quantum stationary interference between the light scattered by imperfections in the optical path and the unscattered light representing the optical signal, a considerable amount of research interests has been directed towards the use of partially coherent illumination.

In case of a spatially incoherent light source, the coherent holographic transform as defined above has to be modified. The canonical differential 2-form on G/C_G has to be replaced by

$$\omega_\nu = \nu dx \wedge dy \quad (\nu \neq 0)$$

inherited from the symplectic form $\omega_{\mathcal{O}_\nu}$ of the coadjoint orbit $\mathcal{O}_\nu$ in the real dual of the Heisenberg Lie algebra. For different center frequencies $\nu \neq 0$, $\nu' \neq 0$, the symplectic affine planes $(\mathcal{O}_\nu, \omega_\nu)$, $(\mathcal{O}_{\nu'}, \omega_{\nu'})$ are different. Therefore the associated irreducible unitary linear representations U_ν , $U_{\nu'}$ of G are non-isomorphic. Consequently the orthogonality relations

$$\langle H_\nu(\psi, \varphi; \cdot, \cdot) | H_{\nu'}(\psi', \varphi'; \cdot, \cdot) \rangle = 0 \quad (\nu \neq \nu')$$

hold on the symplectic plane $\mathbb{R} \oplus \mathbb{R}$ for $\psi, \varphi, \psi', \varphi'$ in $\mathcal{S}(\mathbb{R})$. Instead of the coherent holographic transform, the transform defined by quantum complex linear superposition of the mapping

$$\psi(t')dt' \otimes \varphi(t)dt \longmapsto H(\psi, \varphi; x, y).dx \wedge dy$$

has to be considered. The form coefficient

$$H(\psi, \varphi; x, y) = \int_{\mathbb{R}} \nu H_\nu(\psi, \varphi; x, y) d\nu$$

is performed by integrating over the spatial frequencies ν emitted by the light source.

Figure 10

As a first application of the preceding identity, the various kinds of optical display holograms should be mentioned. A combination of the discretized frequency scale (ν scale dual to the z coordinate axis) with the Bragg frequency selection law for the generated diffractive planar multilayers explains the volume holograms (à la Denisyuk), the rainbow holograms (à la Benton), the multiplex holograms (à la Cross), and the waveguide holograms (à la Caulfield); see Figure 10 for the cone of Bragg angles associated to the multilayered dual manifold of G . The holographic angle decoding procedure can be restated in the following form:

THEOREM 10. *The choice of the hologram plane as a Kirillov orbit $\nu \neq 0$ within the multilayered unitary dual manifold of the Heisenberg nilpotent Lie group G is performed by the Bragg frequency selection law.*

COROLLARY. *The coordinates of a page oriented holographic memory are given by the coordinates $(x, y) \in \mathbb{R} \oplus \mathbb{R}$ of the hologram plane and the reference beam angle to $dx \wedge dy$.*

Since the slit device in processing Benton holograms defines the axis directions of the coordinate system in the hologram plane $\mathbb{R} \oplus \mathbb{R}$, a movement of the illuminating white light source changes the rainbow colours of the optical display hologram.

One of the most common defects in optical display holograms is blurring of the image. It is important to appreciate the fundamental difference between optical holography and photography in this respect. A photograph as a two-dimensional recording of an image formed by a lens can be blurred from the start; the sharpness of the image is not affected by the light source used to illuminate it. The situation, however, is completely reversed for an optical hologram, which is a recording of a stationary quantum interference pattern, not an image. If the interference pattern is blurred at recording, only the brightness of the replay is affected, not the sharpness of the image. The sharpness of the image depends on the direction of the light wavelet packet amplitude densities diffracted by the hologram, which is determined by the spatial frequency of the recorded interference pattern, and also by the direction, size and wavelength of the readout source. In fact it is not possible to record

an optical hologram of a blurred object. If any optical hologram is illuminated with an ideal light source, i.e., a point source at the correct wavelength, angle and distance, then the image will be pin sharp, no matter how it was recorded.

The conclusion that if an optical hologram is read out with a point monochromatic source, then the image would not be blurred at all, is only true if the observer does not look beyond the resolution of the human eye. The eye has only a very small aperture and intercepts only a very small cone of rays from the hologram at any time. If the image in an optical hologram is inspected with larger aperture optics, or if a real image is projected on a screen, the observer will start to see blurring due to the geometric distortion of the light wavelets emanating from each image point. This arises if the replay wavelength is different from the recording wavelength which for a white-light reflection hologram means any change in layer thickness or refractive index, or if the replay angle is not exactly equal to the recording angle, or if a real-image hologram is replayed with an inexact conjugate beam.

Another application of the preceding identity is the superresolution imaging technique ([28], [29], [95], [96], [97], [98]). The superresolution effect is achieved by incoherent to coherent conversion. In this procedure the aperture of the imaging system is reduced and the aperture reduction is compensated by reducing the spatial coherence of the light source. In view of the quantum parallelism, a reduced aperture like a pinhole spatial filter can be inserted also in the reference wavelet channel of the interferometer without limiting the effective resolution of the two-parallel-channel optical imaging system. In the readout step, the stationary quantum interference pattern generated in the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ can be decoded as an optical hologram by using an expanded laser beam.

Neural network models offer a data-driven unsupervised computational approach which is complementary to the algorithm-driven approaches of traditional information processing and artificial intelligence. The fine granularity, massive interconnectivity, and high degree of parallelism set neural network models apart from traditional electronic serial computing. These same features are the hallmarks of optical computing architectures which have led many workers to consider optical implementations of neural network models.

Bernard H. Soffer (1988)

The resonator memory and novelty filter must be considered as prototypes, not merely because they are rather primitive by neural network model standards but also because their relationship to any existing neural

model has yet to be properly established; in several ways, the relationship is a distant one, at best. Many of the features of these devices are nevertheless strikingly reminiscent of neural models. In the resonator memory, for example, it is appropriate to use the term “competition” as it is used in some neural models.

Dana Z. Anderson and Marie C. Erie (1987)

15. The Soffer Optical Resonator. In order to identify explicitly the terms of the Parseval–Plancherel type pixel identity indicated above, we denote by $K_{m,n}$ the complete bichromatic graph of $m + n$ vertices. Define $c(K_{m,n}, 0) := 1$ and let $c(K_{m,n}, l)$ denote the number of choices of $l \geq 1$ disjoint edges in $K_{m,n}$ each linking two vertices of different colours. Then

$$\phi_{m,n}(X) := \sum_{0 \leq l \leq [(m+n)/2]} (-1)^l c(K_{m,n}, l) X^{m+n-2l}$$

denotes the matching polynomial ([45], [58], [75], [159]) of variable X associated to the bipartite graph $K_{m,n}$. For any number $w \in \mathbb{C}$ the radial evaluation of $\phi_{m,n}(X)$ at w is defined by the rule

$$\phi_{m,n}(w) := \sum_{0 \leq l \leq [(m+n)/2]} (-1)^l c(K_{m,n}, l) w^{m+n} (w\bar{w})^{-l}.$$

THEOREM 11. *The coefficients of the matching polynomial $\phi_{m,n}(X)$ are the elementary synaptic strengths $(-1)^l c(K_{m,n}, l)$, $0 \leq l \leq [(m+n)/2]$, where the matching coefficients $c(K_{m,n}, l)$ denote the number of disjoint synaptic interconnects of the local neural network $K_{m,n}$ ($m \geq n \geq 0$) activated by l coherently firing neurons.*

Example 3. In the case $m = n = 3$ the matching coefficients

$$\begin{aligned} c(K_{3,3}, 0) &= 1 \\ c(K_{3,3}, 1) &= 9 \\ c(K_{3,3}, 2) &= 18 \\ c(K_{3,3}, 3) &= 6 \end{aligned}$$

arise. Thus the matching polynomial of the Thomsen graph $K_{3,3}$ is given by

$$\phi_{3,3}(X) = X^6 - 9 X^4 + 18 X^2 - 6.$$

Notice that the local network $K_{3,3}$ forms a non-planar graph.

In terms of Laguerre polynomials of order $m - n \geq 0$ and degree $n \geq 0$, it follows explicitly ([75], [159])

$$\phi_{m,n}(X) = (-1)^n n! X^{m-n} L_n^{m-n}(X^2) \quad (m \geq n \geq 0).$$

By radial evaluation of the matching polynomials $\phi_{m,n}(X)$ defined above, the next theorem describes the relationship between the elementary holograms and the matching polynomials attenuated by the Gaussian $(H_0 \otimes H_0) \in L^2(\mathbb{R} \oplus \mathbb{R})$ with distance: the farther away an input is from a point in the neural network, the less synaptic strength it is given.

THEOREM 12. *Let $m \geq n \geq 0$. Then the elementary holograms admit the form of radially evaluated quasipolynomials*

$$H_1(H_m, H_n; x, y) = \frac{(-1)^n}{\sqrt{m!n!}} e^{-\pi(x^2+y^2)/2} \phi_{m,n}(\sqrt{\pi}(x + iy))$$

for all pairs $(x, y) \in \mathbb{R} \oplus \mathbb{R}$.

Example 4. According to the Corollary of Theorem 5 supra, scaled versions of the elementary hologram $H_1(H_0; \cdot, \cdot) = H_0 \otimes H_0$ can be implemented as diffraction HOEs (cf. Remark 5 supra) for the fundamental transverse mode of a coherent laser light beam. This implementation is performed with a CAD station by projecting layers of constant optical thickness of the rotationally symmetric Gaussian diffraction profile onto the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$. In contrast to the Advanced Systems Analysis Package (ASAP) software procedure, however, the diffraction CGH is based on a quantum holographic description of the diffraction profile and therefore adapted to the purposes of amacronic sensor technology.

COROLLARY. *By quantum complex linear superposition, the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ can be realized as a neural plane of local neural networks.*

In particular the quantum holographic approach to neural networks yields the following result:

THEOREM 13. *The intrinsic holographic interconnection patterns are determined by the representations of linear Schrödinger type whereas the extrinsic holographic interconnects are determined by the representations of linear Fraunhofer type of the Heisenberg nilpotent Lie group G .*

Simulations of the synaptic strength patterns by conventional large-scale digital computers show that the self-organization of excited neural networks results in a cluster pattern of the neurons ([131]); see Figure 11. The American artist Jackson Pollock has been motivated by excited neural networks to create his drip paintings (cf. Figure 12). Notice that the holographic theory of associative memory also leads to according prime of place to the neuroglial cells rather than modifiable synaptic strengths in planar configurations of neurons ([122]).

Figure 11

Jackson Pollock: "Autumn Rhythm", 1950. Metropolitan Museum of Art, New York.

Figure 12

Presently one of the most successful implementations of the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ as a neural plane is the Soffer optical resonator built at the Hughes Research Laboratories ([125], [126], [127], [128], [168], [169]). The optical neurocomputer is formed by a coherent optical resonator cavity consisting of an optical hologram placed between two degenerate four-wavelet-mixing wavefront conjugate mirrors (PCMs). One of the wavefront conjugate mirrors is sesquilinear ([28], [29], [47], [48]) while the other one amplifies higher amplitude density signals more than lower amplitude density signals (see Theorem 5 *supra*). The optical hologram has multiple “example” images stored in it.

The neurocomputer is configured so that each example image is holographically encoded using a reference laser beam that impinges on the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ at a slightly different angle than the reference beams utilized for the other example patterns. After the neural system has been prepared, one can enter any image into the cavity by impinging it onto the optical hologram. The net result is that the holographically encoded image causes partial reconstruction of the reference beams. The complex-valued wavelet packet amplitude density of each reconstructed reference beam is proportional to the L^2 distance between the entered image and the example image associated with the reference. As the reference beams reverberate through the cavity the strongest (highest complex-valued wavelet packet amplitude density in the L^2 sense) one is incrementally amplified and the others are incrementally attenuated so that before long only the reference beam corresponding to the best matching example is left. In other terms, the stored image with the smallest distance to the input pattern survives in the mode competition at the expense of the more distant images. At the output port, i.e., the reconstructed real image port of the optical hologram, the best L^2 fitting example pattern then appears. Thus the optical neurocomputer functions as a nearest neighbour classifier for holographic imagery by recalling through a competitive memory.

The Soffer optical resonator can be viewed as an infinite-dimensional version of the Hopfield network. Or alternatively, if one envisions the optical elements of the neural system as consisting of small discrete optical units, then the Soffer optical resonator can be thought of as simply a large Hopfield network.

The second generation of Soffer optical resonators is based on a self-pumped wavefront conjugate mirror (SP-PCM) in conjunction with a SLM, CCD detector, frame grabber, and host computer ([169]). Similar optical neurocomputers have also been recently built at the Department of Electrical Engineering of Caltech ([142], [143], [144], [145]) and the Joint Institute for Laboratory Astrophysics (JILA) of the University of Colorado ([2], [3], [4], [5], [6]). These neural systems have also successfully demonstrated recording multiple patterns and functioning as a nearest neighbor associative memory. The day-dreaming phenomenon observed in a ring resonator memory reveals the quantum fluctuation as a consequence of the Heisenberg Uncertainty Principle (cf. Section 5 *supra*).

Mathematical models of neural networks are having a profound influence on current research in optical computing. This

trend toward neural computing is motivated by the sophisticated control and information processing that occurs in biological systems. The basic model for a neural network is a large number of simple processing units (i.e., neurons) interacting with one another through weighted interconnections (i.e., synapses). A neural network is the finest grain parallel computer possible where information and program are stored in the weighted interconnections and the processors perform simple thresholding logic. It is this highly parallel nature that gives neural networks their computational power and makes them attractive for optical implementation.

H. John Caulfield (1989)

My interest is, to paraphrase a famous statement, not what mathematics can do for physics, but what physics can do for mathematics. That is my underlying motive.

Stanislaw M. Ulam (1986)

16. Artificial Neural Network Identities. Theorem 12 supra implies the shift register identity ($m \geq n \geq 0$)

$$H_1(H_m, H_n; \mu, \mu') = \langle \delta_1(\mu, \mu', 0) w_1(H_m) | w_1(H_n) \rangle = \frac{(-1)^n}{\sqrt{m!n!}} e^{-\pi(\mu^2 + \mu'^2)/2} \phi_{m,n}(\sqrt{\pi}(\mu + i\mu'))$$

for all points (μ, μ') of the quadratic hololattice $\mathbb{Z} \oplus \mathbb{Z}$. In particular, the following result obtains:

THEOREM 14. For $m \geq n \geq 0$ the identity

$$\sum_{(\mu, \mu') \in \mathbb{Z} \oplus \mathbb{Z}} (-1)^{m+n} e^{-\pi(\mu^2 + \mu'^2)} \phi_{m,m}(\sqrt{\pi}(\mu + i\mu')) \phi_{n,n}(\sqrt{\pi}(\mu + i\mu')) = \sum_{(\mu, \mu') \in \mathbb{Z} \oplus \mathbb{Z}} e^{-\pi(\mu^2 + \mu'^2)} |\phi_{m,n}(\sqrt{\pi}(\mu + i\mu'))|^2$$

holds for the quadratic hololattice $\mathbb{Z} \oplus i\mathbb{Z}$ of Gaussian integers inside the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$.

The preceding theorem gives rise to the following special identities for the odd powers of π in terms of theta-null values $\vartheta(0, 1) = \Sigma e^{-\pi\mu^2}$ ([151], [152], [159]) where

$$\Sigma := \sum_{\mu \in \mathbb{Z}}:$$

$$m = 1, \quad n = 0$$

$$\pi = \frac{\Sigma e^{-\pi \mu^2}}{4 \Sigma \mu^2 e^{-\pi \mu^2}}$$

$$m = 2, \quad n = 1$$

$$\pi^3 = \frac{15 \Sigma (8\pi^2 \mu^4 - 1) e^{-\pi \mu^2}}{32 \Sigma \mu^6 e^{-\pi \mu^2}}$$

$$m = 3, \quad n = 2$$

$$\pi^5 = \frac{45 \Sigma (16\pi^4 \mu^8 - 140\pi^2 \mu^4 + 21) e^{-\pi \mu^2}}{64 \Sigma \mu^{10} e^{-\pi \mu^2}}$$

$$m = 4, \quad n = 3$$

$$\pi^7 = \frac{91 \Sigma (256\pi^6 \mu^{12} - 15840\pi^4 \mu^8 + 166320\pi^2 \mu^4 - 25245) e^{-\pi \mu^2}}{1024 \Sigma \mu^{14} e^{-\pi \mu^2}}$$

Theorem 14 supra shows that the preceding identities for the theta-null values $\vartheta(0, 1)$ are of a combinatorial character.

REMARK 14. The cardinal sine mother wavelet sinc mentioned in Example 1 supra, i.e., the univariate impulse response of the ideal lowpass filter, admits the Euler factorization

$$\text{sinc } x = \prod_{n \geq 1} \left(1 - \frac{x^2}{n^2} \right) \quad (x \in \mathbb{R}).$$

Its logarithmic derivative yields the identity

$$\pi x \cotan \pi x = 1 - 2x^2 \sum_{n \geq 1} \left(\frac{1}{n^2} + \frac{x^2}{n^4} + \frac{x^4}{n^6} + \cdots \right).$$

A comparison with the generating function of the Bernoulli polynomials $B_n(X)$ of degree $n \geq 0$

$$\frac{we^{wX}}{e^w - 1} = \sum_{n \geq 0} \frac{1}{n!} B_n(X) w^n \quad (w \in \mathbb{C})$$

evaluated at $X = 0$ and $w = 2\pi i x$ yields the classical Euler formulae for the even powers of π :

$$\pi^{2n} = (-1)^{n+1} \frac{2(2n)!}{2^{2n} B_{2n}} \zeta(2n) \quad (n \geq 1),$$

where ζ denotes the Riemann zeta-function and $B_{2n} = B_{2n}(0)$ are the Bernoulli numbers. In particular we get the special cases:

$$\begin{array}{ll} n = 1 & \pi^2 = 6\zeta(2) \\ n = 2 & \pi^4 = 90\zeta(4) \\ n = 3 & \pi^6 = 945\zeta(6) \\ n = 4 & \pi^8 = 9450\zeta(8) \end{array}$$

The first identity belongs to the nicest formulae established by Leonhard Euler. It has been explicitly reproduced in the Encyclopedia Britannica (1963).

Hardly a week goes by without an article appearing on the front page of a national magazine or journal trumpeting yet another breakthrough in optical computing.

Lauren P. Silvernail (1990)

Our ability to realize simple neural functions is strictly limited by our understanding of their organizing principles, and not by difficulties in implementation.

Carver A. Mead (1989)

17. Synopsis. The computation of real world phenomena in real time requires computational power that exceeds by many orders of magnitude the capabilities of sequential digital computers presently available. Although the data transfer rate of biological neural networks is merely a few kilocycles, hence not very fast, biological wetware is able to solve tasks such as real-time pattern recognition or sound localization because it operates in analog mode which allows simultaneous summing of many inputs from interconnected units and permits massively parallel data processing without the need for iterative procedures. Extrapolation from simulations of simple neural circuits indicate that a sequential digital computer would have to operate at speeds of more than 10^{18} floating point operations per second in order to match the performance limit of the human brain. The implementation of artificial neural network models based on coherent optical processors and analog electronic circuits of neurons and synapses is currently being pursued in a number of laboratories where several special purpose neurocomputer systems have been fabricated in holographic, optoelectronic, or CMOS VLSI electronic components. In the quantized theory of the electromagnetic field the bosons present in a coherent light beam travelling in a well-defined direction are the optical photons. The Kirillov quantization approach to the theory of the sesquilinear holographic transform $\psi(t')dt' \otimes \varphi(t)dt \mapsto H_1(\psi, \varphi; x, y).dx \wedge dy$ as outlined in this paper implies a link between elementary holograms and artificial neural networks. It allows to rigorously

establish the quantum parallelism as a EPR-type phenomenon (Theorem 7 *supra*) and to recognize the symplectic hologram plane $\mathbb{R} \oplus \mathbb{R}$ as a neural plane (Corollary of Theorem 12 *supra*). It is the quantum theoretical base of the holographic transform $\psi(t')dt' \otimes \varphi(t)dt \mapsto H_1(\psi, \varphi; x, y).dx \wedge dy$ which allows to model three-dimensional planar optics ([79], [87]) by the unitary dual of the Heisenberg nilpotent Lie group G and to establish the universal validity of the quantum holographic concept from amacronic sensor technology to classical SAR image processing.

Wer spricht von Siegen. Überstehen ist alles.

Rainer Maria Rilke (1875–1926)

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